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RESEARCH ARTICLE

Optimal vaccination in a SIRS epidemic model

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Abstract

We propose and solve an optimal vaccination problem within a deterministic compartmental model of SIRS type: the immunized population can become susceptible again, e.g. because of a not complete immunization power of the vaccine. A social planner thus aims at reducing the number of susceptible individuals via a vaccination campaign, while minimizing the social and economic costs related to the infectious disease. As a theoretical contribution, we provide a technical non-smooth verification theorem, guaranteeing that a semiconcave viscosity solution to the Hamilton–Jacobi–Bellman equation identifies with the minimal cost function, provided that the closed-loop equation admits a solution. Conditions under which the closed-loop equation is well-posed are then derived by borrowing results from the theory of *Regular Lagrangian Flows*. From the applied point of view, we provide a numerical implementation of the model in a case study with quadratic instantaneous costs. Amongst other conclusions, we observe that in the long-run the optimal vaccination policy is able to keep the percentage of infected to zero, at least when the natural reproduction number and the reinfection rate are small.

Keywords SIRS model · Optimal control · Viscosity solution · Non-smooth verification theorem · Epidemic · Optimal vaccination

Mathematics Subject Classification 93C15 · 49K15 · 49L25 · 92D30

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1 Introduction

During the recent Covid-19 pandemic, policymakers have been dealing in a first period with the implementation of severe lockdowns, while, in a second phase, with the massive vaccination policy of the susceptible population. Simultaneously, the scientific community experienced a renewed interest for epidemic mathematical models where an agent—typically a social planner—aims at taming the spread of a disease by designing lockdown policies and/or vaccination strategies that minimize social and economic costs. The starting point of the majority of this literature is the classical SIR model (cf. Kermack and McKendrick 1927), where, at each point in time, each person in a population of N individuals is either Susceptible, Infectious, or Recovered from a disease and can dynamically change her/his status according to a deterministic law of motions.

Modeling a social planner's actions usually results into the introduction of control variables in the dynamics of the considered compartmental epidemic model. For example, the transmission rate becomes a control variable in the generalized SIR models considered in Hritonenko and Yatsenko (2022), Kruse and Strack (2020) and Miclo et al. (2020), and a controlled state-variable in the stochastic version of Federico and Ferrari (2021). More in detail, Kruse and Strack (2020) introduces an additional parameter in the dynamics, which is then used by the social planner in order to control the rate at which the disease is transmitted; that parameter is meant to capture the policymakers' measures, such as social distancing, but also lockdown of businesses, schools, universities and other institutions. Analogously, in Miclo et al. (2020) the social planner controls both the instantaneous rate of pairwise meetings between susceptible and infected and the instantaneous probability of contagion, requiring that the percentage of infected individuals does not exceed the ICU constraint, that is the capacity of the health-care system to treat infected patients. Inspired by the previous papers and other different deterministic models, Federico and Ferrari (2021) proposes a stochastic control-theoretic version of the classical SIR model which, considering the transmission rate as a diffusive stochastic state variable, incorporates random fluctuations in the disease's transmission rate. Time-dependent lockdown policies directly affect the dynamics of the models studied in Acemoglu et al. (2020), Alvarez et al. (2020) and Calvia et al. (2022). In particular, Acemoglu et al. (2020) develops a multi-group version of the SIR model and highlights the significant benefits obtained through the optimal targeted lockdown policies; Alvarez et al. (2020) analyzes the optimal lockdown policy in the controlled SIRD (Susceptible-Infected-Recovered-Dead) model, a modified version of SIR; finally, Calvia et al. (2022) provides a first step in the complete theoretical analysis of the dynamic programming approach to a class of controlled compartmental models (such as SIR, SIRD, and SEIR). Among the confinement policies, social distancing also relying on private decisions is considered in Makris (2021), which extends the classical SIR model to incorporate heterogeneity in infection-induced mortality rates of the population. With respect to the literature on optimal confinement policies, that on optimal vaccination has a longer history. The

vaccination strategy that minimizes the social costs arising from the spread of a disease evolving according to the SIR model is studied in Hethcote and Waltman (1973). This contribution was communicated for publication on “Mathematical Biosciences” by Richard Bellman and it perhaps represents one of the earliest applications of the dynamic programming techniques in epidemiology. A related optimization problem is considered in Barrett and Hoel (2007), but in the context of a SIS (Susceptible-Infected-Susceptible) model. The comparison between compulsory vaccination and market allocation is studied in Brito et al. (1991), whereas Goenka and Liu (2012) analyzes the effect of infectious diseases on economic variables and explains how the traditional methods of vaccination and isolation can stabilize the economic fluctuations. The seminal work Geoffard and Philipson (1997) starts observing that “of the roughly 40 vaccines on the market, only the smallpox vaccine has been successful in eradication” by 1997. The study in Geoffard and Philipson (1997) thus aims at understanding which forces prevent the eradication through vaccines of a disease, and the authors conclude that vaccinations yield a drop of infected individuals, which in turn leads to a drop in the demand for vaccines, which finally implies the return of the infectious disease. Disease’s eradication is also treated in Loertscher and Muir (2021) where it is stated that “if eradication is impossible or possible only at tremendous costs, keeping the pandemic under control [...] requires finding a path through territory that is uncharted". The authors propose a way to manage an epidemic providing an optimal lockdown policy which takes into account the maximum capacity of the healthcare system, whose level must not be exceeded. Further, the awareness of a possible future vaccine employment does not substantially impact the optimal policy dynamics until such an event actually occurs. In Ishikawa (2012) the “Vaccinated" (V) compartment is included in the stochastic SIRV model. Later on, Gatto and Schellhorn (2021) proposes a simplified version of SIRV aiming at studying the effect of implementing treatments of uncertain efficacy to control an epidemic; in this case, rather than focusing on the numerical optimal solution of the problem, the authors develop tractable solutions, either analytical or perturbative. In Garriga et al. (2022), the employed epidemiological model is a generalized SIRS models considering an additional V compartment. In this paper, the vaccine arrival, which is considered as a random event with exogenous probability distribution, splits up the epidemic time into two periods, namely Phase I and Phase II. During Phase I the only available policy is a stylized version of a “stay-at-home", whereas Phase II is characterized by the possibility to control the speed at which the population can be vaccinated. In Angeli et al. (2022) both the “Asymptomatic" (A) and the “Vaccinated" compartments are incorporated into the SAIVR model, in which several parameters and initial conditions are set through machine learning techniques; the resulting epidemic’s evolution, obtained for different values of roll-out daily rates and vaccine efficacy, is then deeply analyzed. In the recent Glover et al. (2022) it is studied how to best allocate a given time-varying supply of vaccines across individuals of different ages, and discuss the possible sub-optimality (in terms of economic recovery) of the actual deployment path that prioritized older retired individuals to younger working people. A similar problem of optimal allocation of limited vaccines is considered in Rao and Brandeau (2021) as well.

In this paper we propose and study the problem of optimal vaccination against an infectious disease that evolves according to a generalized SIRS model. Differently to the classical SIR setting, where the compartment of recovered individuals is an absorbing state, in the SIRS model immunized people can become susceptible again at a given rate $\eta > 0$ (due, e.g., to a not complete immunization power of vaccines). We consider a social planner that aims at determining a vaccination strategy which reduces the number of susceptible individuals, while minimizing total social and economic costs. These are due, e.g., to the arrangement of vaccination hubs and to the employment of medical staff.

It is well known that determining an explicit solution to control problems arising in epidemiology is extremely hard, if possible at all. As a matter of fact, those dynamic optimization problems are typically multidimensional and the dynamics of the controlled state-variables are nonlinear. Although we are not able to determine the expression of the optimal vaccination policy in closed form, in this paper we provide a thorough analysis of the optimal vaccination problem, which actually leads to the identification of easily verifiable sufficient conditions for the optimality of a candidate solution and to a numerical implementation in a relevant case study. This is accomplished as we explain in the following. First of all, we show that the assumed semiconcavity property of the instantaneous cost function is inherited by the problem's minimal cost function V, so that the latter is shown to be a semiconcave viscosity solution to the related Hamilton–Jacobi–Bellman (HJB) equation. A delicate technical analysis then allows to prove a verification theorem for non-smooth (viscosity) solutions to the HJB equation (see Theorem 4, our main theoretical result). It is worth noticing that the proof of a verification theorem in the context of viscosity solutions is far to be trivial, and it is typically a remarkable fact. Our proof is inspired by that of Theorem 3.9 in Yong and Zhou (1999). However, in order to achieve the result, the arguments therein needed to be thoroughly adapted and expanded to the present setting, by properly exploiting the semiconcavity of V and argument of convex analysis, namely the properties of supergradient (cf. Proposition 4 below). Since the verification theorem assumes that a solution to the so-called closed-loop equation exists, to proceed further in the analysis we provide sufficient conditions for the well-posedness of the latter (cf. Proposition 6), by suitably employing the theory of Regular Lagrangian Flows (cf. Ambrosio (2004, 2017)). To the best of our knowledge, this is the first paper that combines the theory of Regular Lagrangian Flows with the study an optimal control problem. As a corollary of the verification theorem and of the existence of a solution to the closed-loop equation, we then obtain that the minimal cost function V is indeed the unique semiconcave viscosity solution to the HJB equation.

The latter uniqueness result paves the way for a numerical study of the optimal vaccination problem. Indeed, as a complement to our theoretical analysis, we also provide a numerical implementation which is based on a recursion of the HJB equation, initialized by the null function. For the numerical exercise we assume a specification of the model with quadratic instantaneous costs, under which the conditions of the verification theorem are satisfied. Fixing the values of the model's parameters, we study the evolution of the optimal vaccination policy, the evolution of the instantaneous reproduction number, as well as the dynamics of the (optimally controlled) percentages of susceptible, infected, and immunized (recovered) individuals. A numerical result

suggests that, following the optimal vaccination plan, the social planner is able in the long-run to keep the number of infected individuals equal to zero. In particular, this happens when the reinfection parameter or the natural reproduction number are sufficiently small. However, since the model prescribes reinfection at rate $\eta > 0$, the disease cannot be eradicated in the strict sense, and the vaccination campaign cannot be terminated if the aim is to maintain zero infections. Hence, if the social planner wishes to stop vaccinating the population, other forms of control must be adapted, such as isolation. We also observe that the social planner is allowed to relax the vaccination policy only after a first time period where the maximal possible vaccination effort is made. As expected, the length of such an initial phase increases when the number of initial infected people increases.

The rest of this paper is organized as follows. Section 2 presents the model and the problem's formulation. Section 3 provides the theoretical analysis and the solution to the optimal vaccination problem, while Sect. 4 discusses the numerical results. Finally, conclusions are presented in Sect. 5.

2 Problem formulation

2.1 The generalized SIRS model

We model the spread of the infection by relying on a variation of the classical SIR model that dates back to the work by Kermack and McKendrick Kermack and McKendrick (1927). The society has population N and it consists of three different groups. The first group is formed by those people who are healthy, but susceptible to the disease; the second group contains those who are infected, while the last cohort consists of those who are immunized, that is recovered, dead or vaccinated. However, differently to the classical SIR model, we assume that, once immunized, an individual can become susceptible again, so that we face a SIRS epidemic model. We denote by $S(t)$ the fraction (within a society of N individuals) of individuals who are susceptible at time $t \geq 0$, by $I(t)$ the fraction of infected, and by $R(t)$ the fraction of immunized (which, in the sequel, we will also call recovered). Clearly, $S(t) + I(t) + R(t) = 1$ for all $t \geq 0$.

We briefly review the classical SIRS model and then introduce vaccination policies within it. At time $t \geq 0$, it is assumed that the fraction of infected people $I(t)$ grows at a rate which is proportional to the fraction $S(t)$ of people who are susceptible to the disease. In particular, letting β be the instantaneous transmission rate of the disease, during an infinitesimal interval of time $[t, t + \mathrm{d}t]$, each infected individual generates a fraction $\beta S(t)\mathrm{d}t$ of new infected individuals. It thus follows that the fraction of susceptible individuals that get infected within the interval of time $[t, t + \mathrm{d}t]$ is $\beta I(t)S(t)\mathrm{d}t$. On the other hand, in the same interval $[t, t + \mathrm{d}t]$, the fraction of infected $I(t)$ is reduced at a rate $\gamma > 0$, the rate of recovering from the disease. Hence, it follows the dynamics

$$I'(t) = \beta S(t)I(t) - \gamma I(t).$$

Furthermore, in the infinitesimal interval of time $[t, t+\mathrm{d}t]$, the fraction of susceptible people $S(t)$ naturally decreases at the rate $\beta S(t)I(t)$, because of those people changing their status from being susceptible to being infected, as previously discussed. However, the fraction of susceptible individuals also gains mass, at a rate $\eta > 0$, from the fraction of recovered people $R(t)$ who becomes again susceptible to be re-infected. It thus follows that

$$S'(t) = -\beta S(t)I(t) + \eta R(t).$$

Finally, the fraction of recovered people $R(t)$ increases due to those infected individuals who recover, and decreases because of the transition from being recovered to being again susceptible to the disease. According to the discussion above, we therefore have the dynamics

$$R'(t) = \gamma I(t) - \eta R(t).$$

So far, there is no control in the system, which evolves autonomously according to the differential equations previously derived. We now introduce the possibility that a social planner intervenes on the system through a vaccination policy, described by a function of time $t \mapsto u(t)$, in the sense that we formalize below. Let $U := [0, \overline{U}]$, for some $\overline{U} > 0$, and assume that the vaccination policy $u(\cdot)$ belongs to the set

$$\mathcal{U} := \left\{ u : \mathbb{R}_+ \to U \text{ measurable} \right\}. \tag{1}$$

The quantity $u(t)$ here represents the rate of vaccination of susceptible people at time t; precisely, $u(t)S(t)\mathrm{d}t$ is the fraction of susceptible individuals that, due to vaccination, moves in the interval $[t, t+\mathrm{d}t]$ from the class S to the class R. All in all, the controlled dynamics then become

$$S'(t) = -\beta S(t)I(t) - u(t)S(t) + \eta R(t), \quad t > 0, \qquad S(0) = s, \tag{2}$$

$$I'(t) = \beta S(t)I(t) - \gamma I(t), \quad t > 0, \qquad I(0) = i, \tag{3}$$

and

$$R'(t) = \gamma I(t) - \eta R(t) + u(t)S(t), \quad t > 0, \qquad R(0) = r, \tag{4}$$

for given nonnegative s, i and r such that $s + i + r = 1$. Given that $u(\cdot) \in \mathcal{U}$, the previous system of ODEs is well posed in the Carathéodory sense, that is, there exists a unique triple of absolutely continuous functions S, I, R such that (2), (3), (4) are satisfied a.e.[1] Notice also that for any $t \geq 0$, and for any choice of $u(\cdot) \in \mathcal{U}$, summing

[1] The abbreviation 'a.e.' means *almost everywhere*, which is used to state that a property holds for every element of a set, except for a subset of null (Lebesgue) measure.

up the dynamics of S, I and R we have $(S' + I' + R')(t) = 0$ for all $t > 0$, which then implies that $S(t) + I(t) + R(t) = 1$ for all $t \geq 0$ as $s + i + r = 1$. Given this fact, it is then sufficient to consider only the dynamics of (S, I), being $R(t) = 1 - S(t) - I(t)$. Hence, we obtain

$$S'(t) = -\beta S(t)I(t) - u(t)S(t) + \eta(1 - S(t) - I(t)), \quad t > 0, \quad S(0) = s, \tag{5}$$

and

$$I'(t) = \beta S(t)I(t) - \gamma I(t), \quad t > 0, \qquad I(0) = i. \tag{6}$$

From (6), one has for any $t \geq 0$

$$I(t) = i \exp\left\{ \int_0^t (\beta S(q) - \gamma) \mathrm{d}q \right\},$$

so that $I(t) > 0$ for any $t \geq 0$ and for any $u(\cdot) \in \mathcal{U}$. Lemma 8 in the Appendix shows that also $S(t) > 0$ and $R(t) > 0$ for any $t \geq 0$, so that also $S(t) < 1$, $R(t) < 1$, $I(t) < 1$ for any $t \geq 0$. In the following, we assume that $(s, i) \in (0, 1)^2$ are such that $s + i \in (0, 1)$.[2]

2.2 The social planner problem

The epidemic generates social costs. These might arise because of lost gross domestic product (GDP) due to inability of working for infected, or because of an overwhelming of the national health-care system etc. Also, one can imagine that the more susceptible are, the larger is the probability of an additional wave of the epidemic and, therefore, of additional societal stress. The social planner thus employs a vaccination policy aiming at reducing the number of susceptible individuals. These actions, however, come with a cost, which increases with the amplitude of the effort. The cost is due, e.g., to the arrangement of vaccination hubs and to the employment of medical staff.

In order to tackle the social planner problem with techniques from dynamic programming, it is convenient to keep track of the initial values of $(S(\cdot), I(\cdot))$. We therefore let $x := (s, i)$ and set

$$\mathcal{M} := \left\{ x := (s, i) \in \mathbb{R}^2 : \ x \in (0, 1)^2, \ s + i < 1 \right\}. \tag{7}$$

The social planner aims at minimizing the cost functional

$$\int_0^\infty e^{-rt} C\big(S(t), I(t), u(t)\big) \mathrm{d}t. \tag{8}$$

[2] The choice of considering $s + i < 1$—i.e. of having an initial strictly positive percentage of immunized—is only done in order to deal with an open set in the subsequent mathematical formulation of the problem. As a matter of fact, such a condition is not restrictive from the technical point of view as our results still apply if $s + i < \ell$, for some $\ell > 1$, thus covering the case $s + i = 1$ as well.

Here, $r \geq 0$ measures the social planner's time preferences, and $C : (0,1)^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is a running cost function measuring the negative impact of the disease on the socio-economic state and on the public health, as well as the costs arising because of vaccination policies. The following requirements are satisfied by C. Below, and in the rest of this paper, with a slight abuse of notation, we indicate by $|\cdot|$ both the absolute value and the Euclidean norm in $\mathbb{R}^2$.

Assumption 1 (i) C is continuous and bounded.
(ii) $u \mapsto C(s,i,u)$ is strictly convex for any $(s,i) \in (0,1)^2$.
(iii) C is Lipschitz continuous with respect to (s,i), uniformly in u; that is, there exists $K > 0$ such that for any $u \in U$ we have that

$$|C(s,i,u) - C(s',i',u)| \leq K|(s,i) - (s',i')|, \quad \forall (s,i),(s',i') \in (0,1)^2.$$

(iv) $(s,i) \mapsto C(s,i,u)$ is semiconcave uniformly with respect to $u \in U$; that is, there exists $K > 0$ such that for any $u \in U$ and any $\mu \in [0,1]$ one has $\forall (s,i),(s',i') \in (0,1)^2$

$$\mu C(s,i,u) + (1-\mu)C(s',i',u) - C\big(\mu(s,i) + (1-\mu)(s,i'),u\big) \leq \\ \leq K\mu(1-\mu)|(s,i) - (s',i)'|^2.$$

Without loss of generality, we also take $\inf_{\mathcal{M}\times U} C = 0$. Convexity of $u \mapsto C(s,i,u)$ describes that marginal costs of vaccinations are increasing: E.g., an additional hiring of medical staff for vaccination might have a larger cost in a society that has already devoted many resources to the fight of the epidemics. Finally, the Lipschitz and semiconcavity property of $C(\cdot,\cdot,u)$ are technical requirements that will be important in the next section.

As we already observed, by Lemma 8, the controlled dynamics of $(S(\cdot), I(\cdot))$ evolves within the set $\mathcal{M}$. When needed, we stress the dependency of $(S(\cdot), I(\cdot))$ with respect to $x \in \mathcal{M}$ and $u(\cdot) \in \mathcal{U}$ by writing $(S^{x,u}(\cdot), I^{x,u}(\cdot))$. We shall also simply set $(S^x(\cdot), I^x(\cdot)) := (S^{x,0}(\cdot), I^{x,0}(\cdot))$ to denote the solutions to (5) and (6) when $u(\cdot) \equiv 0$.

Then, we introduce the problem's value function

$$V(x) := \inf_{u(\cdot)\in\mathcal{U}} \int_0^\infty e^{-rt} C\big(S^{x,u}(t), I^{x,u}(t), u(t)\big)\mathrm{d}t, \quad x \in \mathcal{M}. \tag{9}$$

Since C is nonnegative and bounded, we have the following.

Proposition 2 *V is well defined, nonnegative, and bounded.*

In the next section we will show that V is semiconcave, solves the corresponding dynamic programming equation in the viscosity sense, and we will also provide an optimal control in feedback form.

3 The solution to the social planner problem

Let $x = (s, i) \in \mathcal{M}$ and set

$$b(x, u) := (-\beta si - us + \eta(1 - s - i), \beta si - \gamma i), \quad x \in \mathcal{M}. \tag{10}$$

In light of the dynamic programming principle (see, e.g., Yong and Zhou (1999)), we expect that V should identify with a suitable solution to the Hamilton–Jacobi–Bellman (HJB) equation

$$rv(x) = \mathcal{H}(x, Dv(x)), \quad x = (s, i) \in \mathcal{M}, \tag{11}$$

where $Dv := (v_s, v_i)$ denotes the gradient of v (being v_s and v_i the partial derivatives in the s and i direction respectively) and

$$\mathcal{H}(x, p) = \inf_{u \in U} \mathcal{H}_{cv}(x, p; u), \quad x = (s, i) \in \mathcal{M}, \ p = (p_s, p_i) \in \mathbb{R}^2,$$

with

$$\mathcal{H}_{cv}(x, p; u) = \langle b(x, u), p \rangle + C(x, u), \quad x = (s, i) \in \mathcal{M}, \ p = (p_s, p_i) \in \mathbb{R}^2.$$

Here, and in the sequel, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^2$.

Defining the linear operator

$$(\mathcal{L}v)(x) := \beta si(v_i(s, i) - v_s(s, i) + \eta(1 - s - i)v_s(s, i) - \gamma i v_i(s, i), \tag{12}$$

with $x = (s, i) \in \mathcal{M}$, $v \in C^1(\mathcal{M})$, we can separate the linear part in the HJB equation (11) and write

$$(r - \mathcal{L})v(x) = C^\star(x, v_s(x)), \tag{13}$$

where

$$C^\star(s, i, p_s) = \inf_{u \in U} \{C(s, i, u) - usp_s\}, \quad p_s \in \mathbb{R}.$$

For future frequent use, given an open set $\mathcal{O} \subseteq \mathbb{R}^2$ and a function $f : \mathcal{O} \to \mathbb{R}$, we denote by $D^+ f$ (resp., $D^- f$) the supergradient (respectively, subgradient) of f; namely, for any $x \in \mathcal{O}$,

$$D^+ f(x) := \left\{ p \in \mathbb{R}^2 : \liminf_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}, \tag{14}$$

and

$$D^- f(x) := \left\{ p \in \mathbb{R}^2 : \limsup_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}. \tag{15}$$

Then, due to Assumption 1, we have the following preliminary result which is the stationary version of Theorems 7.4.11 and 7.4.14 in Cannarsa and Sinestrari (2014) (by taking $g \equiv 0$ therein).

Theorem 3 *V is semiconcave on $\mathcal{M}$ and solves the HJB equation (11) in the viscosity sense on $\mathcal{M}$; that is,*

$$\textit{(subsolution)} \quad rV(x) \leq \inf_{u \in U} \left\{ \langle b(x,u), p \rangle + C(x,u) \right\}, \quad \forall p := (p_s, p_i) \in D^+ V(x);$$

$$\textit{(supersolution)} \quad rV(x) \geq \inf_{u \in U} \left\{ \langle b(x,u), p \rangle + C(x,u) \right\}, \quad \forall p := (p_s, p_i) \in D^- V(x).$$

Let now

$$U^\star(s, i, p_s) := \operatorname{argmin}_{u \in U} \{ C(s,i,u) - u s p_s \}, \quad (s, i, p_s) \in \mathcal{M} \times \mathbb{R}, \tag{16}$$

which, due to the strict convexity of $C(x, \cdot)$ (cf. Assumption 1), exists unique and is continuous on $\mathcal{M} \times \mathbb{R}$ by the Berge's maximum theorem. In light of the semiconcavity of V and of the proof of the next verification theorem, let us now recall some well-known properties of semiconcave functions for the reader's convenience. Given $\mathcal{O} \subseteq \mathbb{R}^n$, $x \in \mathcal{O}$ and a semiconcave function $f : \mathcal{O} \to \mathbb{R}$, we denote by $D^* f(x)$ the set of *reachable gradients*. We say that a vector $p \in \mathbb{R}^2$ is a reachable gradient of f at x if there exists $\{x_n\}_{n \in \mathbb{N}}$ which converges to x and admits $Df(x_n)$ for each $n \in \mathbb{N}$ such that $p = \lim_{n \to \infty} Df(x_n)$; i.e.

$$D^* f(x) := \left\{ \lim_{n \to \infty} Df(x_n) : \ \exists Df(x_n) \text{ and } x_n \to x \right\}.$$

Also, given and fixed $\xi \in \mathbb{R}^n$, we set

$$f'^{\pm}_{\xi}(x) := \lim_{r \to 0^\pm} \frac{f(x + r\xi) - f(x)}{r}$$

which exists by semiconcavity of f. Further, the notation 'co' denotes the *convex hull*, that is the set of all convex combinations of points of the set.

Proposition 4 *Let $\mathcal{O} \subseteq \mathbb{R}^n$ and $f : \mathcal{O} \subseteq \mathbb{R}^n \to \mathbb{R}$ be semiconcave. Then the following hold true:*

(a) $D^+ f(x)$ is nonempty, closed and convex for each $x \in \mathcal{O}$.
(b) The multi-valued map $x \mapsto D^+ f(x)$ is locally bounded.
(c) f is a.e. differentiable over $\mathcal{O}$.
(d) Given and fixed $x \in \mathcal{O}$, we have

$$D^+ f(x) = co\, D^* f(x).$$

(e) Given and fixed $\xi \in \mathbb{R}^n$ we have

$$f_\xi^+(x) = \min_{p \in D^+ f(x)} \langle p, \xi \rangle, \quad f_\xi^-(x) = \max_{p \in D^+ f(x)} \langle p, \xi \rangle. \tag{17}$$

(f) $D^+ f(x)$ is compact and convex.

Proof We simply provide precise reference for the reader's convenience. Item (a) follows by Proposition 3.1.5-(b) and Proposition 3.3.4-(c) in Cannarsa and Sinestrari (2014); Item (b) is due to Theorem 2.1.7 in Cannarsa and Sinestrari (2014); Item (c) is then implied by Item (b) and Rademacher's Theorem; Items (d) and (e) follow by Theorem 3.3.6 in Cannarsa and Sinestrari (2014); Item (f) is due to Item (d) and the fact that the convex hull of a compact set—such as $D^* f(x)$, by definition—is compact (cf. Corollary A.1.7 in Cannarsa and Sinestrari (2014)). □

We are now ready to state and prove the main result of this section, namely a verification theorem for viscosity solutions to the HJB equation (11). Its proof is inspired by that of Theorem 3.9 in Yong and Zhou (1999), which is, however, thoroughly adapted and expanded to the present setting by suitably exploiting the semiconcavity of V, and thus the subsequent properties of its supergradient (cf. Proposition 4).

In the following, given a semiconcave function $v : \mathcal{M} \to \mathbb{R}$ we set[3]

$$\partial^\pm v_s := v^\pm_{(1,0)}, \qquad \partial^*_s v := \{\partial^+_s v, \partial^-_s v\}.$$

Theorem 5 [Non-smooth Verification Theorem] *Let $v : \mathcal{M} \to \mathbb{R}_+$ be semiconcave, bounded and nonnegative. Then the following hold true:*

1. *If v is a viscosity subsolution to the HJB equation, then $v \le V$ on $\mathcal{M}$.*
2. *Recall (16). Let $x := (s, i) \in \mathcal{M}$, let v be a viscosity supersolution to the HJB equation, and let $u^\star \in \mathcal{U}$ be such that, denoting by $(S^\star(\cdot), I^\star(\cdot))$ the state trajectory associated to $u^\star$, the following holds:*

$$u^\star(t) \in U^\star\left(S^\star(t), I^\star(t), \partial^*_s v(S^\star(t), I^\star(t))\right), \quad \text{for a.e. } t \ge 0. \tag{18}$$

Then $v(s, i) \ge J(s, i; u^\star)$.

Proof Recall b as in (10). Given an admissible control $u(\cdot)$ and the corresponding controlled trajectory $X(\cdot) = (S(\cdot), I(\cdot))$, for future use throughout this proof, define the set

$$\mathcal{T} = \mathcal{T}_{u(\cdot)} := \left\{ t > 0 : v(X(\cdot)) \text{ is differentiable and } \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} b(X(s), u(s)) \mathrm{d}s \right.$$
$$\left. = \lim_{h \to 0^+} \frac{1}{h} \int_{t}^{t+h} b(X(s), u(s)) \mathrm{d}s = b(X(t), u(t)) \right\}.$$

[3] Hereafter, the superscript $\pm$ means, as usual, either $+$ or $-$.

Notice that $\mathcal{T}$ has full measure. Indeed, $b(X(\cdot), u(\cdot)) \in L^1_{\text{loc}}(\mathbb{R}^+)$; furthermore, because of the Lipschitz property of $t \mapsto X(t)$ and the locally Lipschitz property of v, which follows by semiconcavity, we have that $t \mapsto v(X(t))$ is locally Lipschitz continuous, and, hence, it is differentiable a.e. This last assertion simply follows by applying Lebesgue's differentiation theorem, see [Theorem 2.19], to $v(X(\cdot))$.

Step 1. Let $\bar{u}(\cdot)$ be an admissible control and $\bar{X}(\cdot) := (\bar{S}(\cdot), \bar{I}(\cdot))$ be the associated state trajectory. In order to simplify notation, from now on, we set $\bar{b}(t) := b(\bar{X}(t), \bar{u}(t))$.

Let now $t \in \mathcal{T}$ and let $\bar{p}(t) \in D^+ v(\bar{X}(t))$. We then have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} e^{-rt} v(\bar{X}(t)) &= \lim_{h \to 0^+} \frac{-e^{-r(t-h)} v(\bar{X}(t-h)) + e^{-rt} v(\bar{X}(t))}{h} \\
&= e^{-rt} \lim_{h \to 0^+} \frac{e^{rh} v(\bar{X}(t)) - e^{rh} v(\bar{X}(t-h)) + v(\bar{X}(t)) - e^{rh} v(\bar{X}(t))}{h} \\
&= e^{-rt} \Bigg(\lim_{h \to 0^+} e^{rh} \frac{v(\bar{X}(t-h) + h\bar{b}(t) + o(h)) - v(\bar{X}(t-h))}{h} + \\
&\quad - r v(\bar{X}(t)) \Bigg) \geq e^{-rt} \left(\langle \bar{p}(t), \bar{b}(t) \rangle - r v(\bar{X}(t)) \right), \qquad (19)
\end{aligned}$$

where the last inequality follows by the fact that $\bar{p}(t) \in D^+ v(\bar{X}(t))$.

Since now v is a viscosity subsolution to the HJB equation, we find from (19)

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-rt} v(\bar{X}(t)) \geq -r v(\bar{X}(t)) + \langle \bar{p}(t), \bar{b}(t) \rangle \geq -C(\bar{X}(t), \bar{u}(t)). \qquad (20)$$

On the other hand, because of the Lipschitz property of $v(\bar{X}(\cdot))$ we can write, for $\bar{X}(0) = x = (s, i) \in \mathcal{M}$,

$$e^{-rT} v(\bar{X}(T)) - v(x) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} e^{-rt} v(\bar{X}(t)) \mathrm{d}t, \qquad (21)$$

and picking a measurable selection $t \mapsto \bar{p}(t) \in D^+ v(\bar{X}(t))$ (see, e.g., page 277 in Yong and Zhou (1999)), and using (20) in (21), we find

$$v(x) \leq e^{-rT} v(\bar{X}(T)) + \int_0^T e^{-rt} C(\bar{X}(t), \bar{u}(t)) \mathrm{d}t.$$

Since v and C are bounded, we safely take the limit as $T \to \infty$ obtaining

$$v(x) \leq \int_0^\infty e^{-rt} C(\bar{X}(t), \bar{u}(t)) \mathrm{d}t.$$

By the arbitrariness of $\bar{u}$ and x it follows that $v \leq V$ on $\mathcal{M}$, as claimed.

Step 2. For $x \in \mathcal{M}$ given and fixed, let

$$P^+(x) := \operatorname{argmin}_{p \in D^+v(x)} \langle p, (1,0) \rangle, \quad P^-(x) := \operatorname{argmax}_{p \in D^+v(x)} \langle p, (1,0) \rangle$$

and notice that

$$\langle p, (1,0) \rangle = \partial_s^+ v(x), \ \ \forall p \in P^+(x), \quad \langle p, (1,0) \rangle = \partial_s^- v(x), \ \ \forall p \in P^-(x).$$

Hence, since $P^{\pm}(x)$ is closed, bounded, and convex, it must have the structure

$$P^{\pm}(x) = \{\partial_s^{\pm} v(x)\} \times [\underline{p}_i^{\pm}(x), \bar{p}_i^{\pm}(x)],$$

for some $-\infty < \underline{p}_i^{\pm}(x) \leq \bar{p}_i^{\pm}(x) < \infty$. The point $(\partial_s^{\pm} v(x), \underline{p}_i^{\pm}(x))$ is thus an extremal point for the convex compact set $D^+v(x)$ (cf. Proposition 4-(e)), and by Proposition 4-(d) there exists a sequence $x_n^{\pm} \to x$ such that

$$\exists Dv(x_n^{\pm}) \to (\partial_s^{\pm} v(x), \underline{p}_i^{\pm}(x)) \in D^+v(x).$$

Since v is a viscosity supersolution we have

$$rv(x_n^{\pm}) \geq \inf_{u \in U} \left\{ \langle b(x_n^{\pm}, u), Dv(x_n^{\pm}) \rangle + C(x_n^{\pm}, u) \right\},$$

which, taking the limit as $n \to \infty$, yields

$$rv(x) \geq \inf_{u \in U} \{ \langle b(x,u), (\partial_s^{\pm} v(x), \underline{p}_i^{\pm}(x)) \rangle + C(x,u) \}. \tag{22}$$

Using the definition of $U^\star$ as in (16), (22) gives

$$rv(x) \geq \langle b(x, U^\star(x, \partial_s^{\pm} v(x)), (\partial_s^{\pm} v(x), \underline{p}_i^{\pm}(x)) \rangle + C(x, U^\star(x, \partial_s^{\pm} v(x))). \tag{23}$$

Now, let $u^\star(\cdot)$ and $X^\star(\cdot) := (S^\star(\cdot), I^\star(\cdot))$ as in the claim, and take $t \in \mathcal{T}^\star = \mathcal{T}_{u^\star(\cdot)}$. Also, let $b^\star(\cdot) := b(X^\star(\cdot), u^\star(\cdot))$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} e^{-rt} v(X^\star(t)) &= \lim_{h \to 0^+} \frac{e^{-r(t+h)} v(X^\star(t+h)) - e^{-rt} v(X^\star(t))}{h} \\
&= e^{-rt} \left(\lim_{h \to 0^+} e^{-rh} \frac{v(X^\star(t+h)) - v(X^\star(t))}{h} - rv(X^\star(t)) \right) \\
&= e^{-rt} \left(\lim_{h \to 0^+} e^{-rh} \frac{v(X^\star(t) + hb^\star(t) + o(h)) - v(X^\star(t))}{h} + \right. \\
&\quad \left. - rv(X^\star(t)) \right),
\end{aligned}$$

from which, using the definition of superdifferential (14) with $f = v$ and $x = X^\star(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-rt} v(X^\star(t)) \leq e^{-rt} \left(-r v(X^\star(t)) + \langle b^\star(t), (\partial_s^{\pm} v(X^\star(t)), \underline{p}_i(X^\star(t)))\rangle \right), \quad (24)$$

so that by (23)

$$e^{-rt} \left(-r v(X^\star(t)) + \langle b^\star(t), (\partial_s^{\pm} v(X^\star(t)), \underline{p}_i(X^\star(t)))\rangle \right) \leq -e^{-rt} C(X^\star(t), u^\star(t)).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-rt} v(X^\star(t)) \leq -e^{-rt} C(X^\star(t), u^\star(t)). \quad (25)$$

Because of the locally Lipschitz property of $t \mapsto v(X^\star(t))$, we can write

$$e^{-rT} v(X^\star(T)) - v(x) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} e^{-rt} v(X^\star(t)) \mathrm{d}t, \quad (26)$$

which, together with (26) and the assumed nonnegativity of v, yields

$$v(x) \geq e^{-rT} v(X^\star(T)) + \int_0^T e^{-rt} C(X^\star(t), u^\star(t)) \mathrm{d}t \geq \int_0^T e^{-rt} C(X^\star(t), u^\star(t)) \mathrm{d}t.$$

Taking limits as $T \uparrow \infty$ and using the monotone convergence theorem by nonnegativity of C, we finally obtain

$$v(x) \geq \int_0^\infty e^{-rt} C(X^\star(t), u^\star(t)) \mathrm{d}t.$$

Hence, $v \geq V$ on $\mathcal{M}$, by arbitrariness of x. □

Combining the results of the previous theorem, we see that if both the assumptions of Part (1) and Part (2) are satisfied, then we find that $v(x) = V(x)$ and $u^\star(\cdot)$ is optimal starting at x.

Part (2) of Theorem 5 assumes (18), which in turn holds if a solution to the closed-loop differential inclusion

$$\begin{cases} S'(t) \in & -\beta S(t) I(t) - S(t)\, U^\star \left(S(t), I(t), \partial_s^* V(S(t), I(t)) \right) \\ & +\eta(1 - S(t) - I(t)), \\ I'(t) = & \beta S(t) I(t) - \gamma I(t), \end{cases} \quad (27)$$

exists. In order to address this aspect, we consider the closed-loop equation

$$\begin{cases} S'(t) = & -\beta S(t) I(t) - S(t)\, U^\star \left(S(t), I(t), \partial_s^+ v(S(t), I(t)) \right) \\ & +\eta(1 - S(t) - I(t)), \\ I'(t) = & \beta S(t) I(t) - \gamma I(t), \end{cases} \quad (28)$$

and look at it from the point of view of the theory of *Regular Lagrangian Flows*; cf. Ambrosio (2017) and Ambrosio (2004), among others. Such theory has been introduced in order to provide a good notion of flow maps even in situations when the associated vector field exhibits little regularity. For this, we recall the following definition (cf. Definition 1 in Ambrosio (2017)):

Definition 1 Let $d \geq 1$, denote by $\mathcal{L}^d$ the Lebesgue measure on $\mathbb{R}^d$, and fix $T > 0$. We say that $X(t; x)$ is a Regular Lagrangian Flow associated to a vector field $m : \mathbb{R}^d \to \mathbb{R}^d$ if:

1. For $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$, $t \mapsto X(t; x)$ is an absolutely continuous solution on $[0, T]$ to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t; x) = m(X(t; x)), \qquad X(0; x) = x;$$

2. for some constant $C > 0$,

$$\mathcal{L}^d\big(x \in \mathbb{R}^d : X(t; x) \in B\big) \leq C \cdot \mathcal{L}^d\big(B\big) \quad \forall t \in [0, T], \ \forall B \subset \mathbb{R}^d \text{ Borel set.}$$

We say that a Regular Lagrangian Flow is unique if, given $X(t; x)$ and $\widetilde{X}(t; x)$ Regular Lagrangian Flows starting from $\mathcal{L}^d$-measurable sets $B_i \subset \mathbb{R}^d$, $i = 1, 2$, we have that

$$X(t; x) = \widetilde{X}(t; x), \qquad \text{for } \mathcal{L}^d - \text{a.e. } x \in B_1 \cap B_2.$$

Proposition 6 *Let $v : \mathcal{M} \to \mathbb{R}_+$ be a semiconcave, bounded, and nonnegative viscosity solution to (11). Set*

$$\widetilde{U}^\star(s, i) := U^\star\left(s, i, \partial_s^+ v(s, i)\right)$$

and assume that

(i) $(s, i, p_s) \mapsto U^\star(s, i, p_s)$ *is locally Lipschitz-continuous on* $\mathcal{M} \times \mathbb{R}$*;*
(ii) $(s \frac{\partial \widetilde{U}^\star}{\partial s})^+ \in L^\infty(\mathcal{M})$.

Then, the Regular Lagrangian Flow (closed-loop equation)

$$\begin{cases} S'(t) = -\beta S(t) I(t) - \widetilde{U}^\star\left(S(t), I(t)\right) S(t) + \eta(1 - S(t) - I(t)), \\ I'(t) = \beta S(t) I(t) - \gamma I(t), \end{cases} \tag{29}$$

with initial data $(S(0), I(0)) = (s, i) \in \mathcal{M}$ *exists and is unique.*

Proof It is enough to embed the closed-loop equation (29) in the setting of Definition 1, and apply Theorem 7 in Ambrosio (2017) (see also Theorems 6.2 and 6.4 in Ambrosio (2004)), after checking the validity of its hypothesis.

In order to accomplish that, we set

$$x = (x_1, x_2) =: (s, i) \in \mathcal{M}, \quad X(t; x) = (X_1, X_2)(t; x) =: (S(t), I(t))$$

and

$$m(x) := \Big(-\beta x_1 x_2 - x_1 \widetilde{U}^\star(x_1, x_2) + \eta(1 - x_1 - x_2),\ \ \beta x_1 x_2 - \gamma x_2 \Big), \quad x \in \mathcal{M},$$

(being extended to $\mathbb{R}^2$ by defining it equal to $(0, 0)$ on $\mathcal{M}^c$).

Firstly, since v is semiconcave by Theorem 3, it follows by Theorem 3 at page 240 of Evans and Gariepy (1992) that $\partial_s^+ v$ is locally of bounded variation on $\mathcal{M}$. Then, as $(s, i, p_s) \mapsto U^\star(s, i, p_s)$ is locally Lipschitz-continuous by assumption, it follows that $\widetilde{U}^\star$ is locally of bounded variation on $\mathcal{M}$ too by Theorem 4 in Josephy (1981), so that m is locally of bounded variation on $\mathbb{R}^2$.

Secondly, given the fact that $(x_1, x_2) \in \mathcal{M}$ and $\widetilde{U}^\star \in [0, \overline{U}]$, it is readily seen that $\frac{|m(x)|}{1+|x|}$ is integrable over $\mathbb{R}^2$. Furthermore, explicit computations yield in $\mathcal{M}$, for some $K \geq 0$,

$$\big(\operatorname{div} m\big)^- \leq K + \Big(-x_1 \frac{\partial \widetilde{U}^\star}{\partial x_1} \Big)^- = K + \Big(x_1 \frac{\partial \widetilde{U}^\star}{\partial x_1} \Big)^+ .$$

Since $\big(s \frac{\partial \widetilde{U}^\star}{\partial s}\big)^+ \in L^\infty(\mathcal{M})$ by assumption, we conclude. □

As a corollary of Theorems 3 and 5, and of Proposition 6, we obtain uniqueness of viscosity solution to HJB (11).

Corollary 7 *Let Assumption 1 and the assumptions of Proposition 6 hold. Then V is the unique bounded locally semiconcave viscosity solution to (11).*

Proof We need to prove uniqueness. By Proposition 6, a solution to (28) exists for almost every $(s, i) \in \mathcal{M}$. This provides the control $u^\star$ required by Part (1) of Theorem 5. Hence, given any other bounded locally semiconcave viscosity solution v to (11), we get $v(s, i) = V(s, i)$ for a.e. $(s, i) \in \mathcal{M}$. Since semiconcavity implies continuity, we conclude that $v = V$ on $\mathcal{M}$. □

4 A case study with numerical illustrations

In this section we introduce a case study and provide numerical simulations. This section simply aims at illustrating the results of the proposed model and no empirical study will be developed. This in particular means that real data will not be employed in order to provide estimations of the model's parameters and no fit of the model on real data will be performed. Indeed, we believe that, though relevant, a similar empirical analysis falls outside the scopes of the present paper and it is left for future research.

In order to proceed with our illustrations, it is useful to preliminarly consider as benchmark the dynamical system (5)–(6) in absence of vaccination. We assume that $\mathcal{R}_o = \beta/\gamma > 1$. In this case the dynamical system has two equilibria:

$$(S_\infty^{(1,o)}, I_\infty^{(1,o)}) = \left(\frac{\gamma}{\beta}, \frac{\eta}{\gamma + \eta} \left(1 - \frac{\gamma}{\beta} \right) \right), \quad (S_\infty^{(2,o)}, I_\infty^{(2,o)}) = (1, 0). \tag{30}$$

The convergence of the system to the first of the above equilibria means that the disease becomes endogenous; the convergence to the second one means that the disease goes to extinction. It is shown in O'Regan et al. (2010) that $(S_\infty^{(1,o)}, I_\infty^{(1,o)})$ is globally asymptotically stable when $\mathcal{R}_o = \beta/\gamma > 1$, whereas $(S_\infty^{(2,o)}, I_\infty^{(2,o)})$ is globally asymptotically stable when $\mathcal{R}_o = \beta/\gamma < 1$.

We assume that the cost function has the following quadratic form

$$C(s,i,u) = \frac{1}{2}\left(ai^2 + b(us)^2\right), \quad a,b>0, \qquad (s,i)\in\mathcal{M},\ u\in U.$$

The latter can be interpreted as a second-order Taylor approximation of any smooth, convex, separable cost function with global minimum in (0, 0). The parameter a can be taken, for instance, as $a = \bar{\imath}^{-1}$, where $\bar{\imath}\in(0,1)$ represents the maximal percentage of infected people that the health-care system can handle; on the other hand, b represents the sensitivity of the policy maker with respect to the vaccination costs. Under this specification of the cost function, for each $(s,i)\in\mathcal{M}$, given v as in Proposition 6, we have $\widetilde{U}^\star(s,i) = U^\star(s,i,\partial_s^+ v(s,i))$, where

$$U^\star(s,i,p_s) = \begin{cases} 0 & \text{if } p_s \le 0, \\ \frac{p_s}{bs} & \text{if } 0 < p_s < b\overline{U}s, \\ \overline{U} & \text{if } p_s \ge b\overline{U}s, \end{cases}$$

Now, $(s,i,p_s)\mapsto U^\star(s,i,p_s)$ is clearly locally Lipschitz continuous. Moreover, we notice that

$$\frac{\partial U^\star}{\partial s}(s,i,p_s) = \begin{cases} 0, & \text{if } p_s \notin (0, b\overline{U}s), \\ -\frac{p_s}{bs^2}, & \text{if } p_s \in (0, b\overline{U}s), \end{cases}$$
$$\frac{\partial U^\star}{\partial p_s}(s,i,p_s) = \begin{cases} 0, & \text{if } p_s \notin (0, b\overline{U}s), \\ \frac{1}{bs}, & \text{if } p_s \in (0, b\overline{U}s). \end{cases}$$

Therefore, $\frac{\partial U^\star}{\partial s} \le 0$ and we have (in the sense of distributions)

$$\begin{aligned} s\frac{\partial \widetilde{U}^\star}{\partial s}(s,i) &= s\frac{\partial U^\star}{\partial s}(s,i,\partial_s^+ v(s,i)) + s\frac{\partial U^\star}{\partial p_s}(s,i,\partial_s^+ v(s,i))\cdot\frac{\partial}{\partial s}\big(\partial_s^+ v\big)(s,i) \\ &\le \frac{1}{b}\frac{\partial}{\partial s}\big(\partial_s^+ v\big)(s,i). \end{aligned}$$

Then, the semiconcavity of v allows to conclude $(s\frac{\partial\widetilde{U}^\star}{\partial s})^+ \in L^\infty(\mathcal{M})$, so that Corollary 7 applies and V is the unique bounded, locally semiconcave viscosity solution to (11).

4.1 Details of numerical scheme

Our numerical method consists of two parts: the construction of the dynamics of (S, I) (see (5) and (6)) for each initial value $(s, i) \in \mathcal{M}$ and, based on the previous discussion, a recursive routine on the HJB equation (11). More precisely, starting from $v^{[0]} \equiv 0$, we use the recursive algorithm:

$$(r - \mathcal{L})v^{[n+1]} = C^\star(s, i, v_s^{[n]}), \quad n \geq 0.$$

Those equations are then solved using the representation formula

$$v^{[n+1]}(s, i) = \int_0^\infty e^{-rt} C^\star\big(S_t^{s,i}, I_t^{s,i}, v_s^{[n]}(S_t^{s,i}, I_t^{s,i})\big)dt, \quad (s, i) \in \mathcal{M}. \tag{31}$$

Such an approach overcomes the issue of the lack of appropriate boundary conditions for the HJB equation (11). As a matter of fact, the boundary $\partial\mathcal{M}$ is unattainable for the underlying controlled dynamical system, so that no natural conditions for V on $\partial\mathcal{M}$ arise. It is worth noticing that although no evidence on the theoretical convergence of the above recursive procedure is provided in the current analysis, the convergence of iteration (31) has been verified through the performed numerical tests. Indeed, if the procedure converges, then the limit is the unique viscosity solution to (11) (see Corollary 7).

The numerical algorithm is designed using MatLab® and all computations are performed on a Quad-Core Intel Core i5 processor (at 2.3 GHz) running macOS. More in detail, the numerical implementation is performed using a regular grid on the space $(0, 1)^2$ with 80×80 nodes and uniform grid's size δ. We consider a discretization of $\mathcal{M}$ represented by the set

$$\widehat{\mathcal{M}} := \{(k_s\delta, k_i\delta) \mid k_s, k_i \in \mathbb{N} \setminus \{0\} \text{ and } (k_s + k_i)\delta \leq 0.95\}.$$

The evaluation of $\partial_s^+ v^{[n]}$ at a point of $\widehat{\mathcal{M}}$ depends on the position of the point in the set. If $(k_s + k_i + 2)\delta \leq 0.95$, then $\partial_s^+ v^{[n]}((k_s, k_i)\delta)$ is approximated by using the forward finite difference scheme

$$\partial_s^+ v^{[n]}((k_s, k_i)\delta) \approx \frac{-\frac{3}{2}v^{[n]}((k_s, k_i)\delta) + 2v^{[n]}((k_s + 1, k_i)\delta) - \frac{1}{2}v^{[n]}((k_s + 2, k_i)\delta)}{\delta^3} + O(\delta^2);$$

if $(k_s + k_i + 2)\delta > 0.95$ and $(k_s + k_i + 1)\delta \leq 0.95$, then the evaluation of $\partial_s^+ v^{[n]}((k_s, k_i)\delta)$ is performed using the first-order accuracy formula

$$\partial_s^+ v^{[n]}((k_s, k_i)\delta) \approx \frac{-v^{[n]}((k_s, k_i)\delta) + v^{[n]}((k_s + 1, k_i)\delta)}{\delta^2} + O(\delta);$$

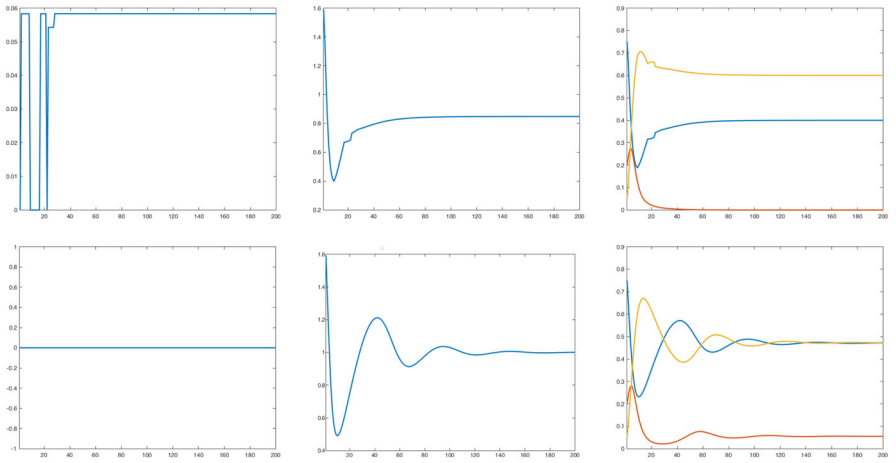

Fig. 1 Comparison between the optimal social planner vaccination policy (upper panel) and the case of no vaccination (lower panel). The figures in the first column show the evolution of the vaccination policy through the optimal control u_t; the ones in the second column show the evolution of the instantaneous reproduction number $\mathcal{R}_t = \frac{\beta}{\gamma} S(t)$; the ones in the third column show evolution of the percentage of susceptible (in blue), infected (in red) and recovered (in green) individuals (color figure online)

finally, if $(k_s + k_i + 1)\delta > 0.95$, then $\partial_s^+ v^{[n]}((k_s, k_i)\delta)$ is replaced by the left derivative approximation

$$\partial_s^- v^{[n]}((k_s, k_i)\delta) \approx \frac{\frac{1}{2} v^{[n]}((k_s - 2, k_i)\delta) - 2v^{[n]}((k_s - 1, k_i)\delta) + \frac{3}{2} v^{[n]}((k_s, k_i)\delta)}{\delta^3} + O(\delta^2).$$

Note that this last choice is justified by Theorem 5, whose statement could indeed have been equivalently given through the notions of viscosity supersolution and subgradient. The algorithm stops when $|v^{[n+1]} - v^{[n]}| < \varepsilon$, with $\varepsilon = 10^{-4}$. Throughout this section, the unit of time will be the week.

4.2 Optimal vaccination versus no vaccination

We set the following values for the parameters. The transmission rate of the disease is $\beta = 0.7$; the average length of infection and reinfection are assumed to be equal to 21 and 180 days respectively, so that $\gamma = \frac{1}{3}$ and $\eta = \frac{7}{180}$; we set $r = \frac{0.005}{52}$ (i.e. a yearly discount rate of 5%) and $a = 0.08$ and $b = 0.016$. We fix $\overline{U} = \frac{7}{120}$. Since $e^{-52\overline{U}} \approx e^{-3}$, one observes that the health-care system vaccinating at the maximal rate $\overline{U}$ is able to vaccinate about 95% of the total population in 1 year. Finally, we initialize the system as

$$S(0) = 75\%, \quad I(0) = 20\%.$$

Table 1 Computational time to approximate the value function V using different values of the stopping criterion's parameter ε

ε	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
Time	0.22 s	0.42 s	0.57 s	0.60 s	0.70 s	1.16 s

Figure 1 provides a comparison between the optimal vaccination policy and the no-vaccination policy. As outcome, we see that in both cases the system oscillates in a first phase and then converges towards an equilibrium.

Our numerical simulations suggest that the optimal policy $u^\star$ always converges towards a limit value $u^\star_\infty$. Assuming that such an equilibrium $u^\star_\infty$ indeed exists, we see that the optimally controlled system has again two long-run equilibria (cf. (30)):

$$(S^{(1)}_\infty, I^{(1)}_\infty) = \left(\frac{\gamma}{\beta}, \frac{\eta}{\eta+\gamma}\left(1-\frac{\gamma}{\beta}\right) - \frac{\gamma}{\eta+\gamma}\frac{u^\star_\infty}{\beta}\right)$$
$$(S^{(2)}_\infty, I^{(2)}_\infty) = \left(\frac{\eta}{\eta+u^\star_\infty}, 0\right),$$

whenever $I^{(1)}_\infty > 0$. Repeating the arguments in O'Regan et al. (2010) (easily adjusted to our setting), it can be proved that $(S^{(1)}_\infty, I^{(1)}_\infty)$ is globally asymptotically stable when $u^\star_\infty < \gamma$.

We can observe that in the case of no vaccination the disease becomes endogenous, whereas the optimal vaccination is able in the long-run to keep the number of infected to zero. More precisely, in the case of no vaccination, the long-run percentage of susceptible and infected individuals, 47% and 6%, respectively, is achieved in about 3 years. On the contrast, if the policy maker adopts the optimal vaccination policy, the vaccination campaign starts with maximum intensity $\overline{U}$ and then it fluctuates for a period of about 28 weeks. After that, it stabilizes at the value $\overline{U}$. In this case, the equilibrium point $(S^{(2)}_\infty, I^{(2)}_\infty)$ is approached counting about 40% of susceptible individuals and almost no infected in less than 1 year. Still, to avoid a new outbreak of the disease, the vaccination policy $\overline{U}$ must be kept to move individuals from the class S to the class R.

Finally, in the case of vaccination, we briefly discuss the computational time of the numerical simulations. In the defined setting, the first part of the numerical method, i.e. the construction of the dynamics of (S, I), approximately takes 55.6 s, whereas the computational speed of the second part, consisting of the recursive routine on the HJB equation (see (31)), is theoretically dependent on the stopping criterion $|v^{[n+1]} - v^{[n]}| < \varepsilon$. In Table 1 we report the algorithmic time taken to compute the value function V for different values of the threshold ε. As expected, we note that the greater the required accuracy is, the slower is the algorithmic procedure to return V, although such an effect appears to be negligible if compared to the running time of the whole numerical procedure.

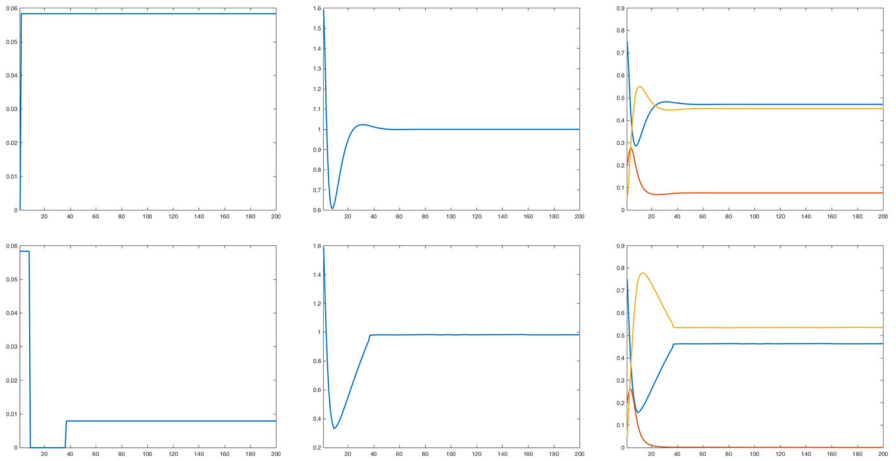

Fig. 2 Comparison between the optimal social planner vaccination policy when the average length of reinfection is 60 days (first row) or 360 days (second row). The figures in the first column show the evolution of the vaccination policy through the optimal control u_t; the ones in the second column show the evolution of the instantaneous reproduction number $\mathcal{R}_t = \frac{\beta}{\gamma} S(t)$; the ones in the third column show the evolution of the percentage of susceptible (in blue), infected (in red) and recovered (in green) individuals (color figure online)

4.3 Variation of the parameter η

In this subsection we study how the optimal vaccination rate, the optimal reproduction number $\mathcal{R}_t$ and the optimally controlled dynamics of susceptible, infected and recovered depend on the reinfection rate η. We assume that the average period of reinfection is equal to 60 or 360 days, so that either $\eta = \frac{7}{360}$ or $\eta = \frac{7}{60}$, respectively. All the other parameters are instead kept fixed to the values assumed in Sect. 4.2.

As expected, from Fig. 2 we observe that the optimal vaccination rate increases when increasing η. However, the more vigorous optimal vaccination rate employed when $\eta = \frac{7}{60}$ is not such to let infections to zero. As a matter of fact, the lower row of Fig. 2 shows that the number of infected stabilizes asymptotically around the level 0.05. On the other hand, the infected population disappears in the long run through a weaker vaccination policy when the reinfection average period is of 1 year circa.

4.4 Variation of the ratio $\frac{\beta}{\gamma}$

In this section we consider strategies corresponding to different values of the natural reproduction number $\mathcal{R}_o = \frac{\beta}{\gamma}$; precisely, we take $\frac{\beta}{\gamma} = 3$ and $\frac{\beta}{\gamma} = \frac{3}{2}$. All the other parameters are instead kept fixed to the values assumed in Sect. 4.2. A comparison of the optimal social planner vaccination policy is shown in Fig. 3.

It is interesting to notice that the two choices of $\mathcal{R}_o$ are such that the dynamical system of susceptible and infected stabilizes around the equilibria in which the disease becomes endogenous or achieves zero infections. As a matter of fact, when $\mathcal{R}_o = 3$, the optimal strategy is constantly equal to $\overline{U} = \frac{7}{120}$ and the dynamical system converges

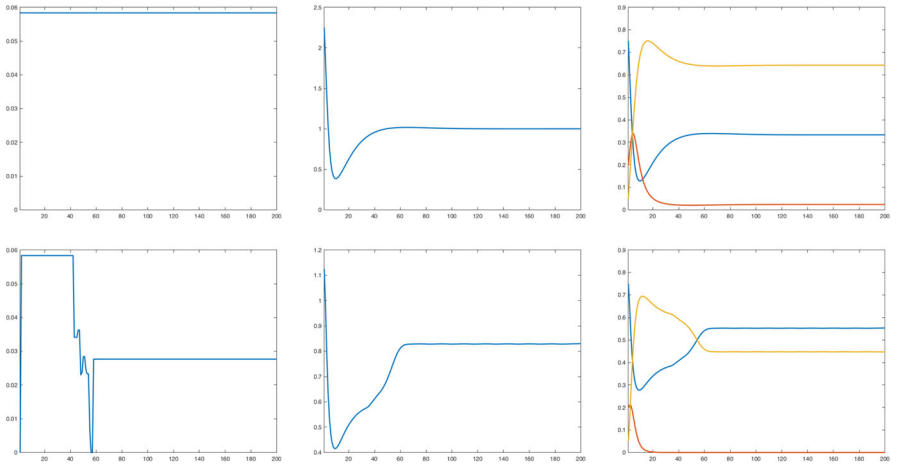

Fig. 3 Comparison between the optimal social planner vaccination policy when $\mathcal{R}_o = 3$ (first row) or $\mathcal{R}_o = 3/2$ (second row). The figures in the first column show the evolution of the vaccination policy through the optimal control u_t; the ones in the second column show the evolution of the instantaneous reproduction number $\mathcal{R}_t = \frac{\beta}{\gamma} S(t)$; the ones in the third column show the evolution of the percentage of susceptible (in blue), infected (in red) and recovered (in green) individuals (color figure online)

to the equilibrium point $(S_\infty^{(1)}, I_\infty^{(1)}) \approx (0.33, 0.11)$. On the other hand, if $\mathcal{R}_o = \frac{3}{2}$, the optimal strategy stabilizes around to the value $u_\infty^\star \approx 0.028$ and the dynamical system converges to the equilibrium point $(S_\infty^{(2)}, I_\infty^{(2)}) \approx (0.56, 0)$. Hence, a lower natural reproduction number $\mathcal{R}_o$ has the effect of making it possible to asymptotically keep the number of infected to zero through the optimal vaccination policy.

4.5 Optimal vaccination policy for different values of initial infected individuals

In this section we study how the optimal vaccination policy reacts to different initial percentages of the infected population. All the parameters are set as in Sect. 4.2, $\frac{\beta}{\gamma} = 1.5$, and the percentage $I(0)$ of the initial infected individuals is 1%, 5% and 10%. The results of the numerical study are presented in Fig. 4.

We observe that, in all the considered cases, the optimal vaccination rate asymptotically stabilizes around the equilibrium level $u_\infty \approx 0.028$, so that, under the optimal vaccination policy, $I_\infty = 0$ after circa 20 weeks. In all the cases, the vaccination policy starts at the maximal rate, then the vaccination campaign is relaxed, and it is finally kept at the constant rate $u_\infty \approx 0.028$. However, the starting date of this final phase of constant vaccination rate is different: it is circa 15 weeks when $I(0) = 1\%$; it is circa 30 weeks when $I(0) = 5\%$; it is circa 54 weeks if $I(0) = 10\%$. That is, the social planner is allowed to stabilize the vaccination rate only after a first time period, whose length increases when the number of initial infected people increases.

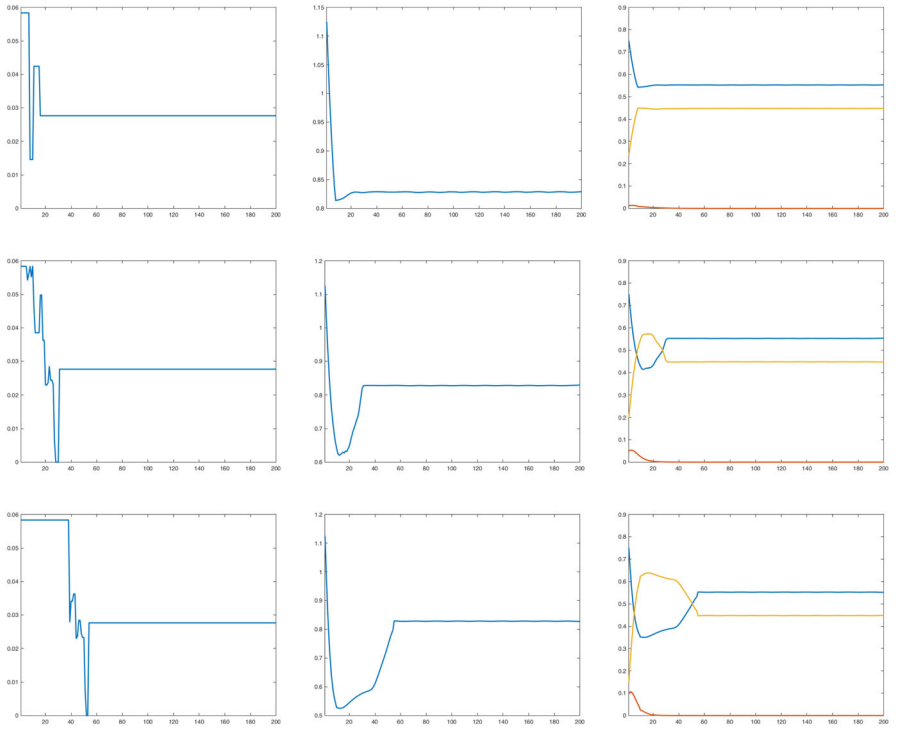

Fig. 4 Comparison between the optimal social planner vaccination policy in the case the initial infected individuals are 1% (first row), 5% (second row) and 10% (third row). The figures in the first column show the evolution of the vaccination policy through the value of the optimal control u_t; the ones in the second column show the evolution of the instantaneous reproduction number $\mathcal{R}_t = \frac{\beta}{\gamma} S(t)$; the ones in the third column show the evolution of the percentage of susceptible (in blue), infected (in red) and recovered (in green) individuals (color figure online)

5 Conclusions

Within a SIR model with reinfection possibility, we have considered the problem of a social planner that aims at reducing the number of individuals susceptible to an infectious disease via a vaccination policy that minimizes social and economic costs. The resulting optimal control problem is nonlinear and nonconvex, due to the nonlinear dynamics of the percentage of susceptible, infected and recovered (immunized) populations. Combining refined techniques of viscosity theory and results from the theory of Regular Lagrangian flows, we are able to provide verifiable sufficient conditions under which the minimal cost function identifies with the unique semiconcave viscosity solution to the corresponding HJB equation. A recursion on the latter is then employed in order to provide a numerical implementation in a case study with quadratic costs. Our experiments show that it is possible to maintain asymptotically the number of infected to zero, at least when the disease is not too infective. However, given that the model allows reinfection with positive probability, this comes at the cost of keeping vaccinating at a constant rate in the long-run. Further, in most of

the numerical results, the optimal control, given in feedback form, fluctuates during the first period (approximately 6–7 months) as it substantially follows the oscillations of the underlying dynamical system. From a practical viewpoint, following the optimal strategy's fluctuations is certainly challenging for a policymaker. Nevertheless, employing a policy that averages those oscillations out may represent an effective guide for the social planner's actions.

There are several directions towards which this research can be extended and continued. First of all, from a technical point of view, it would be interesting to grasp a deeper understanding of how the theory Regular Lagrangian Flows can be helpful for proving the well-posedness of the closed-loop equation arising in nonlinear optimal control problems, as those arising in mathematical epidemiology. Second of all, from a modeling perspective, stochastic and partial observation features should be included in the model, as the evolution of an infectious disease is clearly far to obey completely observable deterministic law of motions. Thirdly, it would be intriguing to study the problem of optimal vaccination problem from a moral-hazard point of view, thus leading to a principal-agent problem where the social planner designs benefits which should induce the agents (the susceptible population) to get vaccinated. These aspects are clearly outside the scope of the present work and are therefore left to future research.

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Declarations

Conflict of interest The authors certify that they have no affiliations with or involvement in any organization or entity with any financial or non-financial interest in the subject matter or materials discussed in this manuscript.

A A Technical result

Lemma 8 *Recall (2), (3) and (4). One has* $S(t) > 0$, $I(t) > 0$ *and* $R(t) > 0$ *for all* $t \geq 0$.

Proof Let $(S(0), I(0), R(0)) =: (s, i, r) \in (0,1)^3$ and define $\tau := \inf\{t \geq 0 : S(t) \leq 0\} \wedge \inf\{t \geq 0 : I(t) \leq 0\} \wedge \inf\{t \geq 0 : R(t) \leq 0\}$. For the sake of contradiction, suppose that $\tau < \infty$. Then, let $t \in [0, \tau)$ and, by using (2), (3) and (4), the chain rule yields:

$$\ln\big(S(t)I(t)R(t)\big) = \ln(s,i,r) + \int_0^t \Big(-\beta I + \eta\frac{R}{S} + \beta S - (\gamma + \eta + u) + \gamma\frac{I}{R} + u\frac{S}{R}\Big)(q)\mathrm{d}q$$

$$\geq \ln(s,i,r) - \beta \int_0^t I(q)\mathrm{d}q - (\gamma + \eta + \overline{U})t$$
$$\geq \ln(s,i,r) - (\gamma + \eta + \overline{U})t - t \max_{q\in[0,t]} I(q), \tag{32}$$

where in the penultimate inequality we have used that $0 \leq u(\cdot) \leq \overline{U}$. The contradiction now follows by taking limits as $t \to \tau$. □

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