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RESEARCH ARTICLE

Stochastic growth, conservation of capital and convergence to a positive steady state

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Abstract
In a general one-sector model of optimal stochastic growth where the productivity of capital is bounded but may vary widely due to technology shocks, we derive a tight estimate of the slope of the optimal policy function near zero. We use this to derive a readily verifiable condition that ensures almost sure global conservation of capital (i.e., avoidance of extinction) under the optimal policy, as well as global convergence to a *positive* stochastic steady state for bounded growth technology; this condition is significantly weaker than existing conditions and explicitly depends on risk aversion. For a specific class of utility and production functions, a strict violation of this condition implies that almost sure long run extinction of capital is globally optimal. Conservation is non-monotonic in risk aversion; conservation is likely to be optimal when the degree of risk aversion (near zero) is either high or low, while extinction may be optimal at intermediate levels of risk aversion.

Keywords Stochastic growth · Conservation · Extinction · Positive steady state · Global stability · Risk aversion

JEL Classification C6 · D9 · O41

Tapan Mitra passed away in 2019.

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1 Introduction

A fundamental issue in the study of economic growth and dynamic capital accumulation is whether the economy converges to a steady state characterized by positive output and consumption in the long run; such an outcome is characterized by conservation of capital or avoidance of extinction where extinction refers to depletion of capital stocks to zero in the long run. Interest in this issue is rooted in the desire to understand and predict the qualitative nature of the so called "long run equilibrium" of the economy.[1] In the study of accumulation of natural resources and other environmental assets that are depleted for economic use, there is an abiding interest in understanding the economics of conservation and extinction that is partly based on the negative environmental externalities associated with extinction.[2]

It is well known that even if technology and endowments in an economy make it feasible to maintain positive consumption and output in the long run, preferences of economic agents and their interaction can create dynamic incentives for extinction. The optimal economic growth model provides a stylized framework to understand the role of intertemporal preferences in determining the long run outcomes of capital accumulation. The one sector stochastic version of this model,[3] where exogenous fluctuations or technology shocks affect the return on investment over time, provides a framework for economists to understand the economic factors that ensure convergence to a *positive* stochastic steady state, i.e., an invariant distribution where consumption and output are strictly positive with probability one; such an outcome is in sharp contrast to almost sure extinction where economic activity pretty much ceases in the long run.

The optimal stochastic growth model has also been used to study optimal harvesting of renewable natural resources whose "natural growth" or biological reproduction (captured by the production function) is subject to exogenous environmental shocks affecting the ecosystem over time; here, the utility from consumption reflects the net social or private benefit from harvesting.[4] Understanding the economic and ecological conditions under which it is (privately or socially) optimal to conserve the resource, i.e., avoid extinction, is helpful for public policy; for instance, the design of policies that affect the cost and revenue associated with harvesting privately managed resources may gain from a nuanced understanding of how the long run outcome depends on the net benefit (i.e., the utility) function. A tight characterization of extinction versus conservation for the stochastic growth model can be very useful here. It is also useful for understanding how the choice of "social discount rate" affects long run outcomes of publicly managed resources under uncertainty.[5] Finally, for applications of the opti-

[1] It may also be partly motivated by a concern that a long run economic outcome with zero consumption may be incompatible with stability of the underlying social structure.

[2] Such as loss of biodiversity and amenity value.

[3] The one sector stochastic optimal growth model was pioneered by Levhari and Srinivasan (1969), Brock and Mirman (1972) and Mirman and Zilcha (1975); see also, Phelps (1962). The survey by Olson and Roy (2006) contains a useful summary of the literature.

[4] See, Clark (2010).

[5] When the natural growth of the resource is stochastic, riskiness of environmental shocks and risk aversion (embedded in the curvature of the utility from resource harvesting) are at least as important as discounting

mal stochastic growth model that involve numerical simulation or computation of the long run empirical distribution of consumption or capital (using specific functional forms of the primitives and calibrated parameter values),[6] it may be useful to know in advance whether or not a globally stable positive invariant distribution exists; readily verifiable conditions on exogenous elements of the model can be helpful in this regard.

Obtaining a tight condition for convergence to a positive steady state or more generally, of conservation of capital, is somewhat challenging in the general version of the stochastic growth model. Among other things, it requires a precise estimate of the optimal policy function near zero in terms of exogenous elements of the model; this has been difficult to obtain outside of some examples with specific functional forms where one can explicitly solve for the optimal policy function. This paper attempts to directly address this gap in the literature.

We consider the general one sector discounted stochastic optimal growth model with independent and identically distributed production shocks. We allow for a fairly general class of concave production functions that may exhibit bounded or unbounded growth. In our framework, technology shocks can cause wide variation in the productivity of capital; in particular, production functions may be globally unproductive (i.e., lie below the 45-degree line) under adverse shocks; however, we assume "bounded shocks" i.e., for any given level of capital input, variation in the realization of the production shock cannot lead to arbitrarily small or arbitrarily large output. We allow for a very general class of strictly concave and smooth utility functions that satisfy a mild restriction on relative risk aversion near zero.

We confine attention to production technology with bounded productivity, i.e., the marginal productivity of capital at zero is finite for all realizations of the random shock.[7,8] In our framework with "bounded shocks", infinite productivity at zero would essentially rule out any possibility of extinction under the optimal policy.[9] As the objective of the paper is to derive conditions that allow us to understand how economic factors such as risk, risk preferences and discounting matter for long run extinction and conservation of capital, it is more useful to focus on production functions with bounded productivity. The latter is also a natural assumption in applications of the

Footnote 5 continued
in determining whether or not it is economically optimal to conserve the resource. For a discussion of issues related to choice of social discount rates in environmental management in the presence of uncertainty see, for instance, Polasky and Dampha (2021).

6 See, for instance, Taylor and Uhlig (1990).

7 The prevalent use of production functions that satisfy the Inada condition (infinite productivity at zero) is largely motivated by technical convenience and in the case of some examples, the ease of obtaining explicit solutions to the dynamic optimization problem.

8 Much of the existing literature on stochastic growth allows for finite productivity at zero; in fact, linear production functions have been used widely in various macroeconomic applications of the stochastic growth model. See, for instance, Rebelo (1991).

9 For instance, Assumption 3.5 in Kamihigashi (2007) would be satisfied in that case and Theorem 3.5 in that paper would then imply global convergence to a positive stochastic steady state. See also, Szeidl (2013).

model to optimal resource management where the production function captures the natural growth of a resource or the biological reproduction of a specie.[10]

We derive an explicit estimate of the optimal propensity to *invest* as output tends to zero in terms of the discount factor, the distribution of productivity and the degree of relative risk aversion (near zero). While this estimate is derived as a lower bound on the optimal propensity, it is the exact limit of this propensity (as output goes to zero) as long as is bounded away from one. This explicit characterization of the behavior of the optimal policy function near zero in a general model (i.e., without using explicit functional forms for the production or utility functions) is the key theoretical contribution of this paper.

We use this tight characterization of the optimal policy function near zero to derive an explicit condition for almost sure conservation of capital under the optimal policy so that loosely speaking, capital, output and consumption paths generated by the optimal policy are strictly positive in the long run (regardless of initial conditions).[11] Note that conservation is consistent with optimal paths visiting neighborhoods of zero infinitely often as long as they rebound every time instead of tending to zero. We demonstrate the tightness of this condition for the widely used family of constant relative risk aversion (CRRA) utility functions and under some restrictions on the general production function; for such economic environments, we show that a reversal of the strict inequality in our condition for almost sure conservation implies that (almost sure) extinction occurs globally under the optimal policy.

Using some of the seminal results on global stability due to Kamihigashi and Stachurski (2014),[12] we show that for the case of bounded growth technology, our condition for almost sure conservation ensures the existence of a globally stable positive stochastic steady state, i.e., from every strictly positive initial stock, the optimal output process converges in distribution to a unique invariant distribution whose support is in the strictly positive real line.

Our condition for almost sure avoidance of extinction is the weakest in the relevant literature and allows us to explicitly highlight the role of risk aversion. We show that if the discount factor and the stochastic technology are such that the condition is satisfied when the relative risk aversion at zero is 1 (for instance, in the case of the log utility function), then it is satisfied for *all* admissible utility functions. If the condition is strictly violated when the relative risk aversion at zero is 1, then conservation is still ensured if relative risk aversion is either small enough or large enough, but almost sure extinction may be optimal when relative risk aversion is in an intermediate range (close to 1). This demonstrates the non-monotone effect of

[10] In this literature, the slope of the production function at zero represents the "intrinsic" growth rate of the resource. Empirical estimates of this growth rate for various biological species are moderate or low (see, for instance, Myers et al. 1999). Much of the bioeconomics literature assumes that the resource growth function is logistic; this function has finite slope at zero. For an exposition of specific models used in the literature see, Lewis (1981), Wilen (1985), Munro and Scott (1985) and Clark (2010).

[11] As extinction is the event that capital stocks converge to zero, avoidance of extinction or *conservation* is the (complementary) event that the upper limit (*limit supremum*) of the sequence of capital stocks is strictly positive.

[12] As a positive stochastic steady state in our model may not be bounded away from zero and because zero is an absorbing state, the usual "mixing" or "splitting" conditions for global stability (see, for instance, Hopenhayn and Prescott (1992) and Bhattacharya and Majumdar (2004)) are not very useful.

increase in risk aversion on conservation and extinction (or, in the case of bounded growth technology, extinction versus convergence to a positive steady state); increased curvature of the utility function increases the incentive to smoothen intertemporal consumption which makes the economy move away from extinction paths but at the same time, higher risk aversion increases the incentive to favor certainty of current consumption against the uncertainty of future consumption that works against the incentive to accumulate and may push the economy towards extinction.

Our framework allows for the so called "unbounded growth" technology that remains productive at all levels of investment; for such technology, it is of interest to understand the conditions under which the optimal policy generates sustained (or long run) growth almost surely;[13] our results are useful for this purpose as avoidance of extinction is a necessary condition for unbounded expansion.

Related Literature. The early literature on optimal stochastic growth imposed strong conditions to ensure that when current output is small, it is optimal to expand output even under the "worst" realization of the exogenous shock; the latter implies that with probability one, long run output and capital stocks lie above a strictly positive lower bound. [14] Brock and Mirman (1972) and Mirman and Zilcha (1975) ensure this by assuming infinite marginal productivity at zero for every realization of the shock and in addition, a strictly positive probability mass on the worst production shock.[15] When the utility function is bounded below, these conditions can be weakened to a requirement that the discounted marginal productivity of capital near zero is greater than one for every realization of the random shock (Hopenhayn and Prescott 1992; Chatterjee and Shukayev 2008). Mitra and Roy (2012a, b) derive weaker conditions for optimal paths to be bounded away from zero that, in particular, depend on the curvature of the utility function[16] and the distribution of shocks. In contrast to this strand of the literature, our paper focuses on avoidance of extinction, i.e., for output and capital stocks to not converge to zero, which is weaker than being bounded away from zero; while we investigate the existence of a globally stable positive steady state whose support is in the strictly positive real line (i.e., has no probability mass at zero), we do not require this support to be bounded away from zero.

Over the last two decades, a growing literature on stochastic growth with "unbounded shocks" has extended the core model to allow for production technologies where for any given level of capital input, variation in realizations of the exogenous shock can lead to arbitrarily small or arbitrarily large output. In this framework, Stachurski (2002a) and Nishimura and Stachurski (2005) use innovative techniques to derive conditions for a globally stable positive stochastic steady state that has no probability mass at zero; somewhat weaker conditions are contained in Zhang (2007) for production functions that satisfy the Inada condition.[17] The weakest condition for

[13] See, for instance, de Hek and Roy (2001).

[14] Assuming infinite productivity at zero is not, by itself, sufficient to ensure that optimal paths are bounded away from zero almost surely; see Mirman and Zilcha (1976) and Mitra and Roy (2012a).

[15] See also, Brock and Majumdar (1978); Majumdar et al. (1989) and Olson (1989).

[16] For an early analysis of the comparative dynamics of the curvature of the utility function, see Danthine and Donaldson (1981). On a somewhat different note, Jones et al. (2004) show that the qualitative relationship between volatility and "mean" growth depends on the curvature of the utility function.

[17] See also, Stachurski (2002b), Nishimura et al. (2006) and Szeidl (2013).

a globally stable positive steady state in the existing literature on optimal stochastic growth is derived by Kamihigashi (2007) in an integrated framework that allows for both bounded and unbounded shocks as well as bounded and unbounded growth. In all of these papers, conditions to ensure avoidance of extinction (necessary for convergence to a positive steady state) do not involve the utility function or risk preference. Though we do not allow for "unbounded shocks", the framework in our paper is closer to the second strand of the literature as we allow capital to be globally unproductive under adverse shocks so that long run output and capital stocks may not be bounded away from zero even if all output is invested every period; a positive stochastic steady state, if it exists, may not be bounded away from zero.

Our key condition for almost sure avoidance of extinction and global convergence to a positive stochastic steady state is weaker than comparable conditions in the existing literature (and in particular, the condition in Kamihigashi 2007) and is fairly tight for a widely used family of utility and production functions; this condition involves, among other factors, the degree of risk aversion near zero.[18,19] Unlike the previous literature, our condition is based on an explicit characterization of the slope of the optimal policy function near zero.

The literature on optimally management of renewable natural resources has also analyzed economic and ecological conditions for conservation and extinction. Reed (1974) provides conditions for conservation and extinction assuming that utility is *linear* in consumption (harvest) which sharply limits the role of role of risk and risk aversion in the behavior of optimal paths near zero.[20] In a more general framework that allows for non-concave production functions and nonlinear utility, Olson and Roy (2000) and Mitra and Roy (2006) provide conditions for optimal resource stocks to be bounded away from zero with probability one[21]. In the context of our model with concave production function and utility function that depends only on consumption, these conditions are stronger than our condition for conservation of capital.

Plan of the paper. Section 2 outlines the model and some basic results. Section 3 contains our key result characterizing the optimal propensity to invest near zero. Section 4 outlines our condition for almost sure avoidance of extinction. Section 5 outlines an upper bound on the optimal investment policy function for a specific family of utility and production functions and uses this to illustrate the tightness of our condition for almost sure conservation. Section 6 discusses the effect of change in relative risk aversion on the optimality of extinction and conservation. Section 7

[18] The sufficient conditions in Mitra and Roy (2012a, b) also involve risk aversion but they are significantly stronger as they ensure outcomes stronger than avoidance of extinction and are not based on any precise estimate of the optimal propensity to invest near zero. In their framework, higher risk aversion near zero makes it less likely that optimal outputs are bounded away from zero; this is in contrast to the non-monotonic effect of risk aversion on conservation in this paper.

[19] A number of papers ensure extinction and conservation (or, convergence to a positive steady state) by imposing conditions directly on the optimal policy function or the kernel of the stochastic process generated by the optimal policy functions. See, for instance, Boylan (1979), Mendelssohn and Sobel (1980) and Athreya (2004). In contrast, the conditions in our paper are on the primitives or exogenous elements of the model such as preferences and technology.

[20] See also, Reed (1978). For analysis of conservation and extinction using a specfic parametric form for the production function see, Lande et al. (1994) and Alvarez and Shepp (1998).

[21] Olson and Roy (2000) also allow for stock-dependent utility.

considers the case of bounded growth technology and shows how our condition for almost sure conservation ensures a globally stable *positive* stochastic steady state. The Appendix contains proofs of all results.

2 Model

We consider an infinite horizon one-good representative agent economy. Let $\mathbb{N}$ denote the set of natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbb{N}_+$ the set of strictly positive natural numbers; let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote respectively the sets of non-negative and strictly positive real numbers. Time is discrete and is indexed by $t \in \mathbb{N}$. The initial stock of output $y_0 \in \mathbb{R}_+$ is given. At each date t, the representative agent observes the current stock of output $y_t \in \mathbb{R}_+$ and chooses the level of current investment x_t, and the current consumption level c_t, such that

$$c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t.$$

This generates y_{t+1}, the output next period through the relation

$$y_{t+1} = f(x_t, r_{t+1}),$$

where f is the "aggregate" production function and r_{t+1} is a random production shock realized at the beginning of period $(t + 1)$.

2.1 Production

The following assumption is made on the sequence of random shocks:

(R.1) $\{r_t\}_{t=1}^{\infty}$ is an independent and identically distributed random process defined on a probability space $(\Omega, \mathcal{F}, P)$, where the marginal distribution function is denoted by Ψ. The support of this distribution is a non-degenerate set $A \subset \mathbb{R}_{++}$.

The production function $f : \mathbb{R}_+ \times A \to \mathbb{R}_+$ is assumed to satisfy the following standard monotonicity, concavity, measurability and smoothness restrictions on the production function:

(T.1) Given any $r \in A$, $f(x, r)$ is assumed to be continuously differentiable and concave in x on $\mathbb{R}_+$, with $f(0, r) = 0$; further, $f'(x, r) = \frac{\partial f(x,r)}{\partial x} > 0$ on $\mathbb{R}_+$ and

$$f'(0, r) = \lim_{x \downarrow 0} f'(x, r) < \infty.$$

For any $x \geq 0$, $f(x, .) : A \to \mathbb{R}_+$ is a (Borel) measurable function.

Note that **(T.1)** implies that for each realization of the random shock, marginal productivity is bounded; we do not allow for production functions where the Inada condition holds at zero.

For each $r \in A$, let $B(r)$ denote the marginal product at zero investment:

$$B(r) = f'(0, r).$$

Observe that for all $x \geq 0$,

$$f(x, r) \leq B(r)x. \tag{1}$$

We assume that

(T.2)

$$\underline{B} = \inf_{r \in A} B(r) > 0, \ \overline{B} = \sup_{r \in A} B(r) < \infty. \tag{2}$$

Note that **(T.2)** allows $\underline{B}$ to be less than one, i.e., we allow for production technologies that are unproductive at all levels of capital input.

For any investment level $x \geq 0$, let the upper and lower bound of the support of output next period be denoted by $\overline{f}(x)$ and $\underline{f}(x)$, respectively. In particular,

$$\overline{f}(x) = \sup_{r \in A} f(x, r), \ \underline{f}(x) = \inf_{r \in A} f(x, r). \tag{3}$$

It is easy to check that $\underline{f}(x)$ and $\overline{f}(x)$ are non-decreasing on $\mathbb{R}_+$, $\underline{f}(0) = \overline{f}(0) = 0$ and $\underline{f}(.)$ is concave on $\mathbb{R}_+$. Further, **(T.1)** and **(T.2)** imply that

$$0 < \underline{f}(x) \leq \overline{f}(x) < \infty, \tag{4}$$

for all $x > 0$. Thus, we assume "bounded shocks".

2.2 Preferences

We denote by u the one period utility function from consumption and we assume that:

(U.1) $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is twice continuously differentiable on $\mathbb{R}_{++}$, $u'(c) > 0$, $u''(c) < 0$ for all $c > 0$.

(**U.2**) $\lim_{c \to 0} u(c) = u(0)$; $\lim_{c \to 0} u'(c) = +\infty$.

Note that we allow for unbounded utility functions. For $c > 0$, let the Arrow-Pratt measure of relative risk aversion at c be defined by:

$$\rho(c) = -\frac{u''(c)c}{u'(c)}.$$

We assume that $\rho(c)$ converges to a strictly positive number as $c \to 0$:

(**U.3**)

$$\lim_{c \to 0} \rho(c) = \rho_0 > 0.$$

2.3 The optimization problem

Given an initial stock $y \in \mathbb{R}_+$, a stochastic process $\{y_t, c_t, x_t\}$ is *feasible* from y if it satisfies $y_0 = y$, and:

(i) $c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t$ for all $t \in \mathbb{N}$,
(ii) $y_t = f(x_{t-1}, r_t)$ for $t \in \mathbb{N}_+$,

and (iii) for each $t \in \mathbb{N}$, $\{c_t, x_t\}$ are $\mathcal{F}_t$ adapted where $\mathcal{F}_t$ is the (sub) σ-field generated by partial history from periods 0 through t.

Let $\delta \in (0, 1)$ denote the time discount factor. The objective of the representative agent is to maximize the expected value of the discounted sum of utilities from consumption:

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right].$$

Given $y \geq 0$, define the stochastic process of consumption $\{c_t^M\}$ by: $c_0^M = y, c_{t+1}^M = f(c_t^M, r_{t+1})$ for all $t \geq 0$. Thus, c_t^M is an upper bound on feasible consumption in period t. We assume that:

(D.1) For all $y \geq 0$,

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t^M)_+\right] < \infty,$$

where $u(c)_+ = \max\{u(c), 0\}$.

Assumption (**D.1**) ensures that for any feasible stochastic process $\{y_t, c_t, x_t\}$ from $y \geq 0$, the objective of the representative agent given by

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right],$$

is well defined though it may equal $-\infty$, and that (see, Kamihigashi 2007)

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right] = \sum_{t=0}^{\infty} \delta^t E[u(c_t)]. \tag{5}$$

Note that (**D.1**) is always satisfied if either u is bounded above or alternatively, if $\limsup_{x\to\infty}[\overline{f}(x)/x] < 1$, i.e., the technology exhibits bounded growth.

Given initial stock $\overline{y} \geq 0$, a feasible stochastic process $\{y_t, c_t, x_t\}$ is *optimal* from $\overline{y}$ if for every feasible stochastic process $\{y'_t, c'_t, x'_t\}$ from $\overline{y}$,

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right] \geq E\left[\sum_{t=0}^{\infty} \delta^t u(c'_t)\right].$$

For $y \geq 0$, let $V(y)$, the value function be defined by

$$V(y) = \sup\left\{E\sum_{t=0}^{\infty} \delta^t u(c_t)) : \{c_t, x_t, y_t\} \text{ is a feasible stochastic process from } y\right\}.$$

We assume:

(**D.2**) $V(y) > -\infty$ for all $y > 0$.

Note that (**D.2**) is always satisfied if $u(0) > -\infty$ or alternatively, if $\underline{B} > 1$, i.e., the worst case production function is productive near zero; if neither of these hold, it is satisfied under some restrictions on the discount factor δ.

Combined with assumption (**D.1**), we now have

$$-\infty < V(y) < +\infty, \text{ for all } y > 0.$$

A *consumption (policy) function* is a function $\widetilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying:

$$0 \leq \widetilde{c}(y) \leq y \quad \text{for all } y \in \mathbb{R}_+.$$

Note that this implies $\widetilde{c}(0) = 0$. Associated with a consumption function $\widetilde{c}(\cdot)$, is an *investment (policy) function* $\widetilde{x} : \mathbb{R}_+ \to \mathbb{R}$, defined by

$$\widetilde{x}(y) = y - \widetilde{c}(y) \text{ for all } y \in \mathbb{R}_+.$$

Thus, the investment function $\widetilde{x}(.)$ satisfies:

$$0 \leq \widetilde{x}(y) \leq y \quad \text{for all } y \in \mathbb{R}_+.$$

A feasible stochastic process $\{y_t, c_t, x_t\}$ is said to be *generated by* a consumption function $\widetilde{c}(y)$ from initial stock $\overline{y} \in \mathbb{R}_+$ if

$$\begin{aligned} y_0 &= \overline{y};\ \ y_{t+1} = f(y_t - \widetilde{c}(y_t), r_{t+1}) \ \text{ for } t \geq 0; \\ c_t &= \widetilde{c}(y_t),\ \ x_t = y_t - \widetilde{c}(y_t) \text{ for } t \geq 0. \end{aligned}$$

A consumption (policy) function $c(y)$ is said to be optimal if for every initial stock $\overline{y} \in \mathbb{R}_+$, the stochastic process $\{y_t, c_t, x_t\}$ generated by the function $c(.)$ is optimal; we refer to the investment policy function $x(y) = y - c(y)$ as the optimal investment function.

Standard dynamic programming arguments (see Theorem 2.1 in Kamihigashi, 2007) imply:

Lemma 1 *The value function $V(y)$ satisfies the functional equation:*

$$V(y) = \max_{0 \leq c \leq y} \left[u(c) + \delta E\left(V(f(y-c, r))\right)\right]. \tag{6}$$

$V(y)$ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$. For each $y \geq 0$, the maximization problem on the right hand side of (6) has a unique solution $c(y)$ and the consumption (policy) function $c(y)$ is the unique optimal consumption (policy) function. For all $y > 0$, $c(y) > 0$ and $x(y) = y - c(y) > 0$. $x(y)$ and $c(y)$ are

continuous and strictly increasing in y *on* $\mathbb{R}_+$. *For all* $y > 0$,*the following Ramsey-Euler equation holds:*

$$u'(c(y)) = \delta E[u'(c(f(x(y), r)))f'(x(y), r)] \\ = \delta \int_A u'(c(f(x(y), r)))f'(x(y), r)d\Psi(r). \tag{7}$$

Let $H : \mathbb{R}_+ \times A \to \mathbb{R}_+$ be *the optimal transition function* defined by:

$$H(y, r) = f(x(y), r).$$

For $y \in \mathbb{R}_+$, the optimal stochastic process of output, consumption and investment $\{y_t(y), c_t(y), x_t(y)\}$ (generated by the optimal policy function) from initial stock y are given by

$$y_0(y) = y,\ \ y_{t+1}(y) = H(y_t(y), r_{t+1}), t \in \mathbb{N}.$$
$$c_t(y) = c(y_t(y)), x_t(y) = x(y_t(y)), t \in \mathbb{N}.$$

Let $\overline{H} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\underline{H} : \mathbb{R}_+ \to \mathbb{R}_+$ denote the upper and lower envelope of the transition functions defined by:

$$\overline{H}(y) = \overline{f}(x(y)), \underline{H}(y) = \underline{f}(x(y)).$$

As $\underline{f}(.)$ is non-decreasing and $x(.)$ is strictly increasing, $\underline{H}(.)$ is non-decreasing on $\mathbb{R}_+$.

Finally, for any feasible stochastic process of capital stocks $\{\overline{x}_t\}$ from initial stock $\overline{y} \in \mathbb{R}_+$, we define extinction (of capital) as the event that $\overline{x}_t$ converges to 0 as $t \to \infty$; we define conservation of capital as the complementary event of extinction, viz., the event that $\limsup_{t\to\infty} \overline{x}_t > 0$. Extinction is said to occur almost surely under the optimal policy (or, extinction is optimal with probability one) if

$$\Pr\left\{\lim_{t\to\infty} x_t(y) = 0\right\} = 1,$$

where $\{x_t(y)\}$ is the optimal stochastic process of capital stocks (i.e., the process generated by the optimal policy function) from initial stock $y \in \mathbb{R}_+$. Conservation of capital is said to occur almost surely under the optimal policy (or, conservation is optimal with probability one) if

$$\Pr\{\limsup{}_{t\to\infty} x_t(y) > 0\} = 1.$$

3 Optimal propensity to invest

The behavior of the optimal policy function near zero is of crucial importance in determining the likelihood of conservation of capital when the capital stock is sufficiently depleted. In this section, we provide an explicit characterization of the limiting optimal propensity to invest as output converges to zero in terms the discount factor, the probability distribution of marginal productivity of capital and the degree of risk aversion.

Recall that $\rho(c) = -\frac{u''(c)c}{u'(c)}$ is the Arrow-Pratt measure of relative risk aversion at $c > 0$. Under assumption (**U.3**), $\rho(c) \to \rho_0 > 0$ as $c \to 0$; thus ρ_0 is the (limiting) risk aversion at zero.

Also, recall that $B(r) = f'(0, r)$ is the marginal product at zero investment corresponding to realization r of the productivity shock. Under our assumptions, $B(r_t)$ is a bounded random variable taking values in $[\underline{B}, \overline{B}] \subset \mathbb{R}_{++}$.With some abuse of notation, we use $B(r)$ to denote the random variable $B(r_t)$.

Define

$$s_0 = \left[\delta E\left\{(B(r))^{1-\rho_0}\right\}\right]^{1/\rho_0} = \left[\delta \int_A (B(r))^{1-\rho_0} d\Psi(r)\right]^{1/\rho_0}. \tag{8}$$

Assumption (**T.2**) ensures that s_0 is well defined and $0 < s_0 < \infty$.

Finally, define $\theta \in (0, 1]$ by

$$\theta = \min\{s_0, 1\}. \tag{9}$$

Note that θ depends only on the discount factor, the probability distribution of the marginal productivity of capital at zero and the degree of relative risk aversion at zero.

We are now ready to state the main result in this section that provides a tight and explicit characterization of the behavior of the optimal policy function near zero.

Proposition 1 (i)

$$\liminf_{y \to 0} \left[\frac{x(y)}{y}\right] \geq \theta = \min\{s_0, 1\}. \tag{10}$$

(ii) *If*

$$\limsup_{y \to 0} \left[\frac{x(y)}{y}\right] < 1,$$

i.e., the optimal propensity to invest $\frac{x(y)}{y}$ *is bounded away from* 1 *as* $y \to 0$, *then* $s_0 < 1$ *and*

$$\lim_{y \to 0} \frac{x(y)}{y} = s_0 = \theta.$$

Part (i) of Proposition 1 provides an explicit lower bound θ for the optimal propensity to invest as output converges to zero; loosely speaking, optimal investment is bounded below by θy for y small enough. This lower bound does not require the production function or the utility function to have any specific functional form. Our subsequent analysis of conservation of capital will use this lower bound. Note that (10) implies that if $s_0 \geq 1$, then $\frac{x(y)}{y} \to 1$ as $y \to 0$.

Part (ii) of the proposition indicates that the lower bound θ is "tight" in the sense that it is the exact limit of the optimal propensity to invest as output converges to zero (loosely, the slope of the policy function at zero) if the optimal propensity to invest is bounded away from 1, i.e., if the optimal propensity to consume $\frac{c(y)}{y}$ is bounded away from zero; in the latter case, the optimal investment function behaves pretty much like the linear function θy for output levels close to zero.[22]

4 Conservation of capital

In this section, we outline our main result on conservation of capital. In particular, we use the bound on the optimal propensity to invest near zero characterized in Proposition 1 to derive an explicit condition on the economic fundamentals under which the optimal policy is such that capital stocks are strictly positive in the long run with probability one.

(**T.3**)

$$\frac{f(x,r)}{x} \to B(r) \text{ as } x \to 0 \text{ uniformly in } r \text{ on } A.$$

Note that (**T.3**) is satisfied if the random shock is multiplicative (for instance, $f(x,r) = rh(x)$) and A, the support of the distribution of random shocks, is a bounded subset of $\mathbb{R}_{++}$.

Our condition for almost sure conservation of capital is as follows:

Condition C:

$$E[\ln(\theta B(r))] > 0. \tag{C}$$

Note that θ and therefore, Condition **C**, depends on the discount factor, the degree of risk aversion, the distribution of random shocks and the marginal productivity of capital. In particular, if $\theta < 1$ so that $\theta = s_0 = \left(\delta E((B(r))^{1-\rho_0}\right)^{1/\rho_0}$, Condition **C**is equivalent to:

$$(1/\rho_0)\left[\ln\delta + \ln E\left((B(r))^{1-\rho_0}\right)\right] + E\ln(B(r)) > 0, \tag{11}$$

which can be written as

$$E[\ln(\delta B(r))] + \left[\left\{\ln E\left((B(r))^{1-\rho_0}\right)\right\} - \left\{E\ln(B(r)^{1-\rho_0})\right\}\right] > 0. \tag{12}$$

Using Jensen's inequality, the second term in square brackets on the left hand side of inequality (12) reflects the interaction between risk aversion (near zero) and riskiness of the productivity shock; it is always non-negative and it is strictly positive if $\rho_0 \neq 1$**.**

[22] It is worth noting that if the production function is linear i.e., $f(x,r) = rx$, the utility function exhibits constant relative risk aversion and $\theta < 1$, then the optimal investment policy function is given by $x(y) = \theta y$ i.e., θ is the optimal propensity to invest at *all* levels of output. See, for instance, de Hek and Roy (2001).

[23]Note that in the deterministic case, the second expression in square brackets on the left hand side of inequality (12) is zero and Condition **C** reduces to the well known "delta-productivity" condition that requires the discounted marginal productivity at zero to be larger than one.

We are now ready to state the main result of this section.

Proposition 2 *Assume (**T.3**) and Condition **C**. Then the following hold for all initial stocks $y \in \mathbb{R}_{++}$:*

(i) *For any $\xi > 0$, there exists $\widehat{\alpha}(y) > 0$ such that*

$$\Pr\{y_t(y) < \widehat{\alpha}(y)\} < \xi \text{ for all } t \in \mathbb{N}. \tag{13}$$

(ii) *Conservation of capital occurs with probability one under the optimal policy, i.e.,*

$$\Pr\{\limsup_{t\to\infty} x_t(y) > 0\} = 1. \tag{14}$$

Further, $\limsup_{t\to\infty} y_t(y) > 0$ and $\limsup_{t\to\infty} c_t(y) > 0$ with probability one, i.e., under the optimal policy, output and consumption levels remain strictly positive in the long run with probability one.

(iii) *Extinction of capital occurs with zero probability under the optimal policy, i.e.,*

$$\Pr\{x_t(y) \to 0\} = 0.$$

Proposition 2 shows that Condition **C** ensures the following. Part (i) of the proposition states that though output (and therefore consumption and capital) may reach arbitrarily small levels from time to time, it is (loosely speaking) bounded away "in probability" from zero. This implication of Condition **C** is used in the proof of parts (ii) and (iii) of the proposition and plays a critical role in the proof of Proposition 7 in Sect. 7, i.e., in establishing the existence of a globally stable positive stochastic steady state. Parts (ii) and (iii) of the proposition state that under the optimal policy, long run capital and consumption are strictly positive with probability one and extinction occurs with probability zero from any strictly positive initial stock.

5 Tightness of Condition C

In this section, we illustrate the tightness of our general condition (Condition **C**) for almost sure conservation of capital. We show that for a class of utility and production functions that are widely used in the literature, strong violation of Condition **C** (in the sense of reversal of the strict inequality in Condition **C**) implies that all optimal paths converge to zero, i.e., extinction occurs with probability one from all positive initial stocks. This result is stated as Proposition 4 in subsection 5.2.

[23] As the log function is concave, Jensen's Inequality ensures that $\ln E(Y) \geq E(\ln Y)$ where $Y = B(r_t)^{1-\rho_0}$ is a positive valued random variable with finite expectation. Note that the curvature of the function $B(r)$ does not matter for this comparison.

The restricted family of utility and production functions that we consider in this section are as follows. First, we confine attention to utility functions that satisfy constant relative risk aversion, i.e., we assume that:

(U.4)

$$u(c) = \frac{c^{1-\rho_0}}{1-\rho_0}, \rho_0 > 0, \rho_0 \neq 1,$$
$$= \ln c \text{ (corresponding to } \rho_0 = 1).$$

Second, we impose the following joint restriction on the set of admissible production and utility functions:

Condition B: For all $r \in A$, $x \in \mathbb{R}_{++}$,

$$\frac{f'(x,r)x^{\rho_0}}{(f(x,r))^{\rho_0}} \leq (B(r))^{1-\rho_0}. \tag{B}$$

Note that

$$\lim_{x\to 0}\left\{\frac{f'(x,r)x^{\rho_0}}{(f(x,r))^{\rho_0}}\right\} = (B(r))^{1-\rho_0},$$

so that Condition **B** essentially requires that the function $\frac{f'(x,r)x^{\rho_0}}{(f(x,r))^{\rho_0}}$ is "maximized at zero". If f is twice differentiable and $\eta_1(x,r)$, $\eta_2(x,r)$ are the first and second elasticities of the production function defined by

$$\eta_1(x,r) = \frac{f'(x,r)x}{f(x,r)}, \eta_2(x,r) = -\frac{f''(x,r)x}{f'(x,r)},$$

then Condition **B** holds if

$$\rho_0 \leq \frac{\eta_2(x,r)}{1-\eta_1(x,r)}, \forall x \in \mathbb{R}_{++}, r \in A.$$

This last condition ensures that $\ln\left(\frac{f'(x,r)x^{\rho_0}}{(f(x,r))^{\rho_0}}\right)$ has a non-positive derivative at every $x > 0$ and is therefore "maximized at $x = 0$".

The required inequality in (B) can be rewritten as:

$$\left(\frac{f'(x,r)}{f'(0,r)}\right)^{1-\rho_0}\left[\frac{f'(x,r)x}{f(x,r)}\right]^{\rho_0} \leq 1, \text{ for all } x \in \mathbb{R}_{++}, r \in A,$$

which always holds if $\rho_0 \in (0, 1]$ but can also hold if $\rho_0 > 1$.

Example 1 Consider the family of production functions:

$$f(x,r) = 0, \text{ if } x = 0,$$
$$= r(x^{1-\eta} + \beta)^{\frac{1}{1-\eta}}, \beta \geq 0, \eta > 1, \text{ if } x > 0.$$

Note that $f(x, r)$ satisfies assumptions (**T.1**)-(**T.3**). If $\beta > 0$, f exhibits bounded growth and Condition **B** is satisfied as long as $\rho_0 \leq \eta$.[24] If $\beta = 0$, f is a linear production function and Condition **B** holds for all $\rho_0 > 0$.

In Sect. 3, we have shown that θ is always a lower bound on the optimal propensity to invest near zero. We now show that for the family of utility and production functions outlined above, θ is also an upper bound on the optimal propensity to invest at all levels of output, i.e., we have an upper bound on the entire optimal investment function. This is an important step towards showing that optimal paths may converge to zero when Condition **C** does not hold. It can be a useful result for other purposes.

Proposition 3 *Assume* (**U.4**) *and Condition B. Then, the optimal propensity to invest is bounded above by* θ *on* $\mathbb{R}_{++}$*, i.e.,*

$$\frac{x(y)}{y} \leq \theta \textit{ for all } y \in \mathbb{R}_{++}. \tag{15}$$

If $\theta = 1$, Proposition 3 is trivial. So, we focus on $\theta < 1$ in which case $\theta = s_0$. The proof first shows that θ is an upper bound on the optimal propensity to invest in the finite horizon version of the infinite horizon dynamic optimization problem in Sect. 2 and then uses policy convergence to extend this to the optimal policy function for the infinite horizon problem. Using Proposition 3, we establish the main result of this section:

Proposition 4 *Assume* (**U.4**) *and Condition B. If*

$$E[\ln(\theta B(r))] < 0, \tag{16}$$

then almost sure extinction is optimal from all initial stocks, i.e., for all $y \in \mathbb{R}_{+}$,

$$\{y_t(y), c_t(y), x_t(y)\} \rightarrow 0 \textit{ with probability one}.$$

Note that the strict inequality in (16) is a reversal of the strict inequality in Condition **C**; Proposition 4 implies that for the specific class of utility and production functions considered in this section, a strong violation of Condition **C** leads to global extinction of capital with probability one which is indicative of the tightness of Condition **C** as a sufficient condition for almost sure conservation.[25]

In a more general framework, Kamihigashi (2006) shows that under bounded productivity, all *feasible* paths may converge to zero almost surely if the production shocks are sufficiently volatile; in particular, Theorem 3.1 in that paper indicates that almost sure extinction occurs if $E[\ln B(r)] < 0$ which is significantly stronger than (16).

---

[24] If $\rho_0 = \eta$, the optimal policy function is linear and the optimal propensity to invest is θ; see, among others, Benhabib and Rustichini (1994); Mitra and Sorger (2014).

[25] If $E[\ln \theta B(r)] = 0$ we have the borderline case between Condition **C** and (16). Here, depending on the specific production function and distribution of shocks, the optimal policy may lead to conservation or extinction.

6 Risk aversion and regeneration

Condition **C** outlined in Proposition 2 allows us to study the effect of change in risk aversion (near zero) on the optimality of conservation of capital (i.e., avoidance of extinction) and positive long run consumption.

We begin by showing that if Condition **C** is satisfied when ρ_0, the (limiting) Arrow-Pratt relative risk aversion at zero consumption, is equal to 1 (such as in the case of the log utility function), then it is satisfied for all admissible utility functions; in this case, change in risk aversion does not affect the optimality of conservation. Note that at $\rho_0 = 1$, $\theta = s_0 = \delta$ so that Condition **C**is equivalent to requiring $E[\ln(\delta B(r))] > 0$.

Proposition 5 *Assume* ***(T.3)****. Suppose that*

$$E[\ln(\delta B(r))] > 0. \tag{17}$$

Then regardless of the degree of risk aversion, i.e., for all $\rho_0 > 0$, it is optimal to conserve capital with probability one and in particular, the conclusions of Proposition 2 always hold.

It is worth noting that (17) is the condition imposed in Kamihigashi (2007) to ensure avoidance of extinction and convergence to a positive steady state; in fact, it is the weakest such condition in the existing literature. Condition **C** is weaker than (17) and they coincide only if $\rho_0 = 1$.

Next, we consider the situation where Condition **C** is *not* satisfied when the risk aversion parameter $\rho_0 = 1$ and in particular, $E[\ln(\delta B(r))] < 0$. Here, change in risk aversion can alter the desirability of conservation. In particular, we show that in certain situations, almost sure conservation may be optimal when the degree of risk aversion is either low or high, but almost sure extinction may be optimal when risk aversion is at an intermediate level.

Proposition 6 *Assume* ***(T.3)****. Suppose that*

$$E[\ln(\delta B(r))] < 0 < E[\ln B(r)].$$

Then the following hold:

(a) *If ρ_0 is close to* 1, *then for utility and production functions that satisfy* (**U.4**)*and Condition B, extinction with probability one is optimal from all initial stocks.*
(b) *There exists $\overline{\rho} > 1$ such that for all $\rho_0 > \overline{\rho}$, almost sure conservation is optimal from all strictly positive initial stocks and in particular, the conclusions of Proposition 2 hold.*
(c) *If, further, $\delta E\,(B(r)) > 1$, there exists $\underline{\rho} \in (0, 1)$ such that for all $\rho_0 \in (0, \underline{\rho})$, almost sure conservation is optimal from all strictly positive initial stocks and in particular, the conclusions of Proposition 2 hold.*

Proposition 6 indicates that the qualitative relationship between conservation and risk aversion can be non-monotonic; as risk aversion (near zero) increases we may move from conservation to extinction to conservation. Note that the range of values of the relative risk aversion parameter that is most likely to be associated with extinction is in the neighborhood of 1 which overlaps with what is often regarded as the empirically relevant range of this parameter for the purpose of quantitative analysis in macroeconomics.

Example 2 Assume that the utility function satisfies constant relative risk aversion, i.e., (**U.4**) holds. Consider the linear production function:

$$f(x, r) = rx.$$

Then, $B(r) = r$. Using (11), Condition **C** can be written as:

$$\phi(\rho_0) = \rho_0 E(\ln r) + \ln \delta + \ln E[r^{1-\rho_0}] > 0.$$

For the chosen utility and production functions, Condition **B** is satisfied for all values of $\rho_0 > 0$. Thus, conservation of capital occurs with probability 1 if $\phi(\rho_0) > 0$, while extinction occurs with probability one if $\phi(\rho_0) < 0$. Assume that the random shock r_t is distributed uniformly on the interval [0.1, 2.5]. Then, $E(r_t) = 1.3$ so that $\delta E(r_t) > 1$ for $\delta > 0.77$. Further,

$$E \ln r = \frac{2.5(\ln 2.5) - 0.1(\ln 0.1)}{2.4} - 1 > 0$$

and

$$\begin{aligned} E[r^{1-\rho_0}] &= \frac{(2.5)^{(2-\rho_0)} - (0.1)^{(2-\rho_0)}}{2.4(2-\rho_0)}, \text{ if } \rho_0 \neq 2, \\ &= \frac{\ln 2.5 - \ln 0.1}{2.4}, \text{ if } \rho_0 = 2. \end{aligned}$$

We plot the function $\phi(\rho_0)$ for $\delta = 0.9$ and $\delta = 0.8$ in Figs. 1 and 2.

Figure 1 shows that when $\delta = 0.9$, $\phi(\rho_0) < 0$ if, and only if, the relative risk aversion parameter ρ_0 lies in an intermediate range (roughly, from 0.5 to 1.42) and it is only for this range of risk aversion that almost sure extinction is globally optimal; $\phi(\rho_0) > 0$, i.e., Condition **C** holds and almost sure conservation of capital is optimal when ρ_0 is outside this range. Figure 2 shows that the intermediate range of risk aversion parameter for which extinction is optimal is larger (roughly, from 0.1 to 1.71) if $\delta = 0.8$

7 Globally stable positive steady state

Our condition for conservation (Condition **C**) ensures that with probability one, optimal paths do not converge to zero and optimal outputs are strictly positive in the long run with probability one. This indicates that if the stochastic process of optimal output

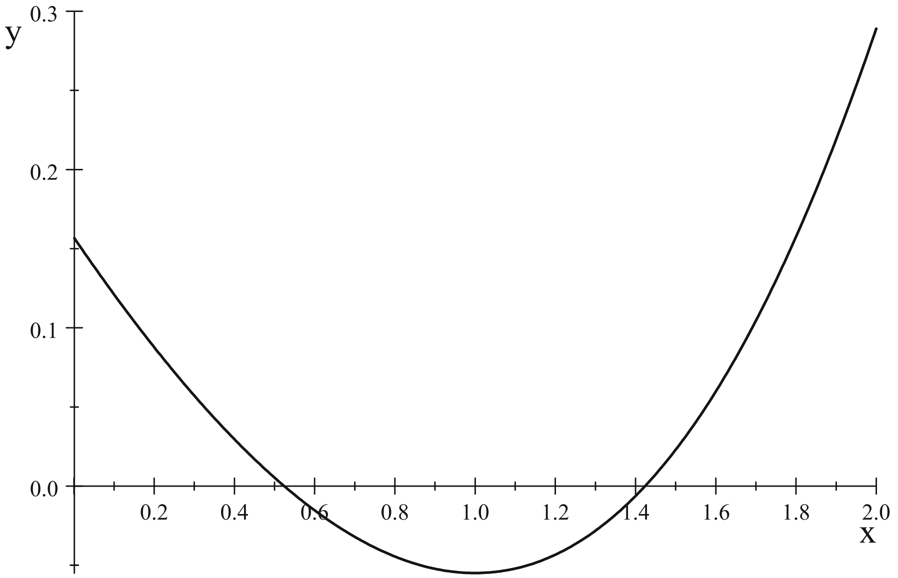

Fig. 1 The function ϕ for $\delta = 0.9$; $y = \phi(\rho_0)$, $x = \rho_0$

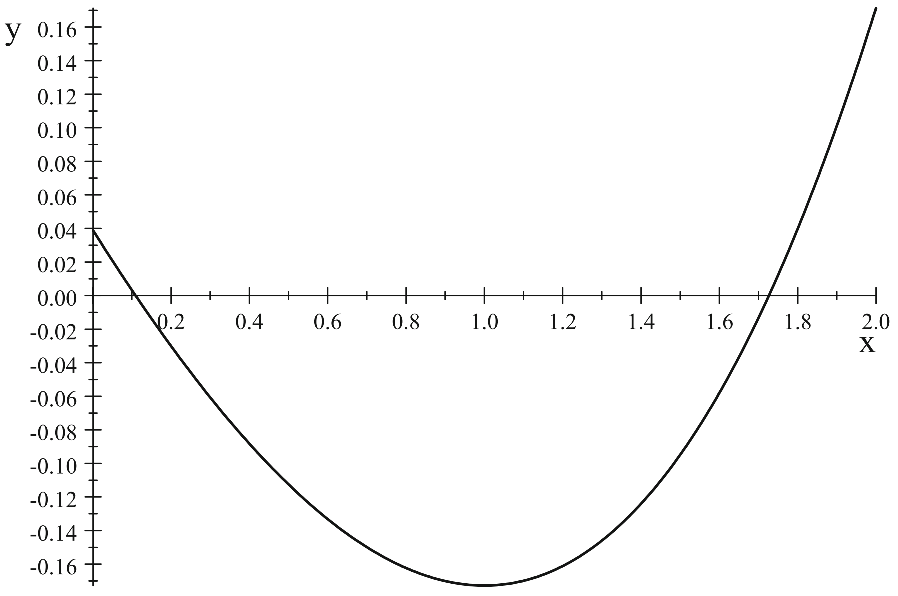

Fig. 2 The function ϕ for $\delta = 0.8$; $x = \rho_0$, $y = \phi(\rho_0)$

from any strictly positive initial stock converges in distribution to an invariant distribution, then the support of the limit distribution is in $\mathbb{R}_{++}$ (it cannot assign strictly positive probability to zero); such an invariant distribution would be the stochastic analogue of a non-zero steady state in the deterministic growth model and we can refer to it as a positive stochastic steady state. Under the convex structure of our model, one may expect the steady state to be globally stable.

As mentioned in the introduction, the existing literature has identified conditions that ensure a globally stable positive stochastic steady state. Our condition for conservation and Proposition 2 can be used to weaken these conditions.

In particular, assume that the production function satisfies:

(T.4)

$$\lim_{x\to\infty} \sup \left[\frac{\overline{f}(x)}{x}\right] < 1.$$

Define the maximum sustainable stock $K \geq 0$ as

$$K = \sup\left\{x \geq 0 : \overline{f}(x) \geq x\right\}. \tag{18}$$

Assumption **(T.4)** ensures that $K < \infty$.

For technical convenience we also assume that:

(T.5) $\underline{f}$ and $\overline{f}$ are continuous and strictly increasing on $\mathbb{R}_+$. For every $x > 0$, $\underline{f}(x) < \overline{f}(x)$ and for any $\upsilon > 1$,

$$\Pr\{f(x, r_t) \leq \upsilon \underline{f}(x)\} > 0, \Pr\{f(x, r_t) \geq \frac{1}{\upsilon}\overline{f}(x)\} > 0.$$

Assumption **(T.5)** ensures that the distribution of output from any current investment is non-degenerate and that $[\underline{f}(x), \overline{f}(x)]$ is the (essential) support of this distribution. Note that continuity of $\overline{f}$ assumed in **(T.5)** implies that $\overline{f}(K) = K$.

We are now ready to state the result on global stability of a positive steady state:

Proposition 7 *Assume* (**T**.**3**), (**T**.**4**), (**T**.**5**)*and Condition* **C**. *Then, there exists a globally stable invariant distribution for the stochastic process of optimal outputs* $\{y_t(y)\}$ *that assigns probability one to* $(0, K]$. *For any initial output* $y \in (0, K]$, *optimal outputs converge in distribution to this positive stochastic steady state.*

The proof of Proposition 7 is entirely based on some recent results on global stability of monotone stochastic processes by Kamihigashi and Stachurski (2014). Condition **C** is the important condition in Proposition 7; it ensures that even though outputs may reach levels arbitrarily close to zero with high probability, the stochastic kernel of the output process is "bounded in probability" on $(0, K]$ in the precise sense of Proposition 2(i).

It should be noted that it is possible to replace **(T.5)** by alternative assumptions that may have weaker requirements in some respects; we choose **(T.5)** for ease of exposition.

One implication of Proposition 7 is that it brings out the important role of risk aversion in convergence to a positive steady state. In particular, Propositions 6 and 7 together indicate that for a bounded growth technology, Condition **C** and therefore, global convergence to a positive stochastic steady state is generally ensured if the degree of relative risk aversion near zero is either sufficiently high or sufficiently low, but if the degree of risk aversion is in an intermediate range, there may be no positive stochastic steady state and the degenerate distribution that puts all probability mass at zero may be the globally stable stochastic steady state.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the contents of this article.

Appendix

A.1 Proof of Proposition 1

We begin by stating a useful result reported in Mitra and Roy (2012a, Lemma 4):

Lemma 2 (Mitra and Roy 2012a) *For any* $c^1 > 0$, $c^2 > 0$, $c^2 \geq c^1$, *if* $\rho(c) \in [\underline{\rho}, \overline{\rho}] \subset \mathbb{R}_{++}$ *for all* $c \in [c^1, c^2]$, *then*

$$\left(\frac{c^2}{c^1}\right)^{-\overline{\rho}} \leq \frac{u'(c^2)}{u'(c^1)} \leq \left(\frac{c^2}{c^1}\right)^{-\underline{\rho}}. \tag{19}$$

Next, we establish some bounds on the limiting behavior of the propensity to consume as output tends to zero. The following lemma shows that the optimal propensity to consume is bounded away from one (i.e., the optimal propensity to invest is bounded away from zero); assumption (**U.3**) which ensures that relative risk aversion is bounded away from zero plays an important role in this result.

Lemma 3

$$\limsup_{y \to 0} \frac{c(y)}{y} < 1. \tag{20}$$

Proof Suppose to the contrary that $\limsup_{y\to 0}\left[\frac{c(y)}{y}\right] = 1$. Fix $\gamma \in (0, 1)$.There exists $\widetilde{y} > 0$ such that

$$\rho(c) \geq \gamma\rho_0 \text{ for all } c \in (0, \overline{f}(\widetilde{y})]. \tag{21}$$

As $\overline{B} < \infty$, there exists $\eta_0 \in (0, 1)$ such that

$$\overline{B}(1-\eta) < 1 \text{ for all } \eta \in (\eta_0, 1).$$

Fix $\eta \in (\eta_0, 1)$. Then, $\overline{f}((1-\eta)y) \leq \overline{B}(1-\eta)y < y$ for all $y \in (0, \widetilde{y}]$. As $\limsup_{y\to 0}\left[\frac{c(y)}{y}\right] = 1$, there exists a sequence $\{y^n\}$ converging to zero, $y^n \in (0, \widetilde{y})$

for all n such that

$$\frac{c(y^n)}{y^n} \geq \eta \text{ for all } n. \tag{22}$$

Then, $f(x(y^n), r) \leq \overline{f}(x(y^n)) \leq \overline{f}((1-\eta)y^n) < y^n$ for all $r \in A$. From the Ramsey-Euler equation (7):

$$\begin{aligned}
\frac{1}{\delta} &= E\left[\frac{u'(c(f(x(y^n), r)))}{u'(c(y^n))} f'(x(y^n), r)\right] \\
&\geq E\left[\left(\frac{c(f(x(y^n), r))}{c(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n), r)\right] \text{ (using Lemma 2 and (21))} \\
&= E\left[\left(\frac{c(f(x(y^n), r))}{f(x(y^n), r)}\right)^{-\gamma\rho_0} \left(\frac{f(x(y^n), r)}{x(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n), r)\right] \left(\frac{\frac{c(y^n)}{y^n}}{1-\frac{c(y^n)}{y^n}}\right)^{\gamma\rho_0} \\
&\geq E\left[\left(\frac{f(x(y^n), r)}{x(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n), r)\right] \left(\frac{\eta}{1-\eta}\right)^{\gamma\rho_0} \text{ (using (22)).}
\end{aligned}$$

Taking the liminf as $n \to \infty$ through the above inequalities and using Fatou's Lemma (see, for instance, section 4.3.3 in Dudley 2002) we have

$$\begin{aligned}
\frac{1}{\delta} &\geq E\left[\liminf_{n\to\infty} \left(\frac{f(x(y^n), r)}{x(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n), r)\right] \left(\frac{\eta}{1-\eta}\right)^{\gamma\rho_0} \\
&= E\left[(B(r))^{1-\gamma\rho_0}\right] \left(\frac{\eta}{1-\eta}\right)^{\gamma\rho_0},
\end{aligned}$$

so that we have

$$\frac{\eta}{1-\eta} \leq \left(\frac{1}{\delta E\left[(B(r))^{1-\gamma\rho_0}\right]}\right)^{\frac{1}{\gamma\rho_0}} \text{ for all } \eta \in (\eta_0, 1).$$

As $\rho_0 > 0$ and the right hand side of the above inequality is independent of η, we have a contradiction for η close enough to 1. □

The next lemma establishes an upper bound on the limiting optimal propensity to consume at zero under the assumption that it is strictly positive.

Lemma 4 *Suppose that*

$$\limsup_{y\to 0} \frac{c(y)}{y} > 0.$$

Then, $s_0 < 1$ *and*

$$\limsup_{y\to 0} \frac{c(y)}{y} \leq 1 - s_0.$$

Proof Let $\bar{z} = \limsup_{y\to 0} \frac{c(y)}{y}$. Under the hypothesis of this lemma and using Lemma 3, $0 < \bar{z} < 1$. We will now show that

$$\bar{z} \leq 1 - s_0. \tag{23}$$

Fix $\lambda \in (0, 1)$ and $\widehat{M} > 0$ such that $\widehat{M} < \frac{\bar{z}}{(1-\bar{z})\overline{B}}$. There exists $\bar{h} \in (0, \min\{\bar{z}, 1 - \bar{z}\})$ such that

$$\widehat{M} \leq \frac{(\bar{z} - h)}{(1 - (\bar{z} - h))\overline{B}} \text{ for all } h \in (0, \bar{h}). \tag{24}$$

Choose any ϵ, h such that $0 < \epsilon < \rho_0$, $h \in (0, \bar{h})$. There exists $\bar{y} > 0$ such that

$$\rho_0 - \epsilon \leq \rho(c) \leq \rho_0 + \epsilon \text{ for all } c \in (0, \overline{f}(\bar{y})), \tag{25}$$

and

$$f'(\bar{y}, r) \geq \lambda \underline{B} \text{ for all } r \in A. \tag{26}$$

By definition of $\bar{z}$ there exists a sequence $\{z^n\}_{n=1}^{\infty}$, $z^n \in (0, \bar{y})$ for all n, $z^n \to 0$ as $n \to \infty$,$\{\frac{c(z^n)}{z^n}\}$ is convergent and

$$\bar{z} + h \geq \frac{c(z^n)}{z^n} \geq \bar{z} - h \text{ for all } n. \tag{27}$$

Note that $\lim_{n\to\infty} \left(\frac{c(z^n)}{z^n}\right) \in [\bar{z} - h, \bar{z}]$. From the Ramsey-Euler equation (7) and using Lemma 2 and (25), we have:

$$\begin{aligned}
\frac{1}{\delta} &= E\left[\frac{u'(c(f(x(z^n), r)))}{u'(c(z^n))} f'(x(z^n), r)\right] \\
&\geq E\left[\left(\frac{c(f(x(z^n), r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} f'(x(z^n), r) I_{[f(x(z^n),r)\geq z^n]}\right] \\
&\quad + E\left[\left(\frac{c(f(x(z^n), r))}{c(z^n)}\right)^{-(\rho_0-\epsilon)} f'(x(z^n), r) I_{[f(x(z^n),r)<z^n]}\right].
\end{aligned} \tag{28}$$

Observe that if $f(x(z^n), r) \geq z^n$, then

$$\begin{aligned}
&\left(\frac{c(f(x(z^n), r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} \\
&= \left(\frac{x(z^n)}{c(z^n)}\right)^{-(\rho_0+\epsilon)} \left(\frac{c(f(x(z^n), r))}{f(x(z^n), r)}\right)^{-(\rho_0+\epsilon)} \left(\frac{f(x(z^n), r)}{x(z^n)}\right)^{-(\rho_0+\epsilon)} \\
&\geq \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \left(\frac{c(f(x(z^n), r))}{f(x(z^n), r)}\right)^{-\rho_0} \left(\frac{f(x(z^n), r)}{x(z^n)}\right)^{-\rho_0} \left(\frac{1-(\bar{z}-h)}{(\bar{z}-h)}\overline{B}\right)^{-\epsilon} \\
&\quad \text{(using (27))} \\
&\geq \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \left(\frac{c(f(x(z^n), r))}{f(x(z^n), r)} \frac{f(x(z^n), r)}{x(z^n)}\right)^{-\rho_0} (\widehat{M})^{\epsilon} \quad \text{(using (24))}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&E\left[\left(\frac{c(f(x(z^n), r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} f'(x(z^n), r) I_{[f(x(z^n), r)\geq z^n]}\right] \\
&\quad \geq (\widehat{M})^{\epsilon} \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \\
&\quad \cdot E\left(\left(\frac{c(f(x(y^n), r))}{f(x(y^n), r)} \frac{f(x(z^n), r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n), r) I_{[f(x(z^n), r)\geq z^n]}\right).
\end{aligned} \tag{29}$$

If $f(x(z^n), r) < z^n$, then

$$\begin{aligned}
\left(\frac{c(f(x(z^n), r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} &\geq \left(\frac{c(f(x(z^n), r))}{c(z^n)}\right)^{-\rho_0} \\
&= \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \left(\frac{c(f(x(z^n), r))}{f(x(z^n), r)} \frac{f(x(z^n), r)}{x(z^n)}\right)^{-\rho_0}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&E\left[\left(\frac{c(f(x(z^n), r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} f'(x(z^n), r) I_{[f(x(z^n), r)<z^n]}\right] \\
&\quad \geq E\left[\left(\frac{c(f(x(z^n), r))}{f(x(z^n), r)} \frac{f(x(z^n), r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n), r) I_{[f(x(z^n), r)<z^n]}\right] \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0}.
\end{aligned} \tag{30}$$

Using (28), (29) and (30), we have

$$\frac{1}{\delta} \geq \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} E\left[\left(\frac{c(f(x(z^n), r))}{f(x(z^n), r)} \frac{f(x(z^n), r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n), r)\right] \min\left\{\widehat{M}^{\epsilon}, 1\right\}, \tag{31}$$

which implies:

$$\left(\frac{c(z^n)}{z^n}\right)^{-\rho_0}\left(\frac{x(z^n)}{z^n}\right)^{\rho_0} \geq \delta E\left[\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\right)^{-\rho_0}\left(\frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r)\right] \cdot \min\left\{\widehat{M}^{\epsilon},1\right\}. \tag{32}$$

For each $r \in A$,

$$\liminf_{n\to\infty}\left[\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\right)^{-\rho_0}\left(\frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r)\right] \geq \overline{z}^{-\rho_0}(B(r))^{1-\rho_0}. \tag{33}$$

Taking the liminf as $n \to \infty$ on both sides of (32) and using Fatou's lemma:

$$\liminf_{n\to\infty}\left(\frac{c(z^n)}{z^n}\right)^{-\rho_0}\left(\frac{x(z^n)}{z^n}\right)^{\rho_0} \geq \delta E\left[\liminf_{n\to\infty}\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\,\frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r))\right]\min\left\{\widehat{M}^{\epsilon},1\right\}. \tag{34}$$

Using (27), (33) and (34), we have

$$\left(\frac{\overline{z}-h}{1-(\overline{z}-h)}\right)^{-\rho_0} \geq \overline{z}^{-\rho_0}\delta E\left[(B(r))^{1-\rho_0}\right]\min\left\{\widehat{M}^{\epsilon},1\right\}.$$

As h, ϵ are arbitrary (and $\widehat{M}$ is independent of h, ϵ), we have

$$\left(\frac{\overline{z}}{1-\overline{z}}\right)^{-\rho_0} \geq \overline{z}^{-\rho_0}\delta E\left[(B(r))^{1-\rho_0}\right],$$

so that

$$(1-\overline{z})^{\rho_0} \geq \delta E\left[(B(r))^{1-\rho_0}\right] = (s_0)^{\rho_0} \tag{35}$$

This establishes (23) and also implies that $s_0 < 1$. The proof is complete. □

The next result is a corollary of the previous lemma:

Corollary 1

$$\frac{c(y)}{y} \to 0, \text{ if } s_0 \geq 1, \tag{36}$$

and

$$\limsup_{y\to 0}\frac{c(y)}{y}\le 1-s_0,\ \textit{if}\ s_0<1. \tag{37}$$

Proof If $s_0 \ge 1$ and $\limsup_{y\to 0}\frac{c(y)}{y} > 0$, we have an immediate contradiction to Lemma 4; this establishes (36). If $s_0 < 1$ and $\limsup_{y\to 0}\frac{c(y)}{y} = 0$, (37) holds immediately; if $\limsup_{y\to 0}\frac{c(y)}{y} > 0$, (37) follows from Lemma 4. □

The expression for $s_0 = \left[\delta E\left\{(B(r))^{1-\rho_0}\right\}\right]^{1/\rho_0}$ is always strictly positive but can be arbitrarily large. The first part of Corollary 1 describes what happens when $s_0 \ge 1$ (so that $\theta = 1$) and shows that in that case, the optimal investment propensity converges to 1 (i.e., to θ).

The next lemma indicates that the upper bound on the limiting optimal propensity to consume at zero outlined in inequality (37) of Corollary 1 is the exact limit as long as the optimal consumption propensity is bounded away from zero.

Lemma 5 *Suppose that*

$$\liminf_{y\to 0}\left[\frac{c(y)}{y}\right] > 0. \tag{38}$$

Then

$$\lim_{y\to 0}\left[\frac{c(y)}{y}\right] = 1-s_0. \tag{39}$$

Proof Let

$$\underline{z} = \liminf_{y\to 0}\frac{c(y)}{y},\ \overline{z} = \limsup_{y\to 0}\frac{c(y)}{y}.$$

Using Lemma 3 and (38),

$$0 < \underline{z} \le \overline{z} < 1. \tag{40}$$

Further, from Lemma 4, $s_0 < 1$ and $\overline{z} \le 1-s_0$. We will show that if (38) holds, then

$$\underline{z} \ge 1-s_0, \tag{41}$$

so that (using (40)), $\underline{z} = \overline{z} = 1-s_0$ and (39) holds. Fix $\widehat{h} \in (0, \min\{\underline{z}, 1-\overline{z}\})$ and $\lambda \in (0,1)$. Choose any ϵ, h such that $0 < \epsilon < \rho_0$, $h \in (0, \widehat{h})$.There exists $\widehat{y} > 0$ such that

$$\rho_0 - \epsilon \le \rho(c) \le \rho_0 + \epsilon \text{ for all } c \in (0, \overline{f}(\widehat{y})), \tag{42}$$

and

$$f'(y, r) \geq \lambda \underline{B} \text{ for all } y \in (0, \widehat{y}). \tag{43}$$

By definition of $\underline{z}$ and without loss of generality, there exists a sequence $\{y^n\}_{n=1}^{\infty}$, such that $y^n \in (0, \widehat{y})$ for all n, $y^n \to 0$ as $n \to \infty$, $\{\frac{c(y^n)}{y^n}\}$ is convergent and for all n

$$\underline{z} + h \geq \frac{c(y^n)}{y^n} \geq \underline{z} - h \text{ for all } n, \tag{44}$$

which also implies

$$1 - \underline{z} - h \leq \frac{x(y^n)}{y^n} \leq 1 - \underline{z} + h \text{ for all } n. \tag{45}$$

Note that $\widehat{h} \in (0, \min\{\underline{z}, 1 - \overline{z}\})$ and $h \in (0, \widehat{h})$ implies that the right hand expression of the second inequality in (44) and the left hand expression of the first inequality in (45) are strictly positive. Then, for $\rho > 0$

$$\left(\frac{\underline{z} + h}{1 - (\underline{z} + h)}\right)^{-\rho} \leq \left(\frac{c(y^n)}{y^n}\right)^{-\rho} \left(\frac{x(y^n)}{y^n}\right)^{\rho} \leq \left(\frac{\underline{z} - h}{1 - (\underline{z} - h)}\right)^{-\rho}. \tag{46}$$

Let M_0, M_1 be as follows

$$M_0 = \frac{(\underline{z} + \widehat{h})}{(1 - (\underline{z} + \widehat{h}))(\underline{z} - \widehat{h})\lambda \underline{B}}, \tag{47}$$

$$M_1 = \frac{[1 - (\underline{z} - \widehat{h})]\overline{B}}{(\underline{z} - \widehat{h})}. \tag{48}$$

$\widehat{h} \in (0, \min\{\underline{z}, 1 - \overline{z}\})$ and (40) implies that $0 < M_0 < \infty$ and $0 < M_1 < \infty$. Further,

$$\frac{(\underline{z} + h)}{(\underline{z} - h)(1 - (\underline{z} + h))\lambda \underline{B}} \leq M_0 \text{ for all } h \in (0, \widehat{h}), \tag{49}$$

and

$$\frac{[1 - (\underline{z} - h)]\overline{B}}{(\underline{z} - h)} \leq M_1 \text{ for all } h \in (0, \widehat{h}). \tag{50}$$

From the Ramsey-Euler equation (7) we have for each n:

$$\begin{aligned}\frac{1}{\delta} &= E\left[\frac{u'(c(f(x(y^n),r)))}{u'(c(y^n))} f'(x(y^n),r)\right]\\ &\leq E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0-\epsilon)} f'(x(y^n),r) I_{[f(x(y^n),r)\geq y^n]}\right]\\ &\quad +E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0+\epsilon)} f'(x(y^n),r) I_{[f(x(y^n),r)<y^n]}\right],\end{aligned} \tag{51}$$

where the inequality follows from Lemma 2 and (42). Observe that if $f(x(y^n), r) \geq y^n$

$$\begin{aligned}&\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0-\epsilon)}\\ &= \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0}\\ &\quad \cdot\left(\frac{\{x(y^n)/y^n\}}{\{c(y^n)/y^n\}}\right)^{\epsilon}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\frac{f(x(y^n),r)}{x(y^n)}\right)^{\epsilon}\\ &\leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0}\left(\frac{1-(\underline{z}-h)}{(\underline{z}-h)}\right)^{\epsilon}\overline{B}^{\epsilon}\\ &\leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0} M_1{}^{\epsilon},\end{aligned}$$

where the last two inequalities follow from (44), (45) and (50); thus,

$$\begin{aligned}&E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0-\epsilon)} f'(x(y^n),r) I_{[f(x(y^n),r)\geq y^n]}\right] \leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\\ &\cdot E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0} I_{[f(x(y^n),r)\geq y^n]}\right] M_1{}^{\epsilon}.\end{aligned} \tag{52}$$

On the other hand, if $f(x(y^n), r) < y^n$,

$$\begin{aligned}&\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0+\epsilon)}\\ &= \left(\frac{x(y^n)}{c(y^n)}\right)^{-(\rho_0+\epsilon)}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-(\rho_0+\epsilon)}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-(\rho_0+\epsilon)}\\ &\leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\left(\frac{1-(\underline{z}+h)}{\underline{z}+h}\right)^{-\epsilon}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}(\underline{z}-h)^{-\epsilon}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0}(\lambda\underline{B})^{-\epsilon}\\ &= \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0}\left(\frac{(1-\underline{z}-h)(\underline{z}-h)}{\underline{z}+h}\lambda\underline{B}\right)^{-\epsilon}\\ &\leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0} M_0^{\epsilon},\end{aligned}$$

where the first inequality follows from (43), (44) and (45) and the second inequality uses (49). Thus,

$$E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0+\epsilon)} f'(x(y^n),r)I_{[f(x(y^n),r)<y^n]}\right]$$
$$\leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0} I_{[f(x(y^n),r)<y^n]}\right] M_0^{\epsilon},$$

and combining this with (51) and (52), we have

$$\frac{1}{\delta} \leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right]\max\{M_0^{\epsilon}, M_1^{\epsilon}\},$$

so that using (46) we have

$$\left(\frac{\underline{z}+h}{1-(\underline{z}+h)}\right)^{-\rho_0} \leq \left(\frac{x(y^n)}{c(y^n)}\right)^{\rho_0}$$
$$\leq \delta E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right]\max\{M_0^{\epsilon}, M_1^{\epsilon}\}. \quad (53)$$

For each $r \in A$,

$$\limsup_{n\to\infty}\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right] \leq \underline{z}^{-\rho_0}(B(r))^{1-\rho_0}. \quad (54)$$

Note that $\{\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\}$ is bounded away from zero so that

$$\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}$$

is uniformly bounded above by an integrable function; taking the limsup as $n \to \infty$ on both sides of (53):

$$\left(\frac{\underline{z}+h}{1-(\underline{z}+h)}\right)^{-\rho_0}$$
$$\leq \delta E\left[\limsup_{n\to\infty}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right]\max\{M_0^{\epsilon}, M_1^{\epsilon}\},$$

(see, for instance, Royden, 1988: Problem 12, Chapter 4), and using (54) we have

$$\left(\frac{\underline{z}+h}{1-(\underline{z}+h)}\right)^{-\rho_0} \leq \underline{z}^{-\rho_0}\delta E\left[(B(r))^{1-\rho_0}\right]\max\{M_0^{\epsilon}, M_1^{\epsilon}\}.$$

As this inequality is shown to hold for all $h \in (0, \widehat{h})$ and all $\epsilon \in (0, \rho_0)$ and as M_0, M_1 are independent of ϵ, h, we have

$$\left(\frac{\underline{z}}{1-\underline{z}}\right)^{-\rho_0} \leq \underline{z}^{-\rho_0} \delta E\left[(B(r))^{1-\rho_0}\right],$$

so that

$$(1-\underline{z})^{\rho_0} \leq \delta E\left[(B(r))^{1-\rho_0}\right] = (s_0)^{\rho_0},$$

which yields (41). This completes the proof the lemma. □

Proof of Proposition 1 Part (i) of the proposition follows from Corollary 1; in particular, (37) implies that if $s_0 < 1$, $\liminf_{y \to 0}\left(\frac{x(y)}{y}\right) \geq s_0 = \theta$. On the other hand, $s_0 \geq 1$ implies $\theta = 1$ and using (36), we have $\frac{x(y)}{y} \to 1 = \theta$ as $y \to 0$. This establishes (10). Part (ii) of the proposition follows directly from Lemma 5.

A.2. Proof of Proposition 2

(i) Note that $0 < \theta \leq 1$. Condition **C** implies that $E[\ln B(r)] > 0$. Also observe that as $x(y) > 0$ for all $y > 0$, using (4) we have $\underline{H}(y) = \underline{f}(x(y)) > 0$ for all $y > 0$. We begin by showing that the following holds:

$$\exists\, \widetilde{\sigma} > 0 \text{ such that for every } y > 0,\ M(y) = \sup_t E\left\{\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}}\right\} < \infty. \tag{55}$$

From Hardy et al (1952, pp. 139, Result 187) or alternatively, Lemma B.1 in Kamihigashi (2007, pp. 494):

$$\lim_{\sigma \downarrow 0} \ln\left[E\left(\frac{1}{\theta B(r)}\right)^{\sigma}\right]^{\frac{1}{\sigma}} = E \ln\left(\frac{1}{\theta B(r)}\right).$$

Using Condition **C**, $E \ln\left(\frac{1}{\theta B(r)}\right) = -E \ln \theta B(r) < 0$ and so there exists $\widetilde{\sigma} > 0$ such that

$$\ln\left[E\left\{\left(\frac{1}{\theta B(r)}\right)^{\widetilde{\sigma}}\right\}\right]^{\frac{1}{\widetilde{\sigma}}} < 0,$$

i.e.,

$$E\left\{\left(\frac{1}{\theta B(r)}\right)^{\widetilde{\sigma}}\right\} < 1. \tag{56}$$

We now show that the sequence $\left\{E\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}}\right\}_{t=0}^{\infty}$ is bounded above. Note that $y_t(y)$ is bounded below and above by $\underline{H}^t(y) > 0$ and $\overline{f}^t(y) < \infty$ so that $0 < E\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}} < \infty$ for every t. Using (56), there exists $\epsilon > 0$ small enough so that

$$\lambda = E\left(\frac{1}{(1-\epsilon)B(r)\theta}\right)^{\widetilde{\sigma}} \in (0, 1). \tag{57}$$

Using assumption (**T.3**) and (10), there exists $a > 0$ such that for all $z \in (0, a)$, $r \in A$,

$$\frac{H(z,r)}{z} = \frac{f(x(z),r)}{x(z)}\frac{x(z)}{z} \geq (1-\epsilon)B(r)\theta.$$

Let

$$m = \left(\frac{1}{\underline{H}(a)}\right)^{\widetilde{\sigma}}.$$

Then, $m < \infty$. Note λ, m do not depend on t or the initial stock y. Then,

$$\begin{aligned}
\left(\frac{1}{y_{t+1}(y)}\right)^{\widetilde{\sigma}} &= \left(\frac{1}{H(y_t(y), r_{t+1})}\right)^{\widetilde{\sigma}} \\
&= \left(\frac{1}{H(y_t(y), r_{t+1})}\right)^{\widetilde{\sigma}} I_{[y_t(y)<a]} + \left(\frac{1}{H(y_t(y), r_{t+1})}\right)^{\widetilde{\sigma}} I_{[y_t(y)\geq a]} \\
&\leq \left(\frac{1}{(1-\epsilon)B(r_{t+1})\theta y_t(y)}\right)^{\widetilde{\sigma}} I_{[y_t(y)<a]} + \left(\frac{1}{\underline{H}(a)}\right)^{\widetilde{\sigma}} I_{[y_t(y)\geq a]} \\
&\leq \left(\frac{1}{(1-\epsilon)B(r_{t+1})\theta y_t(y)}\right)^{\widetilde{\sigma}} + m,
\end{aligned}$$

so that taking expectation (with respect to information at time 0):

$$\begin{aligned}
E\left\{\left(\frac{1}{y_{t+1}(y)}\right)^{\widetilde{\sigma}}\right\} &\leq E\left\{\left(\frac{1}{(1-\epsilon)B(r_{t+1})\theta}\right)^{\widetilde{\sigma}}\right\} E\left\{\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}}\right\} + m, \\
&\quad \text{as } y_t(y) \text{ and } r_{t+1} \text{ are independent} \\
&= \lambda E\left[\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}}\right] + m, \text{ using (57).}
\end{aligned} \tag{58}$$

From (58) it follows that the sequence $\left\{E\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}}\right\}_{t=0}^{\infty}$ is Cauchy and hence, convergent and bounded. Thus, (55) holds. Note that $M(y) > 0$. Now, fix any $y > 0$ and choose any $\xi > 0$. Choose $\widehat{\alpha}(y) \in (0, y)$ such that:

$$\widehat{\alpha}(y) \leq \left(\frac{\xi}{M(y)}\right)^{\frac{1}{\widetilde{\sigma}}}. \tag{59}$$

We will show that for all $t \in \mathbb{N}$,

$$P\{y_t(y) < \widehat{\alpha}(y)\} \leq \xi, \tag{60}$$

so that part (i) of the proposition holds. To see that (60) holds for all t, suppose to the contrary that there is some t for which (60) does not hold, i.e.,

$$P\{y_t(y) < \widehat{\alpha}(y)\} > \xi. \tag{61}$$

Then,

$$\begin{aligned}
E\left\{\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}}\right\} &= E\left[\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}} I_{[y_t(y)<\widehat{\alpha}(y)]} + \left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}} I_{[y_t(y)\geq\widehat{\alpha}(y)]}\right] \\
&\geq E\left[\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}} I_{[y_t(y)<\widehat{\alpha}(y)]}\right] \geq \left(\frac{1}{\widehat{\alpha}(y)}\right)^{\widetilde{\sigma}} P\{y_t(y) < \widehat{\alpha}(y)\} \\
&> \left(\frac{1}{\widehat{\alpha}(y)}\right)^{\widetilde{\sigma}} \xi \text{ , using (61),} \\
&\geq M(y), \text{ using (59),}
\end{aligned}$$

which contradicts (55). Thus (60) holds for all t. This establishes (13).

(ii) As $c(\widetilde{y}) > 0$ and $x(\widetilde{y}) > 0$ for all $\widetilde{y} > 0$, it is sufficient to show that $\Pr\{\limsup_{t\to\infty} y_t(y) > 0\} = 1$. Fix any $\xi > 0$ and let $\widehat{\alpha}(y) > 0$ be chosen so that (13) holds. Observe that

$$\{\omega \in \Omega : \lim_{t\to\infty} y_t(y) = 0\} \subset \cup_{T=0}^{\infty}\{\omega \in \Omega : y_t(y) < \widehat{\alpha}(y) \text{ for all } t \geq T\},$$

and as the sets $\{\omega \in \Omega : y_t(y) < \widehat{\alpha}(y)$ for all $t \geq T\}$ are nested and expanding in T,

$$\begin{aligned}
\Pr\{\lim_{t\to\infty} y_t(y) = 0\} &\leq \lim_{T\to\infty} \Pr\{y_t(y) < \widehat{\alpha}(y) \text{ for all } t \geq T\} \\
&\leq \limsup{}_{T\to\infty} \Pr\{y_T(y) < \widehat{\alpha}(y)\} \leq \xi, \text{ using(13).}
\end{aligned}$$

As ξ is arbitrary, $\Pr\{\lim_{t\to\infty} y_t(y) = 0\} = 0$, i.e., $\Pr\{\limsup_{t\to\infty} y_t(y) > 0\} = 1$.

(iii) Follows immediately from (ii).

A.3 Proof of Proposition 3

Consider finite horizon version of the stationary stochastic dynamic optimization problem outlined in Sect. 2. In particular, for $T \in \mathbb{N}$, and given initial stock $y \geq 0$, the agent maximizes:

$$E\left[\sum_{t=0}^{T} \delta^t u(\widetilde{c}_t)\right]$$

over a feasible stochastic process $\{\widetilde{y}_t, \widetilde{c}_t, \widetilde{x}_t\}_{t=0}^{T}$ where $\widetilde{y}_0 = y$, where $\{\widetilde{c}_t, \widetilde{x}_t\}$ are $\mathcal{F}_t$ adapted where $\mathcal{F}_t$ is the (sub) σ-field generated by partial history from periods 0 through t and:

$$\begin{aligned}&\text{(i) } \widetilde{c}_t \geq 0, \widetilde{x}_t \geq 0 \ \text{ for } t = 0, 1, \ldots T\\ &\text{(ii) } \widetilde{c}_t + \widetilde{x}_t \leq \widetilde{y}_t,\ \ \widetilde{y}_{t+1} = f(\widetilde{x}_t, r_{t+1}) \text{ for } t = 0, 1, \ldots T\end{aligned}$$

Note that there is no terminal stock requirement in period T. Standard arguments can be used to establish that there exists a unique optimal decision rule in each period t and it depends only on the number of periods left till the end of the time horizon.[26]

Lemma 6 *Consider the $T-$period finite horizon problem. There exist (unique) optimal consumption and investment functions denoted by $c^\tau(y)$ and $x^\tau(y)$ that depend only on τ, the number of periods remaining till the end of the time horizon; in any period $t = T - \tau$, it is optimal to consume $c^\tau(y)$ and invest $x^\tau(y)$ if current output is y. For all $y > 0, c^\tau(y) > 0$ for all $\tau \in \mathbb{N}$, $x^\tau(y) > 0$ for all $\tau \in \mathbb{N}_+$. For every $\tau \in \mathbb{N}_+$, $c^\tau(y)$ and $x^\tau(y)$ are continuous and strictly increasing on $\mathbb{R}_+$. The following stochastic Ramsey-Euler equation holds for all $\tau \in \mathbb{N}$ and $y > 0$,*

$$u'(c^{\tau+1}(y)) = \delta E[u'(c^\tau(f(x^{\tau+1}(y), r)))f'(x^{\tau+1}(y), r)]. \tag{62}$$

Proof Using induction on τ and fairly standard arguments as in the infinite horizon case. □

The next lemma establishes a uniform *upper* bound on the optimal propensity to invest in finite horizon problems.

Lemma 7 *Assume (**U.4**) and Condition B. Further, suppose that $\theta < 1$.Then, the following hold:*

(i) *For every $\tau \in \mathbb{N}$ and $y > 0$.*

$$\frac{x^\tau(y)}{y} \leq \theta. \tag{63}$$

[26] See, among others, Majumdar and Zilcha (1987).

(ii) *The finite horizon optimal investment functions converge point-wise to the optimal investment function for the infinite horizon problem, i.e.,*

$$\lim_{\tau\to\infty} x^{\tau}(y) = x(y) \text{ for all } y \geq 0.$$

Proof (i) Note that as $\theta < 1$

$$\theta = s_0 = \left[\delta E((B(r))^{1-\rho_0})\right]^{1/\rho_0} < 1.$$

As $x^0(y) = 0$ for all $y > 0$, (63) holds for $\tau = 0$. Suppose (63) holds for $\tau = t \in \mathbb{N}$. For every $y > 0$, we have from (62) that:

$$(c^{t+1}(y))^{-\rho_0} = \delta E[(c^t(f(x^{t+1}(y), r)))^{-\rho_0} f'(x^{t+1}(y), r)],$$

and as (63) holds for $\tau = t$, $c^t(y) \geq (1-\theta)y$ for all $y > 0$, we have

$$\begin{aligned}(c^{t+1}(y))^{-\rho_0} &= \delta E[(c^t(f(x^{t+1}(y), r)))^{-\rho_0} f'(x^{t+1}(y), r)] \\ &\leq \delta E[((1-\theta)(f(x^{t+1}(y), r)))^{-\rho_0} f'(x^{t+1}(y), r)],\end{aligned}$$

which implies that:

$$\begin{aligned}\left(\frac{c^{t+1}(y)}{y}\right)^{-\rho_0} &\leq \delta(1-\theta)^{-\rho_0} E\left[\left(\frac{f(x^{t+1}(y), r)}{x^{t+1}(y)}\right)^{-\rho_0} f'(x^{t+1}(y), r)\right]\left[\frac{x^{t+1}(y)}{y}\right]^{-\rho_0} \\ &\leq \delta(1-\theta)^{-\rho_0} E\left[(B(r))^{1-\rho_0}\right]\left[\frac{x^{t+1}(y)}{y}\right]^{-\rho_0}, \text{ (using Condition } B) \\ &= (1-\theta)^{-\rho_0}\theta^{\rho_0}\left[\frac{x^{t+1}(y)}{y}\right]^{-\rho_0},\end{aligned}$$

so that

$$\left(\frac{x^{t+1}(y)}{y}\right)\left(1 - \frac{x^{t+1}(y)}{y}\right)^{-1} \leq \frac{\theta}{1-\theta},$$

which yields:

$$\frac{x^{t+1}(y)}{y} \leq \theta.$$

Thus, (63) holds for all $\tau \in \mathbb{N}$.

(ii) The policy convergence follows from the sufficient conditions in Schäl (1975). In particular, one can check that both assumptions (GA) and (C) in Sect. 2 of that paper are satisfied.[27] □

Proposition 3 follows immediately from parts (i) and (ii) of the above lemma.

A.4 Proof of Proposition 4

First, consider the case where $E[\ln(B(r))] < 0$. We will show that in this case, every *feasible* stochastic process must converge to zero with probability one. This is similar to the result reported in Kamihigashi (2006). Given any initial stock $y > 0$, any feasible stochastic process $\{\tilde{y}_t(y), \tilde{c}_t(y), \tilde{x}_t(y)\}$ satisfies $\tilde{y}_t(y) \leq y_t^M(y)$ where $\{y_t^M(y)\}$ is the stochastic process defined by $y_0^M = y$, $y_{t+1}^M = f(y_t^M, r_{t+1})$ for all $t \geq 0$; as $f(y_t^M, r) \leq B(r) y_t^M$ for all $r \in A$, it is easy to check that for all $t \geq 1$:

$$\ln \tilde{y}_t(y) \leq \ln y_t^M \leq \ln y + \sum_{i=1}^{t} \ln B(r_i) = \ln y + t\left[\frac{1}{t}\sum_{i=1}^{t} \ln B(r_i)\right]. \tag{64}$$

and as r_t's are i.i.d., the strong law of large numbers implies $\frac{1}{t}\sum_{i=1}^{t} \ln B(r_t) \to E[\ln(B(r))]$ with probability one as $t \to \infty$; $E[\ln(B(r))] < 0$ then implies that as $t \to \infty$, the right hand side of (64) converges to $-\infty$ and $\tilde{y}_t(y) \to 0$ with probability one.

Next, we consider the case where $E[\ln(B(r))] \geq 0$. Condition (16) of the proposition then implies that $\theta < 1$, i.e., $\theta = s_0 < 1$. Consider the stochastic process $\{y_t(y), c_t(y), x_t(y)\}$ generated by the optimal policy from initial stock $y \geq 0$. The result is trivial if $y = 0$. So, consider $y > 0$. Using Proposition 3 we have

$$H(y, r) = f(x(y), r) = \frac{f(x(y), r)}{x(y)} x(y) \leq B(r) x(y) \leq B(r)\theta y,$$

so that for all $t \geq 1$

$$y_t(y) = H(y_{t-1}(y), r_t) \leq B(r_t)\theta y_{t-1}(y),$$

which can be used to show that

$$\ln y_t(y) \leq \ln y + \sum_{i=1}^{t} \ln \theta B(r_i) = \ln y + t\left[\frac{1}{t}\sum_{i=1}^{t} \ln \theta B(r_i)\right].$$

Using similar arguments as above, (16) implies that as $t \to \infty$, $y_t(y) \to 0$ with probability one.

[27] If $\rho_0 > 1$, we are in the "negative" case in Schäl (1975) and assumption (C) specified in that paper always holds (see discussion in Sect. 2 of that paper). If $\rho_0 \leq 1$, our assumption **(D.1)** implies that assumption (C) in Schäl (1975) holds.

A.5 Proof of Proposition 5

Note that (17) implies $E[\ln B(r)] > 0$ which implies that Condition **C** holds if $\theta = 1$. If $\theta < 1$, then as mentioned in Sect. 4, Condition **C** holds if and only if (12) holds. Using Jensen's inequality,

$$\left[\left\{\ln E\left((B(r))^{1-\rho_0}\right)\right\} - \left\{E\ln(B(r)^{1-\rho_0})\right\}\right] \geq 0,$$

so that (17) implies that (12) holds.

A.6 Proof of Proposition 6

(a) At $\rho_0 = 1$, $\theta = \delta$ so that $E[\ln(\delta B(r))] < 0$ implies that (16) holds for ρ_0 close to 1 (θ being continuous in ρ_0); the result then follows from Proposition 4.

(b) Consider $\rho_0 > 1$. If $\theta = 1$, then $0 < E[\ln B(r)]$ immediately implies Condition **C** holds. So, consider $\theta < 1$. Here, $\theta = s_0$ in which case Condition **C** holds if, and only if, (12) holds, i.e.,

$$E[\ln(\delta B(r))] + q(\rho_0) > 0, \tag{65}$$

where the function $q(\rho)$ is given by

$$q(\rho) = \left[\left\{\ln E\left((B(r))^{1-\rho}\right)\right\} - \left\{E\ln(B(r)^{1-\rho})\right\}\right] \text{ for } \rho > 1,$$

and $q(\rho) > 0$ as the distribution of $B(r)$ is non-degenerate. We will show that $q(\rho) \to +\infty$ as $\rho \to \infty$ so that (65) (and therefore Condition **C**) holds for all ρ_0 large enough. Observe that

$$q(3) = \left[\ln E\left(\frac{1}{B(r)}\right)^2 + 2E\ln B(r)\right] > 0.$$

Consider any $\rho > 3$. Then

$$\begin{aligned}
q(\rho) &= (\rho - 1)\left[\frac{1}{\rho - 1}\left\{\ln E\left((B(r))^{1-\rho}\right)\right\} + E\ln(B(r))\right] \\
&= (\rho - 1)\left[\ln\left\{E\left(\frac{1}{B(r)}\right)^{\rho-1}\right\}^{\frac{1}{\rho-1}} + E\ln(B(r))\right] \\
&\geq (\rho - 1)\left[\frac{1}{2}\ln\left\{E\left(\frac{1}{B(r)}\right)^2\right\} + E\ln(B(r))\right] \\
&= \frac{(\rho - 1)}{2}q(3) \to +\infty \text{ as } \rho \to \infty.
\end{aligned}$$

where the inequality in the third line follows from Liapounov's Inequality.[28] This completes the proof of part (b).

(c) As $s_0 \to \delta EB(r)$ when $\rho_0 \to 0$, $\delta EB(r) > 1$ implies that $\theta = 1$ for all ρ_0 close enough to 0; $0 < E[\ln B(r)]$ then implies that Condition **C** holds for ρ_0 small enough.

A.7 Proof of Proposition 7

Under (**T.5**), the functions H, $\overline{H}$ and $\underline{H}$ defined in Sect. 2 are continuous and strictly increasing in y. Further, for all $y \in (0, K]$ and $r \in A$,

$$K \geq \overline{H}(y) \geq H(y, r) \geq \underline{H}(y) > 0.$$

As $x(y) > 0$ for all $y > 0$, using assumption (**T.4**) and (**T.5**), $\overline{H}(y) > \underline{H}(y)$ for all $y > 0$ and $\overline{H}(y) < y$, for all $y > K$. Define

$$\beta = \inf\{y > 0 : \overline{H}(y) \leq y\}. \tag{66}$$

We begin by stating and proving a useful lemma.

Lemma 8 *Assume* (**T.3**), (**T.4**), (**T.5**) *and Condition* **C**. *Then,* $\beta > 0$. *Further, (i)* $\overline{H}(y) > y$, *for all* $y \in (0, \beta)$, *(ii)* $\overline{H}(\beta) = \beta$ *and (iii)* $\underline{H}(y) < y$ *for all* $y \geq \beta$.

Proof From Proposition 1, we have $\liminf_{y \to 0} \frac{x(y)}{y} \geq \theta$. Observe that $E(\ln \theta B(r)) \leq \ln E(\theta B(r))$ so that Condition **C** implies $\theta E(B(r)) > 1$. Choose $\lambda_0 \in (\frac{1}{\theta E(B(r))}, 1)$. There exists $\epsilon > 0$ such that for all $y \in (0, \epsilon]$ and $r \in A$,

$$\frac{H(y, r)}{y} = \frac{f(x(y), r)}{y} = \frac{f(x(y), r)}{x(y)} \frac{x(y)}{y} \geq B(r)\lambda_0\theta,$$

so that $E(H(y, r)) \geq \lambda_0 E(B(r))\theta y > y$ which implies that $\overline{H}(y) > y$ for all $y \in (0, \epsilon]$. Thus, $K \geq \beta > \epsilon > 0$ and $\overline{H}(\beta) = \beta$. This also implies that $\underline{H}(\beta) < \beta$. From the Ramsey-Euler equation (7):

$$\begin{aligned} u'(c(\beta)) &= \delta E[u'(c(H(\beta, r))) f'(x(\beta), r)] \\ &> \delta E[u'(c(\overline{H}(\beta))) f'(x(\beta), r)] \\ &= \delta E[u'(c(\beta)) f'(x(\beta), r)], \end{aligned}$$

so that $\delta E[f'(x(\beta), r)] < 1$ (the strict inequality in the second line follows from the fact that $c(y)$ is strictly increasing, $H(\beta, r) \leq \overline{H}(\beta)$ with probability one and $H(\beta, r) < \overline{H}(\beta)$ with strictly positive probability using assumption (**T.5**)). As $x(y)$ is strictly increasing in y we have that for all $y > \beta$, $\delta E[f'(x(y), r)] < 1$. Once again using the Ramsey-Euler equation, for all $y > \beta$,

[28] See Chung (1974, pp. 47): $E(|X|^k)^{\frac{1}{k}} \leq E(|X|^m)^{\frac{1}{m}}$ where $1 < k < m < \infty$.

$$u'(c(y)) = \delta E[u'(c(H(y,r)))f'(x(y),r)]$$
$$\leq \delta u'(c(\underline{H}(y)))E[f'(x(y),r)] < u'(c(\underline{H}(y))),$$

and as $c(y)$ is strictly increasing, this implies that $\underline{H}(y) < y$ for all $y > \beta$. This completes the proof of the lemma. □

Proof of Proposition 7 The proof is entirely based on Kamihigashi and Stachurski (2014), hereafter K–S. Let $S = (0, K]$ be the state space and let $\mathcal{F}$ be the set of all Borel subsets of S. Let Q be the associated kernel defined by:

$$Q(x, B) = \Pr\{H(x,r) \in B\}, \forall B \in \mathcal{F}.$$

Define the notion of stationary (invariant) distribution, unique stationary distribution and global stability of Q on S as in Sect. 2.1 in K–S. Note that global stability of Q on S is equivalent to the existence of a unique invariant distribution on S and convergence (in distribution) to this invariant distribution from all $y \in S$. Let P be the probability measure defined on the product space in the usual manner. From Theorem 1 in K–S, global stability of Q is established if: (a) Q is increasing, (b) Q has an excessive distribution, (c) Q is order reversing, and (d) Q is bounded in probability. These concepts are formally defined in Sects. 2.1 and 2.2 of K–S.

From Remark 3 in K–S, it follows that since $H(y, r)$ is strictly increasing in y, Q is increasing. As S has a greatest element (namely, K), Q has an excessive distribution (see Remark 2 in K–S). We now show that Q is order reversing. From Lemma 8, $\underline{H}(y) < y$ for all $y \in [\beta, K]$. As $\underline{H}$ is continuous, there exists $\widetilde{y} \in (0, \beta)$ such that $\underline{H}(y) < y$ for all $y \in [\widetilde{y}, K]$. It is sufficient to show that for any $y^1, y^2 \in (0, K]$, $y^2 \geq y^1$ there exists $t \in \mathbb{N}_+$ such that

$$P\{y_t(y^2) \leq \widetilde{y}\} > 0 \text{ and } P\{y_t(y^1) \geq \widetilde{y}\} > 0.$$

We first show that there exists $\tau_1 \in \mathbb{N}$ such that for any $t \geq \tau_1$, $\underline{H}^t(y^2) < \widetilde{y}$ (where $\underline{H}^i(.) = \underline{H}(\underline{H}^{i-1}(.))$, $\underline{H}^1 = \underline{H}$). To see this, first note that as $\underline{H}$ is strictly increasing (under assumption (**T.5**)), if $\underline{H}^t(y^2) < \widetilde{y}$ for some t, then $\underline{H}^{t+1}(y^2) = \underline{H}(\underline{H}^t(y^2)) < \underline{H}(\widetilde{y}) < \widetilde{y}$ and by induction, $\underline{H}^{t+i}(y^2) < \widetilde{y}$ for all $i \geq 1$. Now, suppose that for all $t = 1, \ldots \infty$, $\underline{H}^t(y^2) \geq \widetilde{y}$. Then, $\underline{H}^t(y^2) \in [\widetilde{y}, K]$ for all t and as $\underline{H}(y) < y$ for all $y \in [\widetilde{y}, K]$, $\underline{H}^{t+1}(y^2) = \underline{H}(\underline{H}^t(y^2)) < \underline{H}^t(y^2)$ i.e., the sequence $\{\underline{H}^t(y^2)\}_{t=1}^{\infty}$ is a strictly decreasing and bounded sequence that converges to some $w \in [\widetilde{y}, K]$. As $\underline{H}^t(y^2) = \underline{H}(\underline{H}^{t-1}(y^2))$ and $\underline{H}$ is continuous, we have $\underline{H}(w) = w$ which contradicts $\underline{H}(y) < y$ for all $y \in [\widetilde{y}, K]$. Thus, there exists $\tau_1 \in \mathbb{N}$ such that for all $t \geq \tau_1$, $\underline{H}^t(y^2) < \widetilde{y}$. Using assumption (**T.5**) we then have,

$$P\{y_t(y^2) < \widetilde{y}\} > 0 \text{ for all } t \geq \tau_1 \tag{67}$$

From Lemma 8, $\overline{H}(y) > y$ for all $y \in (0, \beta)$ and further, $\overline{H}(y) \geq \min\{y, \beta\}$ for all $y \in (0, K]$. We now show that there exists $\tau_2 \in \mathbb{N}$ such that for any $t \geq \tau_2$, $\overline{H}^t(y^1) > \widetilde{y}$ (where $\overline{H}^i(.) = \overline{H}(\overline{H}^{i-1}(.))$, $\overline{H}^1 = \overline{H}$). To see this, first note that as $\overline{H}$ is strictly increasing under assumption (**T.5**), if $\overline{H}^t(y^1) > \widetilde{y}$ for some t, then

$\overline{H}^{t+1}(y^1) = \overline{H}(\overline{H}^t(y^1)) > \overline{H}(\widetilde{y}) > \widetilde{y}$ and by induction, $\overline{H}^{t+i}(y^1) > \widetilde{y}$ for all $i \geq 1$. Now, suppose that for all $t = 1, \ldots \infty$, $\overline{H}^t(y^1) \leq \widetilde{y}$. Then, $\overline{H}^t(y^1) \in (0, \widetilde{y}] \subset (0, \beta)$ for all t so that $\overline{H}^{t+1}(y^1) = \overline{H}(\overline{H}^t(y^1)) > \overline{H}^t(y^1)$ i.e., the sequence $\{\overline{H}^t(y^1)\}$ is a strictly increasing and bounded sequence that converges to some $w' \in (0, \widetilde{y}]$. As $\overline{H}(\overline{H}^t(y^1)) = \overline{H}^t(y^1)$ for each t and $\overline{H}$ is continuous, we have $\overline{H}(w') = w'$ which contradicts $\overline{H}(y) > y$ for all $y \in (0, \beta)$.Thus, there exists $\tau_2 \in \mathbb{N}$ such that for all $t \geq \tau_2$, $\overline{H}^t(y^1) > \widetilde{y}$. Using assumption (**T.5**) we then have,

$$P\{y_t(y^1) > \widetilde{y}\} > 0 \text{ for all } t \geq \tau_2 \tag{68}$$

For $t \geq \max\{\tau_1, \tau_2\}$, both (67) and (68) hold and thus, Q is order reversing. Finally, we show that Q is bounded in probability, i.e., the sequence $\{Q^t(y, .)\}$ is tight for all $y \in S$. Here, Q^t is the t-th order kernel giving the probability of transiting from y to $B \in \mathcal{F}$ in t steps and formally defined by

$$Q^1 = Q, Q^t(y, B) = \int Q^{t-1}(z, B) Q(y, dz).$$

Now, for any $y \in S$, the sequence $\{Q^t(y, .)\}$ is tight if for any $\xi > 0$, there exists a compact set $D \subset S$ such that $Q^t(y, S - D) \leq \xi$ for all t. Proposition 2(i) shows that for any $y \in (0, K]$ and for any $\xi > 0$, there exists $\widehat{\alpha}(y) > 0$ such that

$$P\{y_t(y) < \widehat{\alpha}(y)\} < \xi \text{ for all } t.$$

Defining $D = [\,\widehat{\alpha}(y), K]$, we can see that Q is bounded in probability. The proof is complete.

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