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RESEARCH ARTICLE

Dynamic bargaining with voluntary participation and externalities

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Abstract
Rational and self-interested players are motivated to free-ride on an efficient agreement in economies with externalities. To provide a non-cooperative foundation of the Coase theorem, we consider a dynamic bargaining game for side-payment contracts. Players voluntarily participate in negotiations. If all players do not, then any contract is renegotiated. When the probability of negotiations stopping is sufficiently small, there exists an efficient Markov perfect equilibrium where all players immediately participate in the grand coalition. The agreement converges to the Nash bargaining solution as the stopping probability goes to zero. We further show that for any probability of stopping, all players form the grand coalition in finitely many rounds in every pure strategy Markov perfect equilibrium unless the game stops on the way.

Keywords Coase theorem · Efficiency · Externality · Nash bargaining solution · Side-payment contract · Participation

JEL Classification C72 · C78 · H41

1 Introduction

The Coase theorem (1960) embodies a widely shared view on efficiency among economists. The theorem states that if property rights are well defined and there are no transaction costs, rational and fully informed agents achieve an efficient outcome.[1] Voluntary bargaining is considered to be a decentralized solution to inefficiency in

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[1] For a detailed description of the Coase theorem, see Cooter (1989).

economies with externalities. A simple and powerful idea supporting the theorem is that if an agreement is inefficient, rational agents will renegotiate it to achieve a Pareto-improving outcome. However, the theorem implicitly assumes that all agents participate in bargaining. This assumption needs be justified. For example, in the presence of externalities, economic agents have an incentive to free-ride on an efficient agreement between other agents, and thus not to participate in bargaining. Despite its popularity, the literature has yet to provide a non-cooperative bargaining foundation for the Coase theorem. In order to do this, we need to build a non-cooperative bargaining model that explicitly includes both voluntary participation and renegotiation. This study aims to provide such a model and to show how and when the Coase theorem holds in a dynamic process of multilateral bargaining.

To illustrate the problem, we begin with an example of a public good game.

Example 1 There are three players, each endowed with one money unit. They choose independently how much to contribute to a public good. Let $a_i \in [0, 1]$ be a contribution of player $i = 1, 2, 3$. Player i's payoff is given by $0.6 \sum_{j=1}^{3} a_j + 1 - a_i$. Each player's dominant action is no contribution. If all three players participate in negotiations for the joint provision of the public good, the efficient provision is attained by the full contribution profile $(1, 1, 1)$, with payoffs $(1.8, 1.8, 1.8)$. Suppose that one player, say 1, does not participate and thus does not contribute. Then, the payoff of the remaining players ($i = 2, 3$) is given by $0.6 \sum_{j \neq 1} a_j + 1 - a_i$. The total payoff is $0.2 \sum_{j \neq 1} a_j + 2$, which is maximized by the full contribution profile $(1, 1)$. This generates payoffs of 1.2 to players 2 and 3, and a payoff of 2.2 to player 1.[2] Thus, player 1 has an incentive not to participate in negotiations, resulting in free-riding on the public good provided by the other players. Inefficiency is caused by player 1's non-participation, and thus the Coase theorem seems to fail. This, however, is not the end of a story. Because the total payoff for two participants and one free-rider (4.6) is smaller than the efficient outcome 5.4, there exists a Pareto improvement. Thus, the three players have an incentive to renegotiate their contributions, as the payoff profile $(2.2, 1.2, 1.2)$ will prevail if the renegotiation fails. For example, they might all agree to split the surplus 0.8 equally, which would result in an efficient outcome with payoff profile $(2\frac{7}{15}, 1\frac{7}{15}, 1\frac{7}{15})$. However, by anticipating this renegotiation, players 2 and 3 should recognize that once player 1 opts not to participate, they would be better off disagreeing and, if the bargaining friction is small, could instead renegotiate with player 1 from a symmetric position in order to obtain 1.8. As a result, player 1 would be worse off, and non-participation would turn out to be not beneficial. Efficiency would be attained through voluntary bargaining even under the existence of an incentive to non-participation.

In this paper, we formalize the logic of the efficient bargaining illustrated in the example by presenting a non-cooperative bargaining model with renegotiations.

The Nash bargaining solution is the most widely accepted bargaining solution, applied to many economic, political and environmental problems. As we have seen in

[2] The example is described in a partition function form with transferable utility such that $v(\{1, 2, 3\}) = 5.4$, $v(\{1\}|[\{1\}, \{2, 3\}]) = 2.2$, $v(\{2, 3\}|[\{1\}, \{2, 3\}]) = 2.4$, $v(\{i\}|[\{1\}, \{2\}, \{3\}]) = 1$ for $i = 1, 2, 3$. The values for other coalition structures are symmetrically given.

the example, players may be motivated to free-ride on the Nash bargaining solution, and as a result, efficient bargaining may be deteriorated. The issue of free-riding is particularly relevant to the context of global public goods (Buchholz and Sandler 2021). A prominent example is climate change. Countries are motivated to free-ride on a proposed mechanism to solve climate change. Caparrós (2016) surveys game theoretic papers that apply non-cooperative bargaining models to international environmental agreements.

Our model is briefly described as follows. Negotiations of side-payment contracts take place over (possibly) infinitely many rounds before n players play an underlying strategic-form game. A side-payment contract is contingent on the participants' actions. Each round $k = 1, 2, \cdots$ is characterized by a state variable (S_k, t^k), where S_k denotes the set of players who formed a coalition in previous rounds, and t^k is an "on-going" contract of side payments in S_k. At the beginning of round k, all non-participants decide independently whether to participate in negotiations. Thereafter, negotiations take place between the set S_{k+1} of all incumbent and new participants. The bargaining procedure is of a Rubinstein-type with random proposers. Specifically, one player is selected as a proposer according to a predetermined probability distribution θ over the set of participants. The proposer chooses a side-payment contract t^{k+1} for S_{k+1}, which all other players in S_{k+1} either accept or reject sequentially. If all players accept t^{k+1}, it becomes an "on-going" contract, replacing t^k. If not, the contract t^k remains effective. Thereafter, the negotiations stop with probability ϵ, after which the underlying game is played under the final contract t^{k+1} (and t^k in case of no agreement). Non-participants respond to the final contract optimally. The negotiations continue in the next round $k + 1$ with probability $1 - \epsilon$ and with a new state (S_{k+1}, t^{k+1}), following the same procedure as in round k.

The model has several features. A final contract is binding and can be enforced effectively. Non-participants respond optimally to the contract. Participants negotiate for a contract, anticipating the responses by non-participants. The model captures the strategic interdependence between participants and non-participants. A side-payment contract plays two roles. First, it improves efficiency, defined as surplus maximization. Second, it serves as a device to enforce any action profile by giving players an incentive to play it (see Lemma 1). When participation is mandatory, the model can be reduced to the standard bargaining problem of splitting a cake of fixed size where the size of a cake is given by the maximum total welfare. The model has two restrictive assumptions. A coalition does not shrink but may expand in the process of renegotiation. The formation of multiple coalitions is ruled out. We shall discuss in Section 4 how we can relax these assumptions. The assumption of a single coalition seems reasonable in a context of global public goods where there exists a single kind of public goods such as climate change.

We consider the existence and efficiency of a Markov perfect equilibrium in the bargaining game. We first prove that for every sufficiently small $\epsilon > 0$, there exists an efficient Markov perfect equilibrium where all players participate in negotiations in the first round (Theorem 1). When ϵ vanishes, the agreement converges to an asymmetric Nash bargaining solution, regardless of a proposer. The disagreement point of the Nash bargaining solution is given by a Nash equilibrium of the underlying game. If any round with a state (S_k, t^k) is reached in off-equilibrium play, then all non-participants participate in the negotiations, and the on-going contract t^k is renegotiated to the Nash bargaining solution where the disagreement point is a Nash equilibrium under t^k.

The reason that every player participates in the agreement of the Nash bargaining solution can be intuitively explained as follows. A player is motivated not to participate in the negotiations, provided that all remaining participants make an agreement and that non-participation is better off. It, however, is impossible that the two conditions hold simultaneously. If the remaining players form the coalition, then an agreed-upon allocation is renegotiated to an efficient one involving the non-participant, which has the same total welfare as the Nash bargaining solution. Thus, there exists a trade-off of payoffs between participants and the non-participant. If the non-participant is better off, then the remaining players are worse off than in the Nash bargaining solution, and thus it is optimal for them to break down the negotiations without the non-participant, and to restart the bargaining game from a symmetric position.

We further show the dynamic efficiency of every Markov perfect equilibrium (Theorem 2). Specifically, we prove for any probability $\epsilon > 0$ of stopping that, starting from an inefficient state with non-participants, a coalition expands in every pure strategy Markov perfect equilibrium, under a super-additivity condition of the underlying game. A coalition gradually expands to the largest one unless the game stops on the way. When the bargaining friction vanishes, all players participate in negotiations and form the grand coalition in at most n rounds with probability approaching one.

Finally, we review three strands in the literature closely related to this paper. First, our result contributes to a large body of the literature on the Nash Program which obtains a suitable bargaining solution as an equilibrium for a non-cooperative bargaining game (Nash 1953 and Binmore et al. 1992). Serrano (2021) provides a recent survey of the literature. Since the seminal paper of Rubinstein (1982), a number of non-cooperative sequential bargaining models for the Nash bargaining solution have been proposed not only for bilateral bargaining but also for multilateral bargaining without subcoalitions (Laruelle and Valenciano 2008 and Britz et al. 2010 among others) and with subcoalitions (Okada 2010). To our knowledge, most existing works assume that all players participate in negotiations, and show that the Nash bargaining solution is agreed on in a decentralized bargaining process, given the full participation. Our result extends a non-cooperative support of the Nash bargaining solution to multilateral bargaining situations where players voluntarily participate in negotiations.

Second, this paper shares several features with the literature of non-cooperative coalitional bargaining. It is well-known that an efficient allocation is not always achieved in a super-additive characteristic-function game (Chatterjee et al. 1993 and Okada 1996). Ray and Vohra (1999) extend the analysis to partition-function games with externality. Seidmann and Winter (1998) and Okada (2000) incorporate a phase of renegotiations into their sequential bargaining models, and show that every Markov perfect equilibrium under renegotiations dynamically attains an efficient allocation when the grand coalition is efficient. The result is extended to a history-dependent equilibrium (Hyndman and Ray 2007) and to partition-function games (Gomes 2005, and Gomes and Jehiel 2005). The model in the paper has the feature that a coalition does not break up, as in earlier works (Seidmann and Winter 1998; Okada 2000 and Gomes 2005). Gomes and Jehiel (2005) show asymptotic efficiency in a general coalitional transition process[3] under the condition that there exists an efficient "negative externality-free" state. Their condition does not hold in the public goods game, which is the main example of our model.

Our model and the existing ones mentioned above consider the same problem, that is, efficiency in coalition formation, in different ways. In our view, they are complementary.[4] Most bargaining models in the literature employ proposal-response protocols where a selected player proposes to other players a coalition, which is formed by unanimous consent. Players can join coalitions by invitation only. Speaking metaphorically, these models presume that all players have already come together in a meeting room, and analyze whether and how they can voluntarily reach an efficient agreement. However, if any efficient agreement is expected in the room, some players may not show up, hoping to free-ride. Our model of voluntary participation considers such a possibility of non-participation. Dixit and Olson (2000) forcefully argue the "voluntary" nature of Coasian bargaining in that individuals should have the right to decide freely whether to participate. In many international treaty negotiations, there exists a pre-negotiation stage where countries decide whether to participate in negotiations. Our model is suitable for the analysis of such bargaining situations.

Third, voluntary participation models have been extensively studied in the literature of stable coalitions (D'Aspremont et al. 1983; Barrett 1994; Dixit and Olson 2000; Carraro et al. 2006; Karp and Simon 2013 among others). Most works in the literature consider a two-stage game where all players decide whether to participate in a coalition in the first stage, and they choose their actions non-cooperatively in the second stage.[5] A Nash equilibrium in the first stage of participation corresponds to a stable coalition where no single participant is better-off by opting out (internal stability), and no single outsider is better-off by opting in (external stability). While an equilibrium coalition size can be any integer below the population size, depending on specific forms of payoff functions, it is typically inefficient and, in some cases very small. This inefficiency result is caused by the static nature of the standard model in that participation is only once. Our model provides a dynamic version of it where players have multiple opportunities to participate. We show the efficiency in dynamic coalitional bargaining with participation.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 provides the theorems. Section 4 discusses some extensions and concludes. All proofs are provided in the Appendix.

[3] Gomes and Jehiel (2005) allow a coalition to be broken into sub-coalitions, provided that the transition is approved by all affected players. Hyndman and Ray (2007) also employ a similar transition rule.

[4] In another modeling aspect, the primitive of our model is a strategic form game, whereas most existing models are built on games in characteristic function form and in partition function form.

[5] Usually, a group of participants are assumed to act as a single player who maximizes the group welfare under a fixed allocation rule.

2 The Model

We first introduce our notation. Let $G = (N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$ be an n-person game in strategic form. Here, $N = \{1, \cdots, n\}$ is the set of players, and A_i is a finite set of player i's pure actions a_i. For a non-empty subset S of N (possibly $S = N$), let $A^S = \Pi_{i \in S} A_i$. The product A^S is the set of pure action profiles $a_S = (a_i)_{i \in S}$ for players in S. We denote $A = A^N$. For an action profile $a = (a_i)_{i \in N}$ in A, we often use the notation $a = (a_S, a_{N-S})$, where $a_S \in A^S$ and $a_{N-S} \in A^{N-S}$. When $S = \{i\}$, we also employ the notation $a = (a_i, a_{-i})$, where a_{-i} denotes the action profile other than a_i in a. Player i's payoff function u_i is a real-valued function on A. A probability distribution on A_i is called a mixed action for player i. For a mixed action profile of players, the expected payoff for each player is defined in the usual way. An action profile a is *efficient* if it maximizes the payoff sum $\sum_{i \in N} u_i(a)$ over A. Let M be the maximum payoff sum. Let θ be a function which assigns to every subset S of N a probability distribution $\theta(S)$ over S. The cardinality of a set S is denoted by s. Lastly, R^s denotes the s-dimensional Euclidean space.

For a subset S of N, a *side-payment contract* $t^S = (t_i^S)_{i \in S}$ for S is a vector of functions, $t_i^S : A^S \to R$, which satisfies the feasibility condition

$$\sum_{i \in S} t_i^S(a_S) \le 0, \tag{1}$$

for all action profiles a_S in A^S. Under t^S, player $i \in S$ receives a side payment of $t_i^S(a_S)$ when the members of S choose an action profile a_S. In what follows, we refer to a side-payment contract simply as a contract.

We now describe a dynamic bargaining game. For each $k = 1, 2, \cdots$, let $\omega_k = (S_k, t^k)$ be a state variable in round k, where $S_k \subset N$ and $t^k = (t_i^k)_{i \in S_k}$ is a contract for S_k. We interpret $\omega_k = (S_k, t^k)$ to mean that S_k is the set of all players who have participated and formed a coalition before round k and that t^k is an "on-going" contract agreed upon by S_k. Let the initial state $\omega_1 = (S_1, t^1)$ satisfy $S_1 = \emptyset$ and $t^1 = 0$ (null contract). Each round k consists of the following three stages.

Round k.

Stage 1 (participation)

All non-participants $i \notin S_k$ independently decide whether to participate in negotiations. Let P_k be the set of new participants. Here, P_k may be the empty set.

Stage 2 (negotiation)

Negotiations take place between the incumbent participants S_k and the new participants P_k (if any). Let $S_{k+1} = S_k \cup P_k$. Each player $i \in S_{k+1}$ is selected as a proposer according to the probability distribution $\theta(S_{k+1})$. Proposer i chooses a contract t^{k+1} for S_{k+1}, which all other participants in S_{k+1} either accept or reject sequentially, according to a predetermined order. If all participants accept t^{k+1}, then it becomes an on-going contract, replacing t^k. In this case, we say that coalition S_{k+1} is formed. If t^{k+1} is rejected by any responder, t^k remains as the on-going contract. At the end of stage 2, there is a random choice that determines whether or not the game stops. The negotiations continue in the next round $k+1$ with probability $1-\epsilon$, and the process is repeated for the new state ω_{k+1}, determined by

$$\omega_{k+1} = \begin{cases} (S_{k+1}, t^{k+1}) & \text{if } t^{k+1} \text{ is agreed on by } S_{k+1}, \\ \omega_k & \text{otherwise.} \end{cases}$$

The negotiations stop with probability $\epsilon > 0$, in which case the on-going contract becomes the final one.

Stage 3 (action)

When the negotiations stop, all players independently choose their (pure or mixed) actions under the final contract $t = (t_i)_{i \in N}$. For a pure-action profile $a \in A$, the payoff of each player i is given by

$$u_i(a, t) = u_i(a) + t_i(a), \tag{2}$$

where $t_i(a) = t_i(a_S)$ for each participant i and $t_i(a) = 0$ for each non-participant i.

We denote the bargaining game defined above by Γ^ϵ. The stopping probability ϵ is considered to be bargaining friction. All players perfectly know a history of play whenever they make choices.

A (pure) strategy profile σ for Γ^ϵ is defined in the standard manner. It prescribes a choice to each player, depending on a history of play.[6] For a strategy profile σ, we denote the expected payoff for player i in Γ^ϵ by $Eu_i(\sigma)$.

We consider a Markov perfect equilibrium in the game Γ^ϵ. A strategy profile for Γ^ϵ is a Markov perfect equilibrium if it is a subgame perfect equilibrium where every player's choice for each round k depends only on a state ω_k and on a payoff-relevant history of play within round k. Formally, it is defined as follows.

Definition 1 A strategy profile σ for Γ^ϵ is a Markov perfect equilibrium (MPE) if it is a subgame perfect equilibrium of Γ^ϵ satisfying the following three properties for every round k with a state $\omega_k = (S_k, t^k)$: (i) the participation choices of all non-participants $i \notin S_k$ depend only on ω_k, (ii) a proposer's choice depends only on ω_k and on $S_{k+1} = S_k \cup P_k$ where P_k is the set of new participants, and all responders' choices depend only on ω_k and a contract t proposed for S_{k+1}, and (iii) if a new contract t is agreed on, then the participants' actions depend only on ω_k and t, the non-participants' actions depend only on ω_k and on the participants' actions under t, and otherwise, all players' actions depend only on ω_k.

The definition of an MPE is standard in a repeated game with observable states. It requires that every player's equilibrium strategy depends only on a payoff-relevant history summarized as a state variable. The only difference here is that each round game in Γ^ϵ is modeled as a sequential game with three stages. Accordingly, we require that every player's equilibrium strategy for each round k depends only on a state ω_k and on a payoff-relevant history of play within round k.

[6] We allow players to choose mixed actions in stage 3 when negotiations stop.

A few remarks on condition (iii) for the final stage may be useful. A side-payment among the participants does not affect non-participants' payoffs. The participants' actions only are payoff-relevant to non-participants. Owing to Lemma 1 applied to a group of participants, any action profile of the participants can become their dominant actions under an appropriate contract, and they can attain any feasible payoff profile by contracting. In the public good game where every player has the dominant action of zero contribution, all non-participants' equilibrium actions are simply not to contribute.

In the game, there exists a trivial MPE where, in each round, no players participate in negotiations, and all players play a Nash equilibrium of the underlying game G. Every player is indifferent as to whether or not to participate, given that any other player does not participate. We eliminate this trivial equilibrium from our analysis, focusing instead on an MPE where at least one player participates in the initial round. Note that this does not rule out the possibility that no players may participate when there exist incumbent participants.

We next provide the following result, which is useful for us to construct an MPE.[7]

Lemma 1 *For any pure action profile $a \in A$ and any payoff profile $x \in R^n$ satisfying $\sum_{i \in N} x_i \leq \sum_{i \in N} u_i(a)$, there exists a contract t for N that satisfies the following: (i) $x_i = u_i(a, t)$, for all $i \in N$; and (ii) a is a unique (dominant) Nash equilibrium of G, given t.*

The intuition for the lemma is as follows. When an action profile a is played, a contract t can be designed such that players receive a feasible payoff profile x through payoff transfer, because $\sum_{i \in N} x_i \leq \sum_{i \in N} u_i(a)$. When any action profile $a' \neq a$ is played, the contract t prescribes that each player i who deviates from a must pay a sufficiently large penalty to all other players so that a_i becomes the dominant action for player i under t. Thus, condition (ii) is satisfied. We denote the contract given in Lemma 1 as $t(x, a)$.

Traditionally, cooperative game theory analyzes coalitional bargaining in a strategic form game with transferable utility by formulating the characteristic function of the game. The classic approach uses the maxmin value under the assumption that a coalition S expects that the complementary coalition $N - S$ acts against them in the worst way. For other approaches, see Myerson (1991), for example. In this paper, we analyze coalitional bargaining in the framework of non-cooperative game theory without relying on the characteristic function.[8]

When the bargaining game Γ^ϵ stops with a final contract t, all players, participants and non-participants, play a Nash equilibrium of the underlying game G under t in a subgame perfect equilibrium. In what follows, we arbitrarily fix such a Nash equilibrium for each contract t. In particular, we call a Nash equilibrium played when the game stops without any contract a *disagreement action*, and call its payoff profile a *disagreement payoff*.

[7] Jackson and Wilkie (2005, p.563) show a similar result.

[8] Our approach using Lemma 1 is related to a "defensive-equilibrium representation" of the characteristic function in the Myerson's (1991) terminology where a pair of coalitions choose an equilibrium between them. The way of constructing the worth of a coalition based on a Nash equilibrium is frequently used in the literature of partition function form games.

Let $w = (w_i)_{i \in N} \in R^n$ be a weight vector satisfying $\sum_{i \in N} w_i = 1$ and $w_i > 0$ for all i. A payoff vector $x^* = (x_1^*, \cdots, x_n^*)$ is an *(asymmetric) Nash-bargaining solution* of G with a weight vector w and a disagreement payoff $d = (d_i)_{i \in N} \in R^n$, with $\sum_{i \in N} d_i \leq M$, denoted by $NB(w, d)$, if it is a solution to the following problem:

$$\begin{aligned} \max \quad & \prod_{i \in N} (x_i - d_i)^{w_i} \\ \text{subject to (i)} & \sum_{i \in N} x_i = M \\ \text{(ii)} & \ x_i \geq d_i \quad \text{for all} \quad i = 1, \cdots, n. \end{aligned}$$

Recall that M is the maximum payoff sum for n players. It is straightforward to see that, for all $i \in N$,

$$x_i^* = d_i + w_i \left(M - \sum_{j \in N} d_j \right). \tag{3}$$

The asymmetric Nash bargaining solution $NB(w, d)$ allocates the net surplus $M - \sum_{j \in N} d_j$ to players in proportion to w, in addition to their disagreement payoffs. The weight vector w reflects players' bargaining power.

Before we analyze the bargaining game Γ^ϵ in the next section, we consider a benchmark case that participation is mandatory. In this case, Γ^ϵ is essentially reduced to the standard model of a Rubinstein-type sequential multilateral bargaining game with random proposers where all n players divide the total welfare M. A difference is that, at the end of the game, the non-cooperative game G is played either with or without a contract. Let a^t be a Nash equilibrium of G under a contract t for N.

The following result is well known in the literature on non-cooperative multilateral bargaining games. With a slight abuse of notation, we denote $\theta(N)$ by θ. When the set of participants is N, every player is selected as a proposer according to the probability distribution θ on N.

Proposition 0. Assume that participation is mandatory and that a (pure or mixed) Nash equilibrium a^0 of G is fixed. Then, for every $\epsilon > 0$, there exists a unique MPE of Γ^ϵ, where each player makes an accepted proposal in the first round, and a^0 is chosen in off-equilibrium play when negotiations stop without any contract. The equilibrium payoff profile generated by each player's proposal converges to the Nash bargaining solution $NB(\theta, u(a^0))$ in the limit that ϵ goes to zero.

Since the proof is standard, we omit it (see Laruelle and Valenciano 2008 and Britz et al. 2010, for example). For the sake of analysis, we provide only the MPE strategies for players. Let $x^* = NB(\theta, u(a^0))$ be the Nash bargaining solution with the disagreement payoff $u(a^0)$. Define the payoff vector $y^{i,\epsilon} \in R^n$ for every $\epsilon > 0$ and every $i \in N$, such that

$$y_j^{i,\epsilon} = (1 - \epsilon) x_j^* + \epsilon \cdot u_j(a^0) \ \text{ for all } \ j \neq i,$$

$$y_i^{i,\epsilon} = M - \sum_{j \neq i} y_j^{i,\epsilon}.$$

Note that $y_j^{i,\epsilon}$ is the continuation payoff that player j receives after rejection, and that it is independent of proposer i. Let $e_N \in A$ be the efficient action profile attaining M. In equilibrium, every player i proposes the contract $t(y^{i,\epsilon}, e_N)$ constructed in Lemma 1, and accepts any contract t if $u_i(a^t, t) \geq y_i^{j,\epsilon}$, where a^t is the Nash equilibrium of G chosen under t.

To conclude this section, we suggest another interpretation of bargaining friction ϵ in the game Γ^ϵ. We have introduced the probability of negotiations stopping, ϵ, as bargaining friction. When the negotiations stop, the game G is played only once. Alternatively, we can reformulate Γ^ϵ as a repeated game where G is played infinitely many times in the following way.

A pure strategy profile for Γ^ϵ induces an infinite sequence of states $\{\omega^k\}_{k=1}^\infty$, where $\omega^k = (S_k, t^k)$ for each k, ignoring the random selection of proposers. The expected payoff of player i for this sequence is calculated as

$$\epsilon u_i(t^1) + \epsilon(1-\epsilon)u_i(t^2) + \cdots + \epsilon(1-\epsilon)^{k-1}u_i(t^k) + \cdots,$$

where $u_i(t^k)$ is the payoff for player i in the Nash equilibrium of G under the contract t^k. Setting $\delta = 1 - \epsilon$, the expected payoff is equal to the average discounted payoff sum

$$(1-\delta)\left\{u_i(t^1) + \delta u_i(t^2) + \cdots + \delta^{k-1}u_i(t^k) + \cdots\right\}$$

for the infinite sequence $\{u_i(t^k)\}_{k=1}^\infty$ of payoffs. This observation leads us to the following repeated game reformulation of Γ^ϵ. Each round has the same stages of voluntary participation and of negotiations as those in Γ^ϵ. When a new agreement t^{k+1} is made, the players play the game G under t^{k+1} (or under the default contract t^k in case of no agreement). The next round $k+1$ is played in the new state $\omega^{k+1} = (S_{k+1}, t^{k+1})$. Note that a Markov perfect equilibrium of this repeated game with state variables implies that the players play a Nash equilibrium of G under the contract t^{k+1} in each round k. Thus, our analysis of Γ^ϵ can be applied to the repeated game model above with discounted payoffs.

3 Theorems

We first prove the existence of an efficient MPE in the bargaining game Γ^ϵ when the stopping probability ϵ is sufficiently small. The equilibrium implements the Nash bargaining solution.

Theorem 1 *For every sufficiently small $\epsilon > 0$ and a (pure or mixed) Nash equilibrium a^0 of G, there exists an efficient MPE of Γ^ϵ, where all players participate in negotiations in the first round. Regardless of a proposer, the players agree to the Nash bargaining solution $x^* = NB(\theta, u(a^0))$ in the limit that ϵ goes to zero.*

Even in the case that participation is voluntary, the theorem shows that if the bargaining friction ϵ is sufficiently small, there exists an efficient MPE of Γ^ϵ where the Nash bargaining solution $NB(\theta, u(a^0))$ is immediately agreed on.

The equilibrium is constructed as follows. All players participate in negotiations in the first round, and behave according to the MPE σ^* given in Proposition 0. For every $\epsilon > 0$, the expected payoff profile of players is equal to $NB(\theta, u(a^0))$. When ϵ vanishes, the agreement converges to $NB(\theta, u(a^0))$, regardless of a proposer.

The players behave in off-equilibrium play in the following way. For an on-going contract z, let $x^*(z)$ be the Nash bargaining solution $NB(\theta, u(z))$ where $u(z)$ is the payoff profile generated under z. Suppose that the game reaches any round $k(> 1)$ with a state $\omega_k = (S_k, t^k)$ where $S_k \neq N$. All players outside S_k participate in negotiations in the first stage. In the negotiation stage, all players behave in the same way as σ^* and their expected payoff profile is equal to the Nash bargaining solution $x^*(t^k)$. The only difference is that the disagreement point is $u(t^k)$ instead of $u(a^0)$. The continuation payoff c_i of each player i after rejection is $(1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k)$. Because $M \geq \sum_{i \in N} u_i(t^k)$, it holds that $M \geq \sum_{i \in N} c_i$. Thus, each player's proposal is accepted.

To obtain the intuition for the theorem, we now ask why all players are willing to participate in negotiations, despite their free-riding incentive in G. We can answer the question by examining what would happen if any single player opts out. Suppose that a player h does not participate in the first round. Then, the negotiations take place between the remaining participants in $S = N - \{h\}$. If they choose an action profile a_S, then non-participant h chooses the best response $f(a_S)$ to a_S. Thus, the maximum payoff sum that participants S can attain is given by

$$M^S = \max_{a_S} \sum_{i \in S} u_i(a_S, f(a_S)).$$

The continuation payoff of each player $i \in S$ after rejection is $(1-\epsilon)x_i^* + \epsilon u_i(a^0)$. Note that the Nash bargaining solution $x^* = NB(\theta, u(a^0))$ is agreed on in the following round if the negotiations continue (with probability $1-\epsilon$). On the other hand, if any contract z is agreed on, each player $j \in S$ receives the expected payoff $(1-\epsilon)x_j^*(z) + \epsilon u_j(z)$ because z will be renegotiated to the Nash bargaining solution $x^*(z)$ with the disagreement point $u(z)$ in the following round. Thus, each proposer i's (possibly) optimal contract t is such that all responders j receive payoffs $u_j(t)$ satisfying $(1-\epsilon)x_j^*(t) + \epsilon u_j(t) = (1-\epsilon)x_j^* + \epsilon u_j(a^0)$, which they accept in equilibrium. If the optimal contract t is agreed upon, then proposer i receives payoff $(1-\epsilon)x_i^*(t) + \epsilon(M^S - \sum_{j \in S, j \neq i} u_j(t))$ and non-participant h receives the free-riding payoff $(1-\epsilon)x_h^*(t) + \epsilon u_h(e_S, f(e_S))$, where e_S is the action profile attaining M^S.

A critical point is whether or not it is actually optimal for proposer i to propose the contract t defined above. It is optimal if

$$(1-\epsilon)x_i^*(t) + \epsilon(M^S - \sum_{j \in S, j \neq i} u_j(t)) \geq (1-\epsilon)x_i^* + \epsilon u_i(a^0).$$

Because $\sum_{j\in N} x_j^*(t) = \sum_{j\in N} x_j^* = M$, where M is the maximum payoff sum for all players in the game G, there exists a trade-off of payoffs for the participants and for non-participant h between $x^*(t)$ and x^*. Roughly, if $x_i^*(t) > x_i^*$, such that proposer i makes the optimal proposal t for sufficiently small $\epsilon > 0$, then it holds that $x_h^*(t) < x_h^*$,[9] which means that non-participant h is worse off by the optimal contract t for S. Conversely, if non-participant h is better off by the optimal contract t, then it should be actually optimal for proposer i to make an unacceptable proposal, in which case, non-participant h becomes worse off. Whichever happens, h is worse off by not participating.

The next example illustrates the result of Theorem 1.

Example 2 Consider again the public good game in Example 1. There are three players, each with an endowment of one money unit. For a contribution vector $a = (a_1, a_2, a_3)$, player i's payoff is given by $u_i(a) = 0.6\sum_{j=1}^{3} a_j + 1 - a_i$. The game has a unique equilibrium $a^0 = (0, 0, 0)$. We consider the bargaining game Γ^ϵ, where each player is selected as a proposer with equal probability. The Nash bargaining solution $x^* = NB(u(a^0))$ is given by $x^* = (1.8, 1.8, 1.8)$. In the efficient MPE constructed in Theorem 1, all players participate in negotiations and receive the expected payoff 1.8. Suppose that player 1 does not participate, and thus contributes nothing. Then, players 2 and 3 negotiate for their contributions. As shown in Example 1, the payoff sum for 2 and 3 is maximized by the full contribution (1, 1), and is equal to 2.4. Non-participant 1 enjoys the free-riding payoff 2.2 if players 2 and 3 fully contribute. The payoff sum of the three players is 4.6 and, thus, any contract between 2 and 3 will be renegotiated in round 2 such that the surplus 0.8 is divided equally between the three players. In the negotiations between 2 and 3, the continuation payoff of each $i = 2, 3$ after rejection is given by $c_i = 1.8(1-\epsilon)+\epsilon = 1.8-0.8\epsilon$. Because c_2+c_3 exceeds $2.4+2\times\frac{0.8}{3}$ for sufficiently small ϵ, the negotiations between 2 and 3 fail. As a result, non-participant 1 receives the expected payoff $1.8 - 0.8\epsilon$ smaller than the Nash bargaining solution payoff 1.8. Thus, player 1 is motivated to participate in negotiations, as long as the other two players do so. The same argument is applied to players 2 and 3.

The example can be also analyzed as a cooperative game in partition function form in the following way (see Ray 2007, for example). See footnote 1 for the partition function of the game. If player 1 commits not to participate, players 2 and 3 may want to form a two-person coalition because their paypff sum 2.4 is larger than that of 2 in the no-coalition case. Player 1 can free-ride. However, if renegotiation is possible and players are farsighted, players 2 and 3 do not want to form their coalition allowing player 1 to free-ride, because otherwise their payoff sum will be 2.4 plus $\frac{0.8}{2}$, dividing the surplus 0.8 equally with player 1 in the renegotiation, which is smaller than $\frac{5.4}{3}$ $\times$ 2 in the equal allocation in the grand coalition. Anticipating this, three players immediately agree to form the grand coalition. The MPE constructed in Theorem 1 provides a non-cooperative formulation for this argument.

We next strengthen the efficiency result of Theorem 1. Here, we show that for any bargaining friction ϵ, all players participate in negotiations in at most n rounds in every MPE of Γ^ϵ, provided that the game does not stop on the way to the grand coalition N.

[9] More precisely, we show that the payoff differences $x_j^*(t) - x_j^*$ have the same signs for all participants. See the proof in Appendix.

We introduce some notation. For every subset S of N and action profile $a_S \in A^S$, let $G(N-S, a_S)$ be the $(n-s)$-person game with player set $N-S$, obtained from the game G under the assumption that all players in S choose a_S. We denote the set of Nash equilibria in the game $G(N-S, a_S)$ by $NE(a_S)$.

Assumption 1 For every subset S of N with $S \neq \emptyset, N$, and every $h \notin S$, it holds that

$$\max_{a_T \in A^T} \min_{x \in NE(a_T)} \sum_{i \in T} u_i(a_T, x) > \sum_{i \in S} u_i(a_S, y) + u_h(a_S, y), \tag{4}$$

for every $a_S \in A^S$ and every $y \in NE(a_S)$, where $T = S \cup \{h\}$.

The LHS of (4) is the maxmin value of the payoff sum of group T, including a new participant h, anticipating the worst response by non-participants who choose a Nash equilibrium given an action profile of T. The assumption means that the maxmin value of T is strictly greater than the payoff sum of the incumbent participants S, including the payoff of non-participant h, for every action profile a_S of S and every Nash equilibrium y for non-participants given a_S. Roughly, the payoff sum of participants is increasing if they are joined by a single non-participant.

The next example shows that Assumption 1 holds in the standard model of a linear public good game.

Example 3 Consider the following linear public good game that generalizes Example 1. Let $N = \{1, \cdots, n\}$ be the set of players, each with an endowment of one money unit. Let $a_i \in [0, 1]$ be a contribution of player i. For a contribution vector $a = (a_1, \cdots, a_n)$, player i's payoff is given by $u_i(a) = k \sum_{i=1}^{n} a_i + 1 - a_i$ where $1/n < k < 1$. Since $k < 1$, each player i has the dominant action $a_i^* = 0$. Let S be a subset of N with $S \neq \emptyset, N$ and let $h \notin S$. For every action profile a_S of S, the set $NE(a_S)$ in Assumption 1 consists of the single element y where all players outside S play the dominant actions to contribute nothing. Thus,

$$\begin{aligned} \sum_{i \in S} u_i(a_S, y) + u_h(a_S, y) &= \sum_{i \in S} \left\{ k \sum_{j \in S} a_j + 1 - a_i \right\} + k \sum_{i \in S} a_i + 1 \\ &= (ks - 1 + k) \sum_{i \in S} a_i + s + 1 \end{aligned}$$

where s is the cardinality of S. Let $T = S \cup \{h\}$, $t = s + 1$ and A^T be the set of action profiles for T. Then,

$$\begin{aligned} \max_{a_T \in A^T} \min_{x \in NE(a_T)} \sum_{i \in T} u_i(a_T, x) &= \max_{a_T \in A^T} \sum_{i \in T} \left(k \sum_{j \in T} a_j + 1 - a_i \right) \\ &= (ks - 1 + k) \max_{a_T \in A^T} \sum_{i \in T} a_i + s + 1. \end{aligned}$$

Taking $a_T = (a_S, 1)$ where all players of S choose a_S and player h chooses $a_h = 1$, it holds that $\sum_{i \in T} a_i = \sum_{i \in S} a_i + 1 > \sum_{i \in S} a_i$. Thus, by the two equations above, (4) holds.

Theorem 2 *Under Assumption 1, it holds that for any bargaining friction $\epsilon > 0$, all players participate in negotiations in at most n rounds in every pure strategy MPE of Γ^ϵ, provided that the game does not stop before the grand coalition N forms.*

The theorem shows that the set of participants gradually expands and reaches the grand coalition N in at most n rounds in every MPE. On the way to the grand coalition, the game may stop with an inefficient coalition due to bargaining friction. As the friction vanishes, the welfare loss of the inefficient coalition becomes negligible. Thus, the outcome of Γ^ϵ is efficient in the limit that ϵ goes to zero. We obtained a similar result in Okada (2000) for a super-additive characteristic function form game, in a coalitional bargaining game where a proposer chooses a coalition. The theorem extends it to a strategic form game with participation, and shows that the dynamic efficiency of coalition formation holds through voluntary bargaining in a wide class of economic situations with externalities. A key result in the theorem is that whenever the grand coalition has not formed, there exists at least one non-participant who joins the negotiations in every MPE. The intuition behind the result is as follows.

Let t^k be an on-going contract between participants $S_k (\neq \emptyset)$ in some round k. By way of contradiction, suppose that no players outside S_k participate. The game ends with probability ϵ, and t^k becomes the final contract. In this case, players receive the payoff profile $u(t^k)$, where all players choose a Nash equilibrium of the game G under the contract t^k. With probability $1-\epsilon$, the game continues in the next round $k+1$ with the same state as that in round k. Because the equilibrium satisfies the Markov property in Definition 1, it induces the same play in round $k+1$ as that in round k. Thus, each player i receives the payoff $u_i(t^k)$ in round $k+1$, and thereafter. If any player $i \notin S_k$ deviates from the equilibrium and joins the group S_k, then the maxmin payoff sum of the extended group $S = S_k \cup \{i\}$ is greater than $\sum_{j \in S} u_j(t^k)$, by Assumption 1. This means that some beneficial contract can be agreed upon by negotiations[10] and, thus, each player j in S obtains an expected payoff greater than $u_j(t^k)$. Specifically, the new participant i can be better off than in the equilibrium. This is a contradiction.

Because of the result above, all n players participate in negotiations by round n at least with probability $p(\epsilon) = (1-\epsilon)^{n-1}$. As ϵ reduces to zero, this probability converges to one.

The last example shows that Theorem 2 does not hold without Assumption 1.

Example 4 Consider a three-person prisoner's dilemma game, where every player $i = 1, 2, 3$ has two actions, namely, C and D. The payoff matrix is given in Table 1. In Table 1, player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix. In each cell, the three numbers show payoffs for players 1, 2, and 3, respectively, from top to bottom. Here, D is the dominant action for every player. The game has a unique Nash equilibrium with payoff (3, 3, 3).

[10] More precisely, we should consider players' continuation payoffs after a contract is agreed on. See the formal proof given in Appendix.

Table 1 Three-person prisoner's dilemma

	C	D
C	4 4 4	3 6 3
D	6 3 3	5 5 2
	C	

	C	D
C	3 3 6	2 5 5
D	5 2 5	3 3 3
	D	

Let $S = \{1, 2\}$, $h = 3$, and $T = \{1, 2, 3\}$ in Assumption 1. The maximum total payoffs of T and of S with non-participant 3 are equal to 12. Thus, Assumption 1 is violated. We remark that the payoffs are only inefficient in Table 1 if all three players defect. If inefficiency arises when even one player defects, then Assumption 1 is satisfied, and thus Theorem 2 holds.

Assume that each participant is selected as a proposer with equal probability in Γ^ϵ. We construct an MPE of Γ^ϵ as follows. In round 1, only players 1 and 2 participate in negotiations. When non-participant 3 chooses D, the maximum payoff sum for 1 and 2 is 7, which is given by (C, D, D) and (D, C, D). Their Nash bargaining solution is the payoff profile $(\frac{7}{2}, \frac{7}{2})$, where 1 and 2 split the surplus of 7 equally. Non-participant 3 enjoys the free-riding payoff of 5. In round 2, and thereafter, 3 does not participate. We now claim that the outcome that 1 and 2 participate constitutes a Nash equilibrium of the participation stage in round 1. Suppose that 3 joins $S = \{1, 2\}$. In the negotiations between $T = \{1, 2, 3\}$, the continuation payoffs c_i after rejection for player $i = 1, 2, 3$ are $c_1 = c_2 = \frac{7}{2}(1-\epsilon)+3\epsilon = 3.5-0.5\epsilon$ and $c_3 = 5(1-\epsilon)+3\epsilon = 5-2\epsilon$. Because $c_1+c_2+c_3 = 12-3\epsilon < 12$ for any $\epsilon > 0$, the negotiation is successful. The expected payoff for 3 is given by $\frac{1}{3}(12 - c_1 - c_2) + \frac{2}{3}c_3 = 5 - \epsilon$, which is smaller than the free-riding payoff of 5. Thus, 3 is worse off by participating in the negotiation. Players 1 and 2 receive the continuation payoffs $3.5 - 0.5\epsilon$ if they do not participate. Thus, 1 and 2 are motivated to participate, as long as the other player does so.

4 Discussion and Conclusion

In this paper, we have considered the free-riding problem that is relevant to the Coase theorem. Rational and self-interested players are motivated not to participate in negotiations and to free-ride on an agreement that other players could voluntarily achieve. To solve the problem, we have presented a dynamic bargaining game for an n-person strategic form game where players decide whether to participate in the negotiations. Players are allowed to renegotiate an inefficient agreement to a Pareto-improving one.

We have proved two results on the efficiency of coalitional bargaining. We have first shown that for every sufficiently small friction in the bargaining, there exists an efficient MPE where all players participate in the grand coalition in the first round (Theorem 1). The agreement converges to the Nash bargaining solution as the friction vanishes. The theorem does not hold if the friction is large. In particular, when the game stops in the first round with probability one, only a small group of players participate in a typical case, due to the free-riding incentive. This result is in contrast to that of a proposal-based model in the literature. In the ultimatum bargaining game where a proposer chooses a coalition, the proposer exploits the surplus of the grand coalition.

We have further shown that for every probability of negotiation stopping, a coalition of players gradually expands and reaches the grand coalition in finitely many rounds in every pure strategy MPE, provided that the game does not stop on the way (Theorem 2).

The main results are proved under the two restrictive assumptions. First, participation is binding in that once players participate in a coalition, they cannot leave it. Second, only one coalition forms. We here discuss whether we can relax these assumptions. For simplicity of discussion, we treat them separately. The bargaining game Γ^{ϵ} can be extended to two models, model 1 and model 2, each of which corresponds to one of the assumptions.

Model 1 relaxes the first assumption. It has the same rule as Γ^{ϵ} except that participants can decide whether to leave a coalition. Theorem 1 holds for the extended model as well. The MPE constructed in the proof can be applied, and it remains to be an MPE where the grand coalition forms in all rounds. If any participant leaves it, then the player is not better off for the same reason as any non-participant is not so in Γ^{ϵ}. However, Theorem 2 does not hold for the following reason. Even after a coalition forms, all players decide again whether to participate in it. Thus, the model is substantially equivalent to the repeated game version of Γ^{ϵ} without any contractual state. Then, it holds that the repetition of a subgame perfect equilibrium of the component game is an MPE of the repeated game. Thus, the repetition of an inefficient coalition can be the outcome of an MPE when participants can leave a coalition.

Model 2 relaxes the second assumption. It allows more than one coalition. In the model, there are some "meeting" places for negotiations. Players can choose one of them, or decide not to participate. In each place, the participants negotiate according to the same rule as in Γ^{ϵ}. A state describes a coalitional partition on the player set and the contracts for coalitions. A contract for a coalition specifies an allocation of its surplus, given the contracting decisions of other coalitions. Given a coalitional partition, the members of each coalition decide whether to merge with other coalitions and with which ones. Theorem 1 also holds in the extended model. The MPE constructed in the proof can be applied so that all players choose the identical meeting place. By the same proof as in Theorem 1, it can be shown that the strategy is an MPE where the grand coalition immediately forms. Further, it seems that the proof of Theorem 2 can be applied to the model, although it may depend on a merging process of coalitions. We conjecture that the theorem holds under the super-additivity condition that the merger of two coalitions can increase their surplus, no matter how other coalitions react. A detailed analysis of the model is left to future works.

In conclusion, we comment on the applicability of the Coase theorem. First, to achieve an efficient allocation, players need to have a sufficient number of opportunities to renegotiate inefficient agreements. Second, the possibility of an efficient agreement holds in a general situation where players decide whether to participate in a coalition or leave it, and one and more coalitions form. Third, the strong statement of the theorem that rational players will *necessarily* achieve an efficient allocation through voluntary bargaining is not guaranteed in some bargaining situations, for example, when the members of a coalition are free to leave it. For the statement to hold, the commitment to participate should be binding. In the context of international treaties, signatories need to be constrained not to withdraw from the treaties by morals, political conditions, and public opinion.

Appendix

Proof of Lemma 1 Let $L = 1+\max_{i,a',a''}\{|u_i(a')-u_i(a'')|, |x_i-u_i(a')|\}$. We construct a contract $t = (t_i)_{i\in N}$ for N as follows. For any action profile a' and any i, let $k_i(a')$ be the number of players $j \neq i$ who choose $a'_j \neq a_j$. Define

$$t_i(a') = \begin{cases} x_i - u_i(a) & \text{if } a' = a, \\ k_i(a')L & \text{if } a'_i = a_i \text{ and } k_i(a') > 0 \\ k_i(a')L - (n-1)L & \text{otherwise.} \end{cases}$$

The contract t above is defined according to the rule that any player i who deviates from a_i pays the amount L to every other player. It holds that $\sum_{i\in N} t_i(a) \leq 0$ since $\sum_{i\in N} x_i \leq \sum_{i\in N} u_i(a)$, and that $\sum_{i\in N} t_i(a') = 0$ for every other $a' \in A$ by construction. Thus, t satisfies the feasibility condition (1). Since $x_i = u_i(a) + t_i(a)$ for all $i \in N$, (i) in the lemma holds by (2). To prove (ii), it suffices to see that under t, a_i is the dominant action for each player i. For this, consider the following two cases. Let a'_i be any action of player i other than a_i, and let a'_{-i} be any action profile for all players except i.

Case 1. $a'_{-i} = a_{-i}$: We have $u_i(a,t)-u_i((a'_i, a_{-i}), t) = x_i-u_i(a'_i, a_{-i})+(n-1)L$, which is positive by the definition of L.

Case 2. $a'_{-i} \neq a_{-i}$: We have $u_i((a_i, a'_{-i}), t) - u_i((a'_i, a'_{-i}), t) = u_i(a_i, a'_{-i}) - u_i(a'_i, a'_{-i}) + (n-1)L$, which is also positive by the definition of L. □

Proof of Theorem 1 We first introduce additional notation. Let $e_N \in A$ be the efficient action profile in the game G, and let $x^* = NB(\theta, u(a^0))$. For every $\epsilon > 0$ and every $i \in N$, define the payoff vector $y^{i,\epsilon} \in R^n$, such that

$$y_j^{i,\epsilon} = (1-\epsilon)x_j^* + \epsilon \cdot u_j(a^0) \ \text{ for all } \ j \neq i \tag{A.1}$$

$$y_i^{i,\epsilon} = M - \sum_{j\neq i} y_j^{i,\epsilon}, \tag{A.2}$$

where M is the maximum payoff sum for all n players.

Fix any subset S of N. For every contract t of S, choose a Nash equilibrium $a^t = (a^t_S, a^t_{N-S}) \in A$ in the game G under t. If there exists no pure Nash equilibrium, choose a mixed Nash equilibrium. The following proof is not affected in any critical way.[11] We define a payoff vector $u(t) \in R^n$, such that $u_i(t) = u_i(a^t) + t_i(a^t_S)$ for every $i \in S$ and $u_j(t) = u_j(a^t)$ for every $j \notin S$.

For every $a_S \in A^S$, let $G(N-S, a_S)$ be the $(n-s)$-person game with player set $N-S$ obtained from the game G, under the assumption that all players in S choose

[11] In what follows, whenever we choose a Nash equilibrium in the game G, the same remark applies.

a_S. Choose a Nash equilibrium $f(a_S) \in A^{N-S}$ of $G(N-S, a_S)$. Define

$$M^S = \max_{a_S \in A^S} \sum_{i \in S} u_i(a_S, f(a_S)), \tag{A.3}$$

and let $e_S \in A^S$ be the action profile that attains M^S. Here, M^S is the maximum payoff sum that all participants of S can attain, anticipating the equilibrium responses of non-participants.

For a payoff vector $x \in R^s$ satisfying $\sum_{i \in S} x_i = \sum_{i \in S} u_i(a_S, f(a_S))$, let $t(x, a_S)$ be the contract given by Lemma 1 with respect to the game $G(S, f(a_S))$. For $t = t(x, a_S)$, we define the payoff $u(t)$ to satisfy $u_i(t) = x_i$ for every $i \in S$ and $u_j(t) = u_j(a_S, f(a_S))$ for every $j \notin S$. By Lemma 1.(ii), it holds that $(a_S, f(a_S))$ is a Nash equilibrium of G under t. Also, let $x^*(t) = NB(\theta, u(t))$. Note that $\sum_{i \in N} u_i(t) \leq M$.

We now construct an equilibrium strategy profile σ^* of Γ^ϵ as follows. For every $k = 1, 2, \cdots$, let $\omega_k = (S_k, t^k)$ be a state in round k.

Case 1 $S_k = \emptyset$.

- All players in N participate in the negotiations.
- Let S be a set of participants in either on- or off-equilibrium play. When $S = N$, each player $i \in N$ employs the MPE given by Proposition 0. When $S \neq N$, each player $i \in S$ proposes $t = t(z^{i,\epsilon}, e_S)$, such that the payoff vector $z^{i,\epsilon} \in R^s$ is given by

$$z_j^{i,\epsilon} = \frac{y_j^{i,\epsilon} - (1-\epsilon)x_j^*(t)}{\epsilon} \quad \text{for all } j \in S, j \neq i \tag{A.4}$$

$$z_i^{i,\epsilon} = M^S - \sum_{j \in S, j \neq i} z_j^{i,\epsilon}, \tag{A.5}$$

where $y_j^{i,\epsilon}$ is defined by (A.1), if $(1-\epsilon)x_i^*(t) + \epsilon z_i^{i,\epsilon} \geq y_i^{j,\epsilon}$, and otherwise makes an unacceptable proposal. Note that $y_i^{j,\epsilon}$ is independent of $j (\neq i)$. When a contract w for S is proposed, every responder $j \in S$ accepts it if and only if $(1-\epsilon)x_j^*(w) + \epsilon u_j(w) \geq y_j^{i,\epsilon}$. When negotiations stop with the contract w, the Nash equilibrium a^w of G under w is played.

Case 2 $S_k \neq \emptyset$.
Let t^k be an on-going contract between participants in S^k.

- All players in $N - S_k$ participate in the negotiations.
- Let S be a set of participants, in either on- or off-equilibrium play. When $S = N$, each player $i \in N$ proposes $t(z^{i,\epsilon}, e_N)$, where the payoff vector $z^{i,\epsilon} \in R^n$ is given by

$$z_j^{i,\epsilon} = (1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k) \quad \text{for all } j \in S, j \neq i \tag{A.6}$$

$$z_i^{i,\epsilon} = M - \sum_{j \in N, j \neq i} z_j^{i,\epsilon}. \tag{A.7}$$

When a contract w for N is proposed, every responder $j \in N$ accepts it if and only if $(1-\epsilon)x_j^*(w) + \epsilon u_j(w) \geq z_j^{i,\epsilon}$. When $S \neq N$, each player $i \in S$ proposes $t = t(z^{i,\epsilon}, e_S)$, where the payoff vector $z^{i,\epsilon} \in R^s$ is given by

$$z_j^{i,\epsilon} = \frac{\epsilon u_j(t^k) + (1-\epsilon)(x_j^*(t^k) - x_j^*(t))}{\epsilon} \quad \text{for all } j \in S,\ j \neq i \tag{A.8}$$

$$z_i^{i,\epsilon} = M^S - \sum_{j \in S, j \neq i} z_j^{i,\epsilon}, \tag{A.9}$$

if $(1-\epsilon)x_i^*(t) + \epsilon z_i^{i,\epsilon} \geq (1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k)$, and otherwise makes an unacceptable proposal. When a contract w for S is proposed, every responder $j \in S$ accepts it if and only if $(1-\epsilon)x_j^*(w) + \epsilon u_j(w) \geq (1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k)$. When negotiations stop with the contract w, the Nash equilibrium a^w of G under w is played.

When σ^* is played, all players participate in the negotiations in the first round and behave according to the MPE given by Proposition 0. Each player's proposal is efficient and is accepted. The agreement will not be renegotiated in any future round. Thus, the expected payoff profile of players for σ^* is equal to the Nash bargaining solution $x^* = NB(\theta, u(a^0))$ with the disagreement payoff $u(a^0)$. The payoff profile attained by every player's proposal converges to x^* in the limit that ϵ goes to zero.

When the game starts with state $\omega_k = (S_k, t^k)$ in round $k(> 1)$ off the play of σ^*, all players participate in the negotiations and each player's efficient proposal is accepted. The agreement will not be renegotiated. Thus, the expected payoff of every player $i \in N$, evaluated at the beginning of round k, is given by

$$\begin{aligned}
Eu_i(\sigma^*) &= \sum_{j \in N} \theta_j \cdot z_i^{j,\epsilon} \\
&= \theta_i\{M - \sum_{j \in N, j \neq i} ((1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k))\} + (1-\theta_i)((1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k)) \\
&= \theta_i\{M - \sum_{j \in N}((1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k))\} + (1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k) \\
&= \epsilon\theta_i(M - \sum_{j \in N} u_j(t^k)) + (1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k) \ \ (\text{by } \sum_{j \in N} x_j^*(t^k) = M) \\
&= \epsilon(x_i^*(t^k) - u_i(t^k)) + (1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k) \ \ (\text{by (3)}) \\
&= x_i^*(t^k).
\end{aligned}$$

The expected payoff profile of the players is equal to the Nash bargaining solution $x^*(t^k) = NB(\theta, u(t^k))$ with the disagreement payoff $u(t^k)$.

It is clear that σ^* satisfies the Markov property in Definition 1. When the negotiations stop with a contract w for a set S of participants, σ^* prescribes the Nash equilibrium a^w of $G(S)$ under w.

To prove that σ^* is a subgame perfect equilibrium of Γ^ϵ, it remains to show that σ^* prescribes the optimal choices for all players and for each of their moves in the two stages of participation and negotiation in each round, given that σ^* will be played in all future moves.

Let $\omega_k = (S_k, t^k)$ be a state in every round k.

Case 1 $S_k = \emptyset$.

When the set of participants is N, σ^* prescribes the MPE in Proposition 0 and, thus, it satisfies the optimality of every player's choice in the negotiation stage. Suppose that the set of participants is $S \neq N$. When each player $i \in S$ proposes a contract w for S, all responders j receive the payoffs $(1-\epsilon)x_j^*(w)+\epsilon u_j(w)$ if w is accepted, and receive the payoffs $y_j^{i,\epsilon}$ otherwise. Thus, σ^* prescribes the optimal response rule for j. Given the optimal response rules for all other participants, the optimal proposal for i must be the contract $t = t(z^{i,\epsilon}, e_S)$ defined by (A.4) and (A.5) if its acceptance makes i better off than rejection. If t is accepted, proposer i receives the payoff $(1-\epsilon)x_i^*(t) + \epsilon z_i^{i,\epsilon}$, but receives payoff $y_i^{j,\epsilon}$ in the event of rejection. Thus, it is optimal for i to propose $t = t(z^{i,\epsilon}, e_S)$ if $(1-\epsilon)x_i^*(t) + \epsilon z_i^{i,\epsilon} \geq y_i^{j,\epsilon}$. Otherwise, it is optimal for i to make an unacceptable proposal.

Finally, consider the participation stage. Suppose that a player $h \in N$ does not participate in the negotiations. Then, the set of participants is $S = N - \{h\}$. Now, we examine what happens in the negotiations between S. Each player $i \in S$ proposes the equilibrium contract $t = t(z^{i,\epsilon}, e_S)$ defined by (A.4) and (A.5) if its acceptance makes i better off than rejection. When t is implemented, the payoff profile $u(t) = (z^{i,\epsilon}, u_h(e_S, f(e_S))$ is attained. By the definition of f, $f(e_S)$ is the best response of player h to e_S. For every $j \in S$, with $j \neq i$, (A.4) implies that

$$\begin{aligned} y_j^{i,\epsilon} &= \epsilon z_j^{i,\epsilon} + (1-\epsilon)x_j^*(t) \\ &= \epsilon z_j^{i,\epsilon} + (1-\epsilon)\left\{ z_j^{i,\epsilon} + \theta_j(M - \sum_{m \in S} z_m^{i,\epsilon} - u_h(e_S, f(e_S))) \right\} \\ &= \epsilon z_j^{i,\epsilon} + (1-\epsilon)\left\{ z_j^{i,\epsilon} + \theta_j(M - M^S - u_h(e_S, f(e_S))) \right\}. \end{aligned} \tag{A.10}$$

The last equality holds by (A.5). By the definitions of $y_j^{i,\epsilon}$ in (A.1) and x_j^*, it holds that

$$y_j^{i,\epsilon} = \epsilon u_j(a^0) + (1-\epsilon)\left\{ u_j(a^0) + \theta_j(M - \sum_{m \in N} u_m(a^0) \right\}. \tag{A.11}$$

Thus, it follows from (A.10) and (A.11) that

$$z_j^{i,\epsilon} - u_j(a^0) = (1-\epsilon)\theta_j\left\{ M^S + u_h(e_S, f(e_S)) - \sum_{m \in N} u_m(a^0) \right\}. \tag{A.12}$$

Because

$$\epsilon u_j(a^0) + (1-\epsilon)x_j^* = y_j^{i,\epsilon} = \epsilon z_j^{i,\epsilon} + (1-\epsilon)x_j^*(t), \tag{A.13}$$

(A.12) implies that

$$x_j^* - x_j^*(t) = \epsilon\theta_j\{M^S + u_h(e_S, f(e_S)) - \sum_{m\in N} u_m(a^0)\}.$$

Because j can be any element of $S = N - \{h\}$, the three cases are possible: (i) $x_j^* > x_j^*(t)$, for every $j \in S$; (ii) $x_j^* < x_j^*(t)$, for every $j \in S$; and (iii) $x_j^* = x_j^*(t)$, for every $j \in S$. In case (i), $x_i^* > x_i^*(t)$ for proposer $i \in S$. Then, for any sufficiently small $\epsilon > 0$, it holds that

$$(1-\epsilon)x_i^*(t) + \epsilon z_i^{i,\epsilon} < (1-\epsilon)x_i^* + \epsilon u_i(a^0) = y_i^{i,\epsilon}.$$

Thus, it is not optimal for i to propose t, and the negotiations fail. Non-participant h receives payoff $(1-\epsilon)x_h^* + \epsilon u_h(a^0)$, which is less than or equal to x_h^*. In case (ii), it is optimal for i to propose t for any sufficiently small $\epsilon > 0$, and it is accepted in σ^*. Because $x^*(t) = NB(\theta, u(t))$ and $x^* = NB(\theta, u(a^0))$, it holds that $\sum_{m\in N} x_m^*(t) = \sum_{m\in N} x_m^* = M$. Because $x_j^* < x_j^*(t)$ for every $j \in N$ with $j \neq h$, it must be that $x_h^* > x_h^*(t)$. Therefore, non-participant h receives the payoff $(1-\epsilon)x_h^*(t) + \epsilon u_h(t)$, which is less than x_h^* for sufficiently small $\epsilon > 0$. In case (iii), it holds that $x_j^* = x_j^*(t)$ for all $j \in N$ because $\sum_{m\in N} x_m^*(t) = \sum_{m\in N} x_m^* = M$. In addition, it follows from (A.13) that $z_j^{i,\epsilon} = u_j(a^0)$ for every $j \in N$ with $j \neq h$. Because x^* and $x^*(t)$ are the Nash bargaining solutions with the disagreement points $u(a^0)$ and $u(t) = (z^{i,\epsilon}, u_h(e_S, f(e_S)))$, respectively, it must be that $u_h(a^0) = u_h(e_S, f(e_S))$. Thus, regardless of the outcome of the negotiations in S, non-participant h receives the payoff $(1-\epsilon)x_h^* + \epsilon u_h(a^0)$, which is less than or equal to x_h^*. In all the three cases, non-participant h is not better off by deviating from σ^*. This means that σ^* prescribes a Nash equilibrium in the participation stage.

Case 2. $S_k \neq \emptyset$.

Suppose that the set of participants is N. When each player $i \in N$ proposes a contract w for N, all responders j receive the payoffs $(1-\epsilon)x_j^*(w) + \epsilon u_j(w)$ if w is accepted,[12] and receive the payoffs $z_j^{i,\epsilon} = (1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k)$ otherwise. Thus, σ^* prescribes the optimal response rule for j. It is optimal for i to propose the contract $t(z^{i,\epsilon}, e_N)$, defined by (A.6) and (A.7), because

$$\begin{aligned} z_i^{i,\epsilon} &= M - \sum_{j\in N, j\neq i} z_j^{i,\epsilon} \\ &= M - \sum_{j\in N, j\neq i} \{(1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k)\} \\ &= M - \sum_{j\in N} \left\{(1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k)\right\} + (1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k) \end{aligned}$$

[12] If w is inefficient, then it will be renegotiated to $x^*(w)$ in the following round.

$$= \epsilon \left(M - \sum_{j \in N} u_j(t^k) \right) + (1-\epsilon) x_i^*(t^k) + \epsilon u_i(t^k) \quad \left(\text{by} \sum_{j \in N} x_j^*(t^k) = M \right)$$
$$\geq (1-\epsilon) x_i^*(t^k) + \epsilon u_i(t^k) = z_i^{j,\epsilon}.$$

Next, suppose that the set of participants is $S \neq N$. When each player $i \in S$ proposes a contract w for S, all responders j receive the payoffs $(1-\epsilon)x_j^*(w) + \epsilon u_j(w)$ if w is accepted, and receive the payoffs $(1-\epsilon)x_j^*(t^k) + \epsilon u_j(t^k)$ otherwise. Thus, σ^* prescribes the optimal response rule for j. Given the optimal response rules for all other participants, the optimal proposal for i must be the contract $t = t(z^{i,\epsilon}, e_S)$, defined by (A.8) and (A.9), if its acceptance makes i better off than rejection. If w is accepted, proposer i receives the payoff $(1-\epsilon)x_i^*(w) + \epsilon z_i^{i,\epsilon}$, and receives the payoff $(1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k)$ in the event of rejection. Thus, it is optimal for i to propose $t = t(z^{i,\epsilon}, e_S)$ if $(1-\epsilon)x_i^*(t) + \epsilon z_i^{i,\epsilon} \geq (1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k)$. Otherwise, it is optimal for i to make an unacceptable proposal.

Finally, consider the participation stage. By the same arguments as those used in case 1, we show that σ^* prescribes a Nash equilibrium in the participation stage. Suppose that a player $h \in N$ does not participate in the negotiations. Then, the set of participants is $S = N - \{h\}$. Each player $i \in S$ proposes the equilibrium contract $t = t(z^{i,\epsilon}, e_S)$, defined by (A.8) and (A.9), if its acceptance makes i better off than rejection. For every $j \in S$ with $j \neq i$, (A.8) implies

$$\epsilon z_j^{i,\epsilon} + (1-\epsilon)x_j^*(t) = \epsilon u_j(t^k) + (1-\epsilon)x_j^*(t^k). \tag{A.14}$$

By the definitions of $x_j^*(t)$ and $x_j^*(t^k)$, it thus holds that

$$\epsilon z_j^{i,\epsilon} + (1-\epsilon)\{z_j^{i,\epsilon} + \theta_j(M - M^S - u_h(e_S, f(e_S)))\}$$
$$= \epsilon u_j(t^k) + (1-\epsilon)\left\{ u_j(t^k) + \theta_j \left(M - \sum_{m \in N} u_m(t^k) \right) \right\}.$$

Rearranging the above equation, we obtain

$$z_j^{i,\epsilon} - u_j(t^k) = (1-\epsilon)\theta_j \left\{ M^S + u_h(e_S, f(e_S)) - \sum_{m \in N} u_m(t^k) \right\}. \tag{A.15}$$

Thus, it follows from (A.14) and (A.15) that

$$x_j^*(t^k) - x_j^*(t) = \epsilon\theta_j \left\{ M^S + u_h(e_S, f(e_S)) - \sum_{m \in N} u_m(t^k) \right\}.$$

Similarly to case 1, the three cases are possible: (i) $x_j^*(t^k) > x_j^*(t)$, for every $j \in S$; (ii) $x_j^*(t^k) < x_j^*(t)$, for every $j \in S$; and (iii) $x_j^*(t^k) = x_j^*(t)$, for every $j \in S$. In case (i), $x_i^*(t^k) > x_i^*(t)$ for proposer $i \in S$. Then, for any sufficiently small $\epsilon > 0$, it holds that

$$(1-\epsilon)x_i^*(t) + \epsilon z_i^{i,\epsilon} < (1-\epsilon)x_i^*(t^k) + \epsilon u_i(t^k).$$

Thus, it is not optimal for i to propose t, and the negotiations fail. Non-participant h receives the payoff $(1-\epsilon)x_h^*(t^k) + \epsilon u_h(t^k)$, which is less than or equal to $x_h^*(t^k)$. In case (ii), it is optimal for i to propose t for any sufficiently small $\epsilon > 0$, and it is accepted in σ^*. Because $x^*(t) = NB(\theta, u(t))$ and $x^*(t^k) = NB(\theta, u(t^k))$, it holds that $\sum_{m \in N} x_m^*(t) = \sum_{m \in N} x_m^*(t^k)$. Because $x_j^*(t^k) < x_j^*(t)$ for every $j \in N$ with $j \neq h$, it must be that $x_h^*(t^k) > x_h^*(t)$. Non-participant h receives the payoff $(1-\epsilon)x_h^*(t) + \epsilon u_h(t)$, which is less than $x_h^*(t^k)$ for sufficiently small $\epsilon > 0$. In case (iii), it holds that $x_j^*(t^k) = x_j^*(t)$ for all $j \in N$ because $\sum_{m \in N} x_m^*(t) = \sum_{m \in N} x_m^*(t^k)$. In addition, it follows from (A.14) that $z_j^{i,\epsilon} = u_j(t^k)$ for every $j \in N$ with $j \neq h$. Because $x^*(t^k)$ and $x^*(t)$ are the Nash bargaining solutions with the disagreement points $u(t^k)$ and $u(t) = (z^{i,\epsilon}, u_h(e_S, f(e_S)))$, respectively, it must be that $u_h(t^k) = u_h(e_S, f(e_S))$. Thus, regardless of the outcome of the negotiations in S, non-participant h receives the payoff $(1-\epsilon)x_h^*(t^k) + \epsilon u_h(t^k)$, which is less than or equal to $x_h^*(t^k)$. In all the three cases, non-participant h is not better off by deviating from σ^*. □

Proof of Theorem 2 Let σ be any MPE of Γ^ϵ. We use the same notation as that in the proof of Theorem 1. For a subset S of N and a contract t for S, $u(t) \in R^n$ is the payoff vector for n players, where they choose the Nash equilibrium a^t of the game G under t assigned by σ.

Consider any round k with a state $\omega_k = (S_k, t^k)$ where $S_k \neq \emptyset, N$. We show that at least one player outside S_k participates in S_k on the play of σ, given that σ will be played in the following subgame of Γ^ϵ. By way of contradiction, suppose that no players outside S_k participate on the play of σ. Then, negotiations take place between S_k. Because the on-going contract t^k already attains the maximum payoff sum of S_k on the equilibrium play, no new contract is agreed. Thus, if the game stops with probability ϵ, t^k becomes the final contract and all players receive payoffs $u(t^k)$. With probability $1-\epsilon$, the negotiations continue in the next round $k+1$ with the same state $\omega_{k+1} = (S_k, t^k)$ as ω_k in round k. Because σ has the Markov property in Definition 1, it induces the same play in round $k+1$ as in round k. This means that the expected payoff profile for the players in σ from round $k+1$ (and also from round k) is equal to $u(t^k)$.

Suppose now that some player $h \notin S_k$ deviates from σ and joins S_k. Then, the negotiations take place between $S = S_k \cup \{h\}$. By Assumption 1, we have

$$\max_{a_S \in A^S} \min_{x \in NE(a_S)} \sum_{i \in S} u_i(a_S, x) > \sum_{i \in S_k} u_i(a_{S_k}, y) + u_h(a_{S_k}, y) \tag{A.16}$$

for every $a_{S_k} \in A^{S_k}$ and every $y \in NE(a_{S_k})$. Let $e_S \in A^S$ be the action profile of S that attains the maxmin value on the LHS. Then, (A.16) implies that

$$\min_{x \in NE(e_S)} \sum_{i \in S} u_i(e_S, x) > \sum_{i \in S_k} u_i(t^k) + u_h(t^k). \tag{A.17}$$

Letting

$$r \equiv \min_{x \in NE(e_S)} \sum_{i \in S} u_i(e_S, x) - \sum_{i \in S_k} u_i(t^k) - u_h(t^k),$$

we define the payoff vector $z^h \in R^s$ for S by

$$z_j^h = u_j(t^k) + \frac{r}{s} \text{ for all } j \in S,$$

where s is the cardinality of S. It holds from (A.17) that $z_j^h > u_j(t^k)$ for all $j \in S$.

By the same proof as for Lemma 1, we can prove the following claim. For every action profile $a = (a_S, a_{N-S})$ of N and every payoff profile $x^S \in R^S$ satisfying $\sum_{i \in S} x_i^S \le \sum_{i \in S} u_i(a)$, there exists a contract t for S that satisfies the following: for every $i \in S$, (i) $x_i^S = u_i(a, t)$ and (ii) a_i is the dominant action of player i in G, given t. We denote the contract t by $t(x^S, a)$.

Suppose further that player h, if selected as a proposer, proposes the contract $t = t(z^h, a)$ under which the participants of S choose their dominant actions $a_S = e_S$, and attain z^h when the non-participants choose the actions a_{N-S} assigned by σ after t is agreed. Note that the action profile a_{N-S} is a Nash equilibrium in $NE(e_S)$. Since σ is an MPE, Definition 1.(iii) implies that a_{N-S} depends only on the state ω_k and the participants' actions e_S, not on the payoff allocation z^h among them. We denote it by $a_{N-S}(\omega_k, e_S)$. For the existence of such a contract t, it suffices by the claim above to prove the feasibility of the payoff vector z^h for S when the action profile $a = (e_S, a_{N-S}(\omega_k, e_S))$ is played. In fact, since

$$\sum_{i \in S} u_i(e_S, a_{N-S}(\omega_k, e_S)) \ge \min_{x \in NE(e_S)} \sum_{i \in S} u_i(e_S, x) = \sum_{i \in S} z_i^h,$$

the payoff vector z^h for S is feasible for the action profile a. When t becomes the final contract, the action profile a is played in σ, and each participant i of S receives the payoff z_i^h.

If t is accepted, each responder j receives the payoff $(1-\epsilon)u_j(\sigma|(S,t)) + \epsilon z_j^h$, where $u_j(\sigma|(S,t))$ is the continuation payoff that j will receive in σ from round $k+1$ with the state (S,t). Because $u_j(\sigma|(S,t)) \ge u_j(t) = z_j^h > u_j(t^k)$, it holds that $(1-\epsilon)u_j(\sigma|(S,t)) + \epsilon z_j^h > u_j(t^k)$. Thus, every j accepts t because j receives the payoff $u_j(t^k)$ by rejection. Then, proposer h receives the expected payoff

$$(1-\epsilon)u_h(\sigma|(S,t)) + \epsilon u_h(t) \ge u_h(t) = u_h(t^k) + \frac{r}{s}.$$

The inequality holds because $u_h(\sigma|(S,t)) \geq u_h(t)$. Therefore, proposer h optimally makes an acceptable proposal (which may be different from t) and receives at least the payoff $u_h(t^k) + r/s$.

We have shown that if non-participant h joins the negotiations, then h can receive an expected payoff greater than or equal to $u_h(t^k) + \theta_h(r/s)$ where θ_h is the probability that h becomes a proposer. Note that as a responder, h receives at least the payoff $u_h(t^k)$. Thus, non-participant $h \notin S_k$ is better off by participating in the negotiations in σ. This contradicts that σ prescribes a Nash equilibrium in the participation stage.

By the proof above, it holds that for every $\epsilon > 0$, there exists some player $i \notin S_k$, in every round k with a state $\omega_k = (S_k, t^k)$, where $S_k \neq N$, who participates in the negotiations on the play of σ. Since the number of players is n, this proves the theorem. □

References

Barrett, S.: Self-enforcing international environmental agreements. Oxford Econ. Pap. **46**, 878–894 (1994)

Binmore, K., Osborne, M.J., Rubinstein, A.: Noncooperative models of bargaining. In: Aumann, R.J., Hart, S. (eds.) Handbook of Game Theory with Economic Applications, vol. 1, pp. 179–225. North Holland, Amsterdam (1992)

Britz, V., Herings, P.J.-J., Predtetchinski, A.: Non-cooperative support for the asymmetric Nash bargaining solution. J. Econ. Theory **145**, 1951–1967 (2010)

Buchholz, W., Sandler, T.: Global public goods: a survey. J. Econ. Lit. **59**, 488–545 (2021)

Caparrós, A.: Bargaining and international environmental agreements. Environ. Resour. Econ. **65**, 5–31 (2016)

Carraro, C., Eyckmans, J., Finus, M.: Optimal transfers and participation decisions in international environmental agreements. Rev. Int. Organ. **1**, 379–396 (2006)

Chatterjee, K., Dutta, B., Ray, D., Sengupta, K.: A noncooperative theory of coalitional bargaining. Rev. Econ. Stud. **60**, 463–477 (1993)

Coase, R.H.: The problem of social cost. J. Law Econ. **3**, 1–44 (1960)

Cooter, R.D.: The Coase theorem. In: Eatwell, J., et al. (eds.) The New Palgrave: Allocation, Information and Markets, pp. 64–70. Macmillan, London (1989)

D'Aspremont, C., Jacquemin, A., Gabszewicz, J.J., Weymark, J.A.: On the stability of collusive price leadership. Can. J. Econ. **16**, 17–25 (1983)

Dixit, A., Olson, M.: Does voluntary participation undermine the Coase theorem? J. Public Econ. **76**, 309–335 (2000)

Gomes, A.: Multilateral contracting with externalities. Econometrica **73**, 1329–1350 (2005)

Gomes, A., Jehiel, P.: Dynamic processes of social and economic interactions: on the persistence of inefficiencies. J. Polit. Econ. **113**, 626–667 (2005)

Hyndman, K., Ray, D.: Coalition formation with binding agreements. Rev. Econ. Stud. **74**, 1125–1147 (2007)

Jackson, M.O., Wilkie, S.: Endogenous games and mechanisms: side payments among players. Rev. Econ. Stud. **72**, 543–566 (2005)

Karp, L., Simon, L.: Participation games and international environmental agreements: a non-parametric model. J. Environ. Econ. Manag. **65**, 326–344 (2013)

Laruelle, A., Valenciano, F.: Noncooperative foundations of bargaining power in committees and the Shapley–Shubik index. Games Econ. Behav. **63**, 341–353 (2008)

Myerson, R.B.: Game Theory: Analysis of Conflict. Harvard University Press, Cambridge (1991)

Nash, J.F.: Two-person cooperative games. Econometrica **21**, 128–140 (1953)

Okada, A.: A noncooperative coalitional bargaining game with random proposers. Games Econ. Behav. **16**, 97–108 (1996)

Okada, A.: The efficiency principle in non-cooperative coalitional bargaining. Jpn. Econ. Rev. **51**, 34–50 (2000)

Okada, A.: The Nash bargaining solution in general n-person cooperative games. J. Econ. Theory **145**, 2356–2379 (2010)
Ray, D.: A Game-Theoretic Perspective on Coalition Formation. Oxford University Press, Oxford (2007)
Ray, D., Vohra, R.: A theory of endogenous coalition structures. Games Econ. Behav. **26**, 286–336 (1999)
Rubinstein, A.: Perfect equilibrium in a bargaining model. Econometrica **50**, 97–109 (1982)
Seidmann, D.J., Winter, E.: A theory of gradual coalition formation. Rev. Econ. Stud. **65**, 793–815 (1998)
Serrano, R.: Sixty-seven years of the Nash program: Time for retirement? J. Spanish Econ. Assoc. **12**, 35–48 (2021)