Economic Theory (2024) 77:975–997
https://doi.org/10.1007/s00199-021-01398-x

RESEARCH ARTICLE

Incentive and welfare implications of cross-holdings in oligopoly

Hongkun Ma[1,2] · **Cheng-Zhong Qin**[3] · **Chenhang Zeng**[4]

Received: 28 February 2019 / Accepted: 13 October 2021 / Published online: 2 November 2021

Abstract
Competitive implications of cross-holdings have been extensively analyzed in the literature. Incentives for engaging cross-holdings and welfare effects were however rarely studied. Although a similar logic as with the merger paradox holds for Cournot oligopolies with homogeneous products and symmetric technologies, we show that there are profit incentives for firms to engage cross-holdings with asymmetric technologies. Furthermore, we show that social welfare could be enhanced with cross-holdings even though the market becomes more concentrated. We also discuss the robustness of both the submodularity of the Cournot model with respect to the presence of cross-holdings and our results with respect to product differentiation.

We thank the editor, Nicholas Yannelis, the associate editor, and two anonymous reviewers for their constructive comments and suggestions that have helped to greatly improve the paper. We also thank Rabah Amir, Masaki Aoyagi, Yongmin Chen, Zhiqi Chen, Gautam Gowrisankaran, Patrick Legros and the audience in the 2016 international conference on innovation and industrial economics in Nanjing for insightful comments and suggestions. Financial support from the Humanity and Social Science Planning Foundation of the Ministry of Education of China (Grant No. 20YJA790001), the Humanity and Social Science Youth Foundation of the Ministry of Education of China (Grant No. 18YJC790115), and the National Science Foundation of China (Grant No. 72103210, 71721001) are gratefully acknowledged.

✉ Chenhang Zeng
cz_sdu@163.com

Hongkun Ma
mahongkun1212@163.com

Cheng-Zhong Qin
qin@econ.ucsb.edu

1 Business School, Sun Yat-sen University, Guangzhou 510275, China

2 Zhuhai Huafa Investment Holdings Co. Ltd., Hengqin 519031, China

3 Department of Economics, University of California, Santa Barbara, USA

4 Wenlan School of Business, Zhongnan University of Economics and Law, 182 Nanhu Ave., Wuhan 430073, China

Keywords Cross-holding · Joint profits · Oligopoly · Cournot equilibrium · Submodularity

JEL Classification D43 · L13

1 Introduction

Cross-holding refers to a situation where a firm acquires partial ownership in another firm, which entitles the acquiring firm a share in the acquired firm's profits but not in decision right. They are commonly observed in the real world. Examples include cross-holdings in automobile (Alley 1997), telecommunication (Brito et al. 2014), airline (Airline Business 1998), banking (Dietzenbacher et al. 2000), and IT (Gilo et al. 2006) industries.

A substantial literature on the anti-competitive effects of cross-holdings has emerged. A well-known result in the literature concerning competitive implications of cross-holdings states that with Cournot competition on the output market, the output market equilibrium will become less competitive. As argued in Reynolds and Snapp (1986), after a firm has entered into a long equity position in a rival firm, it is induced to take into consideration the effect of its output decision on the rival's profit. This consideration makes the firm compete less aggressively, because by doing so the firm can increase the profit of the rival and hence its stake in the rival's profit. Thus, if all firms hold partial ownership in each other, they are induced to produce less which leads to greater profits for all.[1]

Most existing analysis of the anti-competitive effects takes cross-holdings as given. However, for reasons as with the *merger paradox* in Salant et al. (1983), depending on the existing structure of cross-holdings, two firms may be better off reducing cross-holdings in each other. Indeed, for Cournot model with symmetric and constant marginal costs, Flath (1991) and Reitman (1994) showed that it is not individually incentive compatible to form cross-holdings. In this paper we investigate incentives to engage cross-holdings and welfare implications with asymmetric technologies. For technical trackability, we focus on cases of cross-holdings in which only one firm acquires partial ownership in rivals. Examples of such "radiation" type of cross-holdings include the multilateral partial ownership arrangements between Microsoft and RealNetworks and between Microsoft and Apple in 1997, which produce the three famous media players (i.e., Windows Media Player, RealPlayer and QuickTime) in the

[1] For empirical evidence and analysis of the anti-competitive effects of cross-holdings, we refer the reader to Alley (1997), Dietzenbacher et al. (2000), Trivieri (2007) and Nain and Wang (2018) among other papers. Alley (1997) studied the effect of cross-holdings on collusion with the data of automobile industry, and found that Japanese automobile manufacturers behave collusively in the domestic market. Dietzenbacher et al. (2000) conducted empirical studies using the data of the Dutch financial sector, and found that the price-cost margins are increased under both Cournot and Bertrand competition. Trivieri (2007) investigated the effects of cross-holdings with the data of Italian banks for the period 1996–2000, and showed that cross-holding reduces market competition. Nain and Wang (2018) collected a sample of 1068 minority share acquisitions in U.S. manufacturing industries, and found a significant increase in both output prices and price-cost margins.

world.[2] We characterize both conditions under which the acquiring and acquired firms have profit incentives to engage cross-holdings and conditions under which total welfare increases with cross-holdings. For the case without pre-existing cross-holdings, Farrell and Shapiro (1990) established necessary and sufficient conditions for bilateral cross-holdings between a large and a small firm to be jointly profitable. We analyze firms' incentives to engage multilateral as well as bilateral cross-holdings with or without pre-existing ones. Our results imply in particular that multilateral cross-holdings can still be profitable even if bilateral cross-holdings are not.

We discuss effects of cross-holdings on the submodularity of Cournot oligopolies. We show that the submodularity of the Curnot model is robust with respect to the presence of cross-holdings.

We also analyze the robustness of competitive and welfare effects of cross-holdings by allowing products to be differentiated. For the model of product differentiation as analyzed in Dixit (1979), Singh and Vives (1984) and Häckner (2000), we show that both the competitive and welfare effects of cross-holdings with homogenous products continue to hold under similar conditions. The strengths of the effects, however, depend on the degree of production differentiation.[3]

The asymmetries in firms' technologies are the key driving force for our results. There are two output effects associated with a firm increasing ownership in the rivals. First, total output falls, which leads to higher market concentration. Second, total output is redistributed across firms, in that all non-acquiring firms increase their outputs while the acquiring firm reduces its output. It follows from these two effects that the non-acquiring firms' profits increase, which could outweigh the loss in the acquiring firm's profit making the increase in partial ownership jointly profitable. Furthermore, cross-holdings are socially desirable when the increase in producer surplus dominates the loss in consumer surplus. The reason is that cross-holdings cause shifts in production similar to cost reductions analyzed in Lahiri and Ono (1988) and Wang and Zhao (2007). These authors found that a marginal cost reduction in a minor firm shifts production from the more efficient firms to this minor firm, which could reduce producer surplus to such a degree that results in a lowered social welfare if this firm is sufficiently inefficient. In our model, if the acquiring firm is sufficiently inefficient, cross holding shifts production from the inefficient acquiring firm to rival firms, which could improve social welfare through the change in rivalry among firms.

The rest of the paper is organized as follows. Section 2 introduces the model, and Sect. 3 covers our analysis and results. In Sect. 4, we extend our discussion to the case of differentiated products. Section 5 concludes. We organize most of the proofs and claims in an "Appendix".

[2] On July 21, 1997, Microsoft acquired a 10% stake in RealNetworks Inc. (then known as Progressive Networks) at the aggregate price of 30 million dollars. See "Microsoft Takes a Stake in Progressive Networks" at http://www.nytimes.com/1997/07/22/business/microsoft-takes-a-stake-in-progressive-networks.html. On August 5, 1997, Microsoft and Apple reached an agreement according to which Microsoft could purchase shares of Apple's nonvoting, convertible, preferred stock at the aggregate price of 150 million dollars. The purchase price was below the market. See "Preferred Stock Purchase Agreement - Apple Computer Inc. and Microsoft Corp.(Aug 04, 1997)" at http://contracts.corporate._ndlaw.com/planning/purchase/971.html.

[3] Dasgupta and Tsui (2004) investigated behavioral implications of cross-holdings between bidders in auctions. We refer the interested reader to their paper for details.

2 The model

Consider an industry with n firms producing and selling a homogeneous good. Denote by x_i the output of firm i, X the total output, and $s_i = x_i/X$ the market share of firm i. Market inverse demand is $P(X)$. To produce output x_i, firm i incurs a variable cost $c_i(x_i)$. We use $c_i'(x_i) = \partial c_i(x_i)/\partial x_i$ and $c_i''(x_i) = \partial c_i'(x_i)/\partial x_i$ to denote the marginal cost and the derivative of the marginal cost of firm i. The following assumptions will be assumed throughout this paper.

Assumption 1 $P'(X) + x_i P''(X) < 0.$

Assumption 2 $c_i''(x_i) \geq 0.$

Assumption 1 is equivalent to $P'(X) + XP''(X) < 0$, and guarantees that firm i's reaction curve is downward-sloping, and hence that the Cournot game is a *submodular* game, or a game with *strategic substitutes*. Together with the property that each payoff depends only on own output and aggregate output, this guarantees the existence of a Cournot equilibrium.[4] With a convex cost function, this condition is also sufficient for the *uniqueness* of equilibrium (see Vives 1999). Assumption 2 implies that the marginal cost of firm i is nondecreasing. These assumptions on the demand and cost functions are standard in Cournot analysis, and they also jointly ensure the *stability* of Cournot equilibrium (see Vives 1999).

Remark 1 Amir and Lambson (2000) studied the effects of entry in a Cournot industry with symmetric firms. They showed that only symmetric equilibrium exists if the demand and cost functions satisfy $c''(x) > P'(X)$ for $0 \leq x \leq X$. This condition is automatically satisfied if the cost function is convex ($c''(x) > 0$). Thus, the result extends the classic McManus (1962, 1964) result by allowing for certain limited concavity of the cost function (increasing returns to scale). Similarly, our results hold if we replace Assumption 2 with $c_i''(x_i) > P'(X)/2$ for all $X \geq x_i \geq 0$. With symmetry, our condition is stronger than Amir and Lambson's (2000) due to the addition of cross-holdings.

We consider cross-holdings under which one firm holds shares in some other firms but none of the other firms holds any ownership shares in its rivals.[5] The cross-holdings between Microsoft, RealNetworks and Apple provide such an example. Figure 1 provides an illustration with firm 1 as the acquiring firm and firms 2–5 as the acquired firms. The arrows in this figure point in the direction of stock flows.[6]

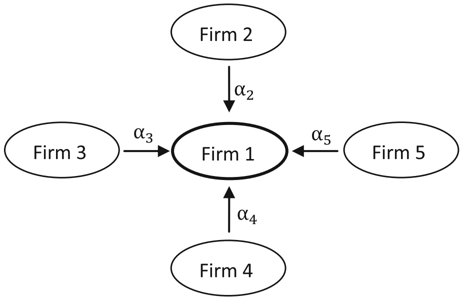
Following the literature, we consider cross-holdings that are *passive*, so that the acquiring firm has *silent* financial interests in acquired firms only. Furthermore, to be consistent with empirical observations, the acquiring firm is restricted to acquire no more than 50% of passive ownership in any rival firm. Consequently, each acquired

[4] For a proof of these results and further discussion, see Novshek (1985) and Amir (1996). The latter also provides an alternative sufficient condition for the same conclusion: $P(X)$ is a log-concave function, i.e. $PP'' - P'^2 \leq 0$.

[5] We say that cross-holdings between two firms are unidirectional if one firm holds partial ownership in the other firm but not conversely. They are the simplest radiation type of cross-holdings.

[6] Note that we may also have outsiders in the market, which are not graphically shown in the figure.

Fig. 1 Cross-holding structure



firm i decides on x_i independently, and keeps operating earnings net of those going to the acquiring firm. In contrast, the acquiring firm receives financial interests in rivals' operating earnings in addition to its own operating earning. As a result, the objective that guides the acquiring firm's output choice is the maximization of the sum of its own operating earning and return on its ownership holdings.

Suppose that firm 1 initially holds a share, α_i, of firm i's ownership for $i = 2, 3, \ldots, k$, where $2 \leq k \leq n$. We are interested in finding conditions for when it will be jointly profitable for firm 1 and firm $j (2 \leq j \leq k)$ to increase α_j without hurting the other acquired firms.

Let α_j change while keep α_i fixed for all $2 \leq i \leq k$ and $i \neq j$. Firm $i \geq 2$ chooses output x_i to maximize its operating earning $\pi_i = P(X)x_i - c_i(x_i)$ with first-order condition:

$$P(X) + x_i P'(X) - c_i'(x_i) = 0. \tag{1}$$

Differentiating both sides of this first-order condition with respect to x_i yields that

$$\left(P'(X) + x_i P''(X)\right) dX + P'(X)dx_i - c_i''(x_i)dx_i = 0. \tag{2}$$

As the acquiring firm, firm 1 chooses x_1 to maximize

$$\pi_1 + \sum_{i=2}^{k} \alpha_i \pi_i.$$

Consequently, firm 1's first-order condition is given by

$$P(X) + P'(X)x_1 - c_1'(x_1) + \sum_{i=2}^{k} \alpha_i P'(X)x_i = 0. \tag{3}$$

As in Farrell and Shapiro (1990), set

$$\lambda_i = \frac{-P'(X) - x_i P''(X)}{c_i''(x_i) - P'(X)} \text{ and } \Lambda = \sum_{i=1}^{n} \lambda_i.$$

Then, by (2),

$$dx_i = -\lambda_i dX, \ i = 2, 3, \ldots, n. \tag{4}$$

It follows that

$$dx_1 = dX - \sum_{i=2}^{n} dx_i = \left(1 + \sum_{i=2}^{n} \lambda_i\right) dX = (1 + \Lambda - \lambda_1) dX. \tag{5}$$

Notice that $\lambda_i > 0$ for all firms. By definition and (4), λ_i measures how its equilibrium output changes in response to changes in price for firm $i \geq 2$. Furthermore, λ_1 has a similar interpretation under condition $\alpha_i = 0$ for all $2 \leq i \leq k$. In contrast, in the presence of cross-holdings (i.e., $\alpha_i > 0$), (3) implies that such interpretation is inappropriate.

3 Analysis and results

3.1 Profitability for forming cross-holdings

An increase of firm 1's ownership in firm j will cause the total market output to change. The following lemma shows that the market will be more concentrated as firm 1 increases its ownership in firm j.

Lemma 1 *The industry output decreases with α_j.*

Intuitively, as firm 1 increases its ownership share in firm j, it tends to be more concerned with financial interests in firm j and thus be more conservative in production. This in turn provides incentives for the other firms to increase production (i.e., output expansion effect) as shown in the next lemma.

Lemma 2 *Firm 1 reduces its output, while all other firms increase theirs as α_j increases.*

Lemmas 1 and 2 together show that all non-acquiring firms increase their outputs as firm 1 increases its partial ownership in firm j, but nonetheless total increase in non-acquiring firms' output falls below the acquiring firm's output decrease. The industry output decreases as a result of an increase in partial ownership of firm 1 in firm j. Our next lemma concerns the effects of these shifts in output on firms' profits.

Lemma 3 *The operating earning of firm 1 decreases but the operating earning of each of other firms increases with α_j.*

Since only α_j changes, for firm $i \geq 2$ and $i \neq j$, an increase in the operating earning means that it becomes more profitable. Thus, Lemma 3 shows that firms other than firms 1 and j will be better off as firm 1 increases its ownership in firm j. For firm 1 and firm j, we need to look at their joint profit as α_j changes. Our next proposition establishes a sufficient and necessary condition for when total profit of firms 1 and j also increases as α_j increases.

Proposition 1 *For $2 \leq j \leq k$, the joint profit of firm 1 and firm j increases with α_j if and only if*

$$s_1(\Lambda - \lambda_1) < \sum_{i=2}^{k} \alpha_i s_i(\lambda_1 + \lambda_i - \Lambda) + (1 - \alpha_j)s_j(1 + \lambda_j). \tag{6}$$

Increasing α_j induces rival firms to produce more, which can be understood as the output effect that curtails the joint profit of firm 1 and firm j. On the other hand, it generates a counteracting effect—price effect, which is due to the reduction in industry output. Recall that $\Lambda - \lambda_1$ on the left-hand side of (6) represents the sum of firm $i \geq 2$'s equilibrium output responsiveness to changes in price. If $\Lambda - \lambda_1$ is small, the output effect for all other firms is relatively small, which implies that the other firms benefit less from such additional acquisition. Furthermore, there is an output redistribution effect between firm 1 and firm j. If firm 1 is relatively inefficient (a small s_1), this output redistribution effect improves joint efficiency because it allows the more efficient firm, firm j, to produce more. As a result, if $\Lambda - \lambda_1$ and/or s_1 is small, it is more likely to realize a joint profit improvement for firm 1 and firm j.

Since the total profit of firm 1 and firm j increases under condition (6), mutually beneficial prices for transferring ownership from firm j to firm 1 are possible.[7] That is, condition (6) guarantees that firms 1 and j will be better off as firm 1 increases its ownership in firm j. Example 1 below provides an illustration of this result.

Example 1 Consider an industry with 3 firms. Suppose market demand is $P(X) = 1 - X$ and firms' cost functions are $c_1(x_1) = 2x_1^2/5$, $c_2(x_2) = x_2^2/2$, $c_3(x_3) = x_3^2$. Simple calculations show $\lambda_1 = 5/9$, $\lambda_2 = 1/2$, $\lambda_3 = 1/3$. Thus, $\Lambda = \sum_{i=1}^{3} \lambda_i = 25/18$. Notice that λ_i for all $1 \leq i \leq 3$ are calculated for Cournot equilibrium without cross-holdings. Since we consider linear market demand and quadratic cost functions, each λ_i remains constant as cross-holdings vary. The same is true for subsequent examples.

Suppose firm 1 holds α_2 amount of partial ownership in firm 2 and α_3 amount in firm 3. Then, in Cournot equilibrium with these cross-holdings,

$$x_1 = \frac{3 - 1.5\alpha_2 - \alpha_3}{12.9 - 1.5\alpha_2 - \alpha_3}, \; x_2 = \frac{2.7}{12.9 - 1.5\alpha_2 - \alpha_3}, \; x_3 = \frac{1.8}{12.9 - 1.5\alpha_2 - \alpha_3}.$$

Accordingly,

$$s_1 = \frac{3 - 1.5\alpha_2 - \alpha_3}{7.5 - 1.5\alpha_2 - \alpha_3}, \; s_2 = \frac{2.7}{7.5 - 1.5\alpha_2 - \alpha_3}, \; s_3 = \frac{1.8}{7.5 - 1.5\alpha_2 - \alpha_3}.$$

With these market shares and $n = 3$, Proposition 1 states that the joint profit of firm 1 and firm 3 increases with α_3 if and only if

$$0.6 - 2.1\alpha_2 + 14.8\alpha_3 < 0.$$

[7] Grossman and Hart (1980) showed that if each of the existing shareholders of a firm holds a small amount of shares, then no takeover will ever take place. They referred to this fundamental problem as the "free-rider" problem. Passive cross-holdings we consider in the present paper do not grant acquiring firm's takeovers. As such, the free-rider problem is not relevant to increases in cross-holdings.

It is clear that there exist values for α_2 and α_3 with which (6) holds (for example, $\alpha_2 = 0.36$ and $\alpha_3 = 0.01$). □

Condition (6) does not hold for Cournot oligopolies with linear demand, constant symmetrical marginal costs, and homogeneous products. Consequently, cross-holdings cannot be jointly profitable to form in such cases. Proposition 1 shows that asymmetries in firms' technologies can help eliminate disincentive for engaging cross-holdings.

The following corollary follows directly from Lemma 3 and Proposition 1.

Corollary 1 *A small increase in firm 1's partial ownership in firm j from α_j satisfying (6) raises firm i's profit as well as the joint profit of firm 1 and firm j, for $2 \leq i \leq k$ and $i \neq j$.*

Corollary 1 shows that as firm 1's partial ownership in firm j increases from α_j satisfying (6), all other acquired firms are better off. An implication of this result is that all other acquired firms should welcome firm 1 to increase its partial ownership in firm j from α_j satisfying (6).

Farrell and Shapiro (1990) investigated when there is no pre-existing partial ownership, whether a firm has profit incentives to acquire partial ownership in a rival firm in a Cournot oligopolistic market. They asserted that given $\alpha_i = 0$ for all $i \geq 2$, the total profit of firm 1 and firm j increases as a result of a small amount of partial ownership acquisition by firm 1 in firm j if and only if[8]

$$s_1(\Lambda - \lambda_1) < s_j(1 + \lambda_j). \tag{7}$$

Note that since there is no pre-existing partial ownership, the failure of (7) does not necessarily imply that it cannot be jointly profitable to increase firm 1's partial ownership in firm j from some pre-existing partial ownership $\alpha_j > 0$. In fact, given $\alpha_j > 0$ and $\alpha_i = 0$ for $2 \leq i \leq k$ and $i \neq j$, (6) implies that it is jointly profitable to increase firm 1's partial ownership in firm j if and only if

$$s_1(\Lambda - \lambda_1) < s_j(1 + \lambda_j) - \alpha_j s_j(1 + \Lambda - \lambda_1). \tag{7'}$$

The reason is that the values for each s_i (and possibly each λ_i) in conditions (7) and (7′) are different. Recall that an increase in α_j shifts production from firm 1 to the rival firms (Lemma 2). As a result, s_1 in condition (7′) is smaller than that in (7), while s_j in condition (7′) is larger than that in (7). It is worth remarking that (7′) reduces to (7) with $\alpha_j = 0$. As such, condition (7′) is an extension of condition (7) to allow for pre-existing partial ownership.

It is also worth pointing out that mutually beneficial multilateral cross-holdings are still possible even if condition (7′) is violated for all $\alpha_j \in [0, 1/2)$ and $j \geq 2$. The following example provides an illustration.

Example 2 Consider a Cournot industry with 3 firms, market demand $P(X) = 1 - X$, and firms' cost functions $c_1(x_1) = x_1^2/40$, $c_2(x_2) = x_2^2/10$, and $c_3(x_3) = x_3^2/12$.

[8] In Farrell and Shapiro (1990, p. 287), it is assumed that $j = 2$.

Simple calculations lead to $\lambda_1 = 20/21$, $\lambda_2 = 5/6$, and $\lambda_3 = 6/7$. Thus, $\Lambda = \sum_{i=1}^{3} \lambda_i = 37/14$.

Consider cross-holdings represented by α_2 and α_3. In Cournot equilibrium,

$$x_1 = \frac{20\,(42 - 35\alpha_2 - 36\alpha_3)}{3213 - 700\alpha_2 - 720\alpha_3},$$
$$x_2 = \frac{735}{3213 - 700\alpha_2 - 720\alpha_3}, x_3 = \frac{756}{3213 - 700\alpha_2 - 720\alpha_3}.$$

It follows that

$$s_1 = \frac{20\,(42 - 35\alpha_2 - 36\alpha_3)}{2331 - 700\alpha_2 - 720\alpha_3},$$
$$s_2 = \frac{735}{2331 - 700\alpha_2 - 720\alpha_3}, s_3 = \frac{756}{2331 - 700\alpha_2 - 720\alpha_3}.$$

Consider bilateral cross-holding as in Farrell and Shapiro (1990). With $j = 2$ and $\alpha_3 = 0$, (7′) reduces to

$$609 + 6671\alpha_2 < 0,$$

which fails for all $\alpha_2 \in [0, 1/2)$. Similarly, with $j = 3$ and $\alpha_2 = 0$, (7′) becomes

$$168 + 8577\alpha_3 < 0,$$

which also fails for all $\alpha_3 \in [0, 1/2)$. This shows that there does not exist any jointly profitable bilateral partial ownership arrangement with firm 1 as the acquiring firm.

Firms' operating earnings in the absence of cross-holdings are $\pi_1 = 0.07$, $\pi_2 = 0.0576$, $\pi_3 = 0.06$. With cross-holdings given by $\alpha_2 = 0.1$ and $\alpha_3 = 0.4$, however, their operating earnings are $\pi_1 = 0.0514$, $\pi_2 = 0.0729$, $\pi_3 = 0.076$. It follows that with the cross-holdings, the joint profit of firm 1 and firm 2 (i.e., $\pi_1 + \pi_2 + \alpha_3\pi_3$) and that of firm 1 and firm 3 (i.e., $\pi_1 + \alpha_2\pi_2 + \pi_3$) as well as the total profit of all three firms (i.e., $\pi_1 + \pi_2 + \pi_3$) are higher than that without any cross-holdings, respectively. As a result, mutually (bilaterally or multilaterally) beneficial prices for transferring partial ownership characterized by the cross-holdings exist. □

It is worth noticing that cross-holdings generate a (directed) network between the firms, with the link from firm i to firm j characterized by the fractional ownership that firm i holds in firm j. A natural stability concept for endogenous cross-holdings would require that firm i and firm j be unable to increase total profits by changing firm i's partial ownership in firm j for all $i \neq j$. This concept of stability is adapted from Jackson and Wolinsky (1996) allowing for continuous links (i.e. links with continuous intensities). The following remark illustrates that pairwise stable cross-holdings as specified above do not automatically exist in our setting and deserve special studies.[9]

[9] We refer the interested reader to Qin et al. (2017) for an analysis of pairwise stable cross-holdings in a linear symmetric Cournot oligopoly.

Remark 2 For the radiation type of cross-holdings analyzed in this paper, necessary and sufficient conditions for the existence of pairwise stable cross-holdings can be derived from Proposition 1: cross-holdings with firm 1 holding fractional ownership α_j in firm j, for $j = 2, \ldots, k$, are pairwise stable if and only if

$$s_1(\Lambda - \lambda_1) = \sum_{i=2}^{k} \alpha_i s_i(\lambda_1 + \lambda_i - \Lambda) + (1 - \alpha_j)s_j(1 + \lambda_j), \quad j = 2, \ldots, k.$$

It follows from simple calculations that the above conditions are satisfied in Example 1 with $\alpha_2 = 0.4189$ and $\alpha_3 = 0.0189$, but no permissible values for α_2 and α_3 can make these conditions satisfied in Example 2.

Farrell and Shapiro (1990, p. 287) concluded that (7) implies that the joint profit falls if firm 1 is bigger than firm 2 (in terms of market share), but joint profit increases if firm 1 is smaller than firm 2. Below we show by example that this conclusion is misleading in two ways. First, the condition in (7) does not necessarily imply that joint profit decreases when firm 1 is bigger than firm 2 as illustrated in the following example.

Example 3 Consider an industry with 3 firms. Suppose market demand is $P(X) = 1 - X$ and firms' production costs are $c_1(x_1) = 2x_1^2/5$, $c_2(x_2) = x_2^2/2$, $c_3(x_3) = x_3^2$. It can be shown that $\lambda_1 = 5/9$, $\lambda_2 = 1/2$, $\lambda_3 = 1/3$. Thus, $\Lambda = \sum_{i=1}^{3} \lambda_i = 25/18$. In Cournot equilibrium, $x_1 = 10/43$, $x_2 = 9/43$, and $x_3 = 6/43$ which implies $s_1 = 2/5$, $s_2 = 9/25$, and $s_3 = 6/25$. Consequently, $s_1(\Lambda - \lambda_1) = 1/3$, $s_2(1 + \lambda_2) = 27/50$, and $s_3(1 + \lambda_3) = 8/25$. It follows that $s_1(\Lambda - \lambda_1) < s_2(1 + \lambda_2)$. Notice that firm 1 is bigger than firm 2. However, by (7), it is mutually beneficial for firm 1 to hold some partial ownership in firm 2. □

Second, it is not always profitable for a small firm to acquire partial ownership of a big firm. Example 4 provides an illustration.

Example 4 Consider a Cournot oligopoly having 4 firms, market demand $P(X) = 1 - X$, and firms' cost functions $c_1(x_1) = 2x_1^2/3$, $c_2(x_2) = x_2^2/2$, $c_3(x_3) = x_3^2/4$, $c_4(x_4) = x_4^2/6$. It can be shown that $\lambda_1 = 3/7$, $\lambda_2 = 1/2$, $\lambda_3 = 2/3$, $\lambda_4 = 3/4$. Thus, $\Lambda = \sum_{i=1}^{4} \lambda_i = 197/84$, $x_1 = 36/281$, $x_2 = 42/281$, $x_3 = 56/281$, and $x_4 = 63/281$. We have $s_1 = 36/197$, $s_2 = 42/197$, $s_3 = 56/197$, and $s_4 = 63/197$. It follows that $s_1 < s_2$. Notice that with $\alpha_3 = \alpha_4 = 0$ and the above market shares, (7) requires that $12 + 245\alpha_2 < 0$. This cannot hold for all $\alpha_2 \in [0, 1/2)$. Consequently, it is not jointly profitable for firm 1 to hold partial ownership in firm 2, even though firm 2 is bigger than firm 1. □

3.2 Submodularity and cross-holdings

In this section, we examine whether and how cross-holdings affect the submodularity of the Cournot model. To this end, we continue to let firm 1 hold a share, α_i, of firm i's ownership for $i = 2, 3, \ldots, k$. The cross partial derivative of firm 1's final profit with respect to x_1 and x_j, where $j \geq 2$, is given by

$$\frac{\partial^2(\pi_1 + \sum_{i=2}^{k} \alpha_i \pi_i)}{\partial x_1 \partial x_j} = \begin{cases} (1+\alpha_j)P'(X) + (x_1 + \Sigma_{i=2}^{k}\alpha_i x_i)P''(X) & 2 \leq j \leq k; \\ P'(X) + (x_1 + \Sigma_{i=2}^{k}\alpha_i x_i)P''(X) & j > k. \end{cases} \quad (8)$$

In comparison, for firm $i \geq 2$, we have

$$\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} = P'(X) + x_i P''(X), \quad \text{where} \quad j \neq i. \quad (9)$$

Using the cross-partial test, it follows that the game is submodular if the partial derivatives in both (8) and (9) are negative (see Novshek 1985 and Amir 1996),[10] which hold under $P'(X) + XP''(X) < 0$ and $P'(X) < 0$. Thus submodularity follows directly.

Proposition 2 *With the presence of cross-holdings, the submodularity of the Cournot model is still valid under Assumption 1.*

Since each firm's cost function is assumed to depend on its own output, but not on the output of any other firm, this submodularity guarantees the existence of Cournot equilibrium with cross-holdings.

3.3 Welfare analysis

We analyze how welfare changes with cross-holdings. As usual, the welfare is the sum of consumer and producer surplus:

$$W = \int_0^X P(z)dz - \sum_{i=1}^{n} c_i(x_i).$$

Therefore, the change in total welfare as firms' quantities change is given by $dW = \sum_{i=1}^{n}(P(X) - c_i'(x_i))dx_i$. From the first-order conditions (1) and (3), we can obtain the price-cost margin for each firm:

$$P(X) - c_1'(x_1) = -P'(X)\left(x_1 + \sum_{i=2}^{k} \alpha_i x_i\right)$$

for firm 1 and

$$P(X) - c_i'(x_i) = -P'(X)x_i$$

[10] We refer the readers to Amir (2005) for an excellent survey on the submodularity of Cournot and supermodularity of Bertrand models.

for firm $i \geq 2$. As a result,

$$dW = -P'(X)\left(x_1 + \sum_{i=2}^{k} \alpha_i x_i\right) dx_1 - \sum_{i=2}^{n} P'(X) x_i dx_i.$$

Substituting $dx_1 = (1 + \Lambda - \lambda_1)dX$ and $dx_i = -\lambda_i dX$, for $i \geq 2$, into the above equation yields that

$$dW = -P'(X)\left(x_1 + \sum_{i=2}^{k} \alpha_i x_i\right)(1 + \Lambda - \lambda_1)dX + \sum_{i=2}^{n} P'(X)\lambda_i x_i dX.$$

Consequently,

$$\frac{dW}{d\alpha_j} = \frac{dX}{d\alpha_j} P'(X)\left(-\left(x_1 + \sum_{i=2}^{k} \alpha_i x_i\right)(1 + \Lambda - \lambda_1) + \sum_{i=2}^{n} \lambda_i x_i\right). \tag{10}$$

Notice that (10) can be rewritten as

$$\frac{dW}{d\alpha_j} = \frac{dX}{d\alpha_j} P'(X)\left(\underbrace{-\left(x_1 + \sum_{i=2}^{k} \alpha_i x_i\right)(1 + \Lambda)}_{\text{negative}} + \underbrace{\lambda_1\left(x_1 + \sum_{i=2}^{k} \alpha_i x_i\right) + \sum_{i=2}^{n} \lambda_i x_i}_{\text{positive}}\right). \tag{11}$$

As in Farrell and Shapiro (1990, p. 288), increasing α_j generates two counteracting effects on social welfare. On the one hand, an increase in α_j reduces the output of firm 1, which is detrimental to social welfare. This negative effect is summarized by $-(x_1 + \sum_{i=2}^{k} \alpha_i x_i)(1 + \Lambda)$. On the other hand, all other firms increase their production as α_j increases, which is favorable to social welfare. This positive effect is captured by $\lambda_1(x_1 + \sum_{i=2}^{k} \alpha_i x_i) + \sum_{i=2}^{n} \lambda_i x_i$. If the negative effect is dominated by the positive effect, an increase in α_j will increase social welfare even though market becomes more concentrated.

The following proposition characterizes conditions for when total welfare increases with α_j.

Proposition 3 *The social welfare increases with α_j if and only if*

$$\sum_{i=2}^{n} \lambda_i s_i > (1 + \Lambda - \lambda_1)\left(s_1 + \sum_{i=2}^{k} \alpha_i s_i\right). \tag{12}$$

The intuitions for Proposition 3 is straightforward. An increase in α_j results in a reduction in total output which is detrimental to social welfare. However, it also creates output shifting from the acquiring firm to all non-acquiring firms. If firm 1 is relatively inefficient, this output shifting increases industry profit in the market, which may even dominate the welfare loss due to total output reduction.

Lahiri and Ono (1988) analyzed the welfare effect of improvement in production efficiency (a reduction in marginal cost) for a firm under Cournot competition with homogeneous products and constant marginal costs. The authors found that helping a minor firm with a sufficiently low market share by reducing its marginal cost reduces social welfare. A cost reduction in an inefficient firm on the one hand increases industry output which has a positive effect on social welfare. On the other hand, it shifts production from other more efficient firms to the less inefficient one, which has a negative effect on social welfare. As long as the market share of the inefficient firm is sufficiently low, social welfare decreases.[11] In comparison, welfare change in our setting is also resulted from production shifts, but production shifts are driven by cross-holdings that do not change firms' production costs.

Observe that condition (12) cannot be implied by condition (6). This means that it is not always socially desirable for firm 1 to increase its ownership in firm j even though it is mutually beneficial for them. Nevertheless, the two conditions do overlap. The following example provides an illustration.

Example 5 Consider an industry with 3 firms. Suppose market demand is $P(X) = 1 - X$ and firms' cost functions are $c_1(x_1) = 6x_1^2$, $c_2(x_2) = x_2^2/2$, $c_3(x_3) = 10x_3^2$. It follows from simple calculations that $\lambda_1 = 1/13$, $\lambda_2 = 1/2$, $\lambda_3 = 1/21$. Thus, $\Lambda = \sum_{i=1}^{3} \lambda_i = 341/546$. Suppose firm 1 holds α_2 amount of partial ownership in firm 2 and α_3 amount in firm 3. Then, in Cournot equilibrium with these cross-holdings,

$$x_1 = \frac{42 - 21\alpha_2 - 2\alpha_3}{887 - 21\alpha_2 - 2\alpha_3},\ x_2 = \frac{273}{887 - 21\alpha_2 - 2\alpha_3},\ x_3 = \frac{26}{887 - 21\alpha_2 - 2\alpha_3}.$$

Accordingly,

$$s_1 = \frac{42 - 21\alpha_2 - 2\alpha_3}{341 - 21\alpha_2 - 2\alpha_3},\ s_2 = \frac{273}{341 - 21\alpha_2 - 2\alpha_3},\ s_3 = \frac{26}{341 - 21\alpha_2 - 2\alpha_3}.$$

We then look at whether firm 1 obtains incentives to increase its share (i.e., α_3) in firm 3, and whether the increase in share is socially desirable. To see that, by condition (6), the joint profit of firm 1 and firm 3 increases with α_3 if and only if

$$63\alpha_2 + 1644\alpha_3 - 178 < 0.$$

By condition (12), the social welfare increases with α_3 if and only if

$$252\alpha_2 + 24\alpha_3 - 47 < 0.$$

[11] By analogy, shifting production from the inefficient firm to other efficient firms could increase social welfare, even though industry output decreases.

Fig. 2 The overlapping area

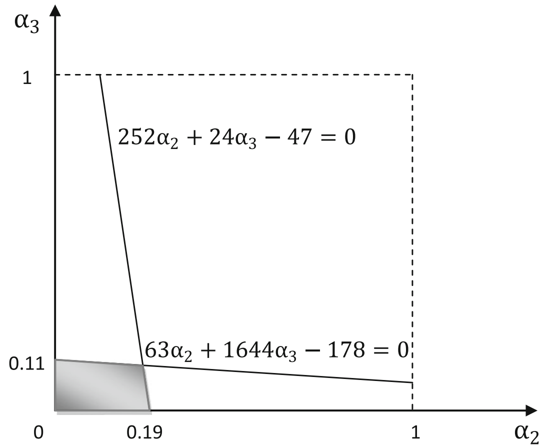


Figure 2 illustrates the two above regions with the horizontal axis representing α_2 and the vertical axis representing α_3. In the region below the line $63\alpha_2 + 1644\alpha_3 - 178 = 0$, joint profit of firm 1 and firm 3 increases with α_3. In the region on the left side of the line $252\alpha_2 + 24\alpha_3 - 47 = 0$, increasing α_3 raises social welfare. As long as both α_2 and α_3 are located in the shaded area, firm 1 obtains incentives to increase its share in firm 3 and social welfare also increases even though the market becomes concentrated.

It is desirable to determine when mutually beneficial cross-holdings can also promote the social welfare. The following corollary provides a characterization of such conditions.

Corollary 2 *For $2 \le j \le k$, raising α_j increases both joint profit and social welfare if (6) and the following condition are satisfied:*

$$\sum_{i=2}^{n} \lambda_i s_i > s_1 + \sum_{i=2}^{k} (1 + \lambda_i)\alpha_i s_i + (1 - \alpha_j) s_j (1 + \lambda_j). \tag{13}$$

With condition (13), (12) is now implied by (6). Thus, when valid, (13) implies that as firm 1's partial ownership in firm j increases, social welfare increases even though market becomes more concentrated.

Corollary 2 provides a sufficient condition for a profitable cross-holding to be welfare enhancing. Consider a special case in which the marginal costs of firms are constant, i.e., $c_i'(x_i) = c_i$ for all $i = 1, 2, \ldots, n$. When the acquiring firm is the most efficient firm, the following corollary shows that it is no longer possible for social welfare to increase with cross-holdings.

Corollary 3 *If firm 1 is the most efficient firm, then for any $j \neq 1$, social welfare decreases as firm 1 increases its ownership in firm j.*

With the increase in partial ownership, the industry output decreases by Lemma 1 which reduces social welfare. Furthermore, firm 1 (the most efficient firm) reduces its output while all other firms (the inefficient firms) expand production by Lemma 2, which further reduces the social welfare.

4 Cross-holdings with product differentiation

In this section, we analyze the robustness of our results in the previous section by allowing firms' products to be differentiated.

Following the IO literature on oligopoly with differentiated products, we assume that the inverse demand function for firm i's product is given by

$$p_i = a_i - x_i - \gamma \sum_{j \neq i} x_j, \tag{14}$$

where $\gamma \in (0, 1)$ measures the substitutability of the firms' products. A microeconomic foundation for the linear demand functions in (14) can be provided using the model of a representative consumers. We refer the readers to Shapley and Shubik (1969), Dixit (1979), Singh and Vives (1984), and Amir et al. (2017) for discussions.

Under Cournot competition with cross-holdings $\alpha_2, \ldots, \alpha_k$, firm $i \geq 2$ chooses output x_i to maximize its profit

$$\pi_i = (p_i - c_i)x_i = \left(a_i - x_i - \gamma \sum_{i \neq j} x_j - c_i \right) x_i,$$

yielding first-order condition:

$$a_i - 2x_i - \gamma \sum_{j \neq i} x_j - c_i = 0. \tag{15}$$

In contrast, as the acquiring firm, firm 1 chooses x_1 to maximize its final profit

$$\pi_1 + \sum_{i=2}^{k} \alpha_i \pi_i = (p_1 - c_1)x_1 + \sum_{i=2}^{k} \alpha_i (p_i - c_i)x_i.$$

Its optimal choice is characterized by the first-order condition:

$$a_1 - 2x_1 - \gamma \sum_{j \neq 1} x_j - c_1 - \gamma \sum_{i=2}^{k} \alpha_i x_i = 0. \tag{16}$$

The following lemma characterizes the competitive effects of cross-holdings as firm 1 increases its ownership in firm j.

Lemma 4 *As α_j increases, (i) the industry output decreases; (ii) firm 1 reduces its output, while all other firms increase theirs; (iii) the price-cost margins for all firms* *are increased; (iv) the operating earning of firm 1 decreases but the operating earning of each of other firms increases.*

Lemma 4 shows that when the acquiring firm raises its ownership in an acquired firm, the same effects on firms' individual and industry equilibrium outputs as with homogenous products continue to hold.

We now consider possible change in the joint profit of firm 1 and firm j as firm 1, the acquiring firm, increases its ownership in firm j.

Proposition 4 *For $2 \leq j \leq k$, the joint profit of firm 1 and firm j increases with α_j if and only if*

$$(n-1)\gamma s_1 + (n-2)\gamma \sum_{i=2}^{k} \alpha_i s_i < 2(1-\alpha_j)s_j. \tag{17}$$

As with homogenous products, (17) is more likely to hold the relatively less efficient firm 1 is. Notice also that the condition is more stringent the bigger γ is. Intuitively, as γ increases, the products are less differentiated, which intensifies market competition. Thus, reduced competition due to cross-holdings are more beneficial the less differentiated the products are.

The next result is concerned with the welfare effect of cross-holdings in the presence of cross-holdings.

Proposition 5 *The social welfare increases as α_j increases if and only if*

$$\gamma \sum_{i=2}^{n} s_i > (2 + (n-2)\gamma)\left(s_1 + \gamma \sum_{i=2}^{k} \alpha_i s_i\right). \tag{18}$$

Similar explanations for the welfare effects in a homogeneous good market in the previous section can be applied here. Furthermore, we observe that the higher the degree of product differentiation (i.e., the smaller of γ), the less likely social welfare is beneficiated by an increase in partial ownership, in the sense that the reduced market competition weakens the positive impacts of cross-holding on social welfare. Also, the results for the Cournot oligopoly with differentiated products reduce to those with homogeneous goods when $\gamma = 1$.

Wang and Zhao (2007) analyzed the welfare effects of cost reductions in differentiated oligopolies. The authors showed that a small cost reduction in a high-cost firm increases the industry output and raises consumer surplus, but leads to the reduction in industry profits. Social welfare decreases if the output share of the high-cost firm is below a critical level determined by the cost and demand parameters. In comparison, we show that increasing partial ownership in a rival firm by an inefficient firm will create a socially desirable output shifting which increases industry profits. If the market share of firm 1 is sufficiently low, this increase in industry profits will outweigh the welfare loss due to the reduction in industry output, which results in an increase in social welfare.

5 Conclusion

Cross-holdings are ubiquitous in the real world. We analyzed a model of multilateral cross-holdings between horizontal firms in homogeneous Cournot oligopolies with a single acquiring firm. We characterized conditions for partial ownership arrangements to be mutually beneficial among participating firms, and conditions for total welfare to increase with cross-holdings. Furthermore, we showed that the submodularity of the Cournot model remains with the presence of cross-holdings, and discussed the robustness of our results with respect to product differentiation. Our paper contributes to the literature on behavioral effects of cross-holdings in oligopolies by analyzing a model of cross-holding structures that participating firms have incentives to form, and further providing welfare implications of the model.

Several areas are worthwhile directions for future research. One direction is to carry out similar analysis for cross-holdings beyond those of the radiation type considered in the present paper. Another direction is to introduce controlling stakes, as in Levy et al. (2018), into the model and analyze their effects on market competition. A third direction is to endogenize the decision to acquire shares in asymmetric rivals, which we believe will generate rich policy implications and greatly enrich the current literature.

Appendix: Proofs

Proof of Lemma 1

Totally differentiating (3) and further manipulating the resulting expression yields

$$dx_1 = -\lambda_1 dX + \mu \sum_{i=2}^{k} \alpha_i s_i E dX - \mu \left(x_j d\alpha_j + \sum_{i=2}^{k} \alpha_i dx_i \right), \tag{19}$$

where

$$\mu = -\frac{P'(X)}{c_1''(x_1) - P'(X)} \text{ and } E = -\frac{XP''(X)}{P'(X)}.$$

Next, by (5) and (19), we get

$$\frac{dX}{d\alpha_j} = \frac{-\mu x_j}{1 + \Lambda - \mu \left(\sum_{i=2}^{k} \alpha_i s_i E + \sum_{i=2}^{k} \alpha_i \lambda_i \right)}. \tag{20}$$

Given our assumptions on demand and cost functions, we have $\lambda_i > 0$, $E < 1$, and $0 < \mu < 1$. Therefore, we have

$$1 + \Lambda - \mu \left(\sum_{i=2}^{k} \alpha_i s_i E + \sum_{i=2}^{k} \alpha_i \lambda_i \right) > 1 - \mu \sum_{i=2}^{k} \alpha_i s_i + \Lambda - \mu \sum_{i=2}^{k} \alpha_i \lambda_i > 0.$$

Since the numerator $-\mu x_j < 0$, the sign of $\frac{dX}{d\alpha_j}$ is negative. That is, $\frac{dX}{d\alpha_j} < 0$.

Proof of Lemma 2

It follows straightforward from (4) and (5) that

$$\frac{dx_1}{d\alpha_j} = (1 + \Lambda - \lambda_1)\frac{dX}{d\alpha_j}; \quad \frac{dx_i}{d\alpha_j} = -\lambda_i \frac{dX}{d\alpha_j} \quad \text{for} \quad i \geq 2.$$

With $\frac{dX}{d\alpha_j} < 0$ and $\lambda_i > 0$, we have $\frac{dx_i}{d\alpha_j} > 0$ and $\frac{dx_1}{d\alpha_j} < 0$.

Proof of Lemma 3

Firstly, we examine how $d\pi_1$ changes with regard to $d\alpha_j$. With $\pi_1 = P(X)x_1 - c_1(x_1)$, we obtain that

$$\begin{aligned}\frac{d\pi_1}{dX} &= P'(X)x_1 + \left(P(X) - c_1'(x_1)\right)\frac{dx_1}{dX} \\ &= -P'(X)\left(x_1(\Lambda - \lambda_1) + \sum_{i=2}^{k} \alpha_i x_i (1 + \Lambda - \lambda_1)\right).\end{aligned}$$

It is obvious that $d\pi_1/dX > 0$. As a result,

$$\frac{d\pi_1}{d\alpha_j} = \frac{d\pi_1}{dX}\frac{dX}{d\alpha_j} < 0.$$

Next, we examine how $d\pi_i$ changes with regard to $d\alpha_j$ for all other firms. For all $i \geq 2$, $\pi_i = P(X)x_i - c_i(x_i)$ and $dx_i = -\lambda_i dX$. So we have

$$\frac{d\pi_i}{dX} = P'(X)x_i + \left(P(X) - c_i'(x_i)\right)\frac{dx_i}{dX} = x_i P'(X)(1 + \lambda_i) < 0.$$

As a result,

$$\frac{d\pi_i}{d\alpha_j} = \frac{d\pi_i}{dX}\frac{dX}{d\alpha_j} > 0.$$

Proof of Proposition 1

Let Π denote firm 1 and firm j's joint profit. Then $\Pi = \pi_1 + \sum_{i=2}^{k} \alpha_i \pi_i + (1 - \alpha_j)\pi_j$, and

$$\frac{d\Pi}{d\alpha_j} = \frac{d\Pi}{dX}\frac{dX}{d\alpha_j} = \frac{d\left(\pi_1 + \sum_{i=2}^{k} \alpha_i \pi_i + (1 - \alpha_j)\pi_j\right)}{dX}\frac{dX}{d\alpha_j}.$$

With the expressions for $\frac{d\pi_1}{dX}$ and $\frac{d\pi_i}{dX}$ in the proof of Lemma 3, we further obtain that

$$\frac{d\Pi}{d\alpha_j} = \frac{dX}{d\alpha_j} P'(X) \left(-x_1(\Lambda - \lambda_1) + \sum_{i=2}^{k} \alpha_i x_i (\lambda_1 + \lambda_i - \Lambda) + (1 - \alpha_j) x_j (1 + \lambda_j) \right).$$

Since $\frac{dX}{d\alpha_j} P'(X) > 0$, the joint profit increases with α_j if and only if

$$x_1(\Lambda - \lambda_1) < \sum_{i=2}^{k} \alpha_i x_i (\lambda_1 + \lambda_i - \Lambda) + (1 - \alpha_j) x_j (1 + \lambda_j).$$

Dividing both sides by X yields $s_1(\Lambda - \lambda_1) < \sum_{i=2}^{k} \alpha_i s_i (\lambda_1 + \lambda_i - \Lambda) + (1 - \alpha_j) s_j (1 + \lambda_j)$.

Proof of Proposition 3

Notice that $\frac{dX}{d\alpha_j} P'(X) > 0$, thus from (10), the social welfare increases in α_j if and only if the second term on the right-hand side is positive, i.e.,

$$\sum_{i=2}^{n} \lambda_i x_i > \left(x_1 + \sum_{i=2}^{k} \alpha_i x_i \right) (1 + \Lambda - \lambda_1).$$

Dividing both sides by X yields (12).

Proof of Lemma 4

Differentiating the first-order condition (15) yields that

$$2dx_i = -\gamma \sum_{j \neq i} dx_j. \tag{21}$$

By (21), we obtain that

$$(2 - \gamma) dx_i = -\gamma dX, \tag{22}$$

$$(2 - \gamma) dx_1 = (2 + \gamma(n - 2))\, dX. \tag{23}$$

Similarly, differentiating firm 1's first-order condition yields that

$$2dx_1 + \gamma \sum_{j \neq 1} dx_j + \gamma \sum_{i=2}^{k} \alpha_i dx_i + \gamma \sum_{i=2}^{k} x_i d\alpha_i = 0. \tag{24}$$

We first prove part (i). From (22), (23) and (24), we have that

$$\left(2+\gamma(n-1)-\frac{\gamma^2}{2-\gamma}\sum_{i=2}^{k}\alpha_i\right)dX=-\gamma\sum_{i=2}^{k}x_i d\alpha_i.$$

Note that we only let α_j change while keep α_i fixed. Therefore, $d\alpha_i = 0$ for all $2 \leq i \leq k$ and $i \neq j$. Simple calculations yield that

$$\frac{dX}{d\alpha_j}=\frac{-\gamma x_j}{2+\gamma(n-1)-\frac{\gamma^2}{2-\gamma}\sum_{i=2}^{k}\alpha_i}.$$

It is easy to see that $dX/d\alpha_j \leq 0$ for all $\gamma \in (0, 1)$.

We next prove part (ii). For all firm i, $i \geq 2$, we obtain from (22) that

$$\frac{dx_i}{d\alpha_j}=\frac{-\gamma}{2-\gamma}\frac{dX}{d\alpha_j}>0.$$

For firm 1, we obtain from (23) that

$$\frac{dx_1}{d\alpha_j}=\frac{2+\gamma(n-2)}{2-\gamma}\frac{dX}{d\alpha_j}<0.$$

For part (iii), we obtain from the inverse demand function that

$$dp_i=-dx_i-\gamma\sum_{j\neq i}dx_j.$$

By (21), it follows that $dp_i = -dx_i + 2dx_i = dx_i$ for $i \geq 2$. Thus,

$$\frac{dp_i}{d\alpha_j}=\frac{dx_i}{d\alpha_j}>0.$$

By (22) and (23),

$$\frac{dp_1}{d\alpha_j}=\frac{dX}{d\alpha_j}\frac{(n-2)\gamma^2-(n-2)\gamma-2}{2-\gamma}>0.$$

Lastly, for firm i, $i \geq 2$, we have that

$$\frac{d\pi_i}{d\alpha_j}=\frac{d\pi_i}{dX}\frac{dX}{d\alpha_j}=-(p_i-c_i+x_i)\frac{\gamma}{2-\gamma}\frac{dX}{d\alpha_j}\geq 0. \tag{25}$$

For firm 1, we have that

$$\frac{d\pi_1}{d\alpha_j} = \frac{d\pi_1}{dX}\frac{dX}{d\alpha_j} = \frac{\gamma}{2-\gamma}\left((n-1)\gamma x_1 + (2+(n-2)\gamma)\sum_{i=2}^{k}\alpha_i x_i\right)\frac{dX}{d\alpha_j} \le 0. \tag{26}$$

Proof of Proposition 4

As before, let Π denote firm 1 and firm j's joint profit. Then $\Pi = \pi_1 + \sum_{i=2}^{k}\alpha_i\pi_i + (1-\alpha_j)\pi_j$, and

$$\frac{d\Pi}{d\alpha_j} = \frac{d\Pi}{dX}\frac{dX}{d\alpha_j} = \frac{d\left(\pi_1 + \sum_{i=2}^{k}\alpha_i\pi_i + (1-\alpha_j)\pi_j\right)}{dX}\frac{dX}{d\alpha_j}.$$

With the expressions for $d\pi_i/dX$ and $d\pi_1/dX$ in (25) and (26), we further obtain that

$$\frac{d\Pi}{d\alpha_j} = \frac{dX}{d\alpha_j}\frac{\gamma}{2-\gamma}\left((n-1)\gamma x_1 + \sum_{i=2}^{k}\alpha_i x_i(n-2)\gamma - 2(1-\alpha_j)x_j\right).$$

Since $dX/d\alpha_j < 0$, the joint profit increases with α_j if and only if

$$(n-1)\gamma x_1 + (n-2)\gamma\sum_{i=2}^{k}\alpha_i x_i < 2(1-\alpha_j)x_j.$$

Dividing both sides by X yields (17).

Proof of Proposition 5

Following the literature, a representative consumer has the following utility function of n goods:

$$U(\mathbf{x}, I) = \sum_{i=1}^{n} a_i x_i - \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 + \gamma\sum_{i\neq j} x_i x_j\right) + I.$$

The social welfare in this section can be obtained as

$$\begin{aligned} W &= U(\mathbf{x}, I) - \left(\sum_{i=1}^{n} p_i x_i + I\right) + \sum_{i=1}^{n}(p_i - c_i)x_i \\ &= \sum_{i=1}^{n}(a_i - c_i)x_i - \frac{1}{2}\sum_{i=1}^{n} x_i^2 - \frac{\gamma}{2}\sum_{i\neq j} x_i x_j. \end{aligned}$$

Totally differentiating W with respect to α_j yields

$$\begin{aligned}\frac{dW}{d\alpha_j} &= \frac{dW}{dX}\frac{dX}{d\alpha_j}\\ &= \left(\sum_{i=1}^{n}(a_i - c_i)\frac{dx_i}{dX} - \sum_{i=1}^{n} x_i \frac{dx_j}{dX} - \frac{\gamma}{2}\sum_{i\neq j} x_i \frac{dx_j}{dX} - \frac{\gamma}{2}\sum_{i\neq j}\frac{dx_i}{dX}x_j\right)\frac{dX}{d\alpha_j}.\end{aligned}$$

With the expressions for dx_i/dX and dx_1/dX in (22) and (23), we obtain that

$$\frac{dW}{d\alpha_j} = \frac{dX}{d\alpha_j}\frac{1}{2-\gamma}\left((2+(n-2)\gamma)\left(x_1 + \gamma\sum_{i=2}^{k}\alpha_i x_i\right) - \gamma\sum_{i=2}^{n} x_i\right).$$

Since $dX/d\alpha_j < 0$, the social welfare increases with α_j if and only if

$$(2+(n-2)\gamma)\left(x_1 + \gamma\sum_{i=2}^{k}\alpha_i x_i\right) < \gamma\sum_{i=2}^{n} x_i.$$

Dividing both sides by X yields (18).

References

Airline Business: Airlines with Ownership by Other Carriers. News, vol. 48 (1998)

Alley, W.A.: Partial ownership arrangements and collusion in the automobile industry. J. Ind. Econ. **45**, 191–205 (1997)

Amir, R.: Cournot oligopoly and the theory of supermodular games. Games Econ. Behav. **15**, 132–148 (1996)

Amir, R., Lambson, V.E.: On the effects of entry in Cournot market. Rev. Econ. Stud. **67**, 235–254 (2000)

Amir, R.: Supermodularity and complementarity in economics: an elementary survey. South. Econ. J. **71**, 636–660 (2005)

Amir, R., Ericksonz, P., Jin, J.: On the microeconomic foundations of linear demand for diferentiated products. J. Econ. Theory **169**, 641–665 (2017)

Brito, D., Cabral, L., Vasconcelos, H.: Divesting ownership in a rival. Int. J. Ind. Organ. **34**, 9–24 (2014)

Dasgupta, S., Tsui, K.: Auctions with cross-shareholdings. Econ. Theory **24**, 163–194 (2004). https://doi.org/10.1007/s00199-003-0407-y

Dixit, A.K.: A model of duopoly suggesting a theory of entry barriers. Bell J. Econ. **10**, 20–32 (1979)

Dietzenbacher, E., Smid, B., Volkerink, B.: Horizontal integration in the Dutch financial sector. Int. J. Ind. Organ. **18**, 1223–1242 (2000)

Farrell, J., Shapiro, C.: Asset ownership and market structure in oligopoly. Rand J. Econ. **21**, 275–292 (1990)

Flath, D.: When is it rational for firms to acquire silent interests in rivals? Int. J. Ind. Organ. **9**, 573–583 (1991)

Gilo, D., Moshe, Y., Spiegel, Y.: Partial cross ownership and tacit collusion. Rand J. Econ. **37**, 81–99 (2006)

Grossman, S.J., Hart, O.D.: Takeover bids, the free-rider problem, and the theory of the corporation. Bell J. Econ. **11**, 42–64 (1980)

Häckner, J.: A note on price and quantity competition in differentiated oligopolies. J. Econ. Theory **93**, 233–239 (2000)

Jackson, M., Wolinsky, A.: A strategic model of social and economic networks. J. Econ. Theory **71**(1), 44–74 (1996)

Lahiri, S., Ono, Y.: Helping minor firms reduces welfare. Econ. J. **98**, 1199–1202 (1988)
Levy, N., Spiegel, Y., Gilo, D.: Partial vertical integration, ownership structure and foreclosure. Am. Econ. J. Microecon. **10**(1), 132–180 (2018)
McManus, M.: Numbers and size in Cournot oligopoly. Yorks. Bull. Soc. Econ. Res. **14**, 14–22 (1962)
McManus, M.: Equilibrium, numbers and size in Cournot oligopoly. Yorks. Bull. Soc. Econ. Res. **16**(2), 68–75 (1964)
Nain, A., Wang, Y.: The product market impact of minority stake acquisitions. Manag. Sci. **64**(2), 825–844 (2018)
Novshek, W.: On the existence of Cournot equilibrium. Rev. Econ. Stud. **52**(1), 85–98 (1985)
Qin, C.-Z., Zhang, S., Zhu, D.: A model of endogenous cross-holdings in oligopoly. Front. Econ. China **12**(3), 465–479 (2017)
Reitman, D.: Partial ownership arrangements and the potential for collusion. J. Ind. Econ. **42**(3), 313–322 (1994)
Reynolds, R.J., Snapp, B.R.: The competitive effects of partial equity interests and joint ventures. Int. J. Ind. Organ. **4**, 141–153 (1986)
Salant, S.W., Switzer, S., Reynolds, R.J.: Losses from horizontal merger: the effects of an exogeneous change in industry structure on Cournot-Nash equilibrium. Q. J. Econ. **98**, 185–199 (1983)
Shapley, L., Shubik, M.: Price strategy oligopoly with product variation. Kyklos **22**, 30–44 (1969)
Singh, N., Vives, X.: Price and quantity competition in a differentiated duopoly. Rand J. Econ. **15**(4), 546–554 (1984)
Trivieri, F.: Does cross-ownership affect competition? Evidence from the Italian Banking Industry. J. Int. Financ. Mark. Inst. Money **17**(1), 79–101 (2007)
Vives, X.: Oligopoly Pricing: Old Ideas and New Tools. MIT Press, Cambridge (1999)
Wang, X.H., Zhao, J.: Welfare reductions from small cost reductions in differentiated oligopoly. Int. J. Ind. Organ. **25**(1), 173–185 (2007)