Economic Theory (2021) 71:1499–1525
https://doi.org/10.1007/s00199-021-01344-x

RESEARCH ARTICLE

Rawls's difference principle and maximin rule of allocation: a new analysis

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Received: 10 October 2020 / Accepted: 25 January 2021 / Published online: 16 February 2021

Abstract

If Rawls's *A Theory of Justice* has achieved fame among economists, this is due to his Difference Principle, which says that inequalities of resources should be to the benefit of the less fortunate, or more operationally, that allocations of resources should be ranked by the maximin criterion. We extend the Rawlsian maximin in two ways: first, by resorting to the more general *min-of-means* formula of decision theory, second, by addressing the case where the resources accruing to each individual are uncertain to society. For the latter purpose, we resort to the ex ante versus ex post distinction of welfare economics. The paper axiomatically characterizes the ex ante and ex post forms of the Rawlsian maximin and compares them in terms of egalitarian criteria. It finally recommends and axiomatizes a compromise egalitarian theory that mixes the two forms.

Keywords Rawls · Maximin · Difference principle · Ex ante egalitarian · Ex post egalitarian · Min-of-means

JEL Classication D63, D71, D81, I30

The authors gratefully acknowledge Marc Fleurbaey's and Itzhak Gilboa's comments on this paper.
M. Pivato: Supported by Labex MME-DII (ANR11-LBX-0023-01) and CHOp (ANR-17-CE26-0003).

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1 Introduction

If Rawls's political philosophy enjoys great fame among economists, this is mostly due to the argument he makes for the "Difference Principle", which he introduces and defends in his landmark book, *A Theory of Justice* (1971–1989). Informally, the Principle claims that social and economic inequalities should be to the advantage of the less fortunate; more operationally, it says that society should evaluate inequalities in terms of the maximin criterion, i.e., it should rank an allocation above another if and only if the amount of resources of the less well off is higher in the former allocation than in the latter. Rawls's book has two lines of defence for the Difference Principle: one that is directly normative and another that is more roundabout but actually more well-known, which involves a detour by the "original position" construction.[1] In both cases, Rawls translates the Difference Principle into the maximin criterion to evaluate social allocations. When he follows the roundabout argument, maximin viewed as a social criterion is deduced from maximin viewed as a criterion of rational decision under uncertainty, with the "veil of ignorance" condition capturing the decision-maker's uncertainty in the original position.

The present paper explores the Difference Principle from a new theoretical angle, which we explain through a brief retrospective. That Rawls could in the 1970s endorse maximin as a criterion of social and individual rationality surprised his fellow economists, who had long been ingrained in the utilitarian tradition of summing individual advantages, and as far as the detour through the original position went, had just absorbed the foundational arguments for expected utility provided by von Neumann and Morgenstern and followers. The resulting debate, which brought Rawls (1974a, b) together with such prominent critics as Arrow (1973), Musgrave (1974) and Harsanyi (1975), is well-known. (Among other participants, Alexander (1974) was also critical of Rawls's adoption of maximin, while Phelps (1973) and Sen (1974) showed more inclination to accept this criterion.) Also well-known is the shift that Sen (1970) imposed on the maximin criterion by applying it recursively, from the less to the more advantaged individuals, to satisfy the strong form of the Pareto principle and thus break the indifferences left by maximin. The leximin criterion, as it came to be called, was to become central to the new form of social choice theory initiated by Sen. For his part, though perhaps not without hesitation, Rawls (1974a) reaffirmed maximin as being his chosen criterion.

After these early reactions to *A Theory of Justice*, economists have pursued the Difference Principle mostly on the technical side. While social choice theorists thoroughly axiomatized leximin social preferences, public economists explored the consequences of adopting maximin to devise optimal taxation and redistribution schemes, to provide an alternative to standard utilitarian calculations.[2] Although it primarily features new lines of thinking on distributional issues, the more recent work on fairness still pays attention to Rawls's analysis of this concept in terms of the Difference Principle, and

[1] In *A Theory of Justice*, the direct normative argument is to be found in §17, and the more famous indirect argument through the original position is the object of the entire ch. III. The same duality can be found in the later summary *Justice as Fairness* (2001).

[2] See Sect. 4 for a survey of axiomatizations of leximin and maximin. Some works on optimal taxation use maximin; see Atkinson (1995), Boadway and Jacquet (2008), and the references listed by the latter writers.

it does use maximin and leximin, either as points of comparison or sometimes even as guiding criteria.[3]

Significant as these developments are, they have not clarified the foundational misgivings the early economists had concerning the Difference Principle. This comes as a surprise because both decision theory and social choice theory have in the intervening decades developed new tools that could have helped bridge the gap that Arrow and others had worryingly discovered between the use of maximin and their theoretical commitments. The present paper will reconstruct the Difference Principle by borrowing contemporary reference work from both of these areas, thus indirectly responding to the old debate. In this way, we hope to establish a more substantial exchange between Rawls's theory of justice and theoretical economics.

There were two major defects in Rawls's position and that of his adversaries. First, both established too sharp a contrast between the egalitarian maximin and additive formulas with a utilitarian interpretation—or equivalently, when the theory postulates an original position, between the extremely risk-averse maximin and the more flexible expected utility rule. This is especially visible in the controversy between Rawls (1974a) and Harsanyi (1975), which strikes one as being fairly crude in retrospect. The by now well-understood min-of-means model of preference under uncertainty introduced by Gilboa and Schmeidler (1989) makes it possible to bridge some of the gap between these apparently irreconciliable orientations. By allowing for multiple expected utility representations on the decision maker's part, this model keeps maximin without imposing extreme risk-aversion, which delivers a more nuanced conception of the original position than Rawls's and Harsanyi's. In parallel fashion, when it is translated into a criterion for social allocations, min-of-means permits grading egalitarianism along a continuum. We will heavily borrow from this elegant construction here.

The second problem is that both Rawls and his adversaries underestimated the relevance of uncertainty over social states of affairs when assessing inequality. In particular, they ignored uncertainty over the total amount and distribution of resources society can allocate; let us call this *resource uncertainty*. They only paid attention to the uncertainty that is inherent to the original position, which concerns which particular member of society the ignorant observer will become. But resource uncertainty would affect even a fully self-cognizant social observer. For instance, the macroeconomic conditions are typically uncertain and influence society's distributive possibilities; and so do certain physical or biological factors that take a stochastic form. Rawls did not address the problems raised for the Difference Principle by these very natural observations. Neither did his fellow economists, which is more disappointing since they had claimed to have the tools to handle any kind of uncertainty; they just overlooked this relevant form. But they were of course limited by the economics of their time. From the 1980s and 1990s onwards, social choice theory developed an apparatus to tackle social uncertainty that can fit our purposes (see Mongin and Pivato 2016, and Fleurbaey 2018 for updated reviews). This apparatus involves a crucial distinction between ex ante and ex post rules of social evaluation, which will be the other significant borrowing of the paper.

[3] See in particular Fleurbaey and Maniquet (2009 and 2011, ch.11). A prominent concept in the fairness literature, i.e., egalitarian equivalence, admits of an interpretation in terms of maximin.

Below, we motivate the Rawlsian use of this distinction by a numerical example that illustrates that the Difference Principle can be reconstructed in two ways when resource uncertainty prevails. The ex ante way first takes an expected value of income for each individual separately, and then applies maximin to these numbers. The ex post way reverses the order, first applying maximin in each state of the world separately, and then taking the expected value of these numbers. Based on this distinction, our results unfold as follows.

A preliminary step, Proposition 1 clarifies the sense in which one min-of-means functional representation can be said to be more egalitarian than another. Then Propositions 2 and 3 axiomatize the ex ante and ex post versions of the Difference Principle; the sets of axioms are based on those available for mathematical expectation and a min-of-means representation. Proposition 4 explains how these two versions conflict with each other. Then Proposition 5 reaches a provisional conclusion, namely that the ex post version is more egalitarian than the ex ante version. But this is not the end of the argument, since another numerical example suggests that the ex post version still fails to recognize certain egalitarian intuitions captured by the ex ante one, and that a compromise between the two can be better than either taken in isolation. Following this heuristic, after the technical interlude of Propositions 6, 7 axiomatizes convex combinations of the ex ante and *ex post* versions. This expresses our final conclusion: we offer such mixed formulas as constituting the most defensible reconstruction of Rawls's Difference Principle in the context of resource uncertainty.

Technically, our results are indebted to those of Ben-Porath et al. (1997) and Gajdos and Maurin (2004), who pioneered the application of the min-of-means model to the evaluation of inequality under uncertainty. The former writers investigate functional forms that amount to applying this model twice over, first across states of the world, and second across individuals, or in the reverse order. They carry their analysis at the level of numerical representations, whereas the latter writers develop qualitative axioms that cover more general representations than the min-of-means ones. The present paper is more restricted in scope that these works. Our Rawlsian representations turn out to be particular cases of those of Ben-Porath et al. (1997), hence also of those of Gajdos and Maurin (2004). Our contribution is to explore such particular cases in full detail, both at the numerical and axiomatic level, and we justify this limitation by its relevance to the Difference Principle.

The paper has more remote connections with a currently active line of research, which deals with the aggregation of individual preferences when subjective uncertainty prevails. Specifically, this line of research aims at resolving the tension created by subjective expected utility theory when it is applied collectively along with standard Pareto conditions. Moving to min-of-means allows for a flexibility that this theory lacks.[4] By itself, this important topic is irrelevant here. We have a single evaluation—the social one—hence no way of developing an aggregative setting and exploring its problems of internal consistency. The paper belongs to the different tradition of assessing inequality by defining a social evaluation function that does not necessarily aggregate individual preferences—a distinction that is clearly emphasized by Sen

4 See Crès et al. (2011), Alon and Gayer (2016), Qu (2017), and Hayashi and Lombardi (2019).

(1992, ch. 6).[5] But despite their different orientations, the two strands of literature have some formal connections, which has facilitated our work; in particular, we have drawn some useful hints from Hayashi and Lombardi (2019).

The paper is organized as follows. Section 2 sets the stage for a formal reconstruction of the Difference Principle and introduces the ex ante and ex post versions at the level of numerical representations. Section 3 contains the main technical results, which derive these representations from qualitative relations. The concluding Sect. 4 returns to Rawls's work to assess what has been done here, and what remains to be done, in order to account for the Difference Principle analytically. An "Appendix" collects the proofs of the results.

2 The Difference Principle with resource uncertainty

In the following very simple example, the allocative possibilities of a two-individual society vary across two uncertain states, and we suppose that two policies **A** and **B** must be evaluated. The indexes i and s refer to individuals and states respectively, and the numbers represent money flows. We allow for negative numbers to represent losses with respect to an unspecified status quo financial situation, which is the same for both agents.

A	$s = 1$	$s = 2$
$i = 1$	-1	4
$i = 2$	$\frac{5}{2}$	$\frac{5}{2}$
	p	$1 - p$

B	$s = 1$	$s = 2$
$i = 1$	1	3
$i = 2$	2	2
	p	$1 - p$

Policies **A** and **B** involve the same total amount of money, but distribute it differently across the two dimensions. Whereas individual 2 gets the same income in each state, individual 1 has a higher one in state 2 than in state 1. However, compared with **A**, policy **B** reduces the variance of 1's income. This involves not only transferring some of 1's income from one state to the other, but also transferring some income from 2 to 1. The interpretation of this toy example is that unlike 2, individual 1 can fall prey to bad luck, and **A** represents the natural situation, while **B** involves some form of insurance against this bad luck. For instance, individual 1 could be exposed to some disease that materializes under a given environment, or to the loss of employment that materializes under adverse macroeconomic circumstances, and both of these shocks would bring about income losses for individual 1, whom society may or may not decide to protect.

One way to make this decision is to apply the Rawlsian maximin as if the two matrices were indivisible objects, thus ignoring the distinction between states and individuals. Society would then conclude that **B**, which guarantees the amount 1, is better than **A**, which only guarantees -1. However, this is too crude a procedure if society is in a position to probabilize individual 1's risk; for instance, if, in the medical interpretation, it can use some reasonably well established statistical law to

[5] In the words of Gajdos and Maurin (2004, p. 97): "The issue is not to aggregate individuals' preferences, but to propose principles for defining a reasonable collective attitude towards inequality under uncertainty".

predict the occurrence of the disease. When this is granted, it becomes apparent that probabilities can enter the maximin rule in accordance with two different methods. Either an expected value is first taken for each individual and maximin is then applied, which is the ex ante method, or maximin is first applied in each state and the expected value is then taken, which is the ex post method.

Different rankings can result from these two methods. Suppose the probability of $s = 1$ in the example is $3/10$. The ex ante and ex post methods are associated with two functions V_{xa} and V_{xp} that deliver opposite evaluations of $\mathbf{A}$ and $\mathbf{B}$:

$$V_{\mathrm{xa}}(\mathbf{A}) \;=\; \min\left\{\frac{5}{2}, \frac{5}{2}\right\} \;=\; \frac{5}{2} > V_{\mathrm{xa}}(\mathbf{B}) \;=\; \min\left\{\frac{12}{5}, 2\right\} = 2,$$
$$V_{\mathrm{xp}}(\mathbf{A}) \;=\; \frac{3}{10}(-1) + \frac{7}{10}\left(\frac{5}{2}\right) = \frac{29}{20} < V_{\mathrm{xp}}(\mathbf{B}) \;=\; \frac{3}{10}(1) + \frac{7}{10}(2) \;=\; \frac{34}{20}.$$

In brief, society is faced with a theoretical choice between two prima facie plausible ways of applying maximin. Our paper assumes that probabilistic information of the kind illustrated here is always available, and addresses the problem of deciding between these two ways.

Formally, there will be a *social evaluation relation* bearing on *state-contingent allocations*, henceforth referred to as *policies*, which specify the amount of resources each individual enjoys in each state of nature. For simplicity, we restrict policies to have scalar values. Admittedly, this amounts to side-stepping Rawls's concern for multiple "primary goods" in *A Theory of Justice* (1971), which rather calls for a vector-valued treatment, but there is a warrant for the present treatment in the later Rawls (1982).

We take two finite sets I and S to be the set of individuals and the set of states of the world, with cardinalities $|I|, |S| \geq 2$. Avoiding any feasibility restriction for simplicity, we define the set of policies to be the Euclidean space $\mathbb{R}^{I\times S}$. It is convenient to view policies as $|I| \times |S|$ matrices $\mathbf{X} = \left[x_i^s\right]_{i\in I}^{s\in S}$, where the index i denotes rows and the index s denotes columns, but policies can also be represented as arrays $[\mathbf{x}_i]_{i\in I}$ of $|I|$ row vectors of dimension $|S|$, or as arrays $\left[\mathbf{x}^s\right]^{s\in S}$ of $|S|$ column vectors of dimension $|I|$. Generally, we will regard elements of $\mathbb{R}^S$ as row vectors and elements of $\mathbb{R}^I$ as column vectors. To distinguish between the two classes of vectors when subscripts i or superscripts s are not used, we use the notation $\mathbf{x}_-$ and $\mathbf{x}^-$ respectively; thus, when a policy has identical rows, it is denoted by $\mathbf{X} = \left[\mathbf{x}_-\right]_{i\in I}$, and when it has identical columns, by $\mathbf{X} = \left[\mathbf{x}^-\right]^{s\in S}$. Given a finite set A, the notation $\Delta(A)$ refers to the set of nonnegative vectors in $\mathbb{R}^{|A|}$ the components of which sum to 1; we will refer to $\Delta(A)$ as to the set of *weight vectors* on A when no probabilistic interpretation is available for this set. The social evaluation relation $\succcurlyeq$ is defined on the set $\mathbb{R}^{I\times S}$ of policies and will be assumed throughout to be a weak ordering, with its indifference and strict evaluation parts being denoted $\succ$ by and $\sim$ respectively.

From $\succcurlyeq$ taken as a primitive, one can derive two new social evaluation relations, i.e., the *conditional* $\succcurlyeq_i$ *of* $\succcurlyeq$ *on* i, and the *conditional* $\succcurlyeq^s$ *of* $\succcurlyeq$ *on* s. By a standard definition, let us say that for all $i \in I$ and all $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^S$, we set $\mathbf{x}_i \succcurlyeq_i \mathbf{y}_i$ if and only if there exist matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{I\times S}$ with $\mathbf{X} \succcurlyeq \mathbf{Y}$ such that $\mathbf{X}$ and $\mathbf{Y}$ respectively have $\mathbf{x}_i$ and $\mathbf{y}_i$ as their i-rows and equal each other on all the other rows. Replacing rows by columns, $\succcurlyeq^s$ conditionals are defined similarly on $\mathbb{R}^I$. Notice that i or s do not denote elements, but subsets of the set $I \times S$ (i.e., they stand for $\{i\} \times S$ and $I \times \{s\}$ respectively). Conditional evaluations play a major role in the axioms below. As they are defined here, they are not necessarily transitive; if transitivity is needed, it will be part of the axioms.

We now review the main technical tool of the paper: Gilboa and Schmeidler's (1989) classic axiomatization of the "min-of-means" rule of individual decision. We state it in an adapted form, which was conveniently introduced by Ben-Porath et al. (1997).[6] Given a finite set A, take the set $\mathbb{R}^A$ of all vector-valued functions on this set to be the set of alternatives (in the standard interpretation of Gilboa and Schmeidler, the $a \in A$ represent states of the world, and the alternatives represent uncertain prospects). Define a weak preference relation R on $\mathbb{R}^A$ that may satisfy the following axioms. (As usual, I and P refer to the indifference and strict preference relations associated with R.)

- A1. R is a *weak ordering* (i.e., a transitive, reflexive and complete relation).
- A2. R is *continuous*, i.e., the upper and lower preference contours of R are closed sets.
- A3. R is *monotonic in each* a, i.e., for all $f, g \in \mathbb{R}^A$, if $f(a) \geq g(a)$ for all $a \in A$, then $f \mathrel{\mathsf{R}} g$, and if $f(a) > g(a)$ for all $a \in A$, then $f \mathrel{\mathsf{P}} g$.
- A4. R is *positively homogeneous*, i.e., for all $f, g \in \mathbb{R}^A$, and for all $\lambda > 0$, if $f \mathrel{\mathsf{R}} g$, then $\lambda f \mathrel{\mathsf{R}} \lambda g$.
- A5. R is *invariant by uniform translation*, i.e., for all $f, g \in \mathbb{R}^A$, and for all $\mu \in \mathbb{R}$, if $f \mathrel{\mathsf{R}} g$, then $f + \mu \mathrel{\mathsf{R}} g + \mu$.
- A6. R is *concave* in the following sense: for all $f, g \in \mathbb{R}^A$, and all $\lambda \in]0, 1[$, if $f \mathrel{\mathsf{I}} g$, then $\lambda f + (1 - \lambda) g \mathrel{\mathsf{R}} f$.

A "min-of-means" representation derives for R if it satisfies Axioms A1–A6. More formally:

Lemma A (adapted from Gilboa and Schmeidler 1989) *If a nontrivial preference relation* R *on the set of alternatives* $\mathbb{R}^A$ *satisfies Axioms* A1 *to* A6, *then there exists a function* V *from* $\mathbb{R}^A$ *to* $\mathbb{R}$ *such that for all* $f, g \in \mathbb{R}^A$,

$$f \mathrel{\mathsf{R}} g \quad \textit{if and only if} \quad V(f) \geq V(g),$$

and

$$V(f) := \min_{\boldsymbol{\pi} \in \boldsymbol{\Pi}} \left\{ \sum_{a \in A} \pi_a f(a) \right\},$$

where $\boldsymbol{\Pi} \subseteq \Delta(A)$ *is a closed and convex set of weight vectors* $\boldsymbol{\pi} = (\pi_a)_{a \in A}$. *Moreover,* $\boldsymbol{\Pi}$ *is unique in this format of representation.*

[6] Gilboa and Schmeidler's (1989) original representation theorem for "min-of-means" involves axioms A1–A6, plus some special assumptions connected with their definition of prospects as being lottery-valued. Here, prospects are real-valued functions, and this means that no more than A1–A6 are needed to get the desired representation. This variant is also used in d'Aspremont and Gevers (2002), Gajdos and Maurin (2004) and Hayashi and Lombardi (2019).

Let us now return to the Rawlsian maximin and the two different ways it can include probabilities, as illustrated by the numerical example above. An evaluation rule according to the ex ante method reads as

$$(1)\ V_{\mathrm{xa}}(\mathbf{X}) \ := \ \min_{i \in I} \left\{ \sum_{s \in S} p_s \, x_i^s \right\},$$

and an evaluation rule according to the ex post method as

$$(2)\ V_{\mathrm{xp}}(\mathbf{X}) \ := \ \sum_{s \in S} p_s \min_{i \in I} \left\{ x_i^s \right\},$$

where $\mathbf{p} = (p_s)_{s \in S}$ is a probability vector on S. The use the min operator makes the two rules generally non-equivalent, with the following inequality always holding by a familiar property of this operator:

$$V_{\mathrm{xp}}(\mathbf{X}) \leq V_{\mathrm{xa}}(\mathbf{X}).$$

The distinction between ex ante and ex post social rules naturally follows when uncertainty is introduced at the social level. It was introduced and has long been explored only in relation to additive rules with an utilitarian interpretation; see Hammond (1982) for an early discussion. More recently, a profusion of research has extended this classic distinction to a wider set of normative commitments, including egalitarianism; see Adler and Sanchirico (2006), Fleurbaey (2010, 2018), and Mongin and Pivato (2016) for discussions of this literature. A common feature of all these works is that they include individual preferences in their primitives and investigate their relations to social preferences in terms of aggregative conditions, most prominently variants of the Pareto principle that are adapted to the social uncertainty context. As we have explained, this paper approaches inequality under uncertainty *without* aggregation of individual preferences, as do those of Ben-Porath et al. (1997) and Gajdos and Maurin (2004). However, from whichever perspective, the contrast between ex ante and ex post has not yet been investigated in connection with the Rawlsian rules (1) and (2).

Nor has it been investigated in the more general forms we now introduce. The standard interpretation of a min-of-means representation is in terms of uncertainty and takes it to generalize subjective expected utility (SEU) theory. But if the set of weight vectors in this representation bears on a set of individuals, rather than of states of the world, a more general form of the Rawlsian maximin is in view. One may replace (1) and (2) by

$$(1^+)\ V_{\mathrm{xa}}^+(\mathbf{X}) := \min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i \sum_{s \in S} p_s \, x_i^s \right\},$$

and

$$(2^+)\ V_{\mathrm{xp}}^+(\mathbf{X}) \ := \ \sum_{s \in S} p_s \min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i x_i^s \right\},$$

where $\mathbf{M}$ is a closed and convex set of weight vectors on I. Note that these formulas become straightforward expected value representations when sets of weight vectors are singletons. It can be shown that only in these degenerate cases can the identity $V^+_{\text{xa}} \equiv V^+_{\text{xp}}$ hold. We skip the easy proof; a less obvious impossibility theorem will be stated in the next section.

To recover the original formulas (1) and (2), it is necessary and sufficient that $\mathbf{M}$ contain the unit vectors $(1, 0, ..., 0)$, $(0, 1, ..., 0)$,..., $(0, 0, ..., 1)$; since it is convex, this means that $\mathbf{M}$ is the maximal set of weight functions (i.e., the simplex of $\mathbb{R}^I$). As we suggested in the introduction, formulas (1^+) and (2^+) are more attractive than (1) and (2) because they are less drastically egalitarian and thus provide society with more flexibility in implementing the Difference Principle. The complaint that maximin is too extreme had already surfaced in the controversy of the 1970's. It had led various economists such as Arrow (1973) and Alexander (1974) to replace the min operation with a summation involving power functions so as to allow for a continuum of attitudes. However, this move lacks axiomatic grounding, unlike (1^+) and (2^+), and it is moreover limited to the certainty case.[7]

The comparison between our formulas and the original Rawlsian ones can be extended, starting from the more general intuition that for a fixed probability vector $\mathbf{p}$, the larger the set of weights $\mathbf{M}$, the more egalitarian the implications of min-of-means. To make this intuition precise, take two functions of the same type, either V^+_{xa} and W^+_{xa} of type (1^+), or V^+_{xp} and W^+_{xp} of type (2^+), assuming they have a common probability vector $\mathbf{p}$, and denoting their respective sets of weight vectors by $\mathbf{M}$ and $\mathbf{N}$. Then define V^+_{xa} (resp. V^+_{xp}) to be *at least as egalitarian as* W^+_{xa} (resp. W^+_{xp}) if for all policies $\mathbf{X}$, and for all equal rows policies $\mathbf{K} = [\mathbf{k}_-]_{i \in I}$,

$$\text{if } W^+_{\text{xa}}(\mathbf{K}) \geq W^+_{\text{xa}}(\mathbf{X}), \text{ then } V^+_{\text{xa}}(\mathbf{K}) \geq V^+_{\text{xa}}(\mathbf{X})$$
$$\text{(respectively) if } W^+_{\text{xp}}(\mathbf{K}) \geq W^+_{\text{xp}}(\mathbf{X}), \text{ then } V^+_{\text{xp}}(\mathbf{K}) \geq V^+_{\text{xp}}(\mathbf{X}).$$

In policies of type $\mathbf{K}$, all individuals face the same risk ex ante and get identical money amounts ex post, so a preference for such policies is indicative of an egalitarian tendency, which makes the chosen definition sensible. The following chain of equivalences holds; compare with Proposition 1 in Hayashi and Lombardi (2019).

Proposition 1 *V^+_{xa} is at least as egalitarian as W^+_{xa} if and only if $\mathbf{N} \subseteq \mathbf{M}$, if and only if $W^+_{\text{xa}}(\mathbf{X}) \geq V^+_{\text{xa}}(\mathbf{X})$ for all $\mathbf{X} \in \mathbb{R}^{I \times S}$. The same holds for V^+_{xp} and W^+_{xp} in place of V^+_{xa} and W^+_{xa}.*

[7] Formally, define x_i to be individual i's resources, assuming $x_i > 0$, and fix the social evaluation rule

$$U((x_i)_{i \in I}) := \left(\sum_{i \in I} x_i^a \right)^{1/a},$$

where a is a non-zero parameter. Then, $U((x_i)_{i \in I}) \to \min_{i \in I} x_i$ as $a \to -\infty$. Less egalitarian distributions result from taking finite negative values for a. This argument is typically presented in terms of utility functions, so as to connect Rawls's maximin with the utilitarian sum; see Arrow (1973, pp. 256–257) and Alexander (1974, pp. 610–611).

By the restatement in terms of set inclusion $\mathbf{N} \subseteq \mathbf{M}$, V_{xa} (resp. V_{xp}) is at least as egalitarian as any V_{xa}^{+} (resp. V_{xp}^{+}) with the same $\mathbf{p}$. This provides a way of characterizing the original Rawlsian formulas in the wider context of the min-of-means representations, and single out two polar cases:

- *For any given* $\mathbf{p}$*, the functions* V_{xa} *(resp.* V_{xp}*) maximize the relation "at least as egalitarian as" within the class of all* V_{xa}^{+} *(resp.* V_{xp}^{+}*) functions.*
- *For any given* $\mathbf{p}$*, a* V_{xa}^{+} *(resp.* V_{xp}^{+}*) function whose set of weight vectors* Π *reduces to a singleton minimizes the relation "at least as egalitarian as" within the class of those* $\widetilde{V}_{\text{xa}}^{+}$ *(resp.* $\widetilde{V}_{\text{xp}}^{+}$*) functions whose sets of weight functions* $\widetilde{\Pi}$ *include this singleton.*

Very different comparisons result if one ignores the difference between states and individuals, as in the following minimizing function:

$$(3)\ V_{\text{tot}}(\mathbf{X}) := \min_{(i,s)\in I\times S} \left\{x_i^s\right\},$$

and its min-of-means generalization:

$$(3^{+})\ V_{\text{tot}}^{+}(\mathbf{X}) := \min_{\rho\in\mathbf{R}} \left\{ \sum_{(i,s)\in I\times S} \rho_{is} x_i^s \right\},$$

where $\mathbf{R}$ is a closed and convex set of weight vectors on $I \times S$. Let us take W_{tot}^{+} to be another function of type (3^{+}) and define V_{tot}^{+} to be *at least as egalitarian in toto as* W_{tot}^{+} if for all policies $\mathbf{X}$, and for all policies $\mathbf{K} = [k]_{i\in I}^{s\in S}$ with identical matrix components,

$$\text{if } W_{\text{tot}}^{+}(\mathbf{K}) \geq W_{\text{tot}}^{+}(\mathbf{X}), \text{ then } V_{\text{tot}}^{+}(\mathbf{K}) \geq V_{\text{tot}}^{+}(\mathbf{X}).$$

An adapted version of Proposition 1 follows from this definition; we do not state it formally. Whether egalitarianism in toto is an appropriate notion will be discussed below.

Returning to formulas (1^{+}) and (2^{+}), we illustrate their relations in terms of the numerical example introduced at the start of this section. Consider first how $\mathbf{A}$ and $\mathbf{B}$ would compare if the minimizing formulas (1) and (2) were maximized. We now let the probability vector $\mathbf{p}$ of these formulas vary across all possible values.

- If p is between 0 and $\frac{1}{5}$, then both rules deliver $\mathbf{A} \succ \mathbf{B}$.
- If p is between $\frac{1}{5}$ and $\frac{2}{5}$, then the ex ante rule (1) still gives $\mathbf{A} \succ \mathbf{B}$, but the ex post rule (2) gives $\mathbf{B} \succ \mathbf{A}$.
- If p is between $\frac{2}{5}$ and 1, then both rules give $\mathbf{B} \succ \mathbf{A}$.

Let us now see what maximin delivers when (1^{+}) and (2^{+}) are substituted for (1) and (2), assuming that both of these formulas hold with the same set of weights

$$\mathbf{M} = \left\{ \boldsymbol{\mu} \in \Delta(\{1,2\}) \mid \frac{1}{3} \leq \mu_1 \leq \frac{2}{3} \right\}.$$

The previous results are changed as follows:

- If p is between 0 and $\frac{1}{3}$, then both rules deliver $\mathbf{A} \succ \mathbf{B}$.
- If p is between $\frac{1}{3}$ and $\frac{7}{12}$, then the ex ante rule (1^+) still gives $\mathbf{A} \succ \mathbf{B}$, but the ex post rule ($2^+$) gives $\mathbf{B} \succ \mathbf{A}$.
- If p is between $\frac{7}{12}$ and 1, then both rules give $\mathbf{B} \succ \mathbf{A}$.

By applying maximin to a set of individual weights, rather than the set of individuals, one increases the range of values of p for which $\mathbf{A} \succ \mathbf{B}$ for both the ex ante and ex post functions. The transparent reason is that some weight now goes to the better incomes. More importantly, this numerical example suggests that the ex post function V^+_{xp} may be more egalitarian than the ex ante function V^+_{xa}, though not in the sense of Proposition 1, since the latter concerns functions of the same type (both of the V^+_{xa} type, or both of the V^+_{xp} type). Here, both V^+_{xp} and V^+_{xa} agree that it pays to hedge against the realization of the bad state 1 only if the probability p of that state is high enough, but the ex ante function tilts the balance in favour of the insurance policy $\mathbf{B}$ against the laissez-faire policy $\mathbf{A}$ at higher threshold values for p than does the ex post function. The next section elaborates on this comparison after providing characterizations for each rule.

3 Axiomatizing, comparing and mixing the ex ante and ex post maximin rules

To characterize formulas (1^+) and (2^+) in terms of the social preference $\succcurlyeq$, we need another technical tool, which is an axiomatization of expected *value* when prospects are defined as real-valued mappings. Suppose A is a finite set of indexes, here viewed as states of the world, $\mathbb{R}^A$ is a set of alternatives, here viewed as uncertain prospects, and R is a preference relation on $\mathbb{R}^A$. We need the following axiom.

A7. R is *invariant under translations*, i.e., for all $f, g \in \mathbb{R}^A$ and all vectors $\beta \in \mathbb{R}^A$,

$$f \;\mathsf{R}\; g \quad \text{if and only if} \quad (f + \beta) \;\mathsf{R}\; (g + \beta).$$

Lemma B (de Finetti) *Suppose that a nontrivial preference relation* R *on the set of alternatives* $\mathbb{R}^A$ *satisfies* A1*,* A2*,* A3 *and* A7*. Then, there exists a probability vector* $\mathbf{p} = (p_a)_{a \in A}$ *such that for all* $f, g \in \mathbb{R}^A$*,*

$$f \;\mathsf{R}\; g \quad \textit{if and only if} \quad W(f) \geq W(g),$$

and

$$W(f) = \sum_{a \in A} p_a\, f(a).$$

Moreover, $\mathbf{p}$ *is unique in this format of representation.*

Proof See e.g. Theorem 10.4 of Gilboa (2009). □

Now to the characterization results.[8] Define as *ex ante risk-free* those policies $\mathbf{X} \in \mathbb{R}^{I \times S}$ which are constant across S, i.e., such that for all $i \in I$, $\mathbf{x}_i$ has identical components, and *ex post inequality-free* those policies $\mathbf{X} \in \mathbb{R}^{I \times S}$ which are constant across I, i.e., such that for all $s \in S$, $\mathbf{x}^s$ has identical components.

Note that *ex ante* risk-free policies and *ex post* inequality-free policies can be identified with elements of $\mathbb{R}^I$ and and $\mathbb{R}^S$, respectively. Under this interpretation, an ordering on the set of *ex ante* risk-free policies and an ordering on the set of *ex post* inequality-free policies can be subjected to axioms (such as A1–A7) that hold for Euclidean spaces.

Proposition 2 *Suppose that the social preference $\succcurlyeq$ satisfies the Gilboa-Schmeidler axioms on the subset of ex ante risk-free policies. Suppose also that for all $i \in I$, $\succcurlyeq_i$ satisfies the axioms for expected value, and these conditional evaluations are all identical. Then, the ex ante rule follows, i.e., $\succcurlyeq$ is represented by*

$$(1^+)\ V^+_{\mathrm{xa}}(\mathbf{X}) = \min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i \sum_{s \in S} p_s\, x_i^s \right\} \textit{ for all } \mathbf{X} \in \mathbb{R}^{I \times S},$$

where $\mathbf{p} = (p_s)_{s \in S} \in \Delta(S)$ is a probability vector and $\mathbf{M} \subseteq \Delta(I)$ is a closed and convex set of weight vectors $\boldsymbol{\mu} = (\mu_i)_{i \in I}$. In this representation, $\mathbf{p}$ and $\mathbf{M}$ are uniquely defined.

Proposition 3 *Suppose that $\succcurlyeq$ satisfies the axioms for expected value on the subset of ex post inequality-free policies. Suppose also that for all $s \in S$, $\succcurlyeq^s$ satisfies the Gilboa-Schmeidler axioms and these conditional evaluations are all identical. Then, the ex post rule follows, i.e., $\succcurlyeq$ is represented by*

$$(2^+)\ V^+_{\mathrm{xp}}(\mathbf{X}) = \sum_{s \in S} p_s \min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i x_i^s \right\} \textit{ for all } \mathbf{X} \in \mathbb{R}^{I \times S},$$

where $\mathbf{p} = (p_s)_{s \in S} \in \Delta(S)$ is a probability vector and $\mathbf{M} \subseteq \Delta(I)$ is a closed and convex set of weight vectors $\boldsymbol{\mu} = (\mu_i)_{i \in I}$. In this representation, $\mathbf{p}$ and $\mathbf{M}$ are uniquely defined.

We complement these two characterizations by an impossibility theorem that clarifies the reasons why they are mutually incompatible. To obtain a conflict between ex ante and ex post evaluations, it is not necessary to combine the full axiomatizations for them. Indeed, by conjoining only weak conditions taken from these axiomatizations, one already gets the undesirable conclusion that the sets of weight vectors in the min-of-means representations degenerate to singletons.

[8] The two formulas (1^+) and (2^+) characterized below may be compared with the ex ante and ex post formulas derived by Hayashi and Lombardi (2019, Theorems 1 and 2) for the aggregative setting. A major difference is that (1^+) and (2^+) do not refer to individual utilities and probabilities. Indeed, unlike Hayashi and Lombardi (and the other papers cited in Footnote 4), we assume that uncertainty about the state of nature is represented by a *single* probability distribution (representing the beliefs of the social observer). We are not concerned with heterogeneity of beliefs amongst the agents because, as already noted, we are not concerned with the aggregation of individual ex ante preferences. For the same reason, unlike Hayashi and Lombardi, we do not require the social beliefs to encode ambiguity.

Proposition 4 *Suppose that the social evaluation $\succsim$ is a continuous weak ordering and is strongly monotonic in each $(i, s) \in I \times S$.*[9] *Suppose also that for all $i \in I$, $\succsim_i$ is a weak ordering, with all $\succsim_i$ being identical, and that for all $s \in S$, $\succsim^s$ is a weak ordering, with all $\succsim^s$ being identical. Then, there exist a probability vector $\mathbf{p} = (p_s)_{s \in S} \in \Delta(S)$ and a weight vector $\boldsymbol{\mu} = (\mu_i)_{i \in I} \in \Delta(I)$ such that $\succsim$ is represented by*

$$U(\mathbf{X}) = \sum_{s \in S} p_s \sum_{i \in I} \mu_i x_i^s \text{ for all } \mathbf{X} \in \mathbb{R}^{I \times S}.$$

In this format of representation, $\mathbf{p}$ *and* $\boldsymbol{\mu}$ *are unique.*

Here as elsewhere, by reducing rules of decision to their qualitative basis, representation theorems can help clarify how attractive these rules are. While the use of the expected value formula can be defended on simplicity grounds, the Gilboa-Schmeidler axioms call for a substantial discussion—a discussion that should take place in light of Rawls's two ways of defending the Difference Principle, as explained in the introduction of this paper. Among the axioms, A1 and A2, i.e., ordering and continuity, are standard, and A3, a monotonicity property, merely conveys the fact that the numbers measure desirable quantities. In decision theory, A6 is usually justified by the decision maker's aversion to uncertainty, and the attraction of hedging for such a decision maker. This interpretation pertains to Rawls's "original position" argument. As far as a directly normative argument is concerned, A6 expresses society's aversion to inequality; more precisely, it says that society can improve on two equivalent income distributions by mixing them, since this smooths up some of the inequalities they contain.

Axiom A4 says that social evaluations should be independent of the units we use. When comparing two resource distributions, it should be irrelevant whether money is denominated in euros or dollars, and whether other resources are measured in kilograms or pounds, litres or quarts. When comparing two income distributions, Axiom A4 also says that timescale is irrelevant: it does not matter whether we compare distributions of hourly wages or of yearly incomes, as long as one is proportional to the other.

Axiom A5 puts the focus on *relative* deprivation rather than *absolute* deprivation. The standard of living that we consider "abject poverty" is much different in a modern industrialized society than it was two hundred years ago. Presumably it will again be different two hundred years hence. Axiom A5 says that social preferences between two resource distributions should be independent of where we set the poverty line, and hence, independent of the overall level of economic or technological development in a society.

A4 and A5 are not entirely innocent. Uniformly adding large amounts of money to two risky prospects may well reverse an agent's preferences between them. By analogy, adding a large amount to everyone's resources in two social distributions could perhaps reverse our social preferences between them. But A4 and A5 are satisfied by many popular social evaluation rules besides min-of-means. Among these are the *comonotonically linear rules*, which assigns weights to the *positions* of individuals on

[9] i.e. for all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{I \times S}$, if $x_s^i \geq y_s^i$ for all $(i, s) \in I \times S$, and $x_s^i > y_s^i$ for some $(i, s) \in I \times S$, then $\mathbf{X} \succ \mathbf{Y}$.

the income ladder rather than to the individuals themselves. These rules have attractive properties from both the decision-theoretic and inequality-measurement perspectives. In the context of inequality measurement, the prominent comonotonic rule is the *generalized Gini* social evaluation function introduced by Weymark (1981). Any such function can be obtained as a min-of-means representation by taking the set $\mathbf{M}$ to be the convex hull of the permutations of the vector of weights it assigns to the ranks.[10]

Granting the case for using Gilboa and Schmeidler's axioms in a context of Rawlsian egalitarian social evaluation, should they be applied in accordance with the ex ante conditions of Proposition 2 or the ex post conditions of Proposition 3? We now return to a direct comparison of formulas (1^+) or (2^+). As already pointed out, this comparison is between two functions of dissimilar type. To circumvent the problem this creates, we will call upon Ben-Porath et al. (1997), who explain how to translate utility functions such as V^+_{xa} and V^+_{xp} into functions of the form V^+_t from (3^+). This will provide a common standard in terms of which V^+_{xa} and V^+_{xp} can be compared. It turns out that the translated min-of-means function for V^+_{xa} involves a set of weight vectors that is *included* in the set of weights of the translated min-of-means function for V^+_{xp}, which makes it possible to compare the inequality aversion implied by these two dissimilar functions.

Proposition 5 *Suppose that the ex ante function* V^+_{xa} *and the ex post function* V^+_{xp} *have identical* $\mathbf{M}$ *and* $\mathbf{p}$. *Then there exist closed and convex sets of weight vectors* $\mathbf{R}_{\mathrm{xa}}, \mathbf{R}_{\mathrm{xp}} \subseteq \Delta(I \times S)$ *such that*

$$V^+_{\mathrm{xa}}(\mathbf{X}) = \min_{\rho \in \mathbf{R}_{\mathrm{xa}}} \left\{ \sum_{(i,s) \in I \times S} \rho_{is} x_i^s \right\} \quad \textit{and} \quad V^+_{\mathrm{xp}}(\mathbf{X}) = \min_{\rho \in \mathbf{R}_{\mathrm{xp}}} \left\{ \sum_{(i,s) \in I \times S} \rho_{is} x_i^s \right\}$$

for all $\mathbf{X} \in \mathbb{R}^{I \times S}$. *Furthermore,* $\mathbf{R}_{\mathrm{xa}} \subseteq \mathbf{R}_{\mathrm{xp}}$, *so* V^+_{xp} *is at least as egalitarian in toto as* V^+_{xa}.

From the perspective of Rawls, who has emphasized that the Difference Principle was "a strongly egalitarian conception" (1971, §13, p. 76), it would seem natural to conclude from this result that the ex post method is more satisfactory than the ex ante one. So much could be foreshadowed from the numerical example, in which the former supported an insurance scheme for more values of $\mathbf{p}$ than the latter did. However, this is not the final word on the Difference Principle, as the next argument shows.

This argument originates in Ben-Porath et al. (1997) and is also taken up in Gajdos and Maurin (2004). In a simplified context of two individuals and two states, the following three policies are to be compared:

$\mathbf{X}$	$s=1$	$s=2$
$i=1$	0	1
$i=2$	0	1

$\mathbf{Y}$	$s=1$	$s=2$
$i=1$	1	0
$i=2$	0	1

$\mathbf{Z}$	$s=1$	$s=2$
$i=1$	1	1
$i=2$	0	0

.

[10] The power functions proposed by Arrow (1973) and Alexander (1974) satisfy A1–A4 (and also A6 if $a \leq 1$), but *not* A5; see footnote 7. Indeed, zero plays a special role in these functions, since we must have $x_i > 0$ for all $i \in I$ or the function value is not even well-defined. Thus, in contrast to min-of-means, these social evaluations put the focus on *absolute* rather than *relative* deprivation.

We will assume that states 1 and 2 have equal probabilities and an anonymity requirement applies to social evaluation (i.e., allocations in each state can be permuted as between the individuals). As the argument goes, $\mathbf{Y}$ is socially better than $\mathbf{Z}$ on ex ante grounds, since it equalizes 1's and 2's expected values, unlike $\mathbf{Z}$, and $\mathbf{X}$ is socially better than $\mathbf{Y}$ on ex post grounds, since it equalizes 1's and 2's realized values in each state, unlike $\mathbf{Y}$. From an overall egalitarian perspective, the ranking $\mathbf{X} \succ \mathbf{Y} \succ \mathbf{Z}$ appears to be the most commendable since it involves both forms of equalization at a time. However, no purely ex ante or purely ex post rule taken in isolation can deliver this ranking; it requires mixing two such rules in some suitable way, for instance by convex combinations.[11]

This general argument nicely applies to the Rawlsian context. Indeed, with the original Rawlsian functions,

$$V_{\mathrm{xa}}(\mathbf{X}) = \frac{1}{2} = V_{\mathrm{xa}}(\mathbf{Y}) > V_{\mathrm{xa}}(\mathbf{Z}) = 0 \text{ and } V_{\mathrm{xp}}(\mathbf{X}) = \frac{1}{2} > V_{\mathrm{xa}}(\mathbf{Y}) = 0 = V(\mathbf{Z}),$$

whereas defining $W = \frac{1}{2}V_{\mathrm{xa}} + \frac{1}{2}V_{\mathrm{xp}}$, one gets

$$W(\mathbf{X}) = \frac{1}{2} > W(\mathbf{Y}) = \frac{1}{4} > W(\mathbf{Z}) = 0,$$

as the desired ranking prescribes. Generally, with min-of-means representations V_{xa}^{+}, V_{xp}^{+} and the average $W = \frac{1}{2}V_{\mathrm{xa}}^{+} + \frac{1}{2}V_{\mathrm{xp}}^{+}$, the same comparisons of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ hold, leaving aside a degenerate case.[12] Thus, if there was an apparently good egalitarian argument for considering V_{xp}^{+} alone, there is now a stronger one in favour of mixing V_{xa}^{+} with V_{xp}^{+}. The weakness of the first conclusion comes out clearly from the comparison of $\mathbf{Y}$ and $\mathbf{Z}$. Consistently, one should relinquish the criterion of egalitarianism in toto that supported this conclusion. What the present argument also teaches is that it is inappropriate to evaluate the misfortune of an individual i in a state s regardless of what happens to i in other states than s, and regardless of what happens to other individuals than i in s.

We therefore submit that Rawls's Difference Principle is best reconstructed in terms of *mixed* rules. They offer the added benefit of allowing for a continuum of attitudes. Society will have to face cases where V_{xa}^{+} and V_{xp}^{+} evaluate allocations in opposite ways, and thanks to a mixed rule, it can express only a relative preference, instead of an absolute preference, for ex ante over ex post equality, or conversely. Here again, we conform with our theoretical preference for a flexible application of maximin reasoning.

The rest of this section investigates mixed rules in the form of convex combinations. We first show that these are in fact min-of-means representations, whose sets of weight

[11] Diamond's (1967) early discussion, which has inspired the present one and many others, only deals with the $\mathbf{Y}$ and $\mathbf{Z}$ matrices. He defended $\mathbf{Y}$ on the ground of ex ante egalitarianism, while neglecting the issue of ex post egalitarianism. However, his main concern was to counter Harsanyi's (1955) method of simply summing up the values of all matrix components.

[12] When the set of weight vectors $\mathbf{M}$ only contains the equiprobable distribution $(\frac{1}{2}, \frac{1}{2})$.

vectors are located between those of the rules that are being combined. This should be compared with the theorem in Ben-Porath et al. (1997).

Proposition 6 *Suppose* $U = \alpha V^+_{\mathrm{xa}} + (1-\alpha)V^+_{\mathrm{xp}}$ *for some* $\alpha \in [0,1]$. *Then* U *is of the form* (3^+), *i.e., there is a closed and convex set of weight vectors* $\mathbf{R} \subseteq \Delta(I \times S)$ *such that for all* $\mathbf{X} \in \mathbb{R}^{I\times S}$,

$$U(\mathbf{X}) = \min_{\rho\in\mathbf{R}} \left\{ \sum_{(i,s)\in I\times S} \rho_{is} x_i^s \right\}$$

and $\mathbf{R}_{\mathrm{xa}} \subseteq \mathbf{R} \subseteq \mathbf{R}_{\mathrm{xp}}$.

Thus, if egalitarian comparisons are ruled by convex combinations of V^+_{xa} and V^+_{xp}, the ideal set of weights for a min-of-means representation capturing these comparisons will lie somewhere between the two extremes of $\mathbf{R}_{\mathrm{xa}}$ and $\mathbf{R}_{\mathrm{xp}}$.

We will now axiomatize the set of convex combinations of V^+_{xa} and V^+_{xp}. This should be compared with Gajdos and Maurin (2004, Theorem 3). Suppose there are three evaluation relations $\succcurlyeq_{\mathrm{xa}}$, $\succcurlyeq_{\mathrm{xp}}$ and $\succcurlyeq$ on $\mathbb{R}^{I\times S}$. In our intended application, $\succcurlyeq_{\mathrm{xa}}$ and $\succcurlyeq_{\mathrm{xp}}$ are represented, respectively, by an ex ante maximin evaluation by V^+_{xa} and an ex post maximin evaluation V^+_{xp}, as in Propositions 2 and 3, while $\succcurlyeq$ is the overall social evaluation. We use this axiom:

A8. (*Ex ante-ex post support* condition) For all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{I\times S}$, if $\mathbf{X} \succcurlyeq_{\mathrm{xa}} \mathbf{Y}$ and $\mathbf{X} \succcurlyeq_{\mathrm{xp}} \mathbf{Y}$, then $\mathbf{X} \succcurlyeq \mathbf{Y}$. If the antecedent holds with either $\mathbf{X} \succ_{\mathrm{xa}} \mathbf{Y}$ or $\mathbf{X} \succ_{\mathrm{xp}} \mathbf{Y}$, then $\mathbf{X} \succ \mathbf{Y}$.

This dominance-type condition is obviously necessary for any combination formula, whether convex or not, between the ex ante and ex post representations. It turns out to be sufficient for convex combinations specifically.

Proposition 7 *Suppose that the relations* $\succcurlyeq_{\mathrm{xa}}$ *and* $\succcurlyeq_{\mathrm{xp}}$ *on* $\mathbb{R}^{I\times S}$ *satisfy the axiomatic conditions of Propositions* 2 *and* 3 *respectively, and these are different relations. Suppose also that the relation* $\succcurlyeq$ *on* $\mathbb{R}^{I\times S}$ *satisfies the Gilboa-Schmeidler axioms, as well as the ex ante–ex post support condition. Then* $\succcurlyeq_{\mathrm{xa}}$ *is represented by* V^+_{xa} *of the ex ante type* (1^+), $\succcurlyeq_{\mathrm{xp}}$ *by* V^+_{xp} *of the ex post type* (2^+), *and* $\succcurlyeq$ *by* U *with the property that*

$$U = \alpha V^+_{\mathrm{xa}} + (1-\alpha)V^+_{\mathrm{xp}}$$

for some $\alpha \in [0,1]$. *In this format of representation for* U, α *is uniquely defined.*

4 Concluding comments

We conclude with a brief discussion of some related literature. Milnor (1954) gave the first axiomatic characterization of the maximin decision criteria for individual decisions under "complete ignorance". This result was extended by Maskin (1979), who also axiomatically characterized the leximin decision criteria for individual decisions, and used this to obtain a characterization of the leximin social welfare order

(SWO). Slightly earlier, Hammond (1976, Theorem 7.2) and d'Aspremont and Gevers (1977, Theorems 5 and 7) had already characterized the leximin SWO. Similar axiomatizations appear in Deschamps and Gevers (1978), Barberà and Jackson (1988), Moulin (1988, Theorem 2.4, p. 40), Roemer (1996, Corollary 1.1, p. 35) d'Aspremont and Gevers (2002, Theorem 4.7), and Bossert and Weymark (2004, Theorem 6.1). These axiomatizations generally involve either a convexity axiom (e.g. Milnor, Barberà and Jackson), or an informational environment or invariance axiom such that only ordinal interpersonal comparisons are meaningful, along with an axiom to focus attention on bad outcomes (e.g. Maskin's *Pessimism*, d'Aspremont and Gevers's *Minimal Equity*) or a preference for equalizing transfers (e.g. Hammond's *Equity* axiom). The choice between the leximin and maximin SWOs is generally a choice between Strong Pareto and Continuity. For example, Hammond (1976) makes the former choice, while Bosmans and Ooghe (2013) make the second, and obtain an axiomatization of the maximin SWO in the spirit of Hammond's original result. Slightly outside the mainstream is an early axiomatic characterization of the maximin SWO by Strasnick (1976), who used an unusual framework in which the preference of one individual over one pair of alternatives can be deemed to have equal "priority" with the the preference of another individual over another pair of alternatives.

More recently, other researchers have discovered very different axiomatic paths leading to the same destination. For example, Tungodden (2000, Theorem 1) has an elegant restatement of maximin in terms of a strengthened Pareto condition and an egalitarian property of "contracting extremes". Segal and Sobel (2002, Theorems 2 and 3) simultaneously characterize the maximin, maximax and utility-summation criteria using a "partial separability" axiom; they then apply this to obtain another characterization of the maximin SWO. In a variable-population framework, Fleurbaey and Tungodden (2010, Propositions 3 and 4) give a partial characterization of the maximin SWO using axioms of Replication Invariance and Pigou-Dalton Equity, along with an axiom blocking the "tyranny of aggregation" (so that a small gain for a large majority cannot offset a large cost for a small minority). From a somewhat different angle, Mariotti and Veneziani (2009, Theorem 1) and Lombardi et al. (2016, Theorem 1) derive the leximin SWO from a liberal principal of noninterference (the *Harm Principle*), thereby establishing a surprising link between libertarian and egalitarian philosophies, which are often perceived to be at cross-purposes. For other philosophical justifications of maximin, see Vallentyne (2000) and Tungodden (2003).

The literature discussed so far considers social welfare evaluations *without* uncertainty. In contrast, Fleurbaey (2010) considers social decisions under uncertainty, in a framework very similar to the present paper. His Theorem 2 characterizes the *ex post leximin* criterion, which applies the leximin SWO to the vector of expected utilities of the individuals in society. In Fleurbaey (2010), as in the present paper, uncertainty is assessed with respect to a *single* probability distribution over the states of nature—in other words, these results do not address the problem of *heterogeneity of beliefs* considered by Mongin (1995, 1998, 2016), Mongin and Pivato (2015, 2020), Crès et al. (2011), Alon and Gayer (2016), Qu (2017), Hayashi and Lombardi (2019) and many others. This is not the place for a review of the extensive literature on this topic (see Mongin and Pivato (2016) and Fleurbaey (2018) for surveys), but it is worth mentioning a recent paper by Ceron and Vergopoulos (2019), which axiomatically characterizes a social welfare function that is a convex combination of an ex ante utilitarian component and an ex post utilitarian component. This formula is somewhat reminiscent of our Proposition 7, except that Ceron and Vergopoulos are concerned with belief heterogeneity, rather than egalitarianism.

The papers in the previous paragraph belong to a literature inspired by Harsanyi's (1955) Social Aggregation Theorem. Meanwhile, Gajdos and Kandil (2008) have obtained an interesting variant of Harsanyi's (1953) Impartial Observer Theorem in situations of ambiguity, where precise probabilities are not available. Their Theorem 1 is an axiomatic characterization of a min-of-means criterion; it differs from the classic axiomatization of Gilboa and Schmeidler (1989) in that the sets of priors are *themselves* an object of choice, encoded in the alternatives being compared, rather than being derived as part of the representation. Meanwhile, Gajdos and Kandil's Theorem 5 characterizes SWOs represented by a convex combination of a maximin social welfare function and a relative utilitarian social welfare function; this formula is again somewhat reminiscent of our Proposition 7. Finally, we reiterate that our approach is greatly influenced by the work of Ben-Porath et al. (1997), Gajdos and Maurin (2004) and Hayashi and Lombardi (2019).

Let us now assess what progress was made here, and what progress remains to be made, in the analysis of the Difference Principle from the present perspective of theoretical economics. We will carry out this discussion by returning to some important claims made by Rawls himself. Informally, his Difference Principle states that "the social order is not to establish and secure the more attractive prospects of those better off unless doing so is to the advantage of those less fortunate" (1971, §13, p. 75), or more briefly, that "social and economic inequalities are to be arranged...to the greatest benefit of the least advantaged" (ibid., p. 83). Rawls has repeatedly emphasized that this principle is only part of a wider conception of justice that is not only concerned with distributive issues as narrowly understood. In the framework of *A Theory of Justice*, the Difference Principle is subordinate to the principle of equal opportunities for all, which is in turn subordinate to the principle of equal liberty for all. These overarching principles are discussed neither in the present paper nor, at least most commonly, elsewhere in economics. Besides the obvious division of labour between economics and other social sciences, there are Rawlsian grounds for this omission. Indeed, the lexical structure of principles has precluded neither Rawlsians nor, on occasions, Rawls himself from investigating the Difference Principle in and for itself. This investigation should be done under the proviso that the higher principles already apply. Given the distributional problems economists usually address, we do not think they should feel too uneasy about the limitations that this proviso entails for the treatment of the Difference Principle in this field.[13]

Rawls has also repeatedly emphasized that his principles of justice were primarily concerned with the "basic structure" of society (e.g., 1971, §2, p. 9; §11, p. 61), a feature that becomes even clearer when he employs the device of the original position to derive these principles. This led him to dimiss applications of the Difference Principle

[13] When commenting on the various interpretations of the Difference Principle, van Parijs (2003) appears to warrant this reassuring conclusion regarding the proviso. Some applied work inspired by Rawls outside economics also exploits the Difference Principle without always considering the whole structure of the theory. See, for example, some of the health ethics applications collected by Anand et al. (2006).

to "small-scale situations", such as those featuring in Harsanyi's (1975) objections to Rawls. However, Rawls's dismissal is too quick, because such situations can be representative of different types of society that the Principle is meant to adjudicate. The example of Sect. 2 is representative in this sense, for it illustrates, though very schematically, the contrast between two abstract models of basic social functionings, laissez-faire versus social insurance. Our formal results guarantee that this kind of vignette can be expanded to more states and individuals and thus made more realistic. Generally, *pace* Rawls, we think that insights into the "basic structure" of society can be gotten from analyzing suitably defined "small-scale situations".[14]

The Difference Principle classically raises the issue of the appropriate *distribuendum*. Rawls appears to take two views on this major question. The first, which is at it were the official one, is that the principle distributes quantities of "primary goods", i.e., of those things which are useful to all individuals whatever they specifically want (1971, §11, p. 62; §15, p. 92). Rawls's list of such all-purpose means includes, but is not limited to income and wealth, and this raises an indexing problem for which he admits having no precise solution. The later Rawls (1982) shifted to a second, pragmatic view, to the effect that income or wealth could serve as proxies for the whole collection of primary goods.[15] Here we have followed this low-profile answer, but for simplicity rather than substantive reasons. Our framework could be complexified so as to accommodate multiple primary goods: technically, in the ex ante and ex post formulas (1^+) and (2^+), the scalar values x_i^s would give way to new scalar values $\phi(\mathbf{x}_i^s)$, where $\mathbf{x}_i^s$ would be a vector of quantities of primary goods and ϕ a real-valued aggregative function. Propositions 2 and 3 could then be extended along this line using more material from decision theory, in particular a full-fledged axiomatization of min-of-means (ours was a simplified one).[16] Importantly, the ϕ function would have the interpretation of an aggregator, not of a utility in the sense of a welfare index. It is now fairly well-understood that the utility interpretations of Rawls's metric that have long prevailed in economics miss this essential point.

As the opening of this paper reminded the reader, *A Theory of Justice* offers two distinct treatments of the Difference Principle. One is directly normative and the other goes through the detour of the "original position"; the two arguments are meant to be mutually supportive (1971, §4, pp. 20–30). That economists have primarily been attracted to the latter is easily understandable, since their theory of individual decision under uncertainty provided them with tools to analyze the "veil of ignorance" that gives meaning to the original position. Our analysis can be understood along this familiar line. Then, the ex ante, ex post and mixed rules of Propositions 2, 3 and 7 capture the rational choices of an observer in a presocial state who does not know which individual he or she will be in the social state. The weights in the min-of-means formulas (1^+) and (2^+) become genuine probabilities, and the axioms underlying these formulas receive a standard decision-theoretic interpretation in terms of rationality conditions. However,

[14] Harsanyi asked how doctors should allocate medical resources, and how universities should allocate scholarships, between individuals having different long-term prospects. One can view these examples more favourably than Rawls did.

[15] See also van Parijs's (2003) review on this Rawlsian problem of the *distribuendum*.

[16] An alternative solution is to define a partial ordering of vectors of quantities of primary goods, as in Gibbard (1979).

one can also understand the formulas and axioms directly in terms of egalitarianism, in which case they belong to a social observer who is fully cognizant of the social state of affairs. This interpretation is not question-begging because egalitarianism is an initially vague motto, just a general inspiration, and the point of the axiomatic exercise is to decompose it and turn it into an precise rule of evaluation. Thus, even though we do not reproduce Rawls's direct normative argument for the Difference Principle, this semantic duality of our results appears to correspond to the duality of philosophical treatment in *A Theory of Justice*.[17]

A Appendix

Proof of Proposition 1 If V^+_{xa} is at least as egalitarian as W^+_{xa}, then for all $\mathbf{X} \in \mathbb{R}^{I\times S}$ and all $k \in \mathbb{R}$, the following implication holds:

$$\text{If } k \geq W^+_{\mathrm{xa}}(\mathbf{X}), \text{ then } k \geq V^+_{\mathrm{xa}}(\mathbf{X}).$$

(This simply follows from taking $\mathbf{K}$ in the definition with all components equal to k.) Fix $\mathbf{X}$ and take $k = W^+_{\mathrm{xa}}(\mathbf{X})$. It follows that for all $\mathbf{X} \in \mathbb{R}^{I\times S}$,

$$W^+_{\mathrm{xa}}(\mathbf{X}) \geq V^+_{\mathrm{xa}}(\mathbf{X}). \tag{1}$$

By definition

$$V^+_{\mathrm{xa}}(\mathbf{X}) := \min_{\boldsymbol{\mu}\in\mathbf{M}} \left\{ \sum_{i\in I} \mu_i \sum_{s\in S} p_s\, x_i^s \right\} \quad \text{and} \quad W^+_{\mathrm{xa}}(\mathbf{X}) := \min_{\boldsymbol{\nu}\in\mathbf{N}} \left\{ \sum_{i\in I} \nu_i \sum_{s\in S} p_s\, x_i^s \right\}.$$

Suppose by way of contradiction there is $\boldsymbol{\nu} \in \mathbf{N}$ such that $\boldsymbol{\nu} \notin \mathbf{M}$. As $\mathbf{M}$ is convex and closed, by the separating hyperplane theorem, there exist $\mathbf{y} \in \mathbb{R}^I$ and $\xi \in \mathbb{R}$ such that $\boldsymbol{\nu}\cdot\mathbf{y} < \xi$ and $\boldsymbol{\mu}\cdot\mathbf{y} \geq \xi$ for all $\boldsymbol{\mu} \in \mathbf{M}$. Hence

$$\min_{\boldsymbol{\mu}\in\mathbf{M}} \{\boldsymbol{\mu}\cdot\mathbf{y}\} > \boldsymbol{\nu}\cdot\mathbf{y} \geq \min_{\boldsymbol{\nu}'\in\mathbf{N}} \left\{\boldsymbol{\nu}'\cdot\mathbf{y}\right\},$$

which contradicts inequality (1) because one can find $\mathbf{X} \in \mathbb{R}^{I\times S}$ such that $y_i = \sum_{s\in S} p_s\, x_i^s$ for each $i \in I$. This establishes one chain of implications in the proposition; the other chain is trivial. When V^+_{xp} is compared with W^+_{xp}, the proof applies *mutatis mutandis* (in the last stage, take $\mathbf{X} \in \mathbb{R}^{I\times S}$ such that $\mathbf{x}^s = \mathbf{y}$ for each $s \in S$). □

Proof of Proposition 2 Consider the set B of policies $\mathbf{X} \in \mathbb{R}^{I\times S}$ such that for all $i \in I$, $\mathbf{x}_i$ has identical components, i.e., there exists $\xi_i \in \mathbb{R}$ such that $\mathbf{x}_i = \xi_i \mathbf{1}_S$, where

[17] Rawls's direct normative argument for the Difference Principle hinges on the claim that one should take "the distribution of talents as a collective asset" (1971, §17, p. 101). This claim has raised active philosophical controversy, but thus far received little attention in normative economics.

$\mathbf{1}_S \in \mathbb{R}^S$ has all its components equal to 1. By Lemma A, $\succcurlyeq$ has a min-of-means representation on B:

$$V(\mathbf{X}) = \min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i \xi_i \right\}, \quad \text{for all } \mathbf{X} \in B,$$

where $\mathbf{M} \subseteq \Delta(I)$ is a closed and convex set of strictly positive weight vectors $\boldsymbol{\mu} = (\mu_i)_{i \in I}$.

By Lemma B, for all $i \in I$, $\succcurlyeq_i$ has an expected value representation:

$$V_i(\mathbf{x}_i) = \sum_{i \in I} p_{is} x_{is}, \quad \text{for all } \mathbf{x}_i \in \mathbb{R}^S,$$

where $\mathbf{p}_i = (p_{is})_{s \in S} \in \Delta(S)$.

Now fix $\mathbf{X} \in \mathbb{R}^{I \times S}$, and for all $i \in I$, define $\xi_i = V_i(\mathbf{x}_i)$ and $\widetilde{\boldsymbol{\xi}}_i = \xi_i \mathbf{1}_S \in \mathbb{R}^S$. The V_i representation entails that $\mathbf{x}_i \sim_i \widetilde{\boldsymbol{\xi}}_i$ for all $i \in I$. Define $\widetilde{\mathbf{X}}$ to be $\left[\widetilde{\boldsymbol{\xi}}_i\right]_{i \in I}$. By a standard argument in decision theory (e.g., Wakker 1989, Lemma II.2.7., p. 32), the $\succcurlyeq_i$ are weak orderings if and only if $\succcurlyeq$ is strictly increasing in each of them. From this monotonicity property, $\mathbf{X} \sim \widetilde{\mathbf{X}}$ follows. Hence

$$V(\mathbf{X}) = V(\widetilde{\mathbf{X}}) = \min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i \xi_i \right\} = \min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i \sum_{i \in I} p_{is} x_{is} \right\}, \quad \text{for all } \mathbf{X} \in \mathbb{R}^{I \times S}.$$

If the $\succcurlyeq_i$ are all identical, the uniqueness part of the expected value representation theorem entails that each representation V_i involves the same probability vector, and representation (1^+) holds for $\succcurlyeq$.

That $\mathbf{p}$ and $\mathbf{M}$ are unique given the format of representation (1^+) easily follows from the uniqueness properties stated in Lemmas A and B. □

Proof of Proposition 3 (This proof is close to the previous one, but we give it for completeness.) Consider the set C of policies $\mathbf{X} \in \mathbb{R}^{I \times S}$ such that for all $s \in S$, $\mathbf{x}^s$ has identical components, i.e., there exists $\xi^s \in \mathbb{R}$ such that $\mathbf{x}^s = \xi^s \mathbf{1}_I$, where $\mathbf{1}_I \in \mathbb{R}^I$ has all its components equal to 1. By Lemma B, $\succcurlyeq$ has an expected value representation on C:

$$V(\mathbf{X}) = \sum_{s \in S} p_s \xi^s, \quad \text{for all } \mathbf{X} = \left[\xi^s \mathbf{1}_I\right]^{s \in S} \in C,$$

where $\mathbf{p} = (p_s)_{s \in S} \in \Delta(S)$.

By Lemma A, for all $s \in S$, $\succcurlyeq^s$ has a min-of-means representation:

$$V_s(\mathbf{x}^s) = \min_{\boldsymbol{\mu}_s \in \mathbf{M}_s} \left\{ \sum_{i \in I} \mu_{is} x_{is} \right\}, \quad \text{for all } \mathbf{x}^s \in \mathbb{R}^I,$$

where $\mathbf{M}_s \subseteq \Delta(I)$ is a closed and convex set of strictly positive weight vectors $\boldsymbol{\mu}_s = (\mu_{is})_{i \in I}$.

Now fix $\mathbf{X} \in \mathbb{R}^{I \times S}$, and for all $s \in S$, define $\xi_s = V_s(\mathbf{x}^s)$ and $\widetilde{\boldsymbol{\xi}}^s = \xi^s \mathbf{1}_I \in \mathbb{R}^I$. The V_s representation entails that $\mathbf{x}^s \sim_s \widetilde{\boldsymbol{\xi}}^s$ for all $s \in S$. Define $\widetilde{\mathbf{X}}$ to be $\left[\widetilde{\boldsymbol{\xi}}^s\right]^{s \in S}$. As the $\succcurlyeq^s$ are weak orderings, $\mathbf{X} \sim \widetilde{\mathbf{X}}$ follows. Hence

$$V(\mathbf{X}) = V(\widetilde{\mathbf{X}}) = \sum_{s \in S} p_s\, \xi^s = \sum_{s \in S} p_s \min_{\boldsymbol{\mu}_s \in \mathbf{M}_s} \left\{ \sum_{i \in I} \mu_{is}\, x_{is} \right\}, \quad \text{for all } \mathbf{X} \in \mathbb{R}^{I \times S}.$$

If the $\succcurlyeq^s$ are all identical, the uniqueness part of Lemma A entails that all V_s representations involve the same set of weight vectors, and representation (2^+) holds for $\succcurlyeq$.

That $\mathbf{p}$ and $\mathbf{M}$ are unique given the format of representation (2^+) easily follows from the corresponding uniqueness properties in the two lemmas. □

Proof of Proposition 4 The statement is formally comparable with that of Corollary 1 in Mongin and Pivato (2015, pp. 161–162) and can be proved in the same way. □

Proof of Proposition 5 For the ex ante representation V^+_{xa}, observe that

$$\sum_{i \in I} \mu_i \left(\sum_{s \in S} p_s\, x_i^s \right) = \sum_{(i,s) \in I \times S} (\mu_i\, p_s) x_i^s$$

and that $[\mu_i\, p_s]_{i \in I}^{s \in S}$ is a weight vector on $I \times S$. Since the p_s are fixed, there is a bijective mapping between $\mathbf{M}$ and the set

$$\mathbf{R}_{\text{xa}} = \left\{ \boldsymbol{\rho} = [\mu_i\, p_s]_{i \in I}^{s \in S} \mid \boldsymbol{\mu} \in \mathbf{M} \right\},$$

so that

$$\min_{\boldsymbol{\mu} \in \mathbf{M}} \left\{ \sum_{i \in I} \mu_i \sum_{s \in S} p_s\, x_i^s \right\} = \min_{\boldsymbol{\rho} \in \mathbf{R}_{\text{xa}}} \left\{ \sum_{(i,s) \in I \times S} \rho_{is}\, x_i^s \right\},$$

which provides the desired restatement of V^+_{xa}.

For the ex post representation V^+_{xp}, we consider the matrices $\boldsymbol{\rho}$ defined by

$$\boldsymbol{\rho} = \left[p_s \boldsymbol{\mu}^s \right]^{s \in S},$$

where $\boldsymbol{\mu}^s \in \mathbf{M}$ for all $s \in S$, and observe that they are weight vectors on $I \times S$. Define $\mathbf{R}_{\text{xp}}$ to be the set of all such matrices, i.e.,

$$\mathbf{R}_{\text{xp}} = \left\{ \left[p_s \boldsymbol{\mu}^s \right]^{s \in S} \mid \boldsymbol{\mu}^s \in \mathbf{M} \text{ for all } s \in S \right\}.$$

Meanwhile, from Ben-Porath et al.'s theorem (1997, p. 200), it is the case that

$$\sum_{s\in S} p_s \min_{\boldsymbol{\mu}\in\mathbf{M}} \left\{\sum_{i\in I} \mu_i x_i^s\right\} = \min_{\rho\in\mathbf{R}_{\mathrm{xp}}} \left\{ \sum_{(i,s)\in I\times S} \rho_{is} x_i^s \right\}, \tag{2}$$

which provides the desired restatement of V_{xp}^{+}. Here is a direct proof of Eq. (2). For all $s \in S$, fix $\widehat{\boldsymbol{\mu}}^s = \arg\min_{\boldsymbol{\mu}\in\mathbf{M}} \left\{\sum_{i\in I} \mu_i x_i^s\right\}$. Observe that $\left[p_s \widehat{\boldsymbol{\mu}}^s\right]^{s\in S} \in \mathbf{R}_{\mathrm{xp}}$, so that

$$\sum_{(i,s)\in I\times S} p_s \widehat{\boldsymbol{\mu}}_i^{\,s} x_i^s \geq \min_{\rho\in\mathbf{R}_{\mathrm{xp}}} \left\{ \sum_{(i,s)\in I\times S} \rho_{is} x_i^s \right\},$$

or equivalently,

$$\sum_{s\in S} p_s \min_{\boldsymbol{\mu}\in\mathbf{M}} \left\{\sum_{i\in I} \mu_i x_i^s\right\} \geq \min_{\rho\in\mathbf{R}_{\mathrm{xp}}} \left\{ \sum_{(i,s)\in I\times S} \rho_{is} x_i^s \right\}.$$

To establish the converse inequality, observe that

$$\min_{[p_s\boldsymbol{\mu}^s]^{s\in S}\in\mathbf{R}_{\mathrm{xp}}} \left\{ \sum_{(i,s)\in I\times S} (p_s\mu_i^s) x_i^s \right\} = \min \left\{ \sum_{s\in S} p_s \sum_{i\in I} \mu_i^s x_i^s \mid \boldsymbol{\mu}^s \in \mathbf{M} \text{ for all } s\in S \right\}$$
$$\geq \sum_{s\in S} p_s \min \left\{ \sum_{i\in I} \mu_i^s x_i^s \mid \boldsymbol{\mu}^s \in \mathbf{M} \text{ for all } s \in S \right\} = \sum_{s\in S} p_s \min_{\boldsymbol{\mu}\in\mathbf{M}} \left\{ \sum_{i\in I} \mu_i x_i^s \right\},$$

where the two equations are mere statements and the inequality follows from the concavity of min. This completes the direct argument for Eq. (2).

Both $\mathbf{R}_{\mathrm{xa}}$ and $\mathbf{R}_{\mathrm{xp}}$ inherit the closedness and convexity properties of $\mathbf{M}$. Observe that any $\rho \in \mathbf{R}_{\mathrm{xa}}$ can be written as

$$\boldsymbol{\rho} = \left[p_s \boldsymbol{\mu}^-\right]^{s\in S}.$$

This is a particular case of $\rho \in \mathbf{R}_{\mathrm{xp}}$ where all $\boldsymbol{\mu}^s$ are identical. Thus, $\mathbf{R}_{\mathrm{xa}} \subseteq \mathbf{R}_{\mathrm{xp}}$. □

Proof of Proposition 6 We first show that there is a closed and convex set of weights $\mathbf{R} \subseteq \Delta(I \times S)$ such that for all $\mathbf{X} \in \mathbb{R}^{I\times S}$,

$$\alpha V_{\mathrm{xa}}^{+}(\mathbf{X}) + (1-\alpha) V_{\mathrm{xp}}^{+}(\mathbf{X}) = \min_{\rho\in\mathbf{R}} \left\{ \sum_{(i,s)\in I\times S} \rho_{is} x_i^s \right\}.$$ [18]

[18] Ben-Porath et al. (1997, p. 202) state that a convex combination of two min-of-means functions is itself a min-of-means function.

From the positive homogeneity and quasi-distributivity of the min operator,[19] we have

$$\begin{aligned} &\alpha \min_{\boldsymbol{\rho}\in\mathbf{R}_{\mathrm{xa}}} \left\{ \sum_{(i,s)\in I\times S} \rho_{is} x_i^s \right\} + (1-\alpha) \min_{\boldsymbol{\rho}'\in\mathbf{R}_{\mathrm{xp}}} \left\{ \sum_{(i,s)\in I\times S} \rho'_{is} x_i^s \right\} \\ &= \min \left\{ \alpha \sum_{(i,s)\in I\times S} \rho_{is} + (1-\alpha) \sum_{(i,s)\in I\times S} \rho'_{is} x_i^s \mid \boldsymbol{\rho}\in\mathbf{R}_{\mathrm{xa}}, \boldsymbol{\rho}'\in\mathbf{R}_{\mathrm{xp}} \right\} \\ &= \min_{\boldsymbol{\rho}''\in\mathbf{R}} \left\{ \sum_{(i,s)\in I\times S} \rho'' x_i^s \right\}, \end{aligned}$$

where $\mathbf{R} = \left\{\alpha\rho + (1-\alpha)\rho' \mid \boldsymbol{\rho}\in\mathbf{R}_{\mathrm{xa}}, \boldsymbol{\rho}'\in\mathbf{R}_{\mathrm{xp}}\right\}$ inherits the closedness and convexity properties of $\mathbf{R}_{\mathrm{xa}}$ and $\mathbf{R}_{\mathrm{xp}}$. Using the fact that $\mathbf{R}_{\mathrm{xa}} \subseteq \mathbf{R}_{\mathrm{xp}}$ (Proposition 4) and the convexity of $\mathbf{R}_{\mathrm{xp}}$, we conclude that $\mathbf{R}_{\mathrm{xa}} \subseteq \mathbf{R} \subseteq \mathbf{R}_{\mathrm{xp}}$. □

Proof of Proposition 7 By Lemma A, $\succcurlyeq$ has a min-of-means representation U, and by Propositions 2 and 3, $\succcurlyeq_{\mathrm{xa}}$ and $\succcurlyeq_{\mathrm{xp}}$ are represented by V^+_{xa} and V^+_{xp}. For any function $\phi : \mathbb{R}^k \to \mathbb{R}$, say that ϕ is *positively homogeneous* if for all $\mathbf{X} \in \mathbb{R}^{I\times S}$ and all $\beta > 0$,

$$\phi(\beta\mathbf{X}) = \beta\phi(\mathbf{X}),$$

and that it is *invariant by uniform translation* if for all $\mathbf{X} \in \mathbb{R}^{I\times S}$ all $\gamma \in \mathbb{R}$,

$$\phi(\mathbf{X} + \gamma\,\mathbf{1}_{I\times S}) = \phi(\mathbf{X}) + \gamma,$$

where $\gamma\,\mathbf{1}_{I\times S}$ denotes the matrix having all coefficients equal to γ. These two properties hold for V^+_{xa}, V^+_{xp} and U. Let **(P)** denote their conjunction.

By the first part of Axiom A8, there exists a mapping $\Phi : Rge(V^+_{\mathrm{xa}}, V^+_{\mathrm{xp}}) \to \mathbb{R}$ such that $U(\mathbf{X}) = \Phi(V^+_{\mathrm{xa}}(\mathbf{X}), V^+_{\mathrm{xp}}(\mathbf{X}))$ for all $\mathbf{X} \in \mathbb{R}^{I\times S}$. We will use the property that for all $\mathbf{X} \in \mathbb{R}^{I\times S}$ and all $\beta > 0$,

$$(\beta(V^+_{\mathrm{xa}}(\mathbf{X}) - V^+_{\mathrm{xp}}(\mathbf{X})), 0) \in Rge(V^+_{\mathrm{xa}}, V^+_{\mathrm{xp}}).$$

Indeed, by **(P)** applied to V^+_{xa} and V^+_{xp}:

$$\begin{aligned} V^+_{\mathrm{xa}}(\beta\left[\mathbf{X} - (V^+_{\mathrm{xp}}(\mathbf{X}))\mathbf{1}_{I\times S}\right]) &= \beta(V^+_{\mathrm{xa}}(\mathbf{X}) - V^+_{\mathrm{xp}}(\mathbf{X})) \text{ and} \\ V^+_{\mathrm{xp}}(\beta\left[\mathbf{X} - (V^+_{\mathrm{xp}}(\mathbf{X}))\mathbf{1}_{I\times S}\right]) &= 0. \end{aligned}$$

Because $V^+_{\mathrm{xa}} \geq V^+_{\mathrm{xp}}$, and $\succcurlyeq_{\mathrm{xa}}$ and $\succcurlyeq_{\mathrm{xp}}$ are different orderings, there exists $\overline{\mathbf{X}} \in \mathbb{R}^{I\times S}$ such that $V^+_{\mathrm{xa}}(\overline{\mathbf{X}}) > V^+_{\mathrm{xp}}(\overline{\mathbf{X}})$. Put $c := V^+_{\mathrm{xa}}(\overline{\mathbf{X}})$, $d := V^+_{\mathrm{xp}}(\overline{\mathbf{X}})$. It follows that $(c-d, 0) \in Rge(V^+_{\mathrm{xa}}, V^+_{\mathrm{xp}})$.

[19] By *quasi-distributivity* we refer to the property $\min\{a, b\} + \min\{c, d\} = \min\{a + c, a + d, b + c, b + d\}$, which is extended here to more than two elements in the bracketed sets.

Meanwhile, applying **(P)** to U, V^+_{xa}, V^+_{xp}, one derives a similar property for Φ, i.e., for all $\mathbf{X} \in \mathbb{R}^{I\times S}$, all $\beta > 0$ and all $\gamma \in \mathbb{R}$,

$$\Phi(\beta V^+_{\text{xa}}(\mathbf{X}) + \gamma, \beta V^+_{\text{xp}}(\mathbf{X}) + \gamma) = \beta\Phi(V^+_{\text{xa}}(\mathbf{X}), V^+_{\text{xp}}(\mathbf{X})) + \gamma). \quad (3)$$

(Proof: $\Phi(\beta V^+_{\text{xa}}(\mathbf{X})+\gamma, \beta V^+_{\text{xp}}(\mathbf{X})+\gamma) = \Phi(V^+_{\text{xa}}(\beta\mathbf{X}+\gamma\,\mathbf{1}_{I\times S}), V^+_{\text{xp}}(\beta\mathbf{X}+\gamma\,\mathbf{1}_{I\times S})) = U(\beta\mathbf{X} + \gamma\,\mathbf{1}_{I\times S}) = \beta U(\mathbf{X}) + \gamma \;= \beta\Phi(V^+_{\text{xa}}(\mathbf{X}), V^+_{\text{xp}}(\mathbf{X})) + \gamma$.)

For all $\mathbf{X} \in \mathbb{R}^{I\times S}$ such that $V^+_{\text{xa}}(\mathbf{X}) > V^+_{\text{xp}}(\mathbf{X})$, Eq. (3) implies that

$$\begin{aligned}\Phi(V^+_{\text{xa}}(\mathbf{X}), V^+_{\text{xp}}(\mathbf{X})) &= \Phi(V^+_{\text{xa}}(\mathbf{X}) - V^+_{\text{xp}}(\mathbf{X}), 0) + V^+_{\text{xp}}(\mathbf{X}) \\ &= \frac{V^+_{\text{xa}}(\mathbf{X}) - V^+_{\text{xp}}(\mathbf{X})}{c-d}\Phi(c-d, 0) + V^+_{\text{xp}}(\mathbf{X}).\end{aligned}$$

Defining $\alpha := \frac{\Phi(c-d,0)}{c-d}$, we have that for all $\mathbf{X} \in \mathbb{R}^{I\times S}$ such that $V^+_{\text{xa}}(\mathbf{X}) > V^+_{\text{xp}}(\mathbf{X})$,

$$U(\mathbf{X}) = \alpha V^+_{\text{xa}}(\mathbf{X}) + (1-\alpha)V^+_{\text{xp}}(\mathbf{X}). \quad (4)$$

Notice that α is uniquely defined since V^+_{xa} and V^+_{xp} are different functions. To show that $\alpha \in [0, 1]$, we take $\mathbf{X}, \mathbf{X}', \mathbf{Y}, \mathbf{Y}' \in \mathbb{R}^{I\times S}$ such that

$$\begin{aligned}&V_{\text{xa}}(\mathbf{X}) > V_{\text{xa}}(\mathbf{X}'), &&V_{\text{xp}}(\mathbf{X}) = V_{\text{xp}}(\mathbf{X}'),\\ &V_{\text{xa}}(\mathbf{Y}) = V_{\text{xa}}(\mathbf{Y}'), \quad \text{and} &&V_{\text{xp}}(\mathbf{Y}) > V_{\text{xp}}(\mathbf{Y}').\end{aligned}$$

Then, the second part of Axiom A8 implies that $U(\mathbf{X}) > U(\mathbf{X}')$, hence that $\alpha > 0$, and that $U(\mathbf{Y}) > U(\mathbf{Y}')$, hence that $\alpha < 1$. It remains to deal with those $\mathbf{X} \in \mathbb{R}^{I\times S}$ which satisfy $V^+_{\text{xa}}(\mathbf{X}) = V^+_{\text{xp}}(\mathbf{X})$. For this, we observe that if $V^+_{\text{xa}}(\mathbf{X}) = k$ and $V^+_{\text{xp}}(\mathbf{X}) = k$ for some $k \in \mathbb{R}$, then $V^+_{\text{xa}}(\mathbf{X}) = V^+_{\text{xa}}(k\mathbf{1}_{I\times S})$ and $V^+_{\text{xp}}(\mathbf{X}) = V^+_{\text{xp}}(k\mathbf{1}_{I\times S})$, so by the first part of Axiom 8, $U(\mathbf{X}) = U(k\mathbf{1}_{I\times S})$, hence $U(\mathbf{X}) = k$, and

$$U(\mathbf{X}) = V^+_{\text{xa}}(\mathbf{X}) = V^+_{\text{xa}}(\mathbf{X}),$$

as Eq. (4) prescribes. We conclude that Eq. (4) holds for all $\mathbf{X} \in \mathbb{R}^{I\times S}$. □

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