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RESEARCH ARTICLE

Propensity to consume and the optimality of Ramsey–Euler policies

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Abstract

In a general one-sector optimal stochastic growth model where the production technology may be globally unproductive or allow for unbounded growth, we outline readily verifiable sufficient conditions for optimality that do not require checking the transversality condition. An interior policy function satisfying the Ramsey–Euler condition may not be optimal even if consumption and investment are continuous and increasing in output; our conditions for optimality require that the policy function must also satisfy a lower bound on the propensity to consume. For the case of production functions with multiplicative shocks, the consumption propensity needs to be bounded away from zero; a similar condition is sufficient for more general production functions if the utility function belongs to a restricted class.

Keywords Stochastic growth · Optimal economic growth · Uncertainty · Unbounded growth · Unproductive technology · Transversality condition · Optimality conditions · Euler equation

JEL Classification C6 · D9 · O41

T. Mitra: He passed away on February 3, 2019.

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1 Introduction

The one sector model of optimal economic growth under uncertainty (Levhari and Srinivasan 1969; Brock and Mirman 1972) has been widely used by economists to examine problems of capital accumulation in stochastic environments including macroeconomic growth under technology or productivity shocks and resource management under environmental uncertainty. Variations of the model have also been used to study business cycles.

In this model, a representative agent allocates the currently available output (of a single good) between investment and consumption where consumption generates immediate utility while investment generates next period's output according to a production function that is subject to exogenous production shocks. In the standard version of the model, the exogenous shocks are independent and identically distributed over time. The agent maximizes expected discounted sum of utility from consumption where the discount factor, the utility function and the production function are invariant over time. In such a stationary framework, the intertemporal economic trade-offs faced by the agent are reflected in the *optimal* consumption policy function. Conditions for optimality play a very important role in understanding the nature of this optimal policy function. In a large class of applications where economists work with specific functional forms for utility and production functions, sufficient conditions for optimality help determine whether an explicitly specified policy function is actually optimal. Even when one cannot derive explicit solutions to the dynamic optimization problem, sufficient conditions for optimality are useful in showing that a certain implicitly defined ("candidate") function is optimal. Optimality conditions for the dynamic optimization problem underlying the one sector stochastic growth model can also be useful in dynamic games of capital accumulation such as dynamic games of common property renewable resource extraction[1].

In a convex framework (strictly concave utility, concave production function), the existing literature has used duality theory to derive a set of conditions that are both necessary and sufficient for a policy function to be optimal and, in fact, to be the unique optimal policy function. In particular, an interior policy function (i.e., one where both consumption and investment are always strictly positive when the current stock of output is strictly positive) is optimal if, and only if, it satisfies the Euler condition (called the Ramsey–Euler equation in this literature) and a transversality condition (Mirman and Zilcha 1975; Zilcha 1976, 1978).[2,3]

The Ramsey–Euler equation is a simple first order condition that captures the trade-off between consumption in any two consecutive time periods, and takes the form of a functional equation. We refer to an interior consumption policy function satisfying

1 See, for instance, Mitra and Sorger (2014).

2 Key contributions emphasizing the importance of the transversality condition in models of intertemporal resource allocation include Malinvaud (1953), Cass (1965), Shell (1969), Peleg and Ryder (1972) and Weitzman (1973).

3 That the Euler and transversality conditions are necessary and sufficient for optimality has been established for more general, convex dynamic optimization problems. See, among others, Stokey and Lucas (1989), Acemoglu (2009). Establishing the necessity of transversality condition for optimality in general has been more challenging; see, Kamihigashi (2001, 2003, 2005).

this Ramsey–Euler equation as a *Ramsey–Euler policy* and this paper contributes to a literature on systematic study of the optimality of such a policy.

Using the characterization results mentioned above, a Ramsey–Euler policy can be shown to be an optimal policy, if it satisfies a transversality condition. The transversality condition essentially requires that the expected present value of capital stocks (valued by a shadow price equal to the discounted marginal utility of current consumption) converges to zero in the long run. It is an asymptotic condition on the entire stochastic process generated by the policy function.[4] Verifying the transversality condition can be a non-trivial task when the stochastic process of output and consumption can reach levels arbitrarily close to zero infinitely often (for instance, on sample paths involving runs of bad realizations of the production shock) and the marginal utility of consumption is infinitely large at zero[5]; it can also be somewhat challenging if output and investment can be arbitrarily large with positive probability.

Mitra and Roy (2017) develop an alternative sufficient condition for optimality of a Ramsey–Euler policy; they show that a Ramsey–Euler policy function is optimal if it is continuous or co-monotone (i.e., consumption and investment are both non-decreasing in current output). They derive their results under two restrictions on the production technology. First, the technology is assumed to be productive for investment levels close to zero even under the worst realization of the random shock i.e., marginal productivity at zero is always greater than one. Second, the technology is assumed to exhibit bounded growth i.e., there is a maximum sustainable capital stock beyond which the technology is unproductive even for the best realization of the random shock. A natural question that arises is whether their result extends to more general environments where the production technology may be unproductive at all levels of investment or alternatively, productive at all levels of investment (thus allowing for unbounded expansion of output and consumption).

In their paper, Mitra and Roy (2017) provide an example to show why their result may not hold if the technology is globally productive; in an economy with a deterministic linear production function where the average productivity is always greater than one, they show that there is a continuous and co-monotone Ramsey–Euler policy function that is not optimal.

In this paper, we provide an example of an economy with a deterministic linear production function that is globally *unproductive* i.e., the average productivity is always less than one; we explicitly derive a non-linear solution to the Ramsey–Euler functional equation that is not optimal; this non-optimal consumption function is smooth, strictly convex and strictly increasing in output; also, investment is strictly increasing in output. Together, these two examples show that a continuous and co-monotone Ramsey–Euler policy function need not be optimal once we allow for production functions that do not exhibit bounded growth or are not productive near zero.

[4] For certain versions of our model, in checking for optimality of a Ramsey–Euler path (from an arbitrary initial stock) the transversality condition may be replaced by an infinite number of "period by period" conditions; see, Brock and Majumdar (1988), Dasgupta and Mitra (1988) and Nyarko (1988). Like the transversality condition, these period-by-period conditions taken together involve the entire stochastic process of consumption and capital and establishing optimality by showing that all of them hold can be difficult to implement.

[5] Mitra and Roy (2017) illustrate this difficulty through examples.

In many macroeconomic applications[6], the structure of technology shocks is such that the net return on investment is always negative under adverse realizations of the shock. On the other hand, understanding sustained or long run economic growth requires analysis of models with production technologies that are productive at all levels of investment.[7],[8] It is important to understand the nature of conditions on a Ramsey–Euler policy function that can ensure it is optimal when we allow for such production technologies. The key contribution of this paper is that we develop easily verifiable conditions for optimality of Ramsey–Euler policy functions in a more general version of the one sector growth model than that considered in Mitra and Roy (2017); in particular, we do not require the production function to exhibit bounded growth or to be productive near zero.

In each of the two examples mentioned above, the non-optimal Ramsey–Euler policy function is such that the propensity to consume can be arbitrarily small. We show that some restriction on the behavior of the propensity to consume can play an important role in ensuring optimality of a Ramsey–Euler policy function.

For production functions where the random shock enters multiplicatively, we show that a Ramsey–Euler policy function is optimal if (i) it is either continuous or co-monotone, and (ii) the propensity to consume is bounded away from zero; condition (ii) is required to hold only if the worst case production function is unproductive near zero or if the best case production allows for unbounded expansion. This result is a generalization of the optimality conditions in Mitra and Roy (2017). Note that production functions with multiplicative shock structure are widely used in macroeconomics and resource economics; further, the deterministic production function can be seen as a special case of multiplicative shock.

For more general production functions where the random shock is not necessarily multiplicative, we show that a Ramsey–Euler policy is optimal if (i) holds and the propensity to consume is bounded below by a generalized lower bound that depends on the extent of variation (due to random shock) in the elasticity of the production function.

We also show that if the utility function belongs to a special family (that includes, for instance, all bounded utility functions), then conditions (i) and (ii) mentioned above continue to be sufficient for optimality of a Ramsey–Euler policy even if the random shock is not multiplicative.

It is well known that in our model, the optimal consumption policy function is unique, continuous, and both the optimal consumption and investment are non-decreasing (in fact, strictly increasing) in current output; further, if the optimal policy is interior it must satisfy the Ramsey–Euler condition.[9] This paper shows that these

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[6] Similarly, renewable resources stocks may not be able to regenerate and grow (regardless of the stock size and the amount of harvesting) if environmental conditions are highly adverse.

[7] For analysis of exogenous growth models where the technology may be “productive at infinity” see, among many others, Gale and Sutherland (1968), Levhari and Srinivasan (1969), Majumdar and Zilcha (1987), Jones and Manuelli (1990), de Hek (1999) and de Hek and Roy (2001).

[8] See, for instance, Jones et al. (2005). In many applcations, the shocks enter the production function multiplicatively and are assumed to have a lognormal distribution. Our framework however assumes that the shocks are bounded.

[9] See, for instance, Kamihigashi (2007).

global properties of the policy function that are necessary for optimality may not be sufficient for optimality once we allow for production technologies that are potentially unproductive at zero or productive at infinity. Optimality is however ensured if one can, in addition, verify a condition on the limiting behavior of the propensity to consume (though this may not be necessary for optimality); taken together, they replace the transversality condition in the set of sufficient conditions for the optimality.

Continuity or monotonicity of the Ramsey–Euler policy can be easily verified; it is also easy to verify whether our condition on the propensity to consume is satisfied (for instance, whether it is bounded away from zero) for a candidate consumption function. Our result allows us to immediately verify optimality of explicit solutions to the Euler equation in certain applications with specific functional forms for the utility and production functions where the policy function is linear so that propensity to consume is constant.[10] Linearity is however an exception, rather than the rule. As new examples are developed in the future with non-linear Ramsey–Euler consumption functions (as in the examples outlined in this paper), our result will continue to be useful as a way to verify optimality.[11] Our main result can also be a useful theoretical tool in proving optimality of an implicitly defined policy function.

The paper is organized as follows. Section 2 outlines the model, the assumptions and some definitions. Section 3 outlines some benchmark results for the classical version of the model with bounded growth technology that is productive near zero. Section 4 outlines two important examples to illustrate the fact that a continuous and co-monotone Ramsey–Euler consumption function may not be optimal and that the main result in Mitra and Roy (2017) may not hold for a more general class of production technologies. Section 5 contains the main results of the paper on sufficient conditions for optimality of a Ramsey–Euler policy. Section 6 concludes. Section 7 is the appendix and contains proofs of all results and some details of the example in Sect. 4.1 (including an explanation of the method by which we arrived at an explicit nonlinear solution to the Ramsey–Euler functional equation).

2 The model

We consider an infinite horizon one-good representative agent economy. Let $\mathbb{R}_+$ ($\mathbb{R}_{++}$) denote the set of all non-negative (strictly positive) real numbers. Time is discrete and is indexed by $t = 0, 1, 2, ...$ At each date $t \geq 0$, the representative agent observes the current stock of output $y_t \in \mathbb{R}_+$ and chooses the level of current investment x_t, and

[10] See, for instance, Benhabib and Rustichini (1994).

[11] Our alternative sufficient condition for optimality of a Ramsey–Euler policy is based on the duality approach to the characterization of optimality. A different approach, based on dynamic programming, involves guessing the value function from the Ramsey–Euler condition and verifying that this "candidate" value function satisfies the the Bellman equation (see, for instance, Stokey and Lucas 1989). This approach is useful if the solution to the Bellman equation is unique (for instance, if the utility function is bounded in the stochastic growth model). Recent advances have extended the applicability of this approach to unbounded utility functions; see, among others, Rincón-Zapatero and Rodriguez-Palmero (2003), Matkowski and Nowak (2011) and Kamihigashi (2014).

the current consumption level c_t, such that

$$c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t.$$

This generates y_{t+1}, the output stock next period through the relation

$$y_{t+1} = f(x_t, r_{t+1})$$

where $f(x, r)$ is the production function and r_{t+1} is a random production shock realized at the beginning of period $(t + 1)$.

2.1 Production

We now describe aspects of the above mentioned production process formally. We begin by specifying the nature of the exogenous shocks to production as follows

(R.1) The sequence of random shocks $\{r_t\}_{t=1}^{\infty}$ is assumed to be an independent and identically distributed random process defined on a probability space $(\Omega, \mathcal{F}, P)$, where the marginal distribution is denoted by μ. The support of this distribution function is a non-empty compact set $A \subset \mathbb{R}$. The distribution function corresponding to μ is denoted by F.

The *production function* is a map f from $\mathbb{R}_+ \times A$ to $\mathbb{R}_+$. We impose the following assumptions[12] on the production function f :

(T.1) Given any $r \in A$, $f(., r)$ is assumed to be continuous, strictly increasing and concave on $\mathbb{R}_+$, with $f(0, r) = 0$, and differentiable on $\mathbb{R}_{++}$, with $f'(\cdot, r) > 0$ on $\mathbb{R}_{++}$. Further, for any $x \geq 0$, $f(x, .) : A \to \mathbb{R}_+$, is a (Borel) measurable function.

Define the lower envelope production function $\underline{f}(x) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\underline{f}(x) = \inf_{r \in A} f(x, r).$$

It is easy to check that $\underline{f}(x)$ is non-decreasing on $\mathbb{R}_+$ and $\underline{f}(0) = 0$. Further, $\underline{f}(x)$ is concave on $\mathbb{R}_+$. It follows that the "worst case" average productivity of investment $[\underline{f}(x)/x]$ is non-increasing in x on $\mathbb{R}_{++}$. The upper envelope production function $\overline{f}(x)$ is defined on $\mathbb{R}_+$ by:

$$\overline{f}(x) = \sup_{r \in A} f(x, r)$$

We assume that:

(T.2)

$$\underline{f}(x) > 0, \overline{f}(x) < \infty \text{ for all } x > 0.$$

12 Note that we do not require the production function to be monotonic or continuous in the realization of the production shocks.

Given an initial stock $y \geq 0$, a stochastic process $\{y_t(y,\omega), c_t(y,\omega), x_t(y,\omega)\}$ is *feasible* from y if it satisfies $y_0 = y$, and:

(i) $c_t(y,\omega) \geq 0, x_t(y,\omega) \geq 0$ for $t \geq 0$
(ii) $c_t(y,\omega) + x_t(y,\omega) \leq y_t(y,\omega),\ \ y_{t+1}(y,\omega) = f(x_t(y,\omega), r_{t+1}(\omega))$ for $t \geq 0$

and if for each $t \geq 0$ $\{c_t(y,\omega), x_t(y,\omega)\}$ are $\mathcal{F}_t$ adapted where $\mathcal{F}_t$ is the (sub) σ -field generated by partial history from periods 0 through t.[13]

2.2 Preferences

Consumption in each period generates an immediate return according to a utility function, $u : \mathbb{R}_{++} \to \mathbb{R}$. The following assumption is imposed on the utility function:

(U.1) u is continuously differentiable, strictly increasing and strictly concave on $\mathbb{R}_{++}$ with $u' > 0$ on $\mathbb{R}_{++}$.

We define

$$u(0) \equiv \lim_{c \downarrow 0} u(c),$$

where the limit is allowed to be finite or $-\infty$.

The agent discounts future utility using a time invariant discount factor denoted by $\rho \in (0,1)$.

2.3 The optimization problem

Given initial stock $y \geq 0$, the representative agent's objective is to maximize the expected value of the discounted sum of utilities from consumption:

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t)\right]$$

subject to feasibility constraints.

Given $y \geq 0$, define the stochastic process of consumption $\{c_t^M(y,\omega)\}$ by: $c_0^M(y,\omega) = y, c_{t+1}^M(y,\omega) = f(c_t^M(y,\omega), r_{t+1}(\omega))$ for all $t \geq 0$. Then, for every ω and t, $c_t^M(y,\omega)$ is an upper bound on feasible consumption in period t. We assume that:

(**D.1**) For all $y \geq 0$,

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t^M(y,\omega))_+\right] < \infty$$

where $u(c)_+ = \max\{u(c), 0\}$.

[13] We skip formal definitions of sigma fields and sub sigma fields as these constructs are standard in the theory of stochastic processes.

Assumption (**D.1**) ensures that for any feasible stochastic process $\{y_t(y, \omega), c_t(y, \omega), x_t(y, \omega)\}$ from $y \geq 0$, the objective of the representative agent

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t(y, \omega))\right]$$

is well defined though it may equal $-\infty$, and that (see, Kamihigashi 2007)

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t(y, \omega))\right] = \sum_{t=0}^{\infty} \rho^t E[u(c_t(y, \omega))] \tag{1}$$

Note that (**D.1**) is always satisfied if either u is bounded above or alternatively, if $\limsup_{x\to\infty}[\overline{f}(x)/x] < 1$ i.e., the technology exhibits bounded growth.

Given initial stock $\overline{y} \geq 0$, a feasible stochastic process $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ is *optimal* from $\overline{y}$ if for every feasible stochastic process $\{y'_t(\overline{y}, \omega), c'_t(\overline{y}, \omega), x'_t(\overline{y}, \omega)\}$ from $\overline{y}$,

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t(\overline{y}, \omega))\right] \geq E\left[\sum_{t=0}^{\infty} \rho^t u(c'_t(\overline{y}, \omega))\right]$$

2.4 The optimal consumption function

A *consumption (policy) function,* is a function $c : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying:

$$0 \leq c(y) \leq y \quad \text{for all } y \in \mathbb{R}_+$$

Note that this implies $c(0) = 0$. Associated with a consumption function $c(\cdot)$, is an *investment (policy) function* $x : \mathbb{R}_+ \to \mathbb{R}$, defined by

$$x(y) = y - c(y) \text{ for all } y \in \mathbb{R}_+$$

Thus, the investment function $x(\cdot)$ satisfies:

$$0 \leq x(y) \leq y \quad \text{for all } y \in \mathbb{R}_+$$

A feasible stochastic process $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ is said to be *generated by* a consumption function $c(y)$ from initial stock $\overline{y} \in \mathbb{R}_+$ if for all $\omega \in \Omega$

$$y_0(\overline{y}, \omega) = \overline{y};\;\; y_{t+1}(\overline{y}, \omega) = f(y_t(\overline{y}, \omega) - c(y_t(\overline{y}, \omega)), r_{t+1}(\omega)) \text{ for } t \geq 0;$$
$$c_t(\overline{y}, \omega) = c(y_t(\overline{y}, \omega)),\;\; x_t(\overline{y}, \omega) = x(y_t(\overline{y}, \omega)) = y_t(\overline{y}, \omega) - c(y_t(\overline{y}, \omega)) \text{ for } t \geq 0.$$

A consumption function $c(y)$ is called an *optimal* consumption function if for every $\overline{y} \in \mathbb{R}_+$, the feasible stochastic process $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ generated by $c(y)$ is optimal from initial stock $\overline{y}$.

A consumption function $c(y)$ is said to be *interior* (or, to satisfy *interiority*) if

$$0 < c(y) < y \text{ for all } y > 0.$$

A consumption function $c(y)$ is said to be *co-monotone* if $c(y)$ and $x(y) = y - c(y)$ are non-decreasing in y on $\mathbb{R}_+$.

2.5 Ramsey–Euler and transversality conditions

An *interior* consumption function $c(y)$ is said to satisfy the Ramsey–Euler condition if

$$u'(c(y)) = \rho \int_A u'(c(f(y - c(y), r))) f'(y - c(y), r) dF(r) \text{ for all } y > 0 \quad \text{(RE)}$$

In this case we refer to the consumption function $c(y)$ as a *Ramsey–Euler* (consumption) *policy*.

For any *interior* consumption function $c(y)$, the feasible stochastic process $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ generated by the consumption function $c(y)$ from any initial stock $\overline{y} > 0$ satisfies:

$$y_t(\overline{y}, \omega) > 0, c_t(\overline{y}, \omega) > 0, x_t(\overline{y}, \omega) > 0 \text{ for all } t \geq 0 \text{ and for all } \omega \in \Omega.$$

An interior consumption function $c(y)$ is said to satisfy the *transversality condition* if for all $\overline{y} > 0$:

$$\lim_{t \to \infty} E\{\rho^t u'(c_t(\overline{y}, \omega)) x_t(\overline{y}, \omega)\} = 0 \quad \text{(TC)}$$

where $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ is the feasible stochastic process generated by the consumption function $c(y)$ from initial stock $\overline{y}$.

3 Optimality of Ramsey–Euler policy: benchmark

It is known that if a consumption function is interior, satisfies the Ramsey–Euler condition (RE) and the transversality condition (TC), then it is an optimal consumption function; in other words, a Ramsey–Euler policy is optimal if it satisfies the transversality condition (TC). This was established by Mirman and Zilcha (1975) in the "bounded growth" case; it has since been established in more general settings. A specific version of this sufficiency result (for the model outlined in Sect. 2) is reported in this paper as Lemma 1 (in the Appendix) and is used in the proof of our main results. It should be mentioned that the transversality condition (TC) has also been shown to be necessary for optimality of a Ramsey–Euler policy.

As mentioned earlier, the transversality condition essentially involves the entire stochastic process of consumption and capital generated by a policy function; it cannot be verified immediately by inspecting the policy function. Depending on the specific utility and production functions, verification of the transversality condition can require some work (see, Mitra and Roy 2017 for some more discussion of this issue). It is therefore interesting to explore whether there are alternative conditions for optimality of a policy function that are easier to verify than the transversality condition; in other words, can the transversality condition be replaced by some fairly apparent properties of the policy function.

Mitra and Roy (2017) establish alternative conditions for optimality of a policy function for the "canonical" version of the one sector optimal stochastic growth model where the technology exhibits bounded growth (i.e., there is a maximum sustainable capital stock) and is productive (for sure) near zero. The main result in that paper is stated below for ease of comparison:

Proposition 1 (Theorem 1, Mitra and Roy 2017) *Assume the following:*

$(E.1)$ *There is* $K > 0$ *such that* $\left[\overline{f}(x)/x\right] < 1$ *for all* $x > K$
$(E.2)$ $\lim_{x \downarrow 0}[\underline{f}(x)/x] > 1$

Suppose that $c(\cdot)$ *is an interior consumption function. Then the following statements are equivalent:*

(a) $c(y)$ *is continuous and satisfies the Ramsey–Euler condition (RE)*
(b) $c(y)$ *and* $y - c(y)$ *are nondecreasing on* $\mathbb{R}_+$ *(i.e.,* $c(y)$ *is co-monotone) and* $c(y)$ *satisfies the Ramsey–Euler condition (RE)*
(c) $c(y)$ *and* $y-c(y)$ *are strictly increasing on* $\mathbb{R}_+$ *and* $c(y)$ *satisfies the Ramsey–Euler condition (RE)*
(d) $c(y)$ *is optimal.*

The key implication of this result is that for the canonical version of the model, a Ramsey–Euler policy function is optimal as long as it is continuous (or alternatively, co-monotone).

The proof of this result in Mitra and Roy (2017) uses the end-point conditions $(E.1)$ and $(E.2)$ on the production technology. This naturally leads to the question whether their result extends to economic environments where either $(E.1)$ or $(E.2)$ does not hold i.e., the production technology is not necessarily productive near zero for all realizations of the shock or alternatively, allows for unbounded expansion of capital and consumption (or both). In the next section, we outline two examples to show that their result may not hold if the production technology does *not* satisfy either condition $(E.1)$ or condition $(E.2)$.

4 Non-optimal continuous and co-monotone Ramsey–Euler policy: two examples

In this section, we outline two examples of economies with *deterministic* production technologies that do not satisfy the endpoint conditions assumed in Proposition 1. In first example, the production function is unproductive at all levels of investment and therefore violates condition $(E.2)$. In the second example, the production function is productive at all levels of investment i.e., allows for unbounded expansion of capital and consumption, and therefore violates condition $(E.1)$. For each example, we explicitly specify an interior consumption function that solves the Ramsey–Euler equation (RE), is continuous and co-monotone, but is not optimal.

4.1 Example 1: unproductive technology

In this example, the production function is deterministic, linear and unproductive at all positive input levels. It is given by

$$f(x) = \frac{x}{2} \quad \text{for all } x \geq 0 \tag{2}$$

We specify the utility function u to be:

$$u(c) = \begin{cases} \ln c & \text{for all } c > 0 \\ -\infty & \text{for } c = 0 \end{cases} \tag{3}$$

Finally, let the discount factor $\rho = \frac{1}{2}$. Then, all of our assumptions in Sect. 2 are satisfied.

The Ramsey–Euler functional equation (RE) for this example reduces to:

$$c\left(\frac{y - c(y)}{2}\right) = \frac{c(y)}{4} \quad \text{for all } y > 0 \tag{4}$$

It is easy to see that the consumption function:

$$c^*(y) = \frac{y}{2} \quad \text{for all } y \geq 0$$

solves the Ramsey–Euler function equation (4) and the path $\{c_t^*, x_t^*, y_t^*\}$ generated by this policy function satisfies the transversality condition (TC).[14] Therefore, (using for instance, Lemma 1 in the appendix), $c^*(y)$ is in fact the optimal consumption policy function. Note that strict concavity of the utility and production functions implies that the optimal consumption function is unique.

We now show that there is a continuous and co-monotone solution to the Ramsey–Euler functional equation (4) that is different from $c^*(y)$ and is therefore, not optimal. Consider the function $\phi(y)$ defined by:

$$\phi(y) = \frac{(1 + 4y) - (1 + 8y)^{\frac{1}{2}}}{8} \quad \text{for all } y \geq 0 \tag{5}$$

Note that $\phi(0) = 0$, and since $(1 + 8y)^{\frac{1}{2}} < (1 + 8y + 16y^2)^{\frac{1}{2}} = (1 + 4y)$ for all $y > 0$, we have $\phi(y) > 0$ for all $y > 0$. Further, since $(1 + 8y)^{\frac{1}{2}} > 1$ for all $y > 0$, we have:

$$\phi(y) < (4y/8) = (y/2) = c^*(y) \text{ for all } y > 0 \tag{6}$$

[14] $\rho^t u'(c_t^*))x_t^* = (1/2)^t \to 0$ as $t \to \infty$.

Thus, ϕ is an interior consumption function. Clearly, ϕ is continuous and differentiable on $\mathbb{R}_+$. By differentiating (5), we see that:

$$8\phi'(y) = 4 - \frac{4}{(1+8y)^{\frac{1}{2}}} > 0 \text{ for all } y > 0$$

so that $\phi'(y) > 0$ for $y > 0$, and ϕ is strictly increasing on $\mathbb{R}_+$. Further, $\phi'(y) < \frac{1}{2}$ for all $y > 0$. Thus, the interior consumption function $\phi(y)$ is continuous and co-monotone on $\mathbb{R}_+$. We now claim that $c(y) = \phi(y)$ is a solution to the Ramsey–Euler functional equation (4). To see this, define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$\psi(c) = 2c + c^{\frac{1}{2}} \text{ for all } c \geq 0 \tag{7}$$

Note that $\psi(0) = 0$, and $\psi(c) > 0$ for all $c > 0$. In fact, $\psi(c) > 2c$ for all $c > 0$. Further, $\psi(c)$ is strictly increasing and strictly concave in c on $\mathbb{R}_+$. One can directly verify that the functions ψ and ϕ are inverses of each other i.e., $\psi(\phi(y)) = y$ for all $y \geq 0$ and $\phi(\psi(c)) = c$ for all $c \geq 0$; details are contained in Sect. 7.1 of the Appendix.

The difficulty in solving the functional equation (4) arises from the composition of the unknown function with itself on the left-hand side. To get around this difficulty, one writes down its conjugate functional equation:

$$g(c/4)) = (1/2)(g(c) - c) \text{ for } c \geq 0 \tag{8}$$

(8) can be rewritten as:

$$g(c) = c + 2g(c/4) \text{ for } c \geq 0 \tag{9}$$

We show that $g(c) = \psi(c)$, where ψ is defined by (7), solves (9) and that its inverse $\phi(y)$, defined in (5), is a solution to the Ramsey–Euler functional equation (4). These results are explicitly established in Sect. 7.1 of the Appendix.

One interesting feature of the Ramsey–Euler consumption function $\phi(y)$ in the above example is that the propensity to consume $[\phi(y)/y] \to 0$ as $y \to 0$. We will see that this is a possible source of non-optimality of $\phi(y)$.

To the best of our knowledge, this is the first explicit example of a smooth, nonlinear and strictly increasing consumption function that solves the Ramsey–Euler equation in the canonical stochastic growth model. Even though it is not an optimal consumption function, it may be useful for researchers in the field to understand how one would "guess at" a solution like (5) to the Ramsey–Euler functional equation. Section 7.2 in the appendix explains this in details for a somewhat more general set of parameters.

4.2 Example 2: unbounded growth technology

We now outline an example of an economy where the production technology allows for unbounded expansion of consumption and output i.e., the end point condition $(E.1)$ in Proposition 1 does not hold. In this economy, there is a non-optimal Ramsey–Euler consumption function that is continuous and co-monotone. The example is contained in Mitra and Roy (2017: Example 3, Sect. 5); key aspects are reproduced below for ease of reference. Define the utility function u to be:

$$u(c) = \frac{\sqrt{c}}{1+\sqrt{c}} \quad \text{for all } c \geq 0$$

Then, u satisfies $(U.1)$. The production technology is deterministic and is given by

$$f(x) = 2x$$

which satisfies (**T.1**). Set $\rho = (1/2)$. Consider the consumption function defined by:

$$c(y) = \begin{cases} (1/2)y & \text{for } 0 \leq y \leq 2 \\ 1 & \text{for } y > 2 \end{cases}$$

Observe that $c(y)$ is interior and continuous; further, $c(y)$ and $y - c(y)$ are non-decreasing in y. For $0 < y \leq 2$, we have $c(y) = (1/2)y$, and $f(y - c(y)) = 2(y - (1/2)y) = y$, so that $c(f(y - c(y))) = (1/2)y = c(y)$. Thus

$$\rho u'(c\{f(y - c(y))\}) f'(y - c(y)) = \frac{1}{2} u'((1/2)y)2 = u'(c(y))$$

verifying (RE) for $y \in (0, 2]$. For $y > 2$, we have $2(y - 1) = 2y - 2 > 2$, and so $c\{f(y - c(y))\} = c\{2(y - 1)\} = 1$. Thus,

$$\begin{aligned} \rho u'(c\{f(y - c(y))\}) f'(y - c(y)) &= (1/2)u'(c\{f(y - c(y))\})2 \\ &= u'(1) = u'(c(y)) \end{aligned}$$

verifying (RE) for $y > 2$. Finally, consider a different consumption function $\gamma(\cdot)$ defined by:

$$\gamma(y) = (1/2)y \quad \text{for all } y \geq 0$$

Starting from $y = 4$, the consumption function $\gamma(\cdot)$ generates a path $(\widetilde{y}_t, \widetilde{c}_t, \widetilde{x}_t)$ where consumption $\widetilde{c}_t = 2$ for all $t \geq 0$. On the other hand, the path (y_t, c_t, x_t) starting from $y = 4$, generated by the consumption function $c(\cdot)$, has $y_t \geq 4$ for all $t \geq 0$ and so $c_t = 1$ for all $t \geq 0$, so that the discounted sum of utilities along the path (y_t, c_t, x_t) is strictly smaller than along the path $(\widetilde{y}_t, \widetilde{c}_t, \widetilde{x}_t)$. Thus, $c(\cdot)$ is *not* an optimal consumption function. This concludes the example.

Observe that somewhat similarly to Example 1, an interesting feature of the non-optimal Ramsey–Euler consumption function $c(y)$ in Example 2 is that the propensity to consume $[c(y)/y] \to 0$ as $y \to \infty$.

5 Optimality of Ramsey–Euler policy: sufficient conditions

In this section, we outline properties of a Ramsey–Euler consumption function that are sufficient to ensure that it is optimal even if the production technology is unproductive or allows for unbounded expansion of capital and consumption.

Recall that $\overline{f}, \underline{f}$ are the upper and lower envelopes of the production function defined in Sect. 2; they correspond to "best" and "worst" possible realizations of the random shock.

Define $\overline{K} \geq 0$ by:

$$\overline{K} = \sup\{x \geq 0 : \overline{f}(x) \geq x\}$$

$\overline{K} = \infty$ if the production technology allows for unbounded growth i.e., $\overline{f}(x) > x$ for all $x > 0$; further, $\overline{K} = 0$ if the technology is unproductive for sure and $\overline{f}(x) < x$ for all $x > 0$.

Define $\underline{K} \geq 0$ by:

$$\begin{aligned} \underline{K} &= \inf\{x > 0 : \underline{f}(x) \leq x\} \\ &= \infty, \text{ if } \underline{f}(x) > x \text{ for all } x > 0 \end{aligned}$$

$\underline{K} > 0$ if the "worst case" technology is productive near zero i.e., $\lim_{x \downarrow 0}[\underline{f}(x)/x] > 1$; $\underline{K} = 0$ if $\lim_{x \downarrow 0}[\underline{f}(x)/x] \leq 1$ so that[15] $\underline{f}(x) \leq x$ for all $x \geq 0$ i.e., the "worst case" technology is globally unproductive. Finally, note that $\underline{K} \leq \overline{K}$.

5.1 Main result

In this subsection, we consider the general model outlined in Sect. 2. For each $x > 0$, $r \in A$, let the inverse elasticity of the production function $\eta(x, r) > 1$ be defined by

$$\eta(x, r) = \frac{f(x, r)}{f'(x, r)x}$$

and let

$$\overline{\eta}(x) = \sup_{r \in A} \eta(x, r), \underline{\eta}(x) = \inf_{r \in A} \eta(x, r).$$

We now specify a technical assumption on the production function that is used in the next proposition and simplifies our analysis considerably:

(T.3) (i) There exists $a, b \in A$, $z_0, z_1 \in \mathbb{R}_{++}$ such that

$$\begin{aligned} \underline{f}(x) &= f(x, a) \text{ for all } x \in [0, z_1] \\ \overline{f}(x) &= f(x, b) \text{ for all } x \geq z_2 \end{aligned}$$

[15] Note that $\underline{f}$ is concave on $\mathbb{R}_+$.

(ii)

$$\tau_0 = \lim_{x \to 0} \sup \frac{\overline{\eta}(x)}{\eta(x, a)} < \infty$$
$$\tau_\infty = \lim_{x \to \infty} \sup \frac{\overline{\eta}(x)}{\eta(x, b)} < \infty$$

Under (**T.3**)(i), there is a specific "worst" case production shock a associated with the lower envelope of the production function for investment levels close to zero, and a specific "best" case production shock b associated with the upper envelope of the production function when investment is large enough. (**T.3**)(ii) essentially requires that the variation due to random shock in the elasticity of the production function at zero and infinity are bounded. If the random shock enters the production function multiplicatively, then $\eta(x, r)$ is independent of r so that $\tau_0 = \tau_\infty = 1$ and assumption (**T.3**)(ii) is satisfied. Note that (**T.3**) is also satisfied by some well known production functions that are not ordered by the shock such as $f(x, r) = x^r, r \in A \subset (0, 1)$.

Recall that $f'(0, r) = \lim_{x \to 0^+} f'(x, r)$ is the marginal productivity at zero for realization r of the random shock ($f'(0, r)$ may equal $+\infty$).

We are now ready to state our main proposition:

Proposition 2 *Assume* $(T.3)$. *Consider a Ramsey–Euler consumption function* $c(y)$ *that is either continuous or co-monotone on* $\mathbb{R}_+$. *Further, suppose that*

$$\lim_{y \to 0} \inf \frac{c(y)}{y} > 1 - \frac{1}{\tau_0}, \text{ if } f'(0, a) \leq \tau_0 \qquad (GP1)$$
$$\lim_{y \to \infty} \inf \frac{c(y)}{y} > 1 - \frac{1}{\tau_\infty}, \text{ if } \overline{K} = \infty \qquad (GP2)$$

Then, $c(y)$ *is optimal (and is, in fact, the unique optimal consumption function)*

Proposition 2 provides a set of verifiable properties of a Ramsey–Euler policy function that ensures it is optimal in environments that allow for unproductive technology as well as unbounded growth.

The proof of Proposition 2 is based on showing that the transversality condition (TC) holds i.e., $\rho^t E\{u'(c_t)x_t\} \to 0$ as $t \to \infty$ where $\{c_t\}$ and $\{x_t\}$ are the consumption and investment processes generated by the continuous (and co-monotone) Ramsey–Euler consumption function $c(y)$. This is trivial if the corresponding output process $\{y_t\}$ lies almost surely in a closed interval that is bounded away from zero. The difficulty arises when output and consumption are not bounded away from zero or infinity with positive probability. Our proof is based on using the fact that $x_t \leq y_t$ with probability one and showing that each term of the sequence $\{\rho^t E(u'(c_t)y_t)\}_{t=0}^{\infty}$ is a *contraction* of its previous term. This is different from the proof of optimality of Ramsey–Euler policy in Mitra and Roy (2017) where the transversality condition is shown to hold without demonstrating such a contraction property; the arguments in that proof cannot be easily extended to production functions that are unproductive near zero or productive at infinity.

The sufficient conditions for optimality in Mitra and Roy (2017) impose no restriction on the propensity to consume. The two examples in the previous section indicate that some restrictions on the propensity to consume are needed for a Ramsey–Euler policy function to be optimal when the technology is unproductive or allows for unbounded growth. Conditions (GP1) and (GP2) impose lower bounds on the propensity to consume; the bounds depend on the extent of variation in the elasticity of the production function due to random shocks. These are sufficient conditions; we are unable to determine whether they are necessary for optimality in such technological environments.

It is worth noting that if the production function satisfies bounded growth i.e., $\overline{K} < \infty$, then condition (GP2) no longer applies. However, condition (GP1) may continue to apply even if the production technology is productive near zero i.e., $\underline{K} > 0$. Thus, the sufficient conditions in Proposition 2 are potentially stronger than and do not reduce to the optimality conditions in Mitra and Roy (2017) for production functions that satisfy the assumptions in that paper (or alternatively, Proposition 1 under assumptions E.1 and E.2).

Proposition 2 also yields the following simpler result:

Corollary 1 *Consider a Ramsey–Euler consumption function $c(y)$ that is continuous or co-monotone on $\mathbb{R}_+$. Further, suppose that*

$$\tau = \sup_{x>0} \frac{\overline{\eta}(x)}{\underline{\eta}(x)} < \infty \text{ and } \inf_{y>0} \frac{c(y)}{y} > 1 - \frac{1}{\tau}.$$

Then, $c(y)$ is optimal and in fact, is the unique optimal consumption function.

5.2 Multiplicative shock

In this subsection, we consider production functions where the random shock is multiplicative. Such production functions are widely used in the literature; further, both examples in the previous section deal with deterministic production functions that can be viewed as special cases of multiplicative shock. In particular, for this subsection we assume:

$$f(x, r) = q(r)h(x), r \in A, x \geq 0 \tag{10}$$

Assumptions **(T.1)**, (**T.2**) and (**T.3**) on $f(x, r)$ hold under the following restrictions on the function h and q:

(M.1) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, concave and strictly increasing, $h(0) = 0$ and h is differentiable on $\mathbb{R}_{++}$, $h'(x) > 0$ for all $x > 0$ and $h'(0) = \lim_{x\to 0} h'(x) \in \mathbb{R}_+ \cup \{+\infty\}$ satisfies

$$h'(0) > 0$$

(M.2) $q : A \to \mathbb{R}_{++}$ is Borel-measurable and there exists $a, b \in A$ such that

$$q(a) \leq q(r) \leq q(b) \text{ for all } r \in A.$$

Once again, a and b are respectively the worst and best shocks. Note that if $q(r) = 1$ for all r, we have a deterministic production function where $f(x, r) = h(x)$.

It is easy to check that

$$\eta(x, r) = \frac{f(x, r)}{f'(x, r)x} = \frac{h(x)}{h'(x)x}$$

is independent of r so that

$$\tau_0 = \tau_\infty = 1.$$

Also, observe that $\underline{K} = 0$ if, and only if,

$$q(a)h'(0) \le 1 = \tau_0.$$

Proposition 2 therefore immediately yields:

Corollary 2 *Consider the class of production functions where the random shock is multiplicative and in particular,* (10) *holds under restrictions (M.1) and (M.2). Let* $c(y)$ *be a Ramsey–Euler consumption function that is either continuous or co-monotone on* $\mathbb{R}_+$*. Further, suppose that the propensity to consume* $(c(y)/y)$ *satisfies:*

$$\liminf_{y \to 0} \frac{c(y)}{y} > 0, \text{ if } \underline{K} = 0 \qquad (C.1)$$

$$\liminf_{y \to \infty} \frac{c(y)}{y} > 0, \text{ if } \overline{K} = \infty \qquad (C.2)$$

Then, $c(y)$ *is optimal.*

Note that under the assumptions of Mitra and Roy (2017), $\underline{K} > 0$ and $\overline{K} < \infty$ so that condition C.1 and C.2 in Proposition 2 do not apply and continuity or co-monotonicity of Ramsey–Euler policy is sufficient for optimality. In other words, for the multiplicative shock case, the sufficient conditions for optimality in Proposition 2 reduce to the optimality conditions in Mitra and Roy (2017) under their assumptions (or alternatively, to those in Proposition 1 under restrictions E.1 and E.2). Within the class of production functions with multiplicative shocks, Proposition 2 generalizes the sufficient conditions in Mitra and Roy (2017) to a larger set of production functions.

5.3 A special class of utility functions

In this subsection, we restrict attention to a class of utility functions while allowing the production function to have a fairly general structure. We show that for this class of utility functions, a Ramsey–Euler policy is optimal as long as the propensity to consume is bounded away from zero. In particular, we assume that in addition to (U.1) and (U.2), the utility function u satisfies:

(**U.3**) $u'(c)c$ is bounded on $\mathbb{R}_{++}$; in particular, there exists $M \in \mathbb{R}_{++}$ such that $u'(c)c < M$ for all $c > 0$.

Note that (**U.3**) is satisfied if u is bounded on $\mathbb{R}_+$.[16] Of course, $u'(c)c$ may be be bounded on $\mathbb{R}_{++}$ even if u is not bounded (for instance, $u(c) = \ln c$).

Proposition 3 *Assume (**U.3**). Consider a Ramsey–Euler consumption function $c(y)$ that is either continuous or co-monotone on $\mathbb{R}_+$; further,*

$$\inf_{y>0} \frac{c(y)}{y} > 0.$$

Then, $c(y)$ is optimal.

Note that unlike Propositions 2, 3 does not require any restriction like (**T.3**) on the production function; unlike Proposition 2, the proof of Proposition 3 is not based on a "contraction" argument.

5.4 Application

The sufficient conditions for optimality of Ramsey–Euler policy can be useful in verifying optimality of explicit solutions to the Ramsey–Euler functional equation for specific utility and production functions. For instance, consider a CES utility function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where $\sigma > 0, \sigma \neq 1$. The production function is given by

$$f(x, r) = rx$$

and $\{r_t\}$ is a sequence of i.i.d. random variable with distribution F with support $[a, b]$, $0 < a < b < \infty$. Note that the production technology may be unproductive (at least for certain realizations of the shock) as well as allow for unbounded expansion with positive probability. It is assumed that

$$k = [\rho E(r_t^{1-\sigma})]^{\frac{1}{\sigma}} < 1.$$

It is easy to check (and fairly well known) that the linear consumption function

$$\widetilde{c}(y) = (1-k)y$$

[16] Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a bounded, strictly concave and continuously differentiable function on $\mathbb{R}_+$ so that in particular $b \leq u(c) \leq B$ for all $c \geq 0$, for some $b, B \in \mathbb{R}$. Then

$$b - u(c) \leq u(0) - u(c) \leq u'(c)(-c) \text{ for all } c > 0$$

and so:

$$u'(c)c \leq u(c) - b \leq B - b \text{ for all } c > 0$$

i.e., the function $u'(c)c$ is bounded on $\mathbb{R}_{++}$.

solves the Ramsey–Euler equation for this problem. To assert that $\widetilde{c}(y)$ is optimal by verifying that the consumption and investment process generated by $\widetilde{c}(y)$ satisfies the transversality condition can take some work. On the other hand, by direct appealing to Corollary 2 and noting that the propensity to consume $c(y)/y = 1 - k$ is a strictly positive constant, we can immediately assert optimality of $\widetilde{c}(y)$.[17]

As new examples are developed in the future with non-linear Ramsey–Euler consumption functions our result will continue to be useful as a way to verify optimality.

6 Conclusion

The classical version of the one sector convex model of stochastic optimal growth assumes that the technology is productive near zero and exhibits bounded growth with probability one. In this framework, it has been shown that a policy function satisfying the Ramsey–Euler condition is optimal as long it is continuous or alternatively, if both consumption and investment are non-decreasing in current output. We outline two counterexamples to show that this result may not hold once the classical model is generalized to accommodate production functions that may be globally unproductive for bad realizations of the shock or allow for unbounded expansion of consumption and output. Our analysis indicates that a probable source of this non-optimality is low propensity to consume exhibited by the candidate policy function. We show that in our more general framework, a Ramsey–Euler policy function is optimal if in addition to continuity or monotonicity properties, we can also verify a condition on the propensity to consume. For production functions with multiplicative shock, our condition simply requires the propensity to consume be bounded away from zero; a generalization of this lower bound is shown to be sufficient for optimality in the case of non-multiplicative shock; weaker conditions are outlined for a restricted class of utility functions that includes bounded utility. The sufficient conditions for optimality outlined in this paper can be significantly easier to verify than the transversality condition.

Our analysis is a step forward in characterizing alternative conditions for optimality in a class of dynamic optimization models that includes the stochastic growth model. It will be useful to extend our analysis to stochastic growth models with "unbounded shocks" (see, Stachurski 2002; Nishimura and Stachurski 2005; Kamihigashi 2007) and irreversible investment (see, Olson 1989).

7 Appendix

7.1 Details of Example 1

(a) ψ and ϕ are inverses of each other:

[17] It is worth noting for this family of utility and production functions and following the steps outlined in Sect. 7.2 ,it is possible to derive non-linear policy functions that satisfy the Ramsey–Euler condition (similar to Example 1 in Sect. 4) but do not satisfy our sufficient conditions for optimality; these non-linear policies are in fact, not optimal.

We first show that $\psi(\phi(y)) = y$ for all $y \geq 0$. To this end, let us note that, by (5), for all $y \geq 0$,

$$\phi(y) = \frac{(1+4y) - (1+8y)^{\frac{1}{2}}}{8} = \left[\frac{(1+8y)^{\frac{1}{2}} - 1}{4}\right]^2$$

which yields:

$$(1+8y) = \{1 + 4[\phi(y)]^{\frac{1}{2}}\}^2 = 1 + 16\phi(y) + 8[\phi(y)]^{\frac{1}{2}}$$

and this implies:

$$y = 2\phi(y) + [\phi(y)]^{\frac{1}{2}} = \psi(\phi(y))$$

by using (7) and noting that $\phi(y) \geq 0$ for all $y \geq 0$. We now show that $\phi(\psi(c)) = c$ for all $c \geq 0$. We start with the following identity for all $c \geq 0$:

$$[1 + 4c^{\frac{1}{2}}]^2 = [1 + 8(2c + c^{\frac{1}{2}})]$$

which can be rewritten as:

$$2 + 8[2c + c^{\frac{1}{2}}] - 2[1 + 8(2c + c^{\frac{1}{2}})]^{\frac{1}{2}} = 16c \tag{11}$$

Using the definition of ψ in (7), we can rewrite (11) as:

$$[2 + 8\psi(c)] - 2[1 + 8\psi(c)]^{\frac{1}{2}} = 16c$$

so that:

$$\frac{[1 + 4\psi(c)] - [1 + 8\psi(c)]^{\frac{1}{2}}}{8} = c$$

and using the definition of ϕ in (5), we obtain:

$$\phi(\psi(c)) = \frac{[1 + 4\psi(c)] - [1 + 8\psi(c)]^{\frac{1}{2}}}{8} = c.$$

(b) $g(c) = \psi(c)$ is a solution to conjugate functional equation (9) and its inverse $\phi(y)$ is a solution to the Ramsey–Euler functional equation (4).

Let us write:

$$\begin{aligned} c + 2g(c/4) &= c + 2\psi(c/4) \\ &= c + 4(c/4) + 2(c/4)^{\frac{1}{2}} \\ &= 2c + c^{\frac{1}{2}} = \psi(c) = g(c) \end{aligned} \tag{12}$$

where we have used the definition of ψ in the second line and again in the last line of (12). Since we have just demonstrated that ψ is a solution to (9), we can write:

$$\psi(c/4) = \frac{\psi(c) - c}{2} \quad \text{for } c \geq 0 \tag{13}$$

Since $\phi(y) \geq 0$ for all $y \geq 0$, we can use (13) to write: $\psi(\phi(y)/4) = \frac{\psi(\phi(y))-\phi(y)}{2}$ for all $y \geq 0$ i.e.,

$$\psi(\phi(y)/4) = \frac{y - \phi(y)}{2} \quad \text{for all } y \geq 0 \tag{14}$$

Since $\phi(y) \leq y$ for all $y \geq 0$, we can apply the function ϕ to both sides of (14) to get: $\phi[\psi(\phi(y)/4)] = \phi\left(\frac{y-\phi(y)}{2}\right)$ for all $y \geq 0$ i.e.,

$$\frac{\phi(y)}{4} = \phi\left(\frac{y - \phi(y)}{2}\right) \quad \text{for all } y \geq 0$$

so that ϕ solves the Ramsey–Euler functional equation (4).

7.2 Non-linear solution to the Ramsey–Euler functional equation

In the interest of finding nonlinear solutions to the Ramsey–Euler functional equation, it is worthwhile to explain how one arrives at an explicit solution like (5) in Example 1. Consider the deterministic linear production function $f(x) = ax$ where $a \in (0, 1)$. Let the discount factor $\rho \in (0, 1)$. (In Example 1, we chose $a = \rho = 1/2$.) The utility function u is as specified in Example 1 i.e., $u(c) = \ln c$. The Ramsey–Euler functional equation is then given by:

$$c(a(y - c(y))) = \rho a c(y) \quad \text{for all } y > 0 \tag{15}$$

In what follows that our aim is to find *some* solution of (15) which is distinct from the well-known linear solution: $c(y) = (1 - \rho)y$ for all $y \geq 0$. We can confine our search to a more restrictive class of solutions [for example, all *continuous and strictly increasing* solutions to the functional equation (15)].

Step 1: [The Conjugate Functional Equation]

As already noted in the text, the difficulty in solving the functional equation (15) arises from the composition of the unknown function with itself on the left-hand side. To get around this difficulty, one writes down its *conjugate* functional equation:

$$g(\rho a c) = a(g(c) - c) \quad \text{for } c \geq 0 \tag{16}$$

If $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying $g(c) \geq c$ for all $c \geq 0$, which solves the conjugate functional equation (16), and there is a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(y) \leq y$ for all $y \geq 0$, and:

$$\left.\begin{array}{l} (i)\ h(g(c)) = c \quad \text{for all } c \geq 0 \\ (ii)\ g(h(y)) = y \ \text{for all } y \geq 0 \end{array}\right\} \tag{17}$$

then h solves the functional equation (15). To see this, note that since $h(y) \geq 0$ for all $y \geq 0$, we can use (16) to write: $g(\rho a h(y)) = a[g(h(y)) - h(y)]$ for all $y \geq 0$, and then using (17)(ii) we have:

$$g(\rho a h(y)) = a[y - h(y)] \quad \text{for all } y \geq 0 \tag{18}$$

Since $h(y) \leq y$ for all $y \geq 0$, we can apply the function h to both sides of (18) to get: $h[g(\rho a h(y))] = h\{a[y - h(y)]\}$ for all $y \geq 0$. Using (17)(i) we then obtain: $\rho a h(y) = h\{a[y - h(y)]\}$ for all $y \geq 0$ so that h solves the Ramsey–Euler functional equation (15).

Step 2: [Solving the Conjugate Functional Equation]

Step 1 above suggests that if we can find a solution g to the conjugate functional equation, and g has an inverse (note that g is increasing, in our context), then the inverse of g would solve the Ramsey–Euler functional equation. As we want the solution $h(y)$ to (15) (a consumption function) to be strictly increasing and continuous in y, we should be looking for strictly increasing and continuous solutions $g(c)$ to the conjugate functional equation. Unlike (15), the conjugate functional equation (16) does not involve the composition of g with itself on either side; it belongs to the class of iterated functional equations and can be solved through iterations. Pick any $c \geq 0$, and write, using (16) repeatedly,

$$\begin{aligned} g(c) &= c + (1/a)g(\rho a c) \\ &= c + (1/a)[\rho a c + (1/a)g(\rho^2 a^2 c)] \\ &= c + \rho c + (1/a^2)g(\rho^2 a^2 c) \end{aligned}$$

This iteration process can be continued to write for $t \geq 2$,

$$g(c) = [c + \rho c + \rho^2 c + \cdots + \rho^t c] + (1/a^{t+1})g(\rho^{t+1} a^{t+1} c) \tag{19}$$

Note that the term in square brackets in (19) converges to $c/(1-\rho)$ as $t \to \infty$. So, if the final term in (19) goes to zero as $t \to \infty$, we get $g(c) = c/(1-\rho)$ for $c \geq 0$, and we arrive at the well-known linear solution to the Ramsey–Euler functional equation: $c(y) = (1-\rho)y$ for all $y \geq 0$.

But, we don't really know whether the final term in (19) converges to zero as $t \to \infty$. This will depend, after all, on the behavior of g near zero, and g is the unknown function we are trying to find. It is perfectly legitimate to restrict our search for solutions to (16) to a narrow class. Let us, then, confine our search to those g for which the final term in (19) *does* have a limit as $t \to \infty$. The limit will itself be a function of c, and we denote it by $q(c)$; we are interested in functions g for which $q(c) \neq 0$. Then, using (19), and letting $t \to \infty$, we can write:

$$g(c) = [c/(1-\rho)] + q(c) \quad \text{for all } c \geq 0 \tag{20}$$

The question then arises: what properties must $q(c)$ satisfy in order for (20) to be a valid solution to the conjugate functional equation (16). Given (16) and (20), we must have:

$$(1/a)\{[\rho ac/(1-\rho)] + q(\rho ac)\} = (1/a)g(\rho ac) = (g(c) - c)$$
$$= [c/(1-\rho)] + q(c) - c \quad \text{for } c \geq 0$$

which can be rewritten as:

$$q(c) = [q(\rho ac)/a] + (1/a)\{[\rho ac/(1-\rho)] + c - [c/(1-\rho)] = [q(\rho ac)/a] \quad (21)$$

Recalling Step 1, we want the solution $g(c)$ to (16), given by (20), to be strictly increasing in c, so it is legitimate to confine our search to those functions $q(c)$ which are strictly increasing in c. This would mean that $q(c)$ would be differentiable almost everywhere on $\mathbb{R}_+$. So, we might as well confine our search to differentiable and strictly increasing functions $q(c)$.

Differentiating (21) with respect to c, we would get:

$$q'(c) = q'(\rho ac)\rho \quad (22)$$

Using (21) and (22), we obtain (when $q(c) \neq 0$),

$$\frac{q'(c)}{q(c)} = \frac{q'(\rho ac)\rho}{q(\rho ac)/a}$$

which yields:

$$\frac{q'(c)c}{q(c)} = \frac{q'(\rho ac)\rho ac}{q(\rho ac)} \quad \text{for all } c \text{ such that } q(c) \neq 0 \quad (23)$$

This suggests that we can further restrict our search for $q(c)$ to the iso-elastic class:

$$q(c) = c^{1-\theta} \quad \text{for all } c \geq 0 \quad (24)$$

where $0 < \theta < 1$ is to be appropriately chosen. That is, our proposed "candidate solution" g to the conjugate functional equation is:

$$g(c) = [c/(1-\rho)] + c^{1-\theta} \quad \text{for all } c \geq 0 \quad (25)$$

Note that this has all the desirable properties of g : it is a map from $\mathbb{R}_+$ to $\mathbb{R}_+$, it is continuous and strictly increasing on $\mathbb{R}_+$ (also differentiable for $c > 0$), with $g(c) > c$ for all $c > 0$, and $g(0) = 0$.

The question then arises: what property must θ satisfy in order for (25) to be a valid solution to the conjugate functional equation (16). Given (16) and (25), we must have:

$$(1/a)\{[\rho ac/(1-\rho)] + (\rho ac)^{1-\theta}\} = (1/a)g(\rho ac) = (g(c) - c)$$

$$= [c/(1-\rho)] + c^{1-\theta} - c \quad \text{for } c \geq 0$$

That is,

$$(1/a)(\rho ac)^{1-\theta} = c^{1-\theta} \quad \text{for all } c \geq 0$$

This entails the parameter restriction:

$$(1/a)(\rho a)^{1-\theta} = 1 \tag{26}$$

Note that as $\theta \to 0$, the left hand side of (26) converges to $\rho \in (0, 1)$. And, as $\theta \to 1$, the left hand-side of (26) converges to $(1/a) > 1$. Thus, there is some $\theta \in (0, 1)$, such that (26) holds. We have now verified that:

$$g(c) = [c/(1-\rho)] + c^{1-\theta} \quad \text{for all } c \geq 0$$

with θ chosen to satisfy (26) solves the conjugate functional equation (16).

Step 3: [Solving the Ramsey–Euler Functional Equation (15)]

Note from the form of g in (25) that g is a continuous and strictly increasing function from $\mathbb{R}_+$ onto $\mathbb{R}_+$. Consequently, given any $y \in \mathbb{R}_+$, there is a unique $c \in \mathbb{R}_+$ such that $g(c) = y$; we denote this unique value c by $h(y)$, so that we have h mapping from $\mathbb{R}_+$ to $\mathbb{R}_+$ and:

$$g(h(y)) = y \quad \text{for all } y \geq 0 \tag{27}$$

For any $c \in \mathbb{R}_+$, $g(c) \in \mathbb{R}_+$ by (15). Let us denote $g(c)$ by x, and noting that $g(c)$ is in the domain of h, let us denote $h(g(c))$ by z. Then, we get $z \in \mathbb{R}_+$, and so we evaluate g at this z, and get $g(h(g(c))) = g(z)$. Since $g(c) \in \mathbb{R}_+$, we can apply (27) to also obtain $g(h(g(c))) = g(c)$. Thus, we must have $g(c) = g(z)$. Since g is one-to-one, this implies that $z = c$. That is, by definition of z,

$$h(g(c)) = c \quad \text{for all } c \geq 0 \tag{28}$$

From (28), we also see that h is a map from $\mathbb{R}_+$ onto $\mathbb{R}_+$. And, from (27), we infer that h is strictly increasing on $\mathbb{R}_+$, since g is strictly increasing on $\mathbb{R}_+$. Next, note that h is continuous on $\mathbb{R}_+$. To see this, let $y_1, y_2 \in \mathbb{R}_+$, with $y_1 > y_2$. Then, we have $c_1 \equiv h(y_1) > h(y_2) \equiv c_2$. By using (25) and (27), we have:

$$y_1 - y_2 = g(c_1) - g(c_2) \geq (c_1 - c_2)/(1-\rho)$$

Thus, $0 < h(y_1) - h(y_2) \leq (1-\rho)(y_1 - y_2)$, so that h is Lipschitz, and hence continuous. Finally note that by (25), $g(c) \geq c$ for all $c \geq 0$, and since h is increasing on $\mathbb{R}_+$, $h(g(c)) \geq h(c)$ for all $c \geq 0$. Thus, using (28), we have $h(c) \leq c$ for all $c \geq 0$. Using Step 1, we can now infer that h (the inverse of g defined by (25)) solves the Ramsey–Euler functional equation (15).

Step 4: [The Numerical Illustration in Example 1]

In the numerical illustration in Example 1 in the text, we have chosen $a = (1/2) = \rho$. The parametric restriction on θ in (26) implies that we must have $\theta = (1/2)$. Consequently, g defined in (25) takes the form: $g(c) = 2c + c^{\frac{1}{2}}$ for all $c \geq 0$.

Inverting this function (see Sect. 7.1) yields $h(y) = \frac{(1+4y)-(1+8y)^{\frac{1}{2}}}{8}$, the formula appearing in (5) in the main text.

7.3 A Useful lemma

Lemma 1 *Consider a Ramsey–Euler consumption function $c(y)$ such that*

R.1 $x(y) = y - c(y)$ *is non-decreasing in* y *on* $\mathbb{R}_+$
R.2 *For any interval* $[y', y''] \subset \mathbb{R}_{++}$, $\inf\{c(z) : z \in [y', y'']\} > 0$.

Further, assume that the consumption and investment processes generated by $c(y)$ satisfy the Transversality Condition (TC). Then, $c(y)$ is optimal.

Proof Let $Y = \mathbb{R}_+$. Fix initial stock $\widetilde{y} \in Y$ with $\widetilde{y} > 0$. Consider the stochastic process of output, consumption and investment $\{y_t(\widetilde{y}, \omega), c_t(\widetilde{y}, \omega), x_t(\widetilde{y}, \omega)\}_{t=0}^{\infty}$ for $\omega \in \Omega$, hereafter written as $\{\mathbf{y}_t, \mathbf{c}_t, \mathbf{x}_t\}$, generated by the consumption function $c(y)$. It is easy to check that $\mathbf{y}_t > 0, \mathbf{c}_t > 0, \mathbf{x}_t > 0$ for all $t \geq 0$. Equality or inequalities involving these random variables should be interpreted as holding for all $\omega \in \Omega$. Note that $\{\mathbf{y}_t, \mathbf{c}_t, \mathbf{x}_t\}$ is *feasible* from $\widetilde{y}$. We have to establish that it is *optimal* from $\widetilde{y}$.

Let $\{\underline{y}_t\}, \{\overline{y}_t\}$ be the deterministic sequences defined by:

$$\underline{y}_0 = \overline{y}_0 = \widetilde{y}, \underline{y}_{t+1} = \underline{f}(x(\underline{y}_t)), \overline{y}_{t+1} = \overline{f}(x(\overline{y}_t)), t \geq 0. \tag{29}$$

Note that $\underline{f}(.), \overline{f}(.)$ are nondecreasing on Y. Further, from R.1, it is easy to check that for all $t \geq 0$:

$$\overline{y}_t \geq \mathbf{y}_t \geq \underline{y}_t \tag{30}$$

As $x(z) > 0$ for all $z > 0$ and $\underline{f}(x) > 0, \overline{f}(x) < \infty$ for all $x > 0$,

$$0 < \underline{y}_t \leq \overline{y}_t < \infty \text{ for all } t \geq 0. \tag{31}$$

Let $\{\underline{c}_t\}, \{\overline{c}_t\}$ be the sequences defined by:

$$\underline{c}_t = \inf\{c(z) : z \in [\underline{y}_t, \overline{y}_t]\}, \ \overline{c}_t = \sup\{c(z) : z \in [\underline{y}_t, \overline{y}_t]\} \text{ for all } t \geq 0. \tag{32}$$

Using R.2, we have, $\underline{c}_t > 0$ for all $t \geq 0$; further, $\overline{c}_t \leq \overline{y}_t < \infty$ for all $t \geq 0$. Using (30) and (32), we have:

$$\infty > \overline{c}_t \geq \mathbf{c}_t = c(\mathbf{y}_t) \geq \underline{c}_t > 0 \text{ for all } t \geq 0. \tag{33}$$

Thus, for every $t \geq 0$:

$$-\infty < u(\underline{c}_t) \leq u(c_t) \leq u(\overline{c}_t) < \infty \tag{34}$$

so that for each t, $u(\mathbf{c}_t)$ is a bounded $\mathcal{F}_t-$ measurable function and has finite expectation.

Using (33), we can define the stochastic price process $\{p_t(\overline{y}, \omega)\}$, hereafter written as $\{\mathbf{p}_t\}$, by:

$$\mathbf{p}_t = \rho^t u'(\mathbf{c}_t) \text{ for } t \geq 0. \tag{35}$$

As before, equality or inequalities involving these random variables should be interpreted as holding for all $\omega \in \Omega$. It follows (from (33)) that for every $t \geq 0$,

$$0 < \rho^t u'(\overline{c}_t) \leq \mathbf{p}_t \leq \rho^t u'(\underline{c}_t) < \infty$$

i.e., $\mathbf{p}_t$ is a bounded $\mathcal{F}_t-$measurable random variable (and hence integrable) for each t.

For all $c \geq 0$, and all $t \geq 0$, we have by concavity of u and (35),

$$\rho^t u(\mathbf{c}_t) - \mathbf{p}_t\mathbf{c}_t \geq \rho^t u(c) - \mathbf{p}_t c \tag{36}$$

so that for each $t \geq 0$, we have:

$$E\rho^t u(\mathbf{c}_t) - E\mathbf{p}_t\mathbf{c}_t \geq E\rho^t u(\widehat{\mathbf{c}}_t) - E\mathbf{p}_t\widehat{\mathbf{c}}_t \tag{37}$$

for every bounded $\mathcal{F}_t$ measurable random variable $\widehat{\mathbf{c}}_t \geq 0$ defined on Ω. Note that (using (34)), $E\rho^t u(\mathbf{c}_t)$ is finite; further, as $\widehat{\mathbf{c}}_t$ is a bounded random variable, $E\rho^t u(\widehat{\mathbf{c}}_t)$ on the right hand side of (37) is well defined though it may be $-\infty$.

Using the Ramsey–Euler condition (RE) and (35), one can see that [18]:

$$\mathbf{p}_t = \rho^t u'(\mathbf{c}_t) = E\{\mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1})|\mathcal{F}_t\} \tag{38}$$

Using the concavity of f (in x) we have for all $x \geq 0$ and all $t \geq 0$,

$$f(x, r_{t+1}) - f(\mathbf{x}_t, r_{t+1}) \leq f'(\mathbf{x}_t, r_{t+1})(x - \mathbf{x}_t)$$

so that:

$$\mathbf{p}_{t+1} f(x, r_{t+1}) - \mathbf{p}_{t+1} f(\mathbf{x}_t, r_{t+1}) \leq \mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1})(x - \mathbf{x}_t) \tag{39}$$

Thus, for every bounded $\mathcal{F}_t$ measurable random variable $\widehat{\mathbf{x}}_t \geq 0$ defined on Ω, taking the conditional expectation with respect to $\mathcal{F}_t$ in (39) with $x = \widehat{\mathbf{x}}_t$ we get:

$$\begin{aligned} &E\{\mathbf{p}_{t+1} f(\widehat{\mathbf{x}}_t, r_{t+1})|\mathcal{F}_t\} - E\{\mathbf{p}_{t+1} f(\mathbf{x}_t, r_{t+1})|\mathcal{F}_t\} \\ &\quad \leq E\{\mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1})(\widehat{\mathbf{x}}_t - \mathbf{x}_t)|\mathcal{F}_t\} \\ &\quad = (\widehat{\mathbf{x}}_t - \mathbf{x}_t)E\{\mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1})|\mathcal{F}_t\} = \mathbf{p}_t(\widehat{\mathbf{x}}_t - \mathbf{x}_t) \end{aligned} \tag{40}$$

where the third line uses the fact that $\widehat{\mathbf{x}}_t$ and $\mathbf{x}_t$ are $\mathcal{F}_t$ measurable and the last line in (40) uses (38). Transposing terms in (40), for every bounded $\mathcal{F}_t$ measurable $\widehat{\mathbf{x}}_t \geq 0$, we have:

$$E\{\mathbf{p}_{t+1} f(\mathbf{x}_t, r_{t+1})|\mathcal{F}_t\} - \mathbf{p}_t\mathbf{x}_t \geq E\{\mathbf{p}_{t+1} f(\widehat{\mathbf{x}}_t, r_{t+1})|\mathcal{F}_t\} - \mathbf{p}_t\widehat{\mathbf{x}}_t \tag{41}$$

[18] Strictly speaking, this involves switching from conditional expectation with respect to the distribution function F to a conditional expectation with respect to a sub sigma field.

so that:

$$E\{\mathbf{p}_{t+1}f(\mathbf{x}_t, r_{t+1})\} - E\{\mathbf{p}_t\mathbf{x}_t\} \geq E\{\mathbf{p}_{t+1}f(\widehat{\mathbf{x}}_t, r_{t+1})\} - E\{\mathbf{p}_t\widehat{\mathbf{x}}_t\} \tag{42}$$

Next, one can show that for *any feasible* stochastic process of output, consumption and investment $\{\widehat{\mathbf{y}}_t, \widehat{\mathbf{c}}_t, \widehat{\mathbf{x}}_t\}$ from initial stock $\widetilde{y}$, and for every $T \in \mathbb{N}$

$$E\left\{\sum_{t=0}^{T}\rho^t u(\widehat{\mathbf{c}}_t)\right\} - E\left\{\sum_{t=0}^{T}\rho^t u(\mathbf{c}_t)\right\} \leq E\{\mathbf{p}_T\mathbf{x}_T\} - E\{\mathbf{p}_T\widehat{\mathbf{x}}_T\} \tag{43}$$

To see (43), note that from (37) we have for $t \geq 1$

$$\begin{aligned}
&E\rho^t u(\widehat{\mathbf{c}}_t) - E\rho^t u(\mathbf{c}_t)\\
&\leq E\mathbf{p}_t\widehat{\mathbf{c}}_t - E\mathbf{p}_t\mathbf{c}_t = [E\mathbf{p}_t\widehat{\mathbf{y}}_t - E\mathbf{p}_t\widehat{\mathbf{x}}_t] - [E\mathbf{p}_t\mathbf{y}_t - E\mathbf{p}_t\mathbf{x}_t]\\
&= [E\mathbf{p}_t\widehat{\mathbf{y}}_t - E\mathbf{p}_{t-1}\widehat{\mathbf{x}}_{t-1}] + [E\mathbf{p}_{t-1}\widehat{\mathbf{x}}_{t-1} - E\mathbf{p}_t\widehat{\mathbf{x}}_t]\\
&\quad -[E\mathbf{p}_t\mathbf{y}_t - E\mathbf{p}_{t-1}\mathbf{x}_{t-1}] - [E\mathbf{p}_{t-1}\mathbf{x}_{t-1} - E\mathbf{p}_t\mathbf{x}_t]\\
&\leq [E\mathbf{p}_{t-1}\widehat{\mathbf{x}}_{t-1} - E\mathbf{p}_t\widehat{\mathbf{x}}_t] - [E\mathbf{p}_{t-1}\mathbf{x}_{t-1} - E\mathbf{p}_t\mathbf{x}_t]
\end{aligned}$$

where the first inequality uses (37) and the second inequality uses (42).

The transversality condition (TC) implies that

$$E\{\mathbf{p}_t\mathbf{x}_t\} \to 0 \text{ as } t \to \infty \tag{44}$$

For any feasible stochastic process of output, consumption and investment $\{\widehat{\mathbf{y}}_t, \widehat{\mathbf{c}}_t, \widehat{\mathbf{x}}_t\}$ from initial stock $\overline{y}$,

$$\begin{aligned}
E\left\{\sum_{t=0}^{\infty}\rho^t u(\widehat{\mathbf{c}}_t)\right\} - E\left\{\sum_{t=0}^{\infty}\rho^t u(\mathbf{c}_t))\right\} &= \lim_{T\to\infty} E\left\{\sum_{t=0}^{T}\rho^t u(\widehat{\mathbf{c}}_t) - \sum_{t=0}^{T}\rho^t u(\mathbf{c}_t))\right\}\\
&\leq \lim_{T\to\infty}\sup\left[E\{\mathbf{p}_T\mathbf{x}_T\} - E\{\mathbf{p}_T\widehat{\mathbf{x}}_T\}\right] \leq 0.
\end{aligned}$$

where the equality follows from (1), the first inequality uses (43) and the second inequality uses the transversality condition (44). Hence, $c(y)$ is optimal. This completes the proof of the lemma. □

7.4 Proof of Proposition 2

Proof We claim that under the hypothesis of Proposition 2, we always have:

R.1 $x(y)$ is non-decreasing on $\mathbb{R}_+$
R.2 For any interval $[y', y''] \subset \mathbb{R}_{++}$, $\underline{c}(y', y'') > 0$.

R.1 and R.2 are obvious if $c(y)$ is co-monotone. On the other hand, if $c(y)$ is continuous, R.2 follows immediately and one can show that R.1 holds i.e., $x(y)$ is non-decreasing in y.[19] The proof will use the properties R.1 and R.2 of the policy function.

Fix any $y_0 > 0$ and let $\{c_t\}$, $\{x_t\}$ and $\{y_t\}$ be the stochastic processes of consumption, investment and output generated by $c(y)$ given y_0. We will show that under the hypothesis of the proposition, $\rho^t E[u'(c_t)y_t] \to 0$ as $t \to \infty$. As $x_t \le y_t$ this implies that the transversality condition (TC) holds. As R.1 and R.2 hold, Lemma 1 then implies that $\{c_t\}$, $\{x_t\}$ and $\{y_t\}$ are optimal from y_0; thus $c(y)$ is an optimal consumption function.

Recall z_1, z_2 as defined in assumption (**T.3**)(i). There are (only) two possibilities regarding the behavior of $x(y)$ near zero:

(A.i) There exists a sequence $\{y^n\}_{n=1}^{\infty} \to 0,\ y^n > 0$ for all n and

$$\frac{\underline{f}(x(y^n))}{y^n} \ge 1 \text{ for all } n$$

(A.ii) There exists $\widehat{\epsilon} \in (0, z_1)$ such that

$$\frac{\underline{f}(x(y))}{y} = \frac{f(x(y), a)}{y} < 1 \text{ for all } y \in (0, \widehat{\epsilon})$$

There are (only) two possibilities regarding the behavior of the policy function for large y:

(B.i) There exists a sequence $\{w^n\}_{n=1}^{\infty}$, $w^n > 0$ for all n, $\{w^n\}_{n=1}^{\infty} \to \infty$ and $\frac{\overline{f}(x(w^n))}{w^n} \le 1$ for all n.

(B.ii) There exists $\widehat{y} > z_2$ such that $\frac{\overline{f}(x(y))}{y} = \frac{f(x(y),b)}{y} > 1$ for all $y \ge \widehat{y}$.

The rest of the proof considers four cases based on combinations of these possibilities.

CASE 1: (A.i) and (B.i) hold.

There exists N such that $y^N \le y_0 \le w^N$. Fix N. Using R.1 and $\frac{\underline{f}(x(y^N))}{y^N} \ge 1$, $\frac{\overline{f}(x(w^N))}{w^N} \le 1$, it is easy to check that $y^N \le y_t \le w^N$ for all t. Further, using R.2, we have $c_t \ge \underline{c}(y^N, w^N) > 0$ with probability one and for all t. Thus,

$$0 \le E[u'(c_t)y_t] \le E[u'(\underline{c}(y^N, w^N))w^N] \text{ for all } t,$$

so that $\rho^t E[u'(c_t)y_t] \to 0$ as $t \to \infty$.

CASE 2. Suppose that the candidate policy function satisfies (A.ii) and (B.ii).

As (A.ii) holds, for all $y \in (0, \widehat{\epsilon})$,

$$\frac{\underline{f}(x(y))}{y} = \frac{f(x(y), a)}{y} < 1 \tag{45}$$

[19] To see this, suppose $x(y^1) > x(y^2)$ for $0 \le y^1 < y^2$. Then, $x(y^1) > 0$ so that $y^1 > 0$. As $x(y) = y - c(y)$ is continuous and $x(0) = 0$, there exists $y^3 \in (0, y^1)$ such that $x(y^3) = x(y^2)$. Then, $c(y^3) < c(y^2)$. $x(y^3) = x(y^2)$ implies that the right hand side of the Ramsey–Euler condition (RE) evaluated $y = y^2$ and $y = y^3$ are equal, implying $u'(c(y^3)) = u'(c(y^2))$ that contradicts $c(y^3) < c(y^2)$.

First, consider the case where $f'(0, a) > \tau_0$. Note that $f'(0, a)$ may equal $+\infty$. Choose $\lambda_0 \in (0, 1)$ such that

$$\frac{\tau_0}{\lambda_0} < f'(0, a) \tag{46}$$

Using (45) and (46),

$$\limsup_{y \to 0} \frac{f(x(y), a)}{y} \frac{\tau_0}{f'(x(y), a)} \leq \limsup_{y \to 0} \frac{\tau_0}{f'(x(y), a)} < \lambda_0 \tag{47}$$

Next, consider the case where $f'(0, a) \leq \tau_0$. Using assumption (**T.3**)(ii), $\tau_0 < \infty$ so that $f'(0, a) < \infty$. The latter implies

$$\lim_{x \to 0} \frac{f(x, a)}{f'(x, a)x} = 1$$

so that (as $\lim_{y \to 0} x(y) = 0$)

$$\frac{f(x(y), a)}{f'(x(y), a)x(y)} \to 1 \text{ as } y \to 0 \tag{48}$$

Let $\underline{\alpha} = \liminf_{y \to 0} \frac{c(y)}{y}$. Using Condition (GP1),

$$\limsup_{y \to 0} \frac{x(y)}{y} = 1 - \left(\liminf_{y \to 0} \frac{c(y)}{y} \right) = 1 - \underline{\alpha} < \frac{1}{\tau_0} \tag{49}$$

so that

$$\begin{aligned}
&\limsup_{y \to 0} \left[\frac{f(x(y), a)}{y} \frac{\tau_0}{f'(x(y), a)} \right] \\
&= \limsup_{y \to 0} \frac{x(y)}{y} \tau_0 \left(\frac{f(x(y), a)}{f'(x(y), a)x(y)} \right) \\
&= (1 - \underline{\alpha})\tau_0 < 1
\end{aligned} \tag{50}$$

where we use (48) and (49) in the last line. Choose λ such that

$$\begin{aligned}
\lambda &\in (\lambda_0, 1), \text{ if } f'(0, a) > \tau_0 \\
&\in ((1 - \underline{\alpha})\tau_0, 1) \text{ if } f'(0, a) \leq \tau_0.
\end{aligned}$$

Using (47) and (50), then there exists σ and ϵ_1, $0 < \epsilon_1 < \widehat{\epsilon}$, $\sigma > 0$, such that

$$\frac{f(x(y), a)}{y} \frac{(1 + \sigma)\tau_0}{f'(x(y), a)} < \lambda \text{ for all } y \in (0, \epsilon_1). \tag{51}$$

Fix such σ, ϵ_1. From the definition of τ_0,there exists ϵ, $0 < \epsilon < \epsilon_1$, such that

$$\frac{\overline{\eta}(x)}{\eta(x, a)} \leq (1 + \sigma)\tau_0 \text{ for all } x \in (0, \epsilon)$$

so that

$$\frac{\eta(x,r)}{\eta(x,a)} \leq (1+\sigma)\tau_0 \text{ for all } x \in (0,\epsilon), r \in A. \quad (52)$$

As (B.ii) holds, it must be the case that

$$\lim_{x\to\infty} \frac{f(x,b)}{x} \geq 1$$

so that

$$\lim_{x\to\infty} \frac{f'(x,b)x}{f(x,b)} = 1. \quad (53)$$

Under assumption (**T.3**)(ii), $\tau_\infty < \infty$ and using condition (GP2)

$$\overline{\alpha} = \lim_{y\to\infty} \inf \frac{c(y)}{y} > 1 - \frac{1}{\tau_\infty}$$

so that

$$\tau_\infty(1-\overline{\alpha}) < 1.$$

Choose $\zeta > 0, 0 < \beta < 1$ such that

$$\widetilde{\lambda} = \tau_\infty(1+\zeta)\frac{1-\beta\overline{\alpha}}{\beta} < 1. \quad (54)$$

Fix ζ, β. Using (53), there exists $\overline{y} > z_2$ such that for all $y \geq \overline{y}$

$$\frac{f'(x(y),b)x(y)}{f(x(y),b)} \geq \beta$$
$$\frac{c(y)}{y} \geq \beta\overline{\alpha},$$

and

$$\frac{\overline{\eta}(x(y))}{\eta(x(y),b)} \leq (1+\zeta)\tau_\infty.$$

which implies that for all $y \geq \overline{y}$

$$\frac{\overline{\eta}(x(y))}{\eta(x(y),b)}\left\{\frac{f(x(y),b)}{f'(x(y),b)x(y)}\right\}\frac{x(y)}{y} \leq (1+\zeta)\tau_\infty(\frac{1-\beta\overline{\alpha}}{\beta}) = \widetilde{\lambda}. \quad (55)$$

where $\widetilde{\lambda}$ is defined in (54).

Let E_t denote the expectation conditional on information available in period t. Observe that

$$\begin{aligned} &E_t[u'(c(y_{t+1}))y_{t+1}] \\ &\quad = E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t<\epsilon\}}] + E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t>\overline{y}\}}] \\ &\qquad + E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t\in[\epsilon,\overline{y}]\}}] \end{aligned}$$

Observe that

$$
\begin{aligned}
&E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t<\epsilon\}}] \\
&= E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t\{\frac{f(x_t,r_{t+1})}{y_t}\frac{1}{f'(x_t,r_{t+1})}\}I_{\{y_t<\epsilon\}}\right] \\
&= E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t\{\frac{x_t}{y_t}\frac{\eta(x_t,r_{t+1})}{\eta(x_t,a)}\eta(x_t,a)\}I_{\{y_t<\epsilon\}}\right] \\
&\le E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t\{\frac{x_t}{y_t}(1+\sigma)\tau_0\frac{1}{\eta(x_t,a)}\}I_{\{y_t<\epsilon\}}\right], \text{ using (52)} \\
&= E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t\{\frac{f(x_t,a)}{y_t}\frac{1}{f'(x_t,a)}(1+\sigma)\tau_0\}I_{\{y_t<\epsilon\}}\right] \\
&\le \lambda E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_tI_{\{y_t<\epsilon\}}\right], \text{ using (51).}
\end{aligned} \tag{56}
$$

Also,

$$
\begin{aligned}
&E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t>\overline{y}\}}] \\
&= E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t\left\{\frac{f(x_t,r_{t+1})}{f'(x_t,r_{t+1})x_t}\right\}\frac{x_t}{y_t}I_{\{y_t>\overline{y}\}}\right] \\
&= E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t\left\{\frac{\eta(x_t,r_{t+1})}{\eta(x_t,b)}\right\}\eta(x_t,b)\frac{x_t}{y_t}I_{\{y_t>\overline{y}\}}\right] \\
&\le E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t\frac{\overline{\eta}(x_t)}{\eta(x_t,b)}\left\{\frac{f(x(y_t),b)}{f'(x(y_t),b)x(y_t)}\right\}\left\{\frac{x(y_t)}{y_t}\right\}I_{\{y_t>\overline{y}\}}\right] \\
&\le \widetilde{\lambda}E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_tI_{\{y_t>\overline{y}\}}\right], \text{ using (55)}
\end{aligned} \tag{57}
$$

Finally, given fixed $\epsilon, \overline{y} \in \mathbb{R}_{++}$as defined above, $y_t \in [\epsilon, \overline{y}]$ implies that with probability one, $y_{t+1} \in [\underline{f}(x(\epsilon)), \overline{f}(\overline{y})] \subset \mathbb{R}_{++}$. Therefore for all t,

$$
\begin{aligned}
&E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t\in[\epsilon,\overline{y}]\}}] \\
&\le u'(\underline{c}(\underline{f}(x(\epsilon)), \overline{f}(\overline{y})))\overline{f}(\overline{y}) = Q'.
\end{aligned} \tag{58}
$$

where $0 < Q' < \infty$, using R.2. Let $\widehat{\lambda} = \max\{\lambda, \widetilde{\lambda}\}$. Then, $\widehat{\lambda} \in (0, 1)$ and using (56),(57) and (58)

$$
\begin{aligned}
&\rho^{t+1}E_t[u'(c(y_{t+1}))y_{t+1}] \\
&= \rho^{t+1}E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t<\epsilon\}}] + \rho^{t+1}E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t>\overline{y}\}}] \\
&\quad +\rho^{t+1}E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t\in[\epsilon,\overline{y}]\}}] \\
&\le \rho^{t+1}\widehat{\lambda}[E_t[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_t(I_{\{y_t<\epsilon\}} + I_{\{y_t>\overline{y}\}})] + \rho^{t+1}Q' \\
&\le \widehat{\lambda}\rho^{t+1}E_t[u'(c_{t+1})f'(x_t,r_{t+1})]y_t + \rho^{t+1}Q' \\
&= \widehat{\lambda}\rho^t u'(c_t)y_t + \rho^{t+1}Q'
\end{aligned} \tag{59}
$$

where the last equality follows from the Ramsey–Euler equation (RE). Taking unconditional expectation in (59) we have:

$$\begin{aligned}&\rho^{t+1}E[u'(c(y_{t+1}))y_{t+1}]\\&\quad\leq\widehat{\lambda}\rho^{t}E[u'(c_t)y_t]+\rho^{t+1}Q'\end{aligned}$$

for all t, which implies that $\rho^t E[u'(c_t)y_t]\to 0$ as $t\to\infty$.

CASE 3. Suppose that the candidate policy function satisfies (A.i) and (B.ii).

As in CASE 1, as (A.i) holds, there exists N such that $y^N\leq y_t$ with probability one for all t. As (B.ii) holds, using an identical argument as in CASE 2 (and with some abuse of notation), condition (GP2) ensures that there exists $0<\beta<1$, $\zeta>0$, $\overline{y}>0$ such that

$$0<\widetilde{\lambda}=(1+\zeta)\tau_{\infty}(\frac{1-\beta\overline{\alpha}}{\beta})<1$$

and

$$\frac{\overline{\eta}(x(y))}{\eta(x(y),b)}\left\{\frac{f(x(y),b)}{f'(x(y),b)x(y)}\right\}\frac{x(y)}{y}\leq\widehat{\lambda}\text{ for all }y>\overline{y}$$

so that

$$\begin{aligned}&E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t>\overline{y}\}}]\\&\quad\leq\widetilde{\lambda}E_t\left[u'(c(y_{t+1}))f'(x_t,r_{t+1})y_tI_{\{y_t>\overline{y}\}}\right]\end{aligned}$$

Further,

$$E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t\in[\epsilon,\overline{y}]\}}]\leq u'(\underline{c}(\underline{f}(x(y^N)),\overline{f}(\overline{y}))))\overline{f}(\overline{y})=Q''.$$

where $0<Q''<\infty$. Thus

$$\begin{aligned}&E_t[u'(c(y_{t+1}))y_{t+1}]\\&\quad=E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t>\overline{y}\}}]+E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t\in[y^n,\overline{y}]\}}]\\&\quad\leq\widetilde{\lambda}E_t[u'(c_{t+1})f'(x_t,r_{t+1})]y_t+Q''\end{aligned}$$

and the rest of the proof is identical to CASE 2.

CASE 4. Suppose that the candidate policy function satisfies (A.ii) and (B.i).

As (B.i) holds, using an identical argument as in the proof for CASE 1, there exists n such that $y_t\leq w^n$ with probability one for all t . As condition (A.ii) holds, arguments identical to those used in Case 2 imply that (with some abuse of notation) there exists ϵ,λ, $0<\epsilon<\widehat{\epsilon}$,$0<\lambda<1$ such that

$$\begin{aligned}\frac{\eta(x(y),r)}{\eta(x(y),a)}&\leq(1+\sigma)\tau_0,\\\frac{f(x(y),a)}{y}\frac{(1+\sigma)\tau_0}{f'(x(y),a)}&<\lambda\text{ for all }y\in(0,\epsilon).\end{aligned}$$

and therefore, using identical arguments leading to (56), we have

$$E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t<\epsilon\}}] \leq \lambda E_t\left[u'(c(y_{t+1}))f'(x_t, r_{t+1})y_t I_{\{y_t<\epsilon\}}\right].$$

Further,

$$\begin{aligned} E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t\geq\epsilon\}}] &= E_t[u'(c(y_{t+1}))y_{t+1}I_{\{\epsilon\leq y_t\leq w^n\}}] \\ &\leq u'(\underline{c}(\underline{f}(x(\epsilon)), \overline{f}(x(w^n)))\overline{f}(w^n) = Q\hat{}. \end{aligned}$$

where $0 < Q\hat{} < \infty$ using R.1. Then,

$$\begin{aligned} &E_t[u'(c(y_{t+1}))y_{t+1}] \\ &\quad = E_t[u'(c(y_{t+1}))y_{t+1}I_{\{y_t<\epsilon\}}] + E_t[u'(c(y_{t+1}))y_{t+1}I_{\{\epsilon\leq y_t\leq w^n\}}] \\ &\quad \leq \lambda E_t[u'(c_{t+1})f'(x_t, r_{t+1})]y_t + Q\hat{}, \end{aligned}$$

and the rest of the proof is identical to Case 2. This completes the proof of Proposition 2. ◻

7.5 Proof of Proposition 3

Proof Using identical arguments as at the beginning of the proof of Proposition 2, one can show that properties R.1 and R.2 in the antecedent of Lemma 1 hold. Fix any $y_0 > 0$ and let $\{c_t\}$, $\{x_t\}$ and $\{y_t\}$ be the stochastic paths of consumption, investment and output generated by $c(y)$ given y_0.We will show that under the hypothesis of the proposition, $\rho^t E[u'(c_t)x_t] \to 0$ as $t \to \infty$. Lemma 1 then implies that $\{c_t\}$, $\{x_t\}$ and $\{y_t\}$ are optimal from y_0; thus $c(y)$ is an optimal consumption function. Let $\widehat{\alpha} > 0$ be defined by

$$\inf_{y>0} \frac{c(y)}{y} = \widehat{\alpha}$$

Then, $\widehat{\alpha} \in (0, 1)$. Observe that for any t:

$$\begin{aligned} &\rho^t E[u'(c_t)x_t] \\ &\quad \leq \rho^t E[u'(c_t)y_t] \leq \rho^t E[u'(\alpha y_t)y_t] \\ &\quad = \rho^t \left[\frac{E(u'(\alpha y_t)\alpha y_t)}{\widehat{\alpha}}\right] \leq \rho^t \left[\frac{M}{\widehat{\alpha}}\right] \end{aligned}$$

which converges to 0 as $t \to \infty$. ◻

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