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RESEARCH ARTICLE

Uniqueness, stability and comparative statics for two-person Bayesian games with strategic substitutes

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Abstract This paper considers a class of two-player symmetric games of incomplete information with strategic substitutes. First, we provide sufficient conditions under which there is *either* a unique equilibrium which is stable (in the sense of best-reply dynamics) and symmetric *or* a unique (up to permutations) asymmetric equilibrium that is stable (together with an unstable symmetric equilibrium). Thus, (i) there is always a unique stable equilibrium, (ii) it is either symmetric or asymmetric, and hence, (iii) a very simple local condition—stability of the symmetric equilibrium (i.e., the slope of the best-response function at the symmetric equilibrium)—identifies which case applies. Using this, we provide a very simple sufficient condition on primitives for when the unique stable equilibrium is asymmetric (and similarly for when it is symmetric). Finally, we show that the conditions guaranteeing the uniqueness described above also yield novel comparative statics results for this class of games.

Keywords Uniqueness of equilibrium · Stability · Symmetry breaking · Monotone comparative statics · Strategic substitutes

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1 Introduction

This paper considers a class of two-player two-action symmetric games of incomplete information with strategic substitutes (Bulow et al. 1985). There are three results. First, we provide sufficient conditions under which there is *either* a unique equilibrium which is stable (in the sense of best-reply dynamics) and symmetric *or* a unique (up to permutations)[1] asymmetric equilibrium that is stable (together with an unstable symmetric equilibrium). Thus, there is always a unique stable equilibrium, and it is either symmetric or asymmetric. Moreover, a very simple local condition—stability of the symmetric equilibrium (i.e., the slope of the best-response function at the symmetric equilibrium)—identifies which case applies. This in turn enables us to provide a very simple sufficient condition for when the unique stable equilibrium is asymmetric (and similarly for when it is symmetric). Finally, these conditions also provide novel comparative statics results for the class of games we study.

Our interest in providing conditions that guarantee this form of uniqueness is threefold. First, the result says that under the identified assumptions, there is a unique relevant equilibrium, implying that predictions are meaningful. Second, as in the literature on symmetry breaking discussed further below, it is of interest to provide conditions under which the only equilibrium is asymmetric. Finally, while there are many general comparative statics results for games with strategic complements, there are very few for games with strategic substitutes.

As mentioned, we consider two-player two-action (say High and Low) symmetric games.[2] The critical payoff parameters are the payoff difference of choosing High versus Low against an opponent playing High, denoted by U^{H}, and similarly this difference against an opponent choosing Low, denoted by U^{L}. The game has strategic substitutes when $U^{\mathrm{L}} > U^{\mathrm{H}}$. There is a continuum of types, with density f, where the type is an additively separable cost, x, to choosing High over Low.[3] This class of games admits several economic applications; we focus on the following three that are explained in more detail in Sect. 2.2: (1) a decision to invest (or enter a market) with private costs followed by subsequent competition (see, e.g., de Frutos and Fabra 2007; Amir 2000); (2) investment in a public good, again with private costs or values; and (3) career choice followed by random matching into couples (see also Becker 1993; Hadfield 1999).

Our main assumption is that the density f is log-concave[4] and single peaked with modal type having sufficiently low costs (so that the modal type would choose High).

[1] By unique up to permutations we mean that, as the game is symmetric, if (x, y) is an equilibrium so is (y, x). Henceforth, we refer to this as unique and drop the clause "up to permutations."

[2] It would be interesting, but beyond the scope of this paper, to extend the results to more players and actions, and also to extend those results that would apply, such as the comparative-statics results, to asymmetric environments.

[3] Obviously, the cost could be a benefit, and one could have both; we focus wlog on the case of costs as it is more natural in some of the examples we consider and simplifies the writing.

[4] Most commonly studied distributions have log-concave densities, see Bagnoli and Bergstrom (2005).

This assumption on the modal type is natural in some examples. For instance in the career choice model, it follows if the modal type is more qualified at the task she likes, and in the market entry example, it implies the modal type would choose to enter even when the opponent enters. These conditions yield our result that there is a unique stable outcome. Moreover, the unique stable outcome is a pair of mirror-image asymmetric equilibria if $U^{\mathrm{L}} - U^{\mathrm{H}} > 1/f\left(U^{\mathrm{L}}\right)$, i.e., if the strategic substitutes are strong enough, and it is a symmetric equilibrium if $U^{\mathrm{L}} - U^{\mathrm{H}} < 1/f\left(U^{\mathrm{H}}\right)$.[5]

There are two comparative statics results corresponding to the case of a stable symmetric or asymmetric outcome. The former is intuitive: starting from a symmetric equilibrium x^e a decrease in U^{H} moves the equilibrium down, i.e., x^e decreases. This is intuitive because as U^{H} decreases, the benefit of playing High decreases. The more interesting case is the latter: starting from an asymmetric equilibrium, say $x_1^e > x_2^e$, as U^{H} increases we have that x_1^e increases while x_2^e decreases. Here, the indirect effect of the strategic substitutes dominates for the player choosing a lower threshold, and the direct effect of increasing the benefit of playing High dominates for the player choosing the higher cutoff. Since we show that the asymmetric equilibrium arises when the strategic substitutes are strong, this is intuitively consistent with the indirect effect dominating for one player (it can never dominate for both); that the indirect effect dominates for the player choosing the lower threshold follows from the structural assumptions and will be proven in the subsequent analysis.

The questions of stability and uniqueness have been studied in various submodular contexts [the first such being Cournot (1897)]. Matsuyama (2008) and Amir et al. (2010) provide an excellent discussion of the importance of obtaining symmetry breaking, that is obtaining conditions under which the only (or only relevant) equilibrium is asymmetric. Amir et al. obtain only (pure strategy) asymmetric equilibria in symmetric games with a nonconcavity along the diagonal, and hence a resulting discontinuity in the best-reply correspondence. They show how this generalizes and unifies other papers with a similar structure. By contrast in the environments, we study the existence of a pure-strategy symmetric equilibrium is not ruled out a priori. In this sense, our approach is closer to the important work of Matsuyama who also explores when the only stable equilibria in symmetric environments are asymmetric. However, he elegantly introduces a strategic complementarity into his models, while our attention is on games of strategic substitutes.[6] Hefti (2016a, b, 2017) is also interested in the connection between stability and uniqueness of equilibrium. His work focusses on how various stability properties lead to uniqueness and conversely. This then gives

[5] Strategic substitutes ($U^{\mathrm{H}} < U^{\mathrm{L}}$) together with the assumptions on f imply that $f\left(U^{\mathrm{H}}\right) > f\left(U^{\mathrm{L}}\right)$, which is why these conditions are sufficient but not necessary.

[6] For games with strategic complementarities (supermodular games), there is a significant body of work on the structure of equilibria and their stability and comparative statics (e.g., Milgrom and Roberts 1990; Vives 1990). However, as noted, those results do not apply in our strategic substitutes (submodular) context. It is true that a two-player game of strategic substitutes can be transformed into one with strategic complements by permuting the actions of one player (specifically by reversing the order). However, the symmetry of the game is not preserved in this permutation, so the results on symmetric submodular games do not apply in our case.

conditions that select the symmetric equilibrium in symmetric games.[7] Our results complement his as we provide conditions such that either there is a symmetric stable equilibrium or there is only one (up to permutations) asymmetric equilibrium. Also, the conditions obtained differ; ours focus on assumptions on the distribution of types. Moreover, we are interested in comparative statics, which brings us to the final class of related work. There are limited results on comparative statics in games with strategic substitutes. Roy and Sabarwal (2010) and Acemoglu and Jensen (2013) provide such results for the case where direct effects dominate indirect effects, as occurs, for example, in the symmetric equilibria. We find the asymmetric equilibria of particular interest, where the indirect effects need not be dominated, and our approach to obtain the comparative statics results is therefore different.[8]

2 Model and applications

2.1 The general model

There are two players, 1 and 2, and two actions, H and L. The game is symmetric. Each player draws, independently, a type $x \in \mathbf{R}$, which is her relative dislike or cost of playing H and is her private information. The payoff from playing L is normalized to 0. The payoff from playing H is the sum of $-x$ plus either $U^{\mathrm{H}} > 0$ if the opponent plays H or $U^{\mathrm{L}} > U^{\mathrm{H}}$ if the opponent plays H:

Agent/opponent	H	L
H	$U^{\mathrm{H}} - x$	$U^{\mathrm{L}} - x$
L	0	0

The distribution of each x has a log-concave density f with support on an interval $\left[\underline{x}, \bar{x}\right]$ (we allow for $\underline{x} = -\infty$ or $\bar{x} = \infty$), where f has a single peak that is below U^{H}. Thus, for the modal type, H is the dominant action. To focus on the interesting cases, we also assume that $\bar{x} > U^{\mathrm{L}}$, so that for some types, L is the dominant action (as noted, the assumptions on the modal type of f imply that for some types, H is dominant).

Remark 1 While the model is described as a two-player Bayesian game, it obviously applies also when each "player" is a population of individuals (perhaps a continuum) and after each chooses her action they are randomly paired (see Sect. 2.2.3).

We now describe three natural applications of the model. In each, we explain how the application's parameters map into the model, and interpret the model's critical assumptions in the context of the application.

[7] Zimper (2007) and Roy and Sabarwal (2012) provide conditions on the best-reply function that guarantee dominance solvability in lattice games with strategic substitutes. Roy and Saberwal in particular relate this to global stability.

[8] It remains an open question to what extent the results herein can be extended to general lattice games, and not only those with a differentiable structure as we assume.

2.2 Applications

2.2.1 Public good

Two agents invest (H) or not (L) in a public good. The gross return from investment has decreasing returns: 0 if none invests, U^{L} if one player invests, and $U^{\mathrm{L}} + U^{\mathrm{H}}$, where $U^{\mathrm{H}} < U^{\mathrm{L}}$, if both invest. Each agent's cost of investment is $x \in \mathbf{R}$ and is the agent's private information. Each x is independently drawn from a distribution with a single-peaked log-concave density f where, for the modal type, investing is a dominant action. The agent's payoff matrix is thus

Agent/opponent	H	L
H	$U^{\mathrm{L}} + U^{\mathrm{H}} - x$	$U^{\mathrm{L}} - x$
L	U^{L}	0

which is best-reply equivalent to that of the abstract model above (i.e., the best-reply functions are the same).

2.2.2 R&D or capacity investment

Two firms decide, in a first stage, whether to invest in developing a product or in a technology that reduces per-unit cost of production. In a second stage, the firms compete in the product market. Each firm's profit in the second-stage product competition is 0 if the firm did not invest, the monopoly profit U^{L} if it is the only firm that invested, and a duopoly profit U^{H} if both firms entered. A firm's cost of investment is x and is its private information. Each x is drawn from a distribution with a single-peaked log-concave density f where the modal type would invest even if the other firm invested for sure ($x < U^{\mathrm{H}}$). The firm's payoff matrix is thus exactly that of the abstract model above.

2.2.3 Gender differences in career choices

There are two equally sized intervals of men (m) and women (w), and two occupations, A and B. Each person draws independently a type (k, x) where k is his/her high-income occupation (HIO or H) and x is his/her **dis**like of working at the HIO relative to the other occupation (L). An individual has income w_h from working in his/her HIO, and $w_l < w_h$ in the other profession.

Individuals first choose a profession and then are randomly paired into households. The utility of agents is the sum of job-satisfaction utility, $-x$ or 0, and utility from household income u. Thus the utility of an individual whose spouse earns w is:

choosing HIO	$u\,(w_h + w) - x$
non-HIO	$u\,(w_l + w)$

As discussed, we denote by U^{H} the increase in utility from the additional income due to choosing the HIO (ignoring job dissatisfaction, x) when the spouse has high income (chooses H). Similarly, U^{L} is this difference when the spouse has low income (chooses L). That is,

$$U^{\mathrm{H}} \equiv u\left(w_h + w_h\right) - u\left(w_l + w_h\right)$$
$$U^{\mathrm{L}} \equiv u\left(w_h + w_l\right) - u\left(w_l + w_l\right)$$

Assuming positive and decreasing marginal utility of money implies $U^{\mathrm{L}} > U^{\mathrm{H}} > 0$. Normalizing $u\left(w_l + w_l\right) = 0$ and assuming that an individual's HIO, k, is either A or B, we obtain the same payoff matrix as in the public good application. Finally, we assume that the agents' relative dislike, $x \in \mathbf{R}$, is (independently) drawn according to a log-concave density f with single peak below 0, i.e., that the modal type prefers to work at his/her HIO.

3 Analysis

3.1 Characterization of the equilibria

In this section, we show our main result, that either there is a unique equilibrium that is stable and symmetric or there is a unique pair of (mirror-image) stable asymmetric equilibria and an unstable symmetric equilibrium. Obviously, an equilibrium has the form of threshold strategies: a pair $\left(x^1, x^2\right)$ such that player j of type x_j chooses H iff $x_j < x^j$. Thus, the probability that j's opponent $(-j)$ plays H is $F\left(x^{-j}\right)$, and j's relative payoff from playing H against the distribution of play by $-j$ is:

$$U^{\mathrm{H}} F\left(x^{-j}\right) + U^{\mathrm{L}}\left(1 - F\left(x^{-j}\right)\right) - x_j$$

Player j's best-reply threshold, x^j, given the other player's threshold, x^{-j}, is then

$$x^j = B\left(x^{-j}\right) \equiv U^{\mathrm{H}} F\left(x^{-j}\right) + U^{\mathrm{L}}\left(1 - F\left(x^{-j}\right)\right)$$

Since $U^{\mathrm{L}} > U^{\mathrm{H}}$, we see immediately that the slope of the best-reply function is negative: if one player chooses H more often, then the other player wants to choose it less often (if x^j increases and $U^{\mathrm{L}} > U^{\mathrm{H}}$, then $U^{\mathrm{H}} F\left(x^j\right) + U^{\mathrm{L}}\left(1 - F\left(x^j\right)\right)$ decreases).

A pair of thresholds $\left(x^1, x^2\right)$ is then an equilibrium if $x^1 = B\left(x^2\right)$ and $x^2 = B\left(x^1\right)$ [note that $B\left(x^j\right)$ is the best-reply function of player $-j$, not j]. In general, there can be two types of equilibria: (1) symmetric, in which case $x^1 = x^2$, where we will denote the common equilibrium threshold by x^s and (2) mirror-image asymmetric equilibria, in which case we focus throughout, wlog, on the equilibrium with $x^1 > x^2$.

We are interested in (dynamically) locally stable equilibria. An equilibrium is stable in this sense if, starting from near enough to an equilibrium, the behavior would converge back to the equilibrium, where the dynamics are given by the best-response functions. An equilibrium is unstable if it locally diverges. It is straightforward that an equilibrium (x, y) is stable if $B'(x) \times B'(y) < 1$ and it is unstable if $B'(x) \times B'(y) >$

1. In general, if $B'(x) \times B'(y) = 1$, an equilibrium may be neither stable nor unstable, but we will see that in our model such equilibria are stable.

Proposition 1 *Depending on the model's parameters, either there is a unique equilibrium x^s which is stable and symmetric with $\left|B'(x^s)\right| \leq 1$, or there are three equilibria: an unstable symmetric equilibrium x^s with $\left|B'(x^s)\right| > 1$ and two stable asymmetric equilibria (x, y) and (y, x) with $B'(x) \times B'(y) < 1$.*

Proof Denote the best-reply function by $B(x) = U^{\mathrm{H}} F(x) + U^{\mathrm{L}}(1 - F(x)) \in \left[U^{\mathrm{H}}, U^{\mathrm{L}}\right]$. Since B is continuous, it has a fixed point in the closed interval $\left[U^{\mathrm{H}}, U^{\mathrm{L}}\right]$, which is a symmetric equilibrium. Consider now the function $R(x) = B(B(x))$. Then in any equilibrium, symmetric or not, $x = R(x)$, i.e., equilibria are intersections of R with the 45-degree line. In a symmetric equilibrium, $x = B(x) = R(x)$. An asymmetric equilibrium is a pair of thresholds (x, y) with $x = B(y) = R(x)$ and $y = B(x) = R(y)$.

We consider R' at intersections $R(x) = x$, since then $R'(x) > 1$ implies instability of equilibrium (symmetric or not) and R increasing with $R'(x) < 1$ implies stability (to see the stability argument, observe that if R is increasing and $R'(x) < 1$ then for $\tilde{x}$ close to x (specifically closer than any other fixed point) but below x, we have $x > R(\tilde{x}) > \tilde{x}$ so the best reply to the best reply of $\tilde{x}$ is closer to x but does not overshoot. (Iterating on R this process must converge and cannot converge to a point below x as then it would converge to a fixed point between $\tilde{x}$ and x while we assumed that $\tilde{x}$ is closer to x than any other fixed point.) A similar argument applies for $\tilde{x}$ close to and greater than x.)

Note first that B is decreasing: since $U^{\mathrm{H}} < U^{\mathrm{L}}$, $B'(x) = \left(U^{\mathrm{H}} - U^{\mathrm{L}}\right) f(x) < 0$. Therefore, if there is an asymmetric equilibrium (x, y) with $y > x$, we must have $y > x^s > x$. It cannot be that $y > x > x^s$ since if $y > x^s$ then $x = B(y) < B(x^s) = x^s$, a contradiction. Similarly, it cannot be that $x^s > y > x$ since if $x < x^s$ then $y = B(x) > B(x^s) = x^s$. Now, since $R(\cdot) = B(B(\cdot))$, it is increasing.

Before continuing with the details, we outline the main parts of the proof. We show that if $R'(x^s) > 1$ then R looks like in Fig. 1 where it is convex to the left of x^s and lies below the 45 degree line near x^s. As it has range in $\left[U^{\mathrm{H}}, U^{\mathrm{L}}\right]$, it must intersect the 45 degree line between U^{H} and x^s so there is an asymmetric equilibrium (x, y). Moreover, at this intersection $R' < 1$ so this equilibrium is stable.

We also show that if $R'(x^s) \leq 1$, then R looks as in Fig. 2 where R is strictly concave and increasing to the right of x^s, and since $R'(x^s) \leq 1$ it is below the 45 degree line to the right of x^s, and hence R cannot intersect the 45 degree line to the right of x^s, so there is no asymmetric equilibrium (x, y).

For $R'(x^s) < 1$, we already noted that x^s is stable so, finally, we show that x^s is also stable if $R'(x^s) = 1$. In this case, R is as in Fig. 3 where it is convex below and concave above x^s, and as it is increasing and crosses the 45 degree line from above at x^s, we have that x^s is stable.

We continue now with the formal arguments. As noted, $B'(x) = \left(U^{\mathrm{H}} - U^{\mathrm{L}}\right) f(x) < 0$. Since f is single peaked with peak below U^{H}, then over the interval $\left[U^{\mathrm{H}}, \bar{x}\right]$ we have $f'(x) < 0$ hence $B'' = \left(U^{\mathrm{H}} - U^{\mathrm{L}}\right) f'(x) > 0$. That f is log-concave is equivalent to $\frac{f'(x)}{f(x)}$ being weakly decreasing, which implies that $\frac{B''(x)}{B'(x)}$ is weakly

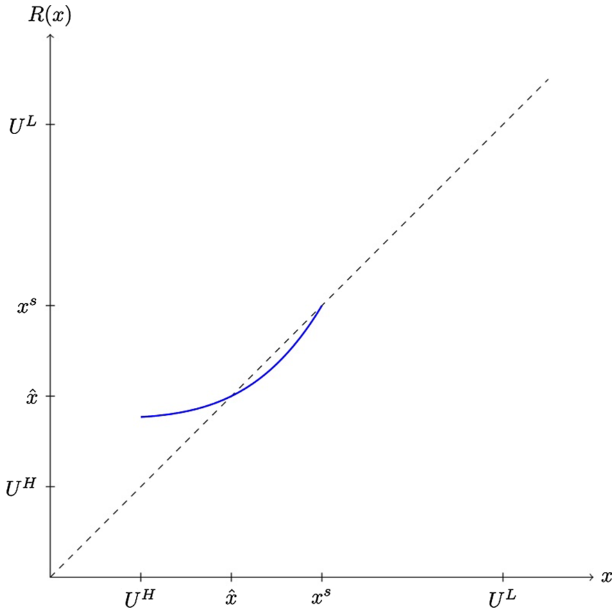

Fig. 1 When $R'(x^s) > 1$, then R is convex below x^s and there exists another fixed point $\hat{x} < x^s$ with $R'(\hat{x}) < 1$

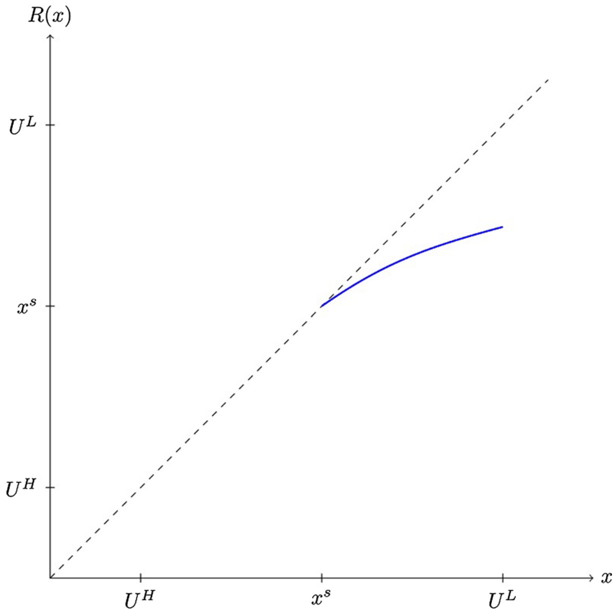

Fig. 2 When $R'(x^s) \leq 1$, then R is concave above x^s and there does not exist another fixed point $\hat{x} > x^s$

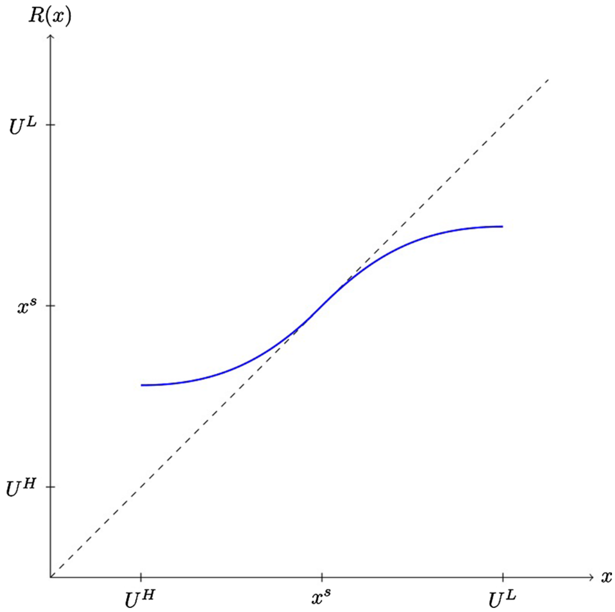

Fig. 3 When $R'(x^s) = 1$, then R is convex below x^s and concave above it. This implies that x^s is stable

decreasing. This implies $0 > B''(x) B'(y) \geq B''(y) B'(x)$ for all $y > x$ (and $B''(x) B'(y) \leq B''(y) B'(x) < 0$ for all $y < x$).

Consider now a symmetric equilibrium $x^s = B(x^s)$. For any x and $y = B(x)$, we have

$$R' = (B(B(x)))' = B'(B(x)) B'(x) = B'(y) B'(x) \tag{1}$$

$$R'' = B''(B(x)) \left(B'(x)\right)^2 + B'(B(x)) B''(x) = B''(y) \left(B'(x)\right)^2 + B'(y) B''(x). \tag{2}$$

Note also that for an asymmetric equilibrium, $R'(x) = R'(y)$. Furthermore, since $B' < 0$ then $R' > 0$.

Consider the case where $R'(x^s) \geq 1$ and recall that

$$B''(y) B'(x) \leq B'(y) B''(x) \iff \tag{3}$$

$$\left|B''(y) B'(x)\right| \geq \left|B'(y) B''(x)\right| \tag{4}$$

for $y > x$. For $x < x^s$ (since $B'' > 0$ and $\left|B'(x^s)\right| = \sqrt{R'(x^s)} \geq 1$) we have $\left|B'(x)\right| > 1$. Hence multiplying the LHS of (3) by $B'(x)$ it becomes positive and by (4) is greater in absolute value than the RHS. Hence, substituting into (2), $R''(x) > 0$. Thus,

$$R'\left(x^s\right) \geq 1 \Rightarrow R''(x) > 0 \quad \forall x < x^s, \tag{5}$$

and similarly one can show

$$R'\left(x^s\right) \leq 1 \Rightarrow R''(x) < 0 \quad \forall x > x^s. \tag{6}$$

First note that if $R'(x^s) = 1$, then x^s is stable. This is because when $R'(x^s) = 1$, we have from the preceding pair of equations that $R'(x) < 1$ for all $x \neq x^s$ which implies stability.

Thus, x^s is stable iff $R'(x^s) \leq 1$ and then, by (6), R does not cross the 45 degree line for any $x > x^s$ so there is no asymmetric equilibrium (recall that if there were an asymmetric equilibrium (x, y) then $R(x) = x$ and $R(y) = y$ and one of them would be greater than x^s and the other would be less).

Also, x^s is unstable iff $R'(x^s) > 1$ and then R must cross the 45-degree line at some $\hat{x} < x^s$ (if not, then for all $x < x^s$ we have $R(x) < x$. But for $x < U^{\mathrm{H}}$, this contradicts that for all $\tilde{x}$ we have $B(\tilde{x}) \in \left[U^{\mathrm{H}}, U^{\mathrm{L}}\right]$ hence $R(x) \geq U^{\mathrm{H}} \geq x$). Thus, $\left(\hat{x}, B\left(\hat{x}\right)\right)$ is an asymmetric equilibrium. Moreover, since $R''(x) > 0$ for all $x < x^s$ this is the only x for which $R(x) = x$ and $R'\left(\hat{x}\right) < 1$ so it is the only asymmetric equilibrium with $x < x^s$ and since $R'\left(\hat{x}\right) < 1$ it is stable (obviously, there exists one other asymmetric equilibrium, its mirror image, $\left(B\left(\hat{x}\right), \hat{x}\right)$). □

Whether the stable equilibrium is asymmetric or symmetric depends on whether the strategic substitutes are strong enough (i.e., whether $\left|B'(x^s)\right|$ is greater than, or weakly less than, 1). The following corollary states sufficient conditions on the model's primitives for that:

Corollary 1 1. *If $U^{\mathrm{L}} - U^{\mathrm{H}} > 1/f\left(U^{\mathrm{L}}\right)$, then the only stable outcome is a pair of (mirror-image) asymmetric equilibria.*
2. *If $U^{\mathrm{L}} - U^{\mathrm{H}} \leq 1/f\left(U^{\mathrm{H}}\right)$, then the only stable outcome is a symmetric equilibrium.*

Proof The symmetric equilibrium is stable and hence by the result above unique iff $\left|B'(x^s)\right| = \left|\left(U^{\mathrm{H}} - U^{\mathrm{L}}\right) f(x^s)\right| \leq 1$ and since $x^s \in \left[U^{\mathrm{H}}, U^{\mathrm{L}}\right]$ and f is decreasing on $\left[U^{\mathrm{H}}, U^{\mathrm{L}}\right]$ this follows if $\left|U^{\mathrm{H}} - U^{\mathrm{L}}\right| = U^{\mathrm{L}} - U^{\mathrm{H}} \leq 1/f\left(U^{\mathrm{H}}\right) \leq 1/f(x^s)$. Similarly, it is unstable, and hence the unique stable equilibrium is asymmetric, iff $\left|B'(x^s)\right| = \left|\left(U^{\mathrm{H}} - U^{\mathrm{L}}\right) f(x^s)\right| > 1$ and again this follows if $\left|U^{\mathrm{H}} - U^{\mathrm{L}}\right| = U^{\mathrm{L}} - U^{\mathrm{H}} > 1/f\left(U^{\mathrm{L}}\right) \geq 1/f(x^s)$. □

3.2 The implication of the characterization in the applications

In the public good case, the difference $U^{\mathrm{L}} - U^{\mathrm{H}}$ measures the decrease in the marginal returns to investment in the good—the difference between the return if one agent invests and the additional return if a second agent invests. If this decrease in returns is sufficiently weak, then both agents have the same threshold of private investment cost below which they invest. If the decrease is sufficiently strong, then a stable equilibrium must be asymmetric: One of the agents invests as long as his cost is below a low threshold, and the second invests below a high threshold (that is, invests more often). Which of the two agents is the one with the low/high threshold is undetermined (i.e., there are two mirror-image stable equilibria).

In the R&D/capacity investment interpretation of the model, $U^{\mathrm{L}} - U^{\mathrm{H}}$ is the difference between monopoly and duopoly profits. If competition decreases profits sufficiently, then the equilibrium outcome is asymmetric—one firm is "aggressive" and enters the market for a wide range of investment costs, while the other invests only as long as its cost is below a low threshold. If instead the gain from being a monopolist versus a duopolist is not too large, then both firms will pick the same investment threshold.

Finally, in the career choice interpretation, what matters is the additional household utility when an individual brings home additional income by working in his/her high-income versus low-income occupation. By the assumption of decreasing marginal utility, this additional household utility is lower if the spouse works in his/her high-income occupation and hence already brings home a high income. If marginal utility is sufficiently decreasing, then there will be an asymmetric equilibrium: Individuals of one gender choose their HIO even if they dislike it quite strongly, while those of the other gender choose their HIO only as long as their dislike is not so strong. While the model does not predict whether men or women will be those choosing their HIO more often, the observed gender wage gap in which men have higher wages corresponds to the first case.

4 Comparative statics

4.1 Theoretical results

In the section, we analyze general properties of the comparative statics of the model. The comparative statics obviously depend on two effects. First, there are the standard direct effects: How each player's choices respond to a parameter change when the other player's behavior does not change. Second, there are the indirect effects: Each player's behavior does change, which further impacts the other player's choices. The results in this section show how the overall equilibrium effect can be determined from the direct effects alone.

To state these results formally, let t be an exogenous parameter affecting both players, with $t = 0$ denoting the initial situation. We thus add the argument t to all functions. So $x^s(t)$ denotes the symmetric equilibrium as a function of t, that is, $x^s(t) = B(x^s(t), t)$. Similarly, an asymmetric equilibrium is a pair $\left(x^1(t), x^2(t)\right)$ that solves $x^j(t) = B\left(x^{-j}(t), t\right)$ for $j = 1, 2$. Denote partial derivatives using subscripts, for example $B_t(x^s(t), t) = \partial B(y, t)/\partial t$ at the point $y = x^s(t)$.

Remark 2 Note that t may be an explicit change in U^{H}, U^{L} or f (e.g., replacing U^{H} by $U^{\mathrm{H}} + t$ or shifting the distribution function F to $F(x + t)$), but t may also be a change in a parameter in an application that affects one or more parameters in the model, e.g., a change in w_l in the career choice application, which affects both U^{L} and U^{H}.

Theorems 1 and 2 formalize the relationships between the direct and indirect effects. Their proofs follow, with elementary algebraic manipulations from Lemmas 1 and 2 that follow Theorem 2.

Theorem 1 states that in the case of a (stable) *symmetric* equilibrium, the combined equilibrium effect turns out to be of the same sign as the direct effect:

Theorem 1 *Consider a stable symmetric equilibrium $x^s(t)$. Then at $t = 0$, $x_t^s(t)$ has the same sign as $B_t(x^s(t), t)$.*

Theorem 2 below considers (stable) *asymmetric* equilibria $\left(x^1, x^2\right)$ where, recall, wlog $x^1 > x^2$. In this case, the relationship depends on the signs of the direct effects and their relative magnitudes. If the direct effects on the two players go in opposite directions (part 1 of the theorem), then the combined equilibrium effect has the same direction as the direct effect for each, and, moreover, the effect on x^1 is larger. If the direct effects are in the same direction, there are two cases: If the direct effect on player 2 is larger than that on player 1 (part 2a) the combined effect on player 2 is the same as the direct effect, while the combined effect on player 1 is the opposite. Otherwise (part 2b) at least one of the combined effects must be the same as the direct effect. However, in this case, which of the three possibilities—whether $x^1(t)$ or $x^2(t)$ or both change in the same direction as the direct effect—cannot be determined without further data.

Theorem 2 *Consider a stable asymmetric equilibrium $\left(x^1(t), x^2(t)\right)$, with the convention that $x^1 > x^2$. Then at $t = 0$:*

1. *If $B_t\left(x^j(t), t\right) < 0 < B_t\left(x^{-j}(t), t\right)$ for $j = 1$ or 2 (where one inequality may be weak), then*

$$x_t^j(t) > 0 > x_t^{-j}(t)$$

   Moreover $\left|x_t^1(t)\right| > \left|x_t^2(t)\right|$.
2. *Otherwise,*
   (a) *If $\left|B_t\left(x^1(t), t\right)\right| \geq \left|B_t\left(x^2(t), t\right)\right| > 0$, then*

$$sign\left(x_t^2(t)\right) = sign\left(B_t\left(x^1(t), t\right)\right) \text{ and } sign\left(x_t^1(t)\right) = -sign\left(B_t\left(x^2(t), t\right)\right).$$

   (b) *If $0 < \left|B_t\left(x^1(t), t\right)\right| < \left|B_t\left(x^2(t), t\right)\right|$ then*

$$sign\left(x_t^2(t)\right) = sign\left(B_t\left(x^1(t), t\right)\right) \text{ or } sign\left(x_t^1(t)\right) = sign\left(B_t\left(x^2(t), t\right)\right).$$

These theorems follow, with elementary algebraic manipulations, from the next two lemmas.

Lemma 1 *In a stable asymmetric equilibrium $\left(x^1(t), x^2(t)\right)$, with the convention that $x^1 > x^2$, at $t = 0$,*

$$\left|B_x\left(x^1(t), t\right)\right| < 1 < \left|B_x\left(x^2(t), t\right)\right|.$$

Proof At $t = 0$, $B_x\left(x^j(t), t\right) = \left(U^{\mathrm{H}} - U^{\mathrm{L}}\right) f\left(x^j\right)$. Recall that $x^1 > x^s > x^2$, and that f is decreasing in this region. Thus $\left|B_x\left(x^1(t), t\right)\right| < \left|B_x\left(x^s(t), t\right)\right| < \left|B_x\left(x^2(t), t\right)\right|$ where x^s denotes the unstable symmetric equilibrium. Since $\left|B_x\left(x^s(t), t\right)\right| > 1$ (by instability) and $\left|B_x\left(x^1(t), t\right)\right| \left|B_x\left(x^2(t), t\right)\right| < 1$ (by stability) we have

$$\left|B_x\left(x^1(t), t\right)\right| < 1 < \left|B_x\left(x^2(t), t\right)\right|.$$

□

Lemma 2 *In a stable equilibrium* $\left(x^1(t), x^2(t)\right)$*, at* $t = 0$*,*

$$sign\left(x_t^1(t)\right) = sign\left(B_t\left(x^2(t), t\right) + B_x\left(x^2(t), t\right) B_t\left(x^1(t), t\right)\right)$$

and likewise

$$sign\left(x_t^2(t)\right) = sign\left(B_t\left(x^1(t), t\right) + B_x\left(x^1(t), t\right) B_t\left(x^2(t), t\right)\right)$$

Proof Taking derivatives of $x^j = B\left(x^{-j}(t), t\right)$ wrt t we obtain:

$$\begin{aligned} x_t^1(t) &= B_t\left(x^2(t), t\right) + B_x\left(x^2(t), t\right) x_t^2(t) \\ x_t^2(t) &= B_t\left(x^1(t), t\right) + B_x\left(x^1(t), t\right) x_t^1(t) \end{aligned}$$

and thus

$$\begin{aligned} x_t^1\left(1 - B_x\left(x^2(t), t\right) B_x\left(x^1(t), t\right)\right) = B_t\left(x^2(t), t\right) \\ + B_x\left(x^2(t), t\right) B_t\left(x^1(t), t\right). \end{aligned}$$

By stability, at $t = 0$, $1 - B_x\left(x^2(t), t\right) B_x\left(x^1(t), t\right) > 0$. Thus we obtain the statement of the Lemma. □

4.2 Applicability of the comparative statics results

The general comparative statics results above yield interesting predictions in some cases—in particular when the stable equilibrium is asymmetric. Consider thus an equilibrium $\left(x^1, x^2\right)$, with the convention that $x^1 > x^2$, and consider an increase in U^{H}. This has an unambiguous (and perhaps surprising) effect on the equilibrium strategies. Player 1's threshold, x^1, unambiguously decreases and player 2's threshold, x^2, increases. The threshold of player 1 decreases because in this case the strategic substitutes effect is so strong that for her the indirect effect—of player 2 choosing H more often—must dominate the direct effect.

Why is the comparative statics on U^{H} unambiguous? Details follow from the proof of Theorem 2, but we provide the basic ideas here. The direct effect of a change in U^{H} is stronger for player 2 than for player 1, as 2 faces an opponent who more often plays H (recall that $x^1 > x^2$). Moreover, we show that player 1 reacts to a change in player 2's threshold more strongly than the change that occurs in player 2's threshold itself (i.e., the slope of the best-reply function is steeper than 1). Combining these two arguments implies that the indirect effect dominates the direct effect for player 1, and thus the overall effect must be a decrease in player 1's threshold. For player 2, the opposite holds since the slope of her best-reply function is less than 1 and player 1's direct effect is smaller than that of player 2's.

To see the above more formally, recall that for $j = 1, 2$, the best-response function is:

$$x^j = B\left(x^{-j}\right) \equiv U^{\mathrm{H}} F\left(x^{-j}\right) + U^{\mathrm{L}}\left(1 - F\left(x^{-j}\right)\right).$$

The derivatives with respect to $t = U^{\mathrm{H}}$ are $B_t^j = F\left(x^{-j}\right) > 0$. Since $F\left(x^1\right) > F\left(x^2\right)$, we thus have $B_t^2\left(x^1\right) > B_t^1\left(x^2\right) > 0$. By Theorem 2 part (2a), the combined effects are $x_t^2 > 0$ and $x_t^1 < 0$.

In our three applications, the above analysis of a change in U^{H} yields the following conclusions. Recall that we consider a stable asymmetric equilibrium in which $x^1 > x^2$. In the public good environment, if the benefit of having a second contribution decreases, then x^2 will become even smaller while x^1 will increase further. That is, the player investing less often will invest even less frequently due to this, but the player investing more often will invest even more despite the benefit of doing so going down. In an asymmetric equilibrium of the R&D example, a decrease in duopoly competition (e.g., an increase in product differentiation) would lead the player to invest more in the asymmetric equilibrium to *decrease* his investment and the one investing less to invest more. Finally, in the gender occupation choice example, a tax increase on households with two high incomes would lower U^{H} and hence further decrease the threshold of the gender choosing the high income less often but would increase the threshold of the one already choosing it often.

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