Econ Theory (2018) 66:557–591
https://doi.org/10.1007/s00199-017-1079-3

RESEARCH ARTICLE

Directional monotone comparative statics

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Received: 20 February 2017 / Accepted: 8 September 2017 / Published online: 16 September 2017

Abstract Many questions of interest in economics can be stated in terms of monotone comparative statics: If a parameter of a constrained optimization problem increases, when does its solution increase as well. We characterize monotone comparative statics in different directions in finite-dimensional Euclidean space by extending the monotonicity theorem of Milgrom and Shannon (Econometrica 62(1):157–180, 1994) to constraint sets ordered in Quah (Econometrica 75(2):401–431, 2007)'s set order. Our characterizations are ordinal and retain the same flavor as their counterparts in the standard theory, showing new connections to the standard theory and presenting new results. The results are highlighted with several applications (in consumer theory, producer theory, and game theory) which were previously outside the scope of the standard theory of monotone comparative statics.

Keywords Monotone comparative statics · i-Directional single crossing property · i-Directional set order · Quasisupermodular function

JEL Classification C61 · C70 · D00

Part of this paper was written when Sabarwal was visiting Université Paris 1 Panthéon-Sorbonne as an invited professor. He is grateful for their warm welcome and hospitality.

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1 Introduction

In economics and game theory, we are frequently interested in how solutions to a constrained optimization problem change when the environment changes. In many cases, the question of interest can be stated in terms of monotone comparative statics: If a parameter of the constrained optimization problem increases, when does its solution increase as well. For example, if a consumer's wealth (or purchasing power) goes up, when does her demand for a particular good go up (the case of a normal good)? Or, if a firm is competing as an oligopoly in multiple markets by producing differentiated products, if plant size increases, when does its output in a given market increase? Or, in the case of a polluting technology, if technological innovation increases, when will pollution abatement and output of the firm both go up? And so on.

We present results that apply to these types of questions. Consider the standard framework of monotone comparative statics. Let X be a set, $f : X \to \mathbb{R}$, and A, B be subsets of X ordered by some relation, $A \sqsubseteq B$. When is it true that $A \sqsubseteq B \Rightarrow \arg\max_A f \sqsubseteq \arg\max_B f$? Intuitively, when is $\arg\max_A f$ increasing in A? Or, more generally, $f : X \times T \to \mathbb{R}$, where T is a partially ordered set. When is it true that $A \sqsubseteq B$ and $t \preceq t' \Rightarrow \arg\max_A f(\cdot, t) \sqsubseteq \arg\max_B f(\cdot, t')$? Intuitively, when is $\arg\max_A f(\cdot, t)$ increasing in (A, t)?

Milgrom and Shannon (1994) show that when X is a lattice[1] and $\sqsubseteq$ is the standard lattice set order, denoted $\sqsubseteq^{lso}$, $\arg\max_A f(\cdot, t)$ *is increasing in* (A, t) *in the standard lattice set order, if, and only if, for every* $t \in T$, $f(\cdot, t)$ *is quasisupermodular on* X *and* f *satisfies single crossing property on* $X \times T$.[2] There are several appealing features of such lattice-theoretic monotone comparative statics results. For example, the sets X and A are not required to be convex and can be finite, the objective function f is not required to be differentiable or continuous, and the results apply even when there are multiple solutions to the optimization problem. Moreover, the notion of quasisupermodularity has a nice economic intuition in terms of complementarities: When X is a product space, when one component variable increases, the "marginal" benefit of another component variable goes up. Some of this standard theory is developed in Topkis (1978), Topkis (1979), LiCalzi and Veinott (1992), Veinott (1992), and Milgrom and Shannon (1994). For a development with partially ordered sets, confer Smithson (1971). These ideas have had many applications in economic theory and game theory, including developing the theory of supermodular games, submodular games, aggregative games, and comparing equilibria.[3]

[1] Recall that a lattice is a partially ordered set in which every two points have a supremum and an infimum. For example, $\mathbb{R}^N$ is a lattice, with the standard product partial order.

[2] Recall: $A \sqsubseteq^{lso} B$, if for every $a \in A, b \in B$, $a \wedge b \in A$ and $a \vee b \in B$. Moreover, $f : X \to \mathbb{R}$ is quasisupermodular, if for every $a, b \in X$, $f(a) \geq (>) f(a \wedge b) \implies f(a \vee b) \geq (>) f(b)$, and $f : X \times T \to \mathbb{R}$ satisfies single crossing property on $X \times T$, if for every $a, b \in X$ with $a \succeq b$ and for every $t, t' \in T$ with $t' \succeq t$, $f(a, t) \geq (>) f(b, t) \Rightarrow f(a, t') \geq (>) f(b, t')$.

[3] Some of this can be seen in Bulow et al. (1985), Vives (1990), Milgrom and Roberts (1990), Zhou (1994), Amir (1996), Amir and Lambson (2000), Echenique (2002), Echenique (2004), Heikkilä and Reffett (2006), Zimper (2007), Roy and Sabarwal (2008, 2010, 2012), Quah and Strulovici (2009), Jensen (2010), Balbus et al. (2014), Monaco and Sabarwal (2016), Amir and Lazzati (2016), Reynolds and Rietzke (2017), Cosandier et al. (2017), and others.

A limitation of these results is that they do not apply to some basic economic problems in which constraint sets are not ordered in the standard lattice set order. For example, consider the standard budget set in consumer theory: $B(p, w) = \{x \in \mathbb{R}_+^N \mid p \cdot x \leq w\}$, where $p \in \mathbb{R}^N$, $p \gg 0$ is a price system, and wealth is $w > 0$. As is well known, for $w < w'$, $B(p, w) \not\sqsubseteq^{lso} B(p, w')$, and therefore, the standard lattice-based monotone comparative statics results cannot be applied directly to the theory of demand.

Quah (2007) develops monotone comparative statics results to include such problems. He considers $f : X \to \mathbb{R}$, where X is a convex sublattice of $\mathbb{R}^N$, and $i \in \{1, \ldots, N\}$ is a direction in $\mathbb{R}^N$. His techniques include new binary relations, denoted Δ_i^λ, ∇_i^λ, a new set order, termed $\mathcal{C}_i$-flexible set order, and a new notion of $\mathcal{C}_i$-quasisupermodular function.[4] In particular, if $w < w'$, then $B(p, w)$ is lower than $B(p, w')$ in the $\mathcal{C}_i$-flexible set order. A main result is: $\arg\max_A f$ *is increasing in* A *in the* $\mathcal{C}_i$*-flexible set order, if, and only if,* f *is* $\mathcal{C}_i$*-quasisupermodular.* Moreover, a sufficient condition for f to be $\mathcal{C}_i$-quasisupermodular is that f is supermodular and i-concave.[5]

Quah (2007) uses some assumptions that are less typical in the standard theory of monotone comparative statics. The domain, X, of the objective function is assumed to be convex. This rules out discrete spaces; in particular, finite games and cases where consumption of some goods is more naturally modeled as discrete, for example, automobiles and homes. Moreover, the notion of $\mathcal{C}_i$-quasisupermodular function uses the binary relations Δ_i^λ, ∇_i^λ and convexity of domain in important ways, and it is less transparent than standard assumptions of quasisupermodularity and single crossing property. Furthermore, the binary relations Δ_i^λ, ∇_i^λ have some counter-intuitive properties—they are non-commutative and their outcomes are not necessarily comparable in the underlying order in $\mathbb{R}^N$. Finally, the framework does not include parameterized objective functions, which rules out cases involving the effect of actions of others on a given agent's payoff, for example, cases with public goods, externalities from other consumers or producers, and more generally, game theoretic strategic effects based on actions of other players.

The framework in this paper includes both parameterized objective functions and budget-type constraint sets and in this sense is an extension of Milgrom and Shannon (1994) to Quah (2007)'s set order. The basic setup is as follows. Consider a sublattice X of $\mathbb{R}^N$, T a partially ordered set, $f : X \times T \to \mathbb{R}$, and a direction $i \in \{1, \ldots, N\}$. A main result is: $\arg\max_A f(\cdot, t)$ *is increasing in* (A, t) *in the* i*-directional set order, if, and only if, for every* $t \in T$, $f(\cdot, t)$ *is* i*-quasisupermodular and satisfies* i*-single crossing property on* X*, and* f *satisfies basic* i*-single crossing property on* $X \times T$. These terms are defined more concretely in the next section, but intuitively, increase in the i-directional set order formalizes the idea of increase in the ith direction in $\mathbb{R}^N$. In our characterization, X is not required to be convex and there is no use of the binary relations Δ_i^λ, ∇_i^λ. The framework allows for parameter effects in the objective

[4] Formal definitions are presented in "Appendix A".

[5] Intuitively, i-concave requires concavity in every direction u, where u is a vector with $u_i = 0$.

function. The new properties i-quasisupermodular, i-single crossing, and basic i-single crossing retain the same flavor as their counterparts in the standard theory of monotone comparative statics. The i-directional set order is a reformulation of Quah's $\mathcal{C}_i$-flexible set order to align more closely with the spirit of monotone methods, and this helps subsume results in Quah (2007).

Our main result is explored in several directions. It is extended to apply to all directions i, it is specialized to consider comparative statics with respect to A only or to t only, and the ordinal nature of the properties allows for increasing transformations of the objective function to also respect the same characterization. Sufficient conditions are explored as well. In particular, Quah's sufficient conditions of supermodular and i-concave remain sufficient in the more general setting here. Furthermore, the characterization here has a natural formulation in terms of cardinal assumptions: i-supermodular and i-increasing differences, and in turn, this has a new and natural formulation in terms of differential conditions using directional derivatives.

Including parameters in the objective function and allowing for more general constraint sets allows our results to apply to cases where standard results in monotone comparative statics are inapplicable.

In consumer theory, we replicate and extend Quah (2007)'s result on normal demand with finitely many divisible goods to allow for up to two discrete goods and more general utility functions. Moreover, we present a new application for parameterized utility functions, using a Stone–Geary-type utility function.

In game theory, we examine a multi-market oligopoly with capacity constraints in which we may conduct monotone comparative statics simultaneously with respect to competitor output and capacity constraint. We also show how a model of auctions with bidding constraints can be analyzed using the results here.

We also show how our results may provide unifying tools for seemingly different applied work. As one example, we show that in a model of emissions standards such as those in Montero (2002) and in Bruneau (2004), a main result that technological innovation can simultaneously increase both pollution abatement and output can be derived by an easy calculation based on our method. As another example, we show that in a discrete choice model of labor supply such as that in Hoynes (1996), our results make it easy to show that both hours worked and leisure hours increase with the overall time constraint and that optimal labor supply depends positively on wage rate and negatively on non-labor income.

The paper proceeds as follows. Section 2 formalizes the constrained optimization problem, the set orders, and properties on objective function. Section 3 presents the main results and corollaries on directional monotone comparative statics. The main results are explored further in subsections formalizing sufficient conditions and differential conditions. Section 4 presents several applications of the main results. "Appendix A" presents some connections to Quah (2007), and "Appendix B" includes details of some proofs.

2 Constrained optimization

Recall that a lattice[6] is a partially ordered set in which every two elements, a and b, have a supremum in the set, denoted $a \vee b$, and an infimum in the set, denoted $a \wedge b$. The supremum and infimum operations are with respect to the partial order. In this paper, we work with finite-dimensional Euclidean space, represented by $\mathbb{R}^N$. This is a lattice in the standard product order on $\mathbb{R}^N$, denoted, as usual, by $\leq$,[7] and in this order, for $a, b \in \mathbb{R}^N$, $a \wedge b = (\min\{a_1, b_1\}, \ldots, \min\{a_N, b_N\})$ and $a \vee b = (\max\{a_1, b_1\}, \ldots, \max\{a_N, b_N\})$. A subset X of a lattice is a sublattice, if for every a and b in X, their supremum in the overall lattice, $a \vee b$, is in X, and their infimum in the overall lattice, $a \wedge b$, is in X.

Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, A be a subset of X, and consider the constrained maximization problem $\max_A f(\cdot, t)$. We are interested in how $\arg\max_A f(\cdot, t)$ changes with (A, t). As the set of maximizers is not necessarily a singleton, this involves a comparison of sets.

2.1 Set orders

There are several set orders on subsets of a lattice (confer Topkis 1998). Two of the more common ones are as follows. Consider a sublattice X of $\mathbb{R}^N$, and subsets A and B of X. A ***is lower than*** B ***in the standard lattice set order***, denoted $A \sqsubseteq^{lso} B$, if for every $a \in A, b \in B$, it follows that $a \wedge b \in A$ and $a \vee b \in B$. A ***is lower than*** B ***in the weak set order***, denoted $A \sqsubseteq^{wso} B$, if for every $a \in A$, there is $b \in B$ such that $a \leq b$, and for every $b \in B$, there is $a \in A$ such that $a \leq b$.[8] Moreover, another set order is of interest when we are considering increases in a particular component of vectors: For $i \in \{1, 2, \ldots, N\}$, A ***is lower than*** B ***in the*** i***-weak set order***, denoted $A \sqsubseteq_i^{wso} B$, if for every $a \in A$, there is $b \in B$ such that $a_i \leq b_i$, and for every $b \in B$, there is $a \in A$ such that $a_i \leq b_i$. As is well known and easy to check: $A \sqsubseteq^{lso} B \implies A \sqsubseteq^{wso} B \implies A \sqsubseteq_i^{wso} B$.

The standard results in monotone comparative statics typically use the standard lattice set order, but that order cannot compare some of the constraint sets of interest here, and therefore, to expand comparability of sets, we work with the following weakenings of the standard lattice set order. Let X be a sublattice of $\mathbb{R}^N$, A and B be subsets of X, and $i \in \{1, 2, \ldots, N\}$. A ***is lower than*** B ***in the*** i***-directional set order***, denoted, $A \sqsubseteq_i^{dso} B$, if for every $a \in A$ and $b \in B$ with $a_i > b_i$, there is $v = s(b - a \wedge b)$ for some $s \in [0, 1]$ such that $a + v \in B$ and $b - v \in A$.[9] In this definition, notice that the vector v satisfies $v \geq 0$, and therefore, $a \leq a + v$ and $b - v \leq b$. Moreover,

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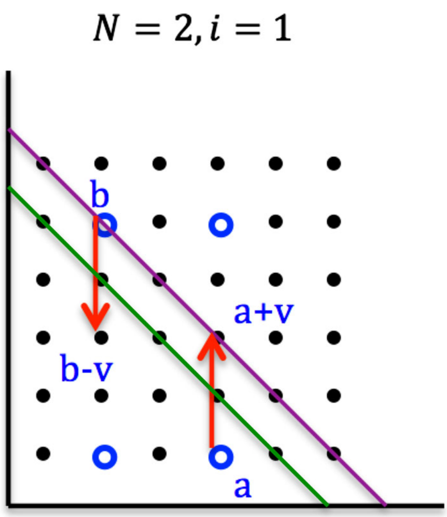
6 This paper uses standard lattice terminology. See, for example, Topkis (1998).

7 For $a, b \in \mathbb{R}^N$, $a \leq b$ means that for every $i = 1, \ldots, N$, $a_i \leq b_i$.

8 In all the set orders considered here, when convenient, we may say A is lower than B equivalently as B is higher than A.

9 The i-directional set order is a reformulation of the C_i-flexible set order in Quah (2007). The definition here retains the spirit of monotone methods, does not require X to be convex, and there is no use of the operators Δ_i^λ, ∇_i^λ. Comparisons to Quah (2007) are presented in "Appendix A".

Fig. 1 *i*-Directional set order



when $a \geq b$, this condition is the same as for the lattice set order, and therefore, a non-trivial application of this order is when $a_i > b_i$ and $a \ngeq b$. Figure 1 shows this idea graphically. For intuition, we can consider the two-good discretized consumption space, and budget-type sets given by the green and the purple lines. For these sets to be ranked in the 1-directional set order, for each a in the lower set and b in the higher set with $a_1 > b_1$, there is $v = s(b - a \wedge b)$ such that $a + v$ is in the higher set and $b - v$ is in the lower set.

Similarly, say that A ***is lower than*** B ***in the directional set order***, denoted $A \sqsubseteq^{dso} B$, if for every $i \in \{1, 2, \ldots, N\}$, A is lower than B in the i-directional set order.

Proposition 1 *Let X be a sublattice of $\mathbb{R}^N$ and A, B be non-empty subsets of X.*

(1) $A \sqsubseteq^{lso} B \;\Rightarrow\; A \sqsubseteq_i^{dso} B \;\Rightarrow\; A \sqsubseteq_i^{wso} B$, *for each $i \in \{1, 2, \ldots, N\}$, and*
(2) $A \sqsubseteq^{lso} B \;\Rightarrow\; A \sqsubseteq^{dso} B \;\Rightarrow\; A \sqsubseteq^{wso} B$.

Proof The proof of (1) is similar to that of (2). To prove (2), suppose first that $A \sqsubseteq^{lso} B$. Fix $i \in \{1, 2, \ldots, N\}$, $a \in A$, and $b \in B$ with $a_i > b_i$. Let $s = 1$. Then $b - v = b - 1(b - a \wedge b) = a \wedge b \in A$ and $a + v = a + 1(a \vee b - a) = a \vee b \in B$. Thus, for every $i \in \{1, 2, \ldots, N\}$, $A \sqsubseteq_i^{dso} B$, whence $A \sqsubseteq^{dso} B$. Now suppose $A \sqsubseteq^{dso} B$. Fix $a \in A$. As B is non-empty, let $b \in B$. If $a \leq b$, then we are done. Otherwise, there is i such that $a_i > b_i$. In this case, there is $v = s(b - a \wedge b)$ for some $s \in [0, 1]$ such that $a + v \in B$. Moreover, $v \geq 0$ implies $a \leq a + v$. The proof is similar for the other case: $b \in B$ implies there is $a \in A$ such that $a \leq b$. □

As shown in this proposition, the i-directional set order is weaker than the standard lattice set order and stronger than the i-weak set order. Similarly, the directional set order is weaker than the standard lattice set order and stronger than the weak set order. One benefit of the i-directional set order is that it can order budget sets for different levels of wealth, whereas the standard lattice set order cannot.

Example 1-1 (Walrasian budget sets) Let $X = \mathbb{R}^N_+$, $N \geq 2$, $p \gg 0$, and $w > 0$. The Walrasian budget set at (p, w) is given by $B(p, w) = \left\{x \in \mathbb{R}^N_+ \mid p \cdot x \leq w\right\}$. We know that in the standard lattice set order when $w < w'$, $B(p, w) \not\sqsubseteq^{lso} B(p, w')$, but these budget sets are comparable in the directional set order: $w < w' \implies B(p, w) \sqsubseteq^{dso} B(p, w')$, as follows. Fix $i \in \{1, 2, \ldots, N\}$, $a \in B(p, w)$ and $b \in B(p, w')$ with $a_i > b_i$. If $p \cdot (a \vee b) \leq w'$, let $s = 1$, and therefore, $v = b - a \wedge b$. In this case, $b - v = a \wedge b \in B(p, w)$, and $a + v = a \vee b \in B(p, w')$. Moreover, if $p \cdot b \leq w$, let $s = 0$, and so, $v = 0$. In this case, $b - v = b \in B(p, w)$, and $a + v = a \in B(p, w')$. In the other cases, let $s \in \left[\frac{p \cdot b - w}{p \cdot (b - a \wedge b)}, \frac{w' - p \cdot a}{p \cdot (b - a \wedge b)}\right] \subset [0, 1]$, and therefore, $v = s(b - a \wedge b)$. In this case, $p \cdot (b - v) \leq w$ and $p \cdot (a + v) \leq w'$. Consequently, $a + v \in B(p, w')$ and $b - v \in B(p, w)$, as desired.

Example 1-2 (Two-good discretized Walrasian budget sets) In the two-good case, the directional set order can be used to order budget sets with discrete consumption. Consider two goods, each consumed in integer amounts. Let $X = \mathbb{Z}^2_+$, $p = (p_1, p_2) \gg 0$, and $w > 0$. The (discretized) Walrasian budget set at (p, w) is given by $B(p, w) = \left\{x \in \mathbb{Z}^2_+ \mid p \cdot x \leq w\right\}$. Consider $w < w'$ and suppose p_1 divides $w' - w$ and p_2 divides $w' - w$. In this case, $w < w' \implies B(p, w) \sqsubseteq^{dso} B(p, w')$, as follows. Fix $i = 1$. Let $a \in B(p, w)$ and $b \in B(p, w')$ with $a_1 > b_1$. As above, if $p \cdot (a \vee b) \leq w'$, let $s = 1$, and if $p \cdot b \leq w$, let $s = 0$. Notice that these cases include the case where $a \geq b$. So suppose $a_1 > b_1$ and $a_2 \not\geq b_2$. Then $b - a \wedge b = (0, b_2 - a_2) > 0$, and $p \cdot (b - a \wedge b) = p_2(b_2 - a_2)$. Let $s = \frac{w' - w}{p \cdot (b - a \wedge b)} = \frac{w' - w}{p_2(b_2 - a_2)}$ and $v = s(b - a \wedge b)$. Notice that $b - v = (b_1, b_2 - s(b_2 - a_2)) = (b_1, b_2 - \frac{w' - w}{p_2}) \in \mathbb{Z}^2_+$, because p_2 divides $w' - w$. Thus $B(p, w) \sqsubseteq^{dso}_1 B(p, w')$. Similarly, $B(p, w) \sqsubseteq^{dso}_2 B(p, w')$, whence $B(p, w) \sqsubseteq^{dso} B(p, w')$.

When there are three or more discrete goods, the discretized Walrasian budget set is not necessarily comparable in the directional set order. Consider $X = \mathbb{Z}^3_+$, $p = (1, 1, 1)$, $w = 1$, $w' = 2$, and $B(p, w) = \{x \in \mathbb{Z}^3_+ \mid p \cdot x \leq 1\}$ and $B(p, w') = \{x \in \mathbb{Z}^3_+ \mid p \cdot x \leq 2\}$. Let $i = 1$, $a = (1, 0, 0) \in B(p, w)$, and $b = (0, 1, 1) \in B(p, w')$. Then $a_1 > b_1$, and for $s \in [0, 1]$ consider $v = s(b - a \wedge b)$. It is easy to check that for $s = 0$, $b - v \notin B(p, w)$, for $s = 1$, $a + v \notin B(p, w')$, and for $s \in (0, 1)$, $b - v \notin \mathbb{Z}^3_+$. Thus, $B(p, w) \not\sqsubseteq^{dso}_1 B(p, w')$.

This does not imply that other sets in higher dimensions are not comparable in the directional set order. For example, consider $A = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $B = \{(0, 2, 0), (1, 1, 0), (0, 1, 1), (1, 1, 1)\}$. In this case, $A \not\sqsubseteq^{lso} B$, because for $a = (1, 0, 0)$ and $b = (0, 2, 0)$, $a \vee b = (1, 2, 0) \notin B$. But it is easy to check that for $i = 1, 2, 3$, $A \sqsubseteq^{dso}_i B$, and therefore, $A \sqsubseteq^{dso} B$.

Moreover, it is easy to see that examples 1-1 and 1-2 can be combined to show that Walrasian budget sets are comparable in the case of finitely many goods, at most two of which are discrete.

Additional classes of sets comparable in the i-directional set order can be derived in a manner analogous to Quah (2007). One such class is presented in “Appendix B”.

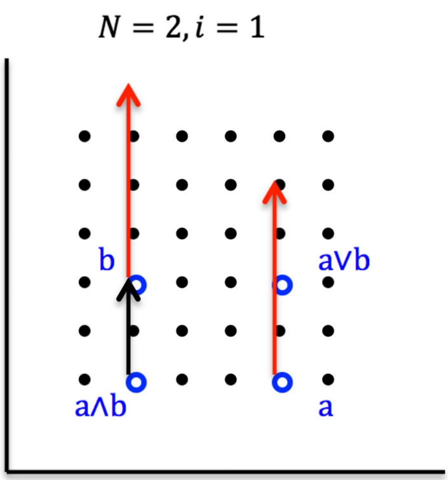


Fig. 2 i-Single crossing property on X

2.2 Objective function

Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f is ***i-quasisupermodular on*** X, if for every $a, b \in X$ with $a_i > b_i$, $f(a) \geq (>) f(a \wedge b) \Longrightarrow f(a \vee b) \geq (>) f(b)$. In this definition, notice that when $a \geq b$, these conditions are satisfied trivially. Therefore, non-trivial application of this definition is when $a_i > b_i$ and $a \ngeq b$. The intuition is the same as in the standard notion of a quasisupermodular function. In other words, if the trade-off between a and $a \wedge b$ is favorable (in the sense that $f(a) \geq f(a \wedge b)$ or $f(a) > f(a \wedge b)$), then the trade-off remains favorable at $a \vee b$ and b, in the same sense. Indeed, recall the definition of a quasisupermodular function: f is ***quasisupermodular on*** X, if for every $a, b \in X$ $f(a) \geq (>) f(a \wedge b) \Longrightarrow f(a \vee b) \geq (>) f(b)$. It is easy to check that *for every i, f is i-quasisupermodular on X, if, and only if, f is quasisupermodular on X.*

Another useful property is the following. Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f satisfies ***i-single crossing property on*** X, if for every $a, b \in X$ with $a_i > b_i$, and for every $v \in \{s(b - a \wedge b) \mid s \in \mathbb{R}, s \geq 0\}$ such that $a + v, b + v \in X$, $f(a) \geq (>) f(b) \Longrightarrow f(a + v) \geq (>) f(b + v)$. In this definition, notice that $v \geq 0$, and $v_i = 0$. Moreover, when $a \geq b$, these conditions are satisfied trivially. Therefore, non-trivial application of this property is when $a_i > b_i$ and $a \ngeq b$. Figure 2 presents a graphical idea.
Notice that the black arrow is $(b - a \wedge b)$ and the red arrow is (translated) $s(b - a \wedge b)$. Intuitively, this property says that if the trade-off between a and b is initially favorable (in the sense that $f(a) \geq f(b)$ or $f(a) > f(b)$), then it remains favorable when we move in the direction $b - a \wedge b$. This intuition is similar to that of the standard single crossing property. In particular, as $v = s(b - a \wedge b)$ satisfies $v \geq 0$ and $v_i = 0$, we may reformulate i-single crossing property as follows: for every $a, b \in X$ with $a_i > b_i$, and for every $v \in \{s(b - a \wedge b) \mid s \geq 0\}$ such that $a + v, b + v \in X$, $f(a_i, a_{-i}) \geq (>) f(b_i, b_{-i}) \Longrightarrow f(a_i, a_{-i} + v_{-i}) \geq (>) f(b_i, b_{-i} + v_{-i})$. This reformulation captures the flavor of the standard single crossing property as follows.

For a, b with $a_i > b_i$, if $f(a_i, a_{-i}) \geq (>)\ f(b_i, b_{-i})$, then when we increase a_{-i} and b_{-i} by a nonnegative $v_{-i} = [s(b - a \wedge b)]_{-i}$, the trade-off remains favorable. Similarly, f satisfies ***directional single crossing property on*** X, if for every $i \in \{1, 2, \ldots, N\}$, f satisfies i-single crossing property on X.

In order to consider parameterized objective functions, let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f satisfies ***basic i-single crossing property on*** $X \times T$, if for every $a, b \in X$ with $a_i > b_i$, and for every $t, t' \in T$ with $t' \succeq t$, $f(a, t) \geq (>)\ f(b, t) \Longrightarrow f(a, t') \geq (>)\ f(b, t')$.[10] The function f satisfies ***basic directional single crossing property on*** $X \times T$, if for every $i \in \{1, 2, \ldots, N\}$, f satisfies basic i-single crossing property on $X \times T$. For convenience of reference, the word "basic" is used in basic i-single crossing property on $X \times T$ to distinguish this definition from that for i-single crossing property on X. It is easy to check that *if f satisfies basic directional single crossing property on $X \times T$, then f satisfies (standard) single crossing property in $(x; t)$.*[11]

3 Directional monotone comparative statics

Some of the main results in this paper concern conditions on f that yield monotone comparative statics, formalized as follows. Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f satisfies ***i-directional monotone comparative statics on*** $X \times T$, if for every A, B subset of X, and for every t, t' in T, $A \sqsubseteq_i^{dso} B$ and $t \preceq t' \Longrightarrow \arg\max_A f(\cdot, t) \sqsubseteq_i^{dso} \arg\max_B f(\cdot, t')$. In other words, f satisfies i-directional monotone comparative statics formalizes the idea that $\arg\max_A f(\cdot, t)$ is increasing in (A, t) in the i-directional set order. Similarly, f satisfies ***directional monotone comparative statics on*** $X \times T$, if for every $i \in \{1, 2, \ldots, N\}$, f satisfies i-directional monotone comparative statics on $X \times T$. With these, we have the following results.

3.1 Main results

Theorem 1 *Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The following are equivalent.*

(1) *f satisfies i-directional monotone comparative statics on $X \times T$.*
(2) *For every $t \in T$, $f(\cdot, t)$ is i-quasisupermodular and satisfies i-single crossing property on X, and f satisfies basic i-single crossing property on $X \times T$.*

Proof Suppose first that (2) holds. Let $A \sqsubseteq_i^{dso} B$ and $t \preceq t'$. Let $a \in \arg\max_A f(\cdot, t)$, $b \in \arg\max_B f(\cdot, t')$, and $a_i > b_i$. Then there is $v = s(b - a \wedge b)$ for some $s \in [0, 1]$ such that $a + v \in B$ and $b - v \in A$.

[10] Notice that this is a strong property, but as shown in the characterization in the main theorem, this is necessary and sufficient for i-directional monotone comparative statics, as defined.

[11] For every $a, b \in X$ with $a \geq b$ and for every $t, t' \in T$ with $t' \succeq t$, $f(a, t) \geq (>)\ f(b, t) \Longrightarrow f(a, t') \geq (>)\ f(b, t')$.

As case 1, suppose $s = 1$. Then $a \wedge b = b - b + a \wedge b = b - v \in A$, and $a \vee b = a + s(a \vee b - a) = a + v \in B$. As $a \in \arg\max_A f(\cdot, t)$, it follows that $f(a, t) \geq f(a \wedge b, t)$, and then i-quasisupermodularity on X implies $f(a \vee b, t) \geq f(b, t)$, and then basic i-single crossing property on $X \times T$ implies $f(a \vee b, t') \geq f(b, t')$. As $b \in \arg\max_B f(\cdot, t')$, it follows that $a + v = a \vee b \in \arg\max_B f(\cdot, t')$. Therefore, $f(a \vee b, t') = f(b, t')$. In particular, $f(a \vee b, t') \not> f(b, t')$, and again i-quasisupermodularity implies $f(a, t') \not> f(a \wedge b, t')$, and then basic i-single crossing property on $X \times T$ implies $f(a, t) \not> f(a \wedge b, t)$. Consequently, $f(a, t) \leq f(a \wedge b, t)$, and it follows that $b - v = a \wedge b \in \arg\max_A f(\cdot, t)$.

As case 2, suppose $s < 1$. Then $a \in \arg\max_A f(\cdot, t)$ and $b - v \in A$ imply $f(a, t) \geq f(b - v, t)$. Moreover, when looking at the ith component, $a_i > b_i \geq (b - v)_i$, because $v = s(b - a \wedge b) \geq 0$. Applying i-single crossing property on X to a and $b - v$, with the directional vector $w = \frac{s}{1-s}[(b - v) - a \wedge (b - v)]$ implies $f(a + w, t) \geq f(b - v + w, t)$. Notice that $v = s(b - a \wedge b) = s[(b - v) - a \wedge b] + sv = s[(b - v) - a \wedge (b - v)] + sv$, and therefore, $v = \frac{s}{1-s}[(b - v) - a \wedge (b - v)] = w$. In other words, $f(a + v, t) \geq f(b, t)$, and then basic i-single crossing property on $X \times T$ implies $f(a + v, t') \geq f(b, t')$, whence $a + v \in \arg\max_B f(\cdot, t')$. Thus, $f(a + v, t') = f(b, t')$, whence $f(a + v, t') \not> f(b, t')$, or equivalently, $f(a + w, t') \not> f(b - v + w, t')$ and then using i-single crossing property on X, $f(a, t') \not> f(b - v, t')$, and then using basic i-single crossing property on $X \times T$, $f(a, t) \not> f(b - v, t)$. Thus, $b - v \in \arg\max_A f(\cdot, t)$, as desired.

In the other direction, suppose f satisfies i-directional monotone comparative statics on $X \times T$. Let's first see that for every t, $f(\cdot, t)$ is i-quasisupermodular on X. Fix t, and a, b with $a_i > b_i$. Form the sets $A = \{a, a \wedge b\}$ and $B = \{b, a \vee b\}$. Notice that $A \sqsubseteq_i^{dso} B$. (Consider $a \in A$ and $b \in B$. Let $v = b - a \wedge b$. Then $a + v = a \vee b \in B$ and $b - v = a \wedge b \in A$. The other cases are satisfied vacuously, because in those cases the ith component of the element from A is not greater than the ith component of the element from B.)

Suppose $f(a, t) \geq f(a \wedge b, t)$. Then $a \in \arg\max_A f(\cdot, t)$. Suppose to the contrary that $f(a \vee b, t) < f(b, t)$. Then $\arg\max_B f(\cdot, t) = \{b\}$. Applying f satisfies i-directional monotone comparative statics to (A, t) and (B, t), there is $s \in [0, 1]$ such that $a + s(a \vee b - a) \in \arg\max_B f(\cdot, t) = \{b\}$. But the ith component of $a + s(a \vee b - a)$ is a_i which is strictly greater than b_i, a contradiction. Therefore, $f(a \vee b, t) \geq f(b, t)$, as desired.

Now suppose $f(a, t) > f(a \wedge b, t)$. Then $\{a\} = \arg\max_A f(\cdot, t)$. Suppose to the contrary that $f(a \vee b, t) \leq f(b, t)$. Then $b \in \arg\max_B f(\cdot, t)$. By i-directional monotone comparative statics, there is $s \in [0, 1]$ such that $b - s(b - a \wedge b) \in \arg\max_A f(\cdot, t) = \{a\}$. But the ith component of $b - s(b - a \wedge b)$ is b_i which is strictly less than a_i, a contradiction. Therefore, $f(a \vee b, t) > f(b, t)$, as desired.

Let's now check that for every t, $f(\cdot, t)$ satisfies i-single crossing property on X. Fix t, and $a, b \in X$ with $a_i > b_i$. Fix $v = s(b - a \wedge b)$ with $s \geq 0$ such that $a + v, b + v \in X$. Before we proceed further, consider the following calculations. Let $y = b + v$, and let $u = y - a \wedge y = a \vee y - a$. Notice that $u = y - a \wedge y = y - a \wedge b = (1 + s)(b - a \wedge b)$. This implies that $v = s(b - a \wedge b) = \frac{s}{1+s}u$. Let $s' = \frac{s}{1+s} \in [0, 1)$ and write $v = s'u$. In particular, $y - s'(y - a \wedge y) = y - v$, and $a + s'(a \vee y - a) = a + v$. Now let $A = \{a, y - v\}$ and $B = \{y, a + v\}$. Then $A \sqsubseteq_i^{dso} B$, because for $a \in A$, and

$y \in B$, there is $s' \in [0,1]$, as above such that $a + s'(a \vee y - a) = a + v \in B$ and $y - s'(y - a \wedge y) = y - v \in A$. The other comparisons are vacuously true, because when considering the ith components, $(y - v)_i \leq y_i = b_i < a_i \leq (a + v)_i$.

Suppose $f(a,t) \geq f(b,t) = f(y - v, t)$. Then $a \in \arg\max_A f(\cdot, t)$. Suppose to the contrary that $f(a+v,t) < f(b+v,t) = f(y,t)$. Then $\{y\} = \arg\max_B f(\cdot,t)$. As f satisfies i-directional monotone comparative statics on $X \times T$, there is $\hat{s} \in [0,1]$ such that $a + \hat{s}(a \vee y - a) \in \arg\max_B f(\cdot,t) = \{y\}$. But considering the ith components, $(a + \hat{s}(a \vee y - a))_i = a_i > b_i = y_i$, a contradiction. Thus $f(a+v,t) \geq f(b+v,t)$, as desired.

Now suppose $f(a,t) > f(b,t) = f(y - v, t)$. Then $\{a\} = \arg\max_A f(\cdot,t)$. Suppose to the contrary that $f(a+v,t) \leq f(b+v,t) = f(y,t)$. Then $y \in \arg\max_B f(\cdot,t)$. As f satisfies i-directional monotone comparative statics, there is $\hat{s} \in [0,1]$ such that $y - \hat{s}(y - a \wedge y) \in \arg\max_A f(\cdot,t) = \{a\}$. But considering the ith components, $(y - \hat{s}(y - a \wedge y))_i = y_i = b_i < a_i$, a contradiction. Thus $f(a+v,t) > f(b+v,t)$, as desired.

Finally, let's check that f satisfies basic i-single crossing property in $X \times T$. Fix a, b with $a_i > b_i$, and fix $t' \succeq t$. Let $A = \{a, b\}$. Then $A \sqsubseteq_i^{dso} A$. Suppose $f(a,t) \geq f(b,t)$. Then $a \in \arg\max_A f(\cdot,t)$. As f satisfies i-directional monotone comparative statics on $X \times T$, there is $s \in [0,1]$ such that $a + v = a + s(b - a \wedge b) \in \arg\max_A f(\cdot,t')$. Notice that $(a + s(b - a \wedge b))_i = a_i > b_i$, and therefore, $a + v = a$, whence $f(a,t') \geq f(b,t')$. Now suppose $f(a,t) > f(b,t)$. Then $\{a\} = \arg\max_A f(\cdot,t)$. Suppose to the contrary that $f(a,t') \leq f(b,t')$. Then $b \in \arg\max_A f(\cdot,t')$. By i-directional monotone comparative statics, there is $s \in [0,1]$ such that $b - v = b - s(b - a \wedge b) \in \arg\max_A f(\cdot,t) = \{a\}$, a contradiction. Thus, $f(a,t') > f(b,t')$. □

This proof uses the same framework as in Milgrom and Shannon (1994). It shows how their approach can be used to extend Quah (2007) without using the additional apparatus in Quah (2007). Some implications of this theorem are formalized in the following corollaries.

Corollary 1 *Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, and $f : X \times T \to \mathbb{R}$. The following are equivalent.*

(1) *f satisfies directional monotone comparative statics on $X \times T$.*
(2) *For every $t \in T$, $f(\cdot,t)$ is quasisupermodular and satisfies directional single crossing property on X, and f satisfies basic directional single crossing property on $X \times T$.*

Proof For this equivalence, notice that f satisfies directional monotone comparative statics on $X \times T$ means that for every $i \in \{1, 2, \ldots, N\}$, f satisfies i-directional monotone comparative statics on $X \times T$, which is equivalent to, for every $i \in \{1, 2, \ldots, N\}$, for every $t \in T$, $f(\cdot,t)$ is i-quasisupermodular and satisfies i-single crossing property on X, and f satisfies basic i-single crossing property on $X \times T$, and this is equivalent to (2). □

Corollary 2 *Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, \ldots, N\}$.*

(1) *If f satisfies i-directional monotone comparative statics on $X \times T$, then*

$$A \sqsubseteq_i^{dso} B \text{ and } t \preceq t' \quad \Rightarrow \quad \arg\max_A f(\cdot, t) \sqsubseteq_i^{wso} \arg\max_B f(\cdot, t').$$

(2) *If f satisfies directional monotone comparative statics on $X \times T$, then*

$$A \sqsubseteq^{dso} B \text{ and } t \preceq t' \quad \Rightarrow \quad \arg\max_A f(\cdot, t) \sqsubseteq^{wso} \arg\max_B f(\cdot, t').$$

Proof Statement (1) follows from relations between i-directional set order and i-weak lattice set order (proposition 1). For statement (2), suppose f satisfies directional monotone comparative statics on $X \times T$. Consider $A \sqsubseteq^{dso} B$ and $t \preceq t'$. Then for every $i \in \{1, 2, \ldots, N\}$, $A \sqsubseteq_i^{dso} B$, and by the theorem, for every $i \in \{1, 2, \ldots, N\}$, $\arg\max_A f(\cdot, t) \sqsubseteq_i^{dso} \arg\max_B f(\cdot, t')$, whence $\arg\max_A f(\cdot, t) \sqsubseteq^{dso} \arg\max_B f(\cdot, t')$, and consequently, $\arg\max_A f(\cdot, t) \sqsubseteq^{wso} \arg\max_B f(\cdot, t')$. □

In other words, under (1), f satisfies i-directional monotone comparative statics on $X \times T$ implies that when $A \sqsubseteq_i^{dso} B$ and $t \preceq t'$, then no matter which maximizer of $f(\cdot, t)$ we take from A, we can find a maximizer of $f(\cdot, t')$ from B that is larger in the ith component, and symmetrically, no matter which maximizer of $f(\cdot, t')$ we take from B, we can find a maximizer of $f(\cdot, t)$ from A that is smaller in the ith component. In particular, when the set of maximizers is a singleton, we conclude that the solution to the optimization problem is increasing in the ith component, in the standard order in the real numbers.[12]

Similarly, f satisfies directional monotone comparative statics on $X \times T$ implies that when $A \sqsubseteq^{dso} B$ and $t \preceq t'$, then no matter which maximizer of $f(\cdot, t)$ we take from A, we can find a larger maximizer of $f(\cdot, t')$ from B, and symmetrically, no matter which maximizer of $f(\cdot, t')$ we take from B, we can find a smaller maximizer of $f(\cdot, t)$ from A. In particular, when the set of maximizers is a singleton, we conclude that the solution to the optimization problem is increasing in the standard vector order in $\mathbb{R}^N$.

These results are useful to exhibit monotone increasing selections. Of course, in the case of unique maximizers, the corollary above provides increasing selections. To consider the case of multiple maximizers, let $\pi_i : \mathbb{R}^N \to \mathbb{R}$ be the ith projection. Let $\mathcal{O}(A, t) = \arg\max_A f(\cdot, t)$ be the non-empty and compact[13] set of maximizers (or optimizers) at (A, t) and consider a selection $(A, t) \mapsto x(A, t) \in \mathcal{O}(A, t)$. A selection $x(A, t)$ is an ***i-directional monotone selection***, if for every $A \sqsubseteq_i^{dso} B$ and $t \preceq t'$, $\pi_i(x(A, t)) \leq \pi_i(x(B, t'))$. Extremal selections are defined as follows. For (A, t), let $\underline{x}_i(A, t) = \inf \pi_i(\mathcal{O}(A, t)) \in \pi_i(\mathcal{O}(A, t))$ and let $\overline{x}_i(A, t) = \sup \pi_i(\mathcal{O}(A, t)) \in \pi_i(\mathcal{O}(A, t))$. These are well defined, because $\mathcal{O}(A, t)$ is compact and the projection is continuous. The *i-upper extremal selection* is defined by (any) $\overline{x}(A, t) \in \mathcal{O}(A, t)$

[12] Notice that the results here are different from Spence–Mirrlees-type conditions, as discussed in Milgrom and Shannon (1994). Those results use path-connected indifference sets and additional assumptions about richly parameterized families of functions, neither of which is assumed here.

[13] This is guaranteed when the constraint set is compact and the objective function is upper semi-continuous, as usual.

such that $\pi_i(\overline{x}(A,t)) = \overline{x}_i(A,t)$ and the *i-lower extremal selection* is defined by (any) $\underline{x}(A,t) \in \mathcal{O}(A,t)$ such that $\pi_i(\underline{x}(A,t)) = \underline{x}_i(A,t)$.

Corollary 3 *Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, \ldots, N\}$. Suppose for every (A,t), $\mathcal{O}(A,t)$ is non-empty and compact.*
If f satisfies i-directional monotone comparative statics, then the i-upper and i-lower extremal selections are both i-directional monotone selections.

Proof Consider the case for the i-upper extremal selection, $\overline{x}(A,t)$, the other case being similar. Suppose $A \sqsubseteq_i^{dso} B$ and $t \preceq t'$. Then $\overline{x}(A,t) \in \mathcal{O}(A,t)$ is such that $\pi_i(\overline{x}(A,t)) = \sup \pi_i(\mathcal{O}(A,t)) = \overline{x}_i(A,t)$. By i-directional monotone comparative statics, there is $x' \in \mathcal{O}(B,t')$ such that $\overline{x}_i(A,t) \le x'_i$. Moreover, $x'_i \le \sup \pi_i(\mathcal{O}(B,t')) = \overline{x}_i(B,t')$, yielding $\pi_i(\overline{x}(A,t)) = \overline{x}_i(A,t) \le \overline{x}_i(B,t') = \pi_i(\overline{x}(B,t'))$. □

The technique in this corollary is not directly applicable to show monotone selections in all directions simultaneously. The main limitation is that the set of maximizers is not necessarily a complete sublattice; in general, $\sup \mathcal{O}(A,t) \notin \mathcal{O}(A,t)$.[14] In this case, using theorem 1 and induction, monotone selections in the partial order on $\mathbb{R}^N$ can still be exhibited for monotone sequences of parameters, say $(A_n, t_n)_{n=0}^{\infty}$ with $m \le n \Rightarrow A_m \sqsubseteq^{dso} A_n$ and $t_m \preceq t_n$, and more generally, using transfinite induction, for chains of parameters indexed by a well-ordered set.

The framework in theorem 1 can be specialized naturally to the case of non-parameterized objective functions. Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f satisfies ***i-directional monotone comparative statics on** X*, if for every A, B subset of X, $A \sqsubseteq_i^{dso} B \Longrightarrow \arg\max_A f \sqsubseteq_i^{dso} \arg\max_B f$. In other words, f satisfies i-directional monotone comparative statics on X formalizes the idea that $\arg\max_A f(\cdot)$ is increasing in A in the i-directional set order.

Corollary 4 *Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, \ldots, N\}$. The following are equivalent.*

(1) *f satisfies i-directional monotone comparative statics on X*
(2) *f is i-quasisupermodular and satisfies i-single crossing property on X*

Proof Apply theorem with singleton $T = \{t\}$. □

Similarly, say that f satisfies ***directional monotone comparative statics on** X*, if for every $i \in \{1, 2, \ldots, N\}$, f satisfies i-directional monotone comparative statics on X. It follows immediately that *f satisfies directional monotone comparative statics on X, if, and only if, f is quasisupermodular and satisfies directional single crossing property on X.*

When X is a convex sublattice of $\mathbb{R}^N$, the corresponding result in Quah (2007) shows that f satisfies i-directional monotone comparative statics on X, if, and only if, f is $\mathcal{C}_i$-quasisupermodular. This yields the equivalence that *f is $\mathcal{C}_i$-quasisupermodular, if,*

[14] Requirements of this type also arise in, for example, Smithson (1971).

and only if, f is i-quasisupermodular and satisfies i-single crossing property on X.[15] For completeness, a direct proof of this equivalence is provided in the appendix.

Theorem 1 can be used to inquire separately about comparative statics with respect to the parameter in the objective function, holding fixed the constraint set. In this case, the condition i-single crossing property on X may be dropped, as follows.

Corollary 5 *Let X be a sublattice of $\mathbb{R}^N$, A be a subset of X, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$.*
If f is i-quasisupermodular on X and satisfies basic i-single crossing property on $X \times T$, then $t \preceq t' \Longrightarrow \arg\max_A f(\cdot, t) \sqsubseteq_i^{dso} \arg\max_A f(\cdot, t')$.

Proof Follow the proof in the corresponding direction in theorem 1, setting $s = 0$ and note that i-directional set order is reflexive. □

In this corollary, A is an arbitrary subset of X. Therefore, under the conditions in this corollary, for an arbitrary constraint set A, as long as the set of maximizers is non-empty, i-directional monotone comparative statics holds with respect to the parameter.[16]

Finally, the ordinal nature of the conditions in theorem 1 implies that i-directional (and directional) monotone comparative statics property is preserved under increasing transformations of the objective function. This is useful in applications.

Corollary 6 *Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f, g : X \times T \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. Suppose g is a strictly increasing transformation of f.*[17]
f satisfies i-directional (respectively, directional) monotone comparative statics on $X \times T$, if, and only if, g satisfies i-directional (respectively, directional) monotone comparative statics on $X \times T$.

Proof If f satisfies i-directional monotone comparative statics on $X \times T$, then f is i-quasisupermodular and satisfies i-single crossing property on X, and satisfies basic i-single crossing property on $X \times T$. As these properties are ordinal, g satisfies these as well, and another application of the theorem yields the result. The other direction is similar. Moreover, the proof for directional monotone comparative statics is similar. □

3.2 Sufficient conditions

Quah (2007) shows that when X is a convex sublattice (a sublattice that is also a convex set) of $\mathbb{R}^N$, if $f : X \to \mathbb{R}$ is supermodular and i-concave, then $\arg\max_A f$ is increasing in A in the $\mathcal{C}_i$-flexible set order. In particular, if f is supermodular and concave, then this condition is satisfied for every i. This is useful, because supermodular and concave are conditions that are easy to check.

15 A similar characterization follows for f is $\mathcal{C}$-quasisupermodular.

16 Of course, if the set of maximizers is empty, i-directional monotone comparative statics holds trivially.

17 That is, there is strictly increasing $h : \mathbb{R} \to \mathbb{R}$ such that $g = h \circ f$, as usual.

We show that these conditions are also sufficient for f to satisfy i-single crossing property on X. Therefore, we can use the same conditions here, apply them to some additional potentially discrete problems, and extend them naturally to include parameterized objective functions, as follows.

Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $u \in \mathbb{R}^N$, $u \neq 0$. The function f ***is (relatively) concave in direction*** u, if for every $a \in X$, the function $f(a + su)$, when viewed as a real-valued function of a real variable s, is a concave function relative to its domain in the real numbers. It is easy to check that *f is (relatively) concave on X,*[18] *if, and only if, for every $u \in \mathbb{R}^N$, $u \neq 0$, f is (relatively) concave in direction u.*

For $i \in \{1, 2, \ldots, N\}$, f is ***(relatively) i-concave on*** X, if for every $u \in \mathbb{R}^N \backslash \{0\}$ with $u_i = 0$, f is (relatively) concave in direction u, and f is ***(relatively) directionally concave on*** X, if for every $i \in \{1, 2, \ldots, N\}$, f is (relatively) i-concave on X. Notice that *if f is (relatively) concave on X, then f is directionally concave on X.*

Theorem 2 *Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$.*
If for every $t \in T$, $f(\cdot, t)$ is i-supermodular and (relatively) i-concave on X, and f satisfies basic i-single crossing property on $X \times T$, then f satisfies i-directional monotone comparative statics on $X \times T$.

Proof Suppose for every $t \in T$, $f(\cdot, t)$ is i-supermodular and (relatively) i-concave on X, and f satisfies basic i-single crossing property on $X \times T$. It is sufficient to show that for every $t \in T$, $f(\cdot, t)$ satisfies i-single crossing property on X and then invoke theorem 1. To do so, fix $t \in T$, $a, b \in X$ with $a_i > b_i$, and $v = s(b - a \wedge b)$ with $s \geq 0$ such that $a + v, b + v \in X$.

Consider the following computations. Let $b' = b+v$, $a' = a+v$ and $u = a \vee b' - a'$. It is easy to check that $(a \vee b') - v = a \vee (b + v) - v = a \vee b$, and therefore, $u = a \vee b - a = b - a \wedge b$. Consequently, $v = su$. Moreover, notice that $u_i = 0$ and $a \vee b' = a' + u = a + (1 + s)u$.

Now, i-concavity in direction u implies that $f(a', t) - f(a \vee b', t) = f(a \vee b' - u, t) - f(a \vee b', t) \geq f(a \vee b' - u - su, t) - f(a \vee b' - su, t) = f(a, t) - f(a \vee b, t)$, and i-supermodularity implies $f(a \vee b', t) - f(b', t) \geq f(a \vee b, t) - f(b, t)$. Consequently, $f(a', t) - f(b', t) = f(a', t) - f(a \vee b', t) + f(a \vee b', t) - f(b', t) \geq f(a, t) - f(a \vee b, t) + f(a \vee b, t) - f(b, t) = f(a, t) - f(b, t)$. It follows that $f(a, t) \geq (>)\ f(b, t) \Rightarrow f(a', t) \geq (>)\ f(b', t)$, as desired. □

The corollaries below follow immediately.

Corollary 7 *Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, and $f : X \times T \to \mathbb{R}$.*
If for every $t \in T$, $f(\cdot, t)$ is supermodular and (relatively) directionally concave on X, and f satisfies basic directional single crossing property on $X \times T$, then f satisfies directional monotone comparative statics on $X \times T$.

[18] With the standard definition, $f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha) f(y)$, with $\alpha \in [0, 1]$ and with the quantifier "relative" applied to mean the points are in the domain of f, as usual.

Proof The hypothesis implies that for every $i \in \{1, \ldots, N\}$, for every $t \in T$, $f(\cdot, t)$ is i-supermodular and (relatively) i-concave on X, and f satisfies basic i-single crossing property on $X \times T$, and the theorem then shows that for every $i \in \{1, \ldots, N\}$, f satisfies i-directional monotone comparative statics on $X \times T$, as desired. □

Corollary 8 *Let X be a sublattice of $\mathbb{R}^N$ and $f : X \to \mathbb{R}$.*

(1) *If f is i-supermodular and (relatively) i-concave on X, then f satisfies i-directional monotone comparative statics on X.*
(2) *If f is supermodular and (relatively) directionally concave on X, then f satisfies directional monotone comparative statics on X.*
(3) *If f is supermodular and (relatively) concave on X, then f satisfies directional monotone comparative statics on X.*

Proof Apply the previous theorem with singleton $T = \{t\}$. □

Moreover, corollary 5 implies that in each of these sufficient conditions, if g is a strictly increasing transformation of f, then g also satisfies the corresponding i-directional (or directional) monotone comparative statics.

3.3 Differential conditions

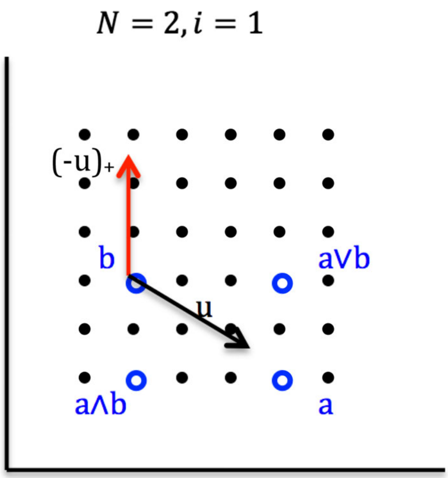
An appealing feature of the single crossing properties defined here is that they are closely aligned to their counterparts in the standard theory. In particular, they possess natural extensions to cardinal properties and can also be formulated in terms of differential conditions in a manner similar to the standard case.

Consider the following cardinal property naturally suggested by the i-single crossing property on X. Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f satisfies ***i-increasing differences on*** X, if for every $a, b \in X$ with $a_i > b_i$, and for every $v \in \{s(b - a \wedge b) \mid s \geq 0\}$ such that $a + v, b + v \in X$, $f(a) - f(b) \leq f(a + v) - f(b + v)$. As earlier, when $a \geq b$, $v = 0$, and this condition is satisfied trivially. Non-trivial application of this definition is when $a_i > b_i$ and $a \ngeq b$. Similarly, f satisfies ***directional increasing differences on*** X, if for every $i \in \{1, 2, \ldots, N\}$, f satisfies i-increasing differences on X. It is easy to check that *if f satisfies i-increasing differences on X, then f satisfies i-single crossing property on X*, and it follows immediately that *if f satisfies directional increasing differences on X, then f satisfies directional single crossing property on X.*

Recall that in the standard theory, f satisfies (standard) increasing differences on $\mathbb{R}^N$, if, and only if, f satisfies increasing differences for every pair of component indices i, j with $i \neq j$. Thus, f satisfies increasing differences on $\mathbb{R}^N$, if, and only if, f is supermodular. Moreover, assuming differentiability, f is supermodular, if, and only if, every pair of cross partials is nonnegative (for every $i \neq j$, $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$). The notion of i-increasing differences can be characterized similarly, using directional derivatives, as follows.

Notice that for $u \in \mathbb{R}^N$, if we let $a = b + u$, then $b - a \wedge b = (b - a)_+ = (-u)_+$. Say that a function $f : X \to \mathbb{R}$ satisfies *i-increasing differences* (u) *on* X, if for every

Fig. 3 Cross partial directional derivatives



$b \in X, u \in \mathbb{R}^N$ with $u_i > 0$, for every $s \geq 0$, such that $b + u, b + s(-u)_+, b + u + s(-u)_+ \in X$, $f(b+u) - f(b) \leq f(b+u+s(-u)_+) - f(b+s(-u)_+)$. Notice that for $u \geq 0$, $(-u)_+ = 0$, and this condition is satisfied trivially. Therefore, non-trivial application of this definition is when $u_i > 0$ and $u \ngeq 0$. It is easy to check that *f satisfies i-increasing differences on X, if, and only if, f satisfies i-increasing differences* (u) *on X*. This recasts i-increasing differences in terms of differences in f based on changes in direction u (where $u_i > 0$). Figure 3 presents the graphical intuition.

The graphical intuition suggests a potential "cross partial" characterization based on directions u and $(-u)_+$. This is achieved as follows. Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. Say that f satisfies *i-increasing differences* (*) *on X*, if for every $b \in X, u \in \mathbb{R}^N$ with $u_i > 0$, and for every $\sigma \geq 0$, $f(b + \sigma u + s(-u)_+) - f(b + s(-u)_+)$ is (weakly) increasing in s. As earlier, we consider only points $b + \sigma u + s(-u)_+, b + s(-u)_+ \in X$. As shown in "Appendix B", *f satisfies i-increasing differences* (u) *on X, if, and only if, f satisfies i-increasing differences (*) on X*.

These formulations show that *i-increasing differences on X is equivalent to i-increasing differences (*) on X*. A benefit of this equivalence is that the condition i-increasing differences (*) on X has the same mathematical structure as the one used to show that a supermodular function can be characterized by the sign of its cross partials (confer Topkis 1978). The only difference is that this definition uses a more general vector u whereas supermodularity uses the basis vectors. This connection can be seen more clearly as follows.

Recall the definition of a directional derivative. Let X be an open set in $\mathbb{R}^N$, $b \in X$ and $u \in \mathbb{R}^N$, and suppose $f : X \to \mathbb{R}$ is continuously differentiable. The *directional derivative of f at b in the direction u* is $D_u f(b) = \lim_{\sigma \to 0} \frac{f(b+\sigma u) - f(b)}{\sigma}$. Recall from the standard theory of supermodular functions (confer Topkis 1978, page 310, for the submodular case) that if u^i is the ith basis vector, then a function f is supermodular on X (assuming X is an open set and a sublattice in $\mathbb{R}^N$, and f is twice continuously differentiable), if, and only if, for all $b \in X$, for all $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$, and for all $\sigma \geq 0$, $f(b + \sigma u^i) - f(b)$ is (weakly) increasing in the jth component (that is, in direction u^j). This is equivalent to: for all $b \in X$, for all $j \neq i$, $D_{u^i} f(b)$ is (weakly) increasing in the jth component (that is, in direction u^j), which is further equivalent to: for all $b \in X$, for all $j \neq i$, $D_{u^j} D_{u^i} f(b) \geq 0$. Using the same logic yields the following result.

Proposition 2 *Let X be an open set and a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$ be twice continuously differentiable, and $i \in \{1, 2, \ldots, N\}$. The following are equivalent.*

(1) *f satisfies i-increasing differences on X.*
(2) *For every $b \in X$, $u \in \mathbb{R}^N$ with $u_i > 0$, $D_{(-u)_+} D_u f(b) \geq 0$.*

Proof We know that f satisfies i-increasing differences on $X \Longleftrightarrow f$ satisfies i-increasing differences (*) on X. In other words, (1) is equivalent to: for every $b \in X$, $u \in \mathbb{R}^N$ with $u_i > 0$, and for every $\sigma \geq 0$, $f(b + \sigma u + s(-u)_+) - f(b + s(-u)_+)$ is (weakly) increasing in s (that is, in the direction $(-u)_+$). Using the fundamental theorem of calculus, this is equivalent to: $\forall b \in X$, $\forall u \in \mathbb{R}^N$ with $u_i > 0$, $D_u f(b + s(-u)_+)$ is (weakly) increasing in s (that is, in direction $(-u)_+$). This, in turn, is equivalent to $\forall b \in X$, $\forall u \in \mathbb{R}^N$ with $u_i > 0$, $D_{(-u)_+} D_u f(b) \geq 0$. □

The second statement can be given a convenient name in terms of nonnegative cross derivatives, as follows. Let X be an open set and a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$ be twice continuously differentiable, and $i \in \{1, 2, \ldots, N\}$. The function f ***has nonnegative i-cross derivative property on*** X, if for every $b \in X$, $u \in \mathbb{R}^N$ with $u_i > 0$, $D_{(-u)_+} D_u f(b) \geq 0$, and f ***has nonnegative directional cross derivative property on*** X, if for every $i \in \{1, 2, \ldots, N\}$, f has nonnegative i-cross derivative property on X. This proposition shows that *i-increasing differences on X is equivalent to nonnegative i-cross derivative property on X*, and it follows immediately that *directional increasing differences on X is equivalent to nonnegative directional cross derivative property on X*.

Similarly, consider the following cardinal property naturally suggested by the basic i-single crossing property on $X \times T$. Let X be a sublattice of $\mathbb{R}^N$, $(T, \preceq)$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f satisfies ***basic i-increasing differences on*** $X \times T$, if for every $a, b \in X$ with $a_i > b_i$, and for every $t, t' \in T$ with $t \preceq t'$, $f(a, t) - f(b, t) \leq f(a, t') - f(b, t')$. The function f satisfies ***basic directional increasing differences on*** $X \times T$, if for every $i \in \{1, 2, \ldots, N\}$, f satisfies basic i-increasing differences on $X \times T$. As earlier, it is easy to check that *if f satisfies basic i-increasing differences on $X \times T$, then f satisfies basic i-single crossing property on $X \times T$*, and it follows immediately that *if f satisfies basic directional increasing differences on $X \times T$, then f satisfies basic directional single crossing property on $X \times T$*. The following result now obtains easily.

Proposition 3 *Let X be an open set and a sublattice of $\mathbb{R}^N$, T be an open subset of $\mathbb{R}^M$, $f : X \times T \to \mathbb{R}$ be twice continuously differentiable, and $i \in \{1, 2, \ldots, N\}$. The following are equivalent.*

(1) *f satisfies basic i-increasing differences on $X \times T$.*

(2) *For every* $b \in X, u \in \mathbb{R}^N$ *with* $u_i > 0$, $D_t D_u f(b,t) \geq 0$.

Proof It is easy to check that f satisfies basic i-increasing differences on $X \times T \Longleftrightarrow$ for every $b \in X, u \in \mathbb{R}^N$ with $u_i > 0$, $f(b+u,t) - f(b,t)$ is (weakly) increasing in t. This is equivalent to: $\forall b \in X, \forall u \in \mathbb{R}^N$ with $u_i > 0$, $D_u f(b,t)$ is (weakly) increasing in t, and this is further equivalent to: $\forall b \in X, \forall u \in \mathbb{R}^N$ with $u_i > 0$, $D_t D_u f(b,t) \geq 0$. □

The second statement can be given a convenient name in terms of nonnegative cross derivatives, as follows. Let X be an open set and a sublattice of $\mathbb{R}^N$, T be an open subset of $\mathbb{R}^M$, $f : X \times T \to \mathbb{R}$ be twice continuously differentiable, and $i \in \{1, 2, \ldots, N\}$. The function f ***has nonnegative basic*** i***-cross derivative property on*** $X \times T$, if for every $b \in X, u \in \mathbb{R}^N$ with $u_i > 0$, $D_t D_u f(b,t) \geq 0$, and f ***has nonnegative basic directional cross derivative property on*** $X \times T$, if for every $i \in \{1, 2, \ldots, N\}$, f has nonnegative basic i-cross derivative property on X. The above proposition shows that *basic* i*-increasing differences on* $X \times T$ *is equivalent to nonnegative basic* i*-cross derivative property on* $X \times T$, and it follows immediately that *basic directional increasing differences on* $X \times T$ *is equivalent to basic nonnegative directional cross derivative property on* $X \times T$. We have the following result.

Theorem 3 *Let* X *be an open set and a sublattice of* $\mathbb{R}^N$, T *be an open subset of* $\mathbb{R}^M$, $f : X \times T \to \mathbb{R}$ *be twice continuously differentiable, and* $i \in \{1, 2, \ldots, N\}$. *If for every* $t \in T$, $f(\cdot, t)$ *is* i*-supermodular*[19] *and has nonnegative* i*-cross derivative property on* X, *and* f *has nonnegative basic* i*-cross derivative property on* $X \times T$, *then* f *satisfies* i*-directional monotone comparative statics on* $X \times T$.

Proof The hypothesis in this statement, combined with the propositions above, implies that f satisfies basic i-single crossing property on $X \times T$, and for every $t \in T$, $f(\cdot, t)$ is i-quasisupermodular and satisfies i-single crossing property on X, and the conclusion follows from an application of theorem 1. □

It follows immediately that *if for every* $t \in T$, $f(\cdot, t)$ *is supermodular and has nonnegative directional cross derivative property on* X, *and* f *has nonnegative basic directional cross derivative property on* $X \times T$, *then* f *satisfies directional monotone comparative statics on* $X \times T$.

The following corollaries help specialize this theorem to the case of comparative statics with respect to A or to t separately.

Corollary 9 *Let* X *be an open set and a sublattice of* $\mathbb{R}^N$, $f : X \to \mathbb{R}$ *is twice continuously differentiable, and* $i \in \{1, 2, \ldots, N\}$.
If f *is* i*-supermodular and has nonnegative* i*-cross derivative property on* X, *then* f *satisfies* i*-directional monotone comparative statics on* X.

Proof Apply the theorem with singleton $T = \{t\}$. □

[19] For every $a, b \in X$ with $a_i > b_i$, $f(a,t) - f(a \wedge b, t) \leq f(a \vee b, t) - f(b,t)$.

This corollary implies immediately that *if f is supermodular and has nonnegative directional cross derivative property on X, then f satisfies directional monotone comparative statics on X.*

In order to understand more concretely the nonnegative i-cross derivative property on X, let's compute $D_{(-u)_+} D_u f(b)$. For convenience, we use subscripts for partial derivatives. Notice that $D_u f(b) = \sum_{j=1}^{N} f_j(b) u_j$, where $f_j(b) \equiv \frac{\partial f}{\partial x_j}(b)$ and u_j is the jth component of u. Therefore,

$$D_{(-u)_+} D_u f(b) = \sum_{k=1}^{N} \sum_{j=1}^{N} f_{k,j}(b) u_j (-u)_{+,k}.$$

Here $f_{k,j}(b)$ is the k, jth cross partial of f evaluated at b, u_j is the jth component of u, and $(-u)_{+,k}$ is the kth component of $(-u)_+$. This is easier to understand if we let $L = \{\ell \mid u_\ell < 0\}$. In this case,

$$(-u)_{+,k} = \begin{cases} -u_k & \text{if } k \in L, \text{ and} \\ 0 & \text{if } k \notin L, \end{cases}$$

and therefore,

$$\begin{aligned}
D_{(-u)_+} D_u f(b) &= \sum_{k=1}^{N} \sum_{j=1}^{N} f_{k,j}(b) u_j (-u)_{+,k} \\
&= \sum_{k \in L} \sum_{j=1}^{N} f_{k,j}(b) u_j (-u_k) \\
&= \sum_{k \in L} \sum_{j \notin L} f_{k,j}(b) u_j (-u_k) + \sum_{k \in L} \sum_{j \in L} f_{k,j}(b) u_j (-u_k) \\
&= \sum_{k \in L} \sum_{j \notin L} f_{k,j}(b) u_j (-u_k) - \sum_{k \in L} \sum_{j \in L} f_{k,j}(b) (-u_j)(-u_k) \\
&= \sum_{k \in L} \sum_{j \notin L} f_{k,j}(b) u_j (-u_k) - \left[w_L' \, D^2 f_L(b) \, w_L \right],
\end{aligned}$$

where f_L is the restriction of f to the components in L, $D^2 f_L(b)$ is the second derivative of f_L evaluated at b, w_L is the restriction of $(-u)_+$ to L, and w_L' is the transpose of w_L.

Notice that for $k \in L$, $-u_k > 0$ and for $j \notin L$, $u_j \geq 0$. In this case, the sign of the term $f_{k,j}(b) u_j (-u_k)$ is determined by the sign of the cross partial $f_{k,j}(b)$. Similarly, for $k \in L$, $-u_k > 0$ and for $j \in L$, $u_j < 0$. In this case, the sign of the term $f_{k,j}(b) u_j (-u_k)$ is determined by the sign of $-f_{k,j}(b)$. In particular, if f is supermodular, then the first term, $\sum_{k \in L} \sum_{j \notin L} f_{k,j}(b) u_j (-u_k) \geq 0$. Moreover, if f is concave in direction $(-u)_+$, then the matrix of second derivatives is negative semidefinite, and therefore, the second term, $-\left[w_L' \, D^2 f_L(b) \, w_L \right] \geq 0$.

Corollary 10 *Let X be an open set and a sublattice of $\mathbb{R}^N$, T be an open subset of $\mathbb{R}^M$, $f : X \times T \to \mathbb{R}$ be twice continuously differentiable, and $i \in \{1, 2, \ldots, N\}$. If for every $t \in T$, $f(\cdot, t)$ is i-supermodular on X, and f has nonnegative basic i-cross derivative property on $X \times T$, then f satisfies i-directional monotone comparative statics on $X \times T$.*

Proof The conditions in this statement imply that for every $t \in T$, $f(\cdot, t)$ is i-quasisupermodular on X, and f satisfies basic i-single crossing property on $X \times T$, and the conclusion follows from an application of corollary 4. □

In order to understand more concretely the nonnegative basic i-cross derivative property on $X \times T$, let's compute $D_t D_u f(b, t)$. Recall that $D_u f(b, t) = \sum_{j=1}^{N} f_{x_j}(b, t)u_j$, where $f_{x_j}(b, t) \equiv \frac{\partial f}{\partial x_j}(b, t)$ and u_j is the jth component of u. Therefore,

$$D_t D_u f(b, t) = \left[\sum_{j=1}^{N} f_{t_1, x_j}(b, t)u_j, \cdots, \sum_{j=1}^{N} f_{t_M, x_j}(b, t)u_j \right],$$

where $f_{t_m, x_n}(b, t) \equiv \frac{\partial^2 f}{\partial t_m \partial x_n}(b, t)$, for $m = 1, \ldots, M$, $n = 1, \ldots, N$. This may be written in standard matrix form as

$$D_t D_u f(b, t) = \begin{bmatrix} f_{t_1, x_1}(b, t) & \cdots & f_{t_1, x_N}(b, t) \\ \vdots & & \vdots \\ f_{t_M, x_1}(b, t) & \cdots & f_{t_M, x_N}(b, t) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}.$$

A useful sufficient condition for nonnegative basic i-cross derivative property on $X \times T$ is the following. Let X be an open set and a sublattice of $\mathbb{R}^N$, T be an open subset of $\mathbb{R}^M$, $f : X \times T \to \mathbb{R}$ be twice continuously differentiable, and $i \in \{1, 2, \ldots, N\}$. *If for some subset M' of $\{1, \ldots, M\}$, $f_{t_m, x_i}(b, t) \geq 0$ for $m \in M'$, and $f_{t_m, x_j}(b, t) = 0$ otherwise, then f has nonnegative basic i-cross derivative property on $X \times T$.* To see that this is true, fix $u \in \mathbb{R}^N$ with $u_i > 0$, and notice that the mth component of $D_t D_u f(b, t)$ is $f_{t_m, x_i}(b, t)u_i \geq 0$ for $m \in M'$ and zero otherwise. This condition retains the flavor of standard increasing differences in $(x; t)$ by working with nonnegative cross partials, and it is useful in applications, as detailed in the next section.

4 Examples

The usefulness of these results is highlighted with several applications in consumer theory, producer theory, and game theory, including applications to consumer demand, theory of competition, environmental emissions standards, labor-leisure decisions with discrete choices, and auctions.

Example 2 (Consumer demand) Consider a consumption space X that is a sublattice of $\mathbb{R}^L_+$, a partially ordered parameter space $(T, \preceq)$, a utility function $u : X \times T \to \mathbb{R}$

and a subset A of X, and consider the utility maximization problem, $\max_A u(\cdot, t)$. When utility is continuous on X and A is a non-empty compact set, this problem has a solution, termed consumer demand. Let's denote it by $D(A, t) = \arg\max_A u(\cdot, t)$. Theorem 1 provides conditions characterizing when $D(A, t)$ is increasing in (A, t) in the i-directional set order and in the directional set order. Some special cases are notable.

Example 2-1 (Normal Walrasian demand) Let's first replicate and extend the result on normal demand in Quah (2007), where parameters in the utility function are excluded. Let the consumption space be $X = \mathbb{R}^L_+$ or $X = \mathbb{R}^L_{++}$, a price vector $p \in \mathbb{R}^L_{++}$, wealth $w > 0$, and let $B(p, w) = \{x \in X \mid p \cdot x \leq w\}$ be the Walrasian budget set and let $D(p, w) = \arg\max_{B(p,w)} u(\cdot)$ be Walrasian demand. We know that $w \leq w' \Rightarrow (\forall i)\ B(p, w) \sqsubseteq^{dso}_i B(p, w')$. Say that demand for good i is *normal*, if $w \leq w' \Rightarrow D(p, w) \sqsubseteq^{dso}_i D(p, w')$. In this setting, the result on sufficient conditions implies that *if u is i-supermodular and i-concave, then Walrasian demand for good i is normal*, and *if u is supermodular and directionally concave, then Walrasian demand for all goods is normal*, replicating the result in Quah (2007). Moreover, corollary 5 implies that strictly increasing transformations of u yield the same conclusions. This implies, for example, that standard cases such as general Cobb-Douglas preferences,[20] constant elasticity of substitution, and taking logarithms of standard preferences are all admissible. Furthermore, we can go beyond Quah (2007) to allow for up to two discrete goods (confer the conditions in example 1-2) and obtain the same result. This can be helpful in applied work, which may need to assume only one or two discrete goods.

Example 2-2 (Law of demand) Our techniques may be used to provide a direct derivation of a version of the law of demand. For each good i, say that *good i satisfies law of demand*, if $p'_i < p_i \Rightarrow D(p, w) \sqsubseteq^{dso}_i D(p', w)$, where, p' is the price system formed by replacing p_i in p by $p'_i > 0$. This formalizes the statement that when price of good i goes down, demand for good i goes up.

We can show that when price of good i goes down, the Walrasian budget set increases in the i-directional set order: $p'_i < p_i \Rightarrow B(p, w) \sqsubseteq^{dso}_i B(p', w)$. Fix $a \in B(p, w)$, $b \in B(p', w)$ with $a_i > b_i$. As case 1, suppose $p' \cdot (a \vee b) \leq w$. In this case, let $s = 1$ and so, $v = b - a \wedge b$. Then $p \cdot (b - v) \leq p \cdot a \leq w$ and $p' \cdot (a + v) = p' \cdot a \vee b \leq w$. As case 2, suppose $p \cdot b \leq w$. In this case, let $s = 0$, and so, $v = 0$. This implies that $p \cdot (b - v) \leq w$ and $p' \cdot (a + v) \leq p \cdot a \leq w$. In the remaining cases, suppose $p' \cdot (a \vee b) > w$ and $p \cdot b > w$. Notice that $\frac{w - p' \cdot a}{p' \cdot (b - a \wedge b)} = \frac{w - p' \cdot a}{p' \cdot (a \vee b) - p' \cdot a} \leq 1$ by the first condition, and $\frac{p \cdot b - w}{p \cdot (b - a \wedge b)} \geq 0$ by the second condition. Moreover, $p \cdot (b - a \wedge b) = p' \cdot (b - a \wedge b)$, because p and p' differ only in the ith component and $b - a \wedge b$ is zero in the ith component. Furthermore,

$$
\begin{aligned}
p \cdot b + p' \cdot a &= p_i b_i + p_{-i} \cdot b_{-i} + p'_i a_i + p_{-i} \cdot a_{-i} \\
&\leq p_i b_i + w - p'_i b_i + p'_i a_i + w - p_i a_i \\
&< w + w,
\end{aligned}
$$

[20] We don't need the restriction that the sum of all-but-one of the exponential parameters is less than or equal to 1, as mentioned in Quah (2007), p. 406, footnote 7.

where the strict inequality follows from $(p_i - p'_i)(b_i - a_i) < 0$. Consequently, $p \cdot b - w < w - p' \cdot a$. Let $s \in \left[\frac{p \cdot b - w}{p \cdot (b - a \wedge b)}, \frac{w - p' \cdot a}{p' \cdot (b - a \wedge b)}\right] \subset [0, 1]$, and let $v = s(b - a \wedge b)$. Then $p' \cdot (a + v) \leq p' \cdot a + \frac{w - p' \cdot a}{p' \cdot (b - a \wedge b)}[p' \cdot (b - a \wedge b)] = w$, whence $a + v \in B(p', w)$, and $p \cdot (b - v) \leq p \cdot b - \frac{p \cdot b - w}{p \cdot (b - a \wedge b)}[p \cdot (b - a \wedge b)] = w$, whence $b - v \in B(p, w)$.

Using theorem 1, it now follows that *good i satisfies the law of demand, if utility is i-quasisupermodular and satisfies i-single crossing property*.[21] Notably, this derivation is in terms of ordinal conditions on the utility function, it does not impose differentiable (or continuous) utility, it does not rely on any computation of income and substitution effects, it does not posit new orders on the consumption space, and it is valid when demand is multi-valued. Moreover, theorem 2 implies that *if u is i-supermodular and i-concave, then good i satisfies law of demand*. In particular, if u is supermodular and concave, then every good satisfies the law of demand.

Example 2-3 (Stone–Geary utility) New applications can be considered when the utility function has parameters.[22] Corollary 4 implies that if u is i-quasisupermodular and satisfies basic i-single crossing property on $X \times T$, then $t \preceq t' \implies \arg\max_A u(\cdot, t) \sqsubseteq_i^{dso} \arg\max_A u(\cdot, t')$. Notably, A can be an arbitrary subset of $\mathbb{R}^L$. This can be seen concretely with a Stone–Geary-type utility function.

Consider consumption space $X = \mathbb{R}^L_+$ or $X = \mathbb{R}^L_{++}$, a bundle $b \in \mathbb{R}^L_+$, and utility given by $u(x, b) = \Pi_{j=1}^L (x_j + b_j)^{\alpha_j}$, where $\alpha_j > 0$ for all j. The bundle b is sometimes viewed as a survival bundle available as an outside option, perhaps through a government program, or through a soup kitchen, or through a charity, and so on, although other interpretations are available.[23] Theoretically, it is a parameter in the utility function. Notice that for each b, $u(\cdot, b)$ is quasisupermodular and quasiconcave. Moreover, when $b = 0$, Stone–Geary specializes to Cobb-Douglas preferences.[24]

In order to use derivatives, let $a \in \mathbb{R}^L_{--}$, and write $u(x, a) = \Pi_{j=1}^L (x_j - a_j)^{\alpha_j}$, where $\alpha_j > 0$ for all j, and consider the monotonic transformation, $v(x, a) = \sum_{j=1}^L \alpha_j \log(x_j - a_j)$. Then for each $a \in \mathbb{R}^L_{--}$, $v(\cdot, a)$ is supermodular and concave on X. Moreover, for fixed $i \in \{1, \ldots, L\}$, and for every $u \in \mathbb{R}^L$ with $u_i > 0$, $D_u v(x, a) = \sum_{j=1}^L \frac{\alpha_j}{x_j - a_j} u_j$ and therefore, $D_{a_i} D_u v(x, a) = \frac{\alpha_i}{(x_i - a_i)^2} u_i > 0$. Consequently, v satisfies basic i single crossing property on $X \times \mathbb{R}_{--}$, where $\mathbb{R}_{--}$ indexes a_i. By theorem 3, v (and u) satisfies i-directional monotone comparative statics on $X \times \mathbb{R}_{--}$. In particular, when a_i goes up (and as long as the corresponding budget sets (weakly) increase in the i-directional set order), demand for good i goes up.

[21] In general, price effects have been hard to accommodate using monotone methods. Antoniadu (2007) and Mirman and Ruble (2008) develop some results using a different approach that is more specialized. Of course, we may invoke Quah (2007) as well and then standard consumer theory that shows that normal goods satisfy the law of demand. This example documents a direct derivation.

[22] Recall that Quah (2007)'s framework does not include parameterized objective functions, and Milgrom and Shannon (1994)'s framework does not include budget-type constraint sets. Our framework includes both.

[23] For example, in models of charitable giving, b may be viewed as a consumer's or donor's intrinsic benefit from donation, as in Harbaugh (1998).

[24] As earlier, there is no restriction that the sum of all-but-one α_j is less than or equal to 1.

In terms of the original problem with nonnegative b, this implies that when a component of the survival bundle is increased, a consumer's optimal response is to decrease the same component of her demand. This is consistent with results in public economics on the effect of more generous social welfare options. It follows here from an easy calculation on the objective function and is valid for an arbitrarily fixed compact budget set, and therefore, includes piece-wise linear and other non-Walrasian budget sets common in applications.

Example 3 (Multi-market competition) Parameters in the payoff function arise naturally in game theory models and many games may naturally have a budget set-type trade-off among feasible strategies. Our results may apply to such models.

Example 3-1 (A two markets, multiple firms case) Let's first consider a concrete two-market, multiple-firm game with capacity (or budget) type constraints. Consider a firm that is competing in two markets; market 1 is imperfectly competitive, say, an oligopoly, and market 2 is perfectly competitive. (For example, the firm might produce a generic product for the competitive market and a differentiated version to have some market power.) Suppose the firm's profits are given by $\Pi(x_1, y; x_2, \ldots, x_M) = \Pi_1(x_1, x_2, \ldots, x_M) + p \cdot y - c(y)$, where Π_1 is the firm's profit in market 1, which has $M \geq 2$ firms, and $p \cdot y - c(y)$ is its profit in market 2. The firm's choice variables are $(x_1, y) \in \mathbb{R}^2_+$. Suppose the actions of other firms in market 1 are complementary to the actions of the firm; that is, $\frac{\partial^2 \Pi_1}{\partial x_j \partial x_1} \geq 0$, for $j = 2, \ldots, M$. (For example, this follows if market 1 competition is of a standard differentiated Bertrand variety. It could also follow if there are production externalities or network externalities in market 1.) The firm's problem is to $\max_A \Pi(x_1, y; x_2, \ldots, x_M)$. It is easy to check that Π is supermodular in (x_1, y). In order to check that Π has the basic 1-single crossing property in $(x_1, y; x_2, \ldots, x_M)$, observe that for u with $u_1 > 0$, $D_{x_{-1}} D_u \Pi(x_1, y; x_{-1}) = [\frac{\partial^2 \Pi_1}{\partial x_2 \partial x_1} u_1, \ldots, \frac{\partial^2 \Pi_1}{\partial x_M \partial x_1} u_1] \geq 0$. Therefore, when competitor action goes up, firm 1's best response in market 1 goes up as well. Notably, this result holds for arbitrary constraint set A.

We can also inquire about comparative statics with respect to A. For motivation, suppose market 1 is subject to production or network externalities. (For example, when other firms produce more, a given firm's marginal cost goes down, either because of a direct upstream or downstream production externality or an indirect one, perhaps through the availability of more skilled labor, more efficient supply chains, and so on.) In other words, suppose x_1 is the firm's production in market 1, and suppose again that $\frac{\partial^2 \Pi_1}{\partial x_j \partial x_1} \geq 0$ for $j = 2, \ldots, M$. The firm faces a capacity constraint for producing both outputs, $A(k) = \{(x_1, y) \mid x_1 + y \leq k\}$. This can be generalized by considering $A(k) = \{(x_1, y) \mid \alpha_1 x_1 + \alpha_2 y \leq k\}$, where α_1 and α_2 are arbitrary positive constants, perhaps indexing different production requirements. (For example, it may be that some more factory space is needed to produce a unit of the differentiated good, as compared to the competitive good, and the "weighted" output is constrained by the "base" plant size of k units.) Consider the firm's problem: $\max_{A(k)} \Pi(x_1, y; x_2, \ldots, x_M)$. For fixed $A(k)$, we still have the earlier result that the firm's supply in market 1 is (weakly) increasing with respect to supply of other firms. Moreover, the framework here allows us to inquire about the effects of an increase in plant capacity *simultaneously* with an increase in other firm actions. To do so, we need to check for x_1-concavity of Π.

This holds when the cost function in the competitive market, $c(y)$ is convex (which follows from concave production technology). In this case, $(k, x_{-1}) \leq (k', x'_{-1}) \Rightarrow \arg\max_{A(k)} \Pi(x_1, y; x_{-1}) \sqsubseteq_1^{dso} \arg\max_{A(k')} \Pi(x_1, y; x'_{-1})$.[25]

We can also inquire about monotone comparative statics for the competitive market. Notice that $\frac{\partial^2 \Pi}{\partial p \partial y} = 1$, and therefore, an increase in the competitive price increases output in the competitive market, regardless of the constraint set. Moreover, y-concavity of Π follows from convex cost in market 1 (again, following from concave production technology) and concave total revenue function in market 1 (which follows if demand is linear, and also if demand has constant elasticity less than or equal to 1, which includes Cobb-Douglas preferences).

Example 3-2 (General case) The two-market example generalizes to multiple markets and multiple firms. Consider a firm producing outputs in $N \geq 2$ markets, with profit in market i given by $\Pi_i(x_i, t_i)$, where $x_i \in \mathbb{R}_+$ is the firm's output in market i, and $t_i \in \mathbb{R}^{M_i}$ is a vector of parameters for market i, including choices of other firms in market i.

Consider first the case where total profit of the firm has the form $\Pi(x; t) = \sum_{i=1}^{N} \Pi_i(x_i, t_i)$ and the firm's problem is to $\max_A \Pi(x; t)$ for x in some constraint set A. In this case, it is easy to check that Π is supermodular in $x = (x_1, \ldots, x_N)$. Moreover, if we assume that each $\Pi_i(x_i, t_i)$ is concave in x_i (a typical assumption) and satisfies basic i-single crossing property in (x_i, t_i) (for example, if $\frac{\partial^2 \Pi_i}{\partial t_{i,j} \partial x_i} \geq 0$ for $j = 1, \ldots, M_i$; consistent with market i having strategic complementarities), then conditions in theorem 1 are satisfied and we may conclude that for each i, $A \sqsubseteq_i^{dso} A'$ and $t_i \leq t'_i$ implies $\arg\max_A \Pi(x; t_i, t_{-i}) \sqsubseteq_i^{dso} \arg\max_{A'} \Pi(x; t'_i, t_{-i})$.

In the more general case, we may consider more flexible trade-offs in profits across markets. For example, consider a Cobb-Douglas type cross market trade-off, given by $\Pi(x; t) = \times_{i=1}^{N} \Pi_i(x_i, t_i)^{\alpha_i}$, where for each i, $\alpha_i > 0$.[26] It is easy to check that Π is quasisupermodular in x, and if each $\Pi_i(x_i, t_i)$ satisfies the same conditions as above (concave in x_i, and satisfies basic i-single crossing property in (x_i, t_i)), then conditions for theorem 1 are satisfied and we have i-directional monotone comparative statics.

Similarly, consider the case of constant elasticity of substitution across markets, given by $\Pi(x; t) = \left(\sum_{i=1}^{N} \Pi_i(x_i, t_i)^{\sigma}\right)^{\frac{\gamma}{\sigma}}$, with $0 < \sigma < 1$ and $\gamma > 0$. It is easy to check that Π is quasisupermodular in x, and if each $\Pi_i(x_i, t_i)$ satisfies the same conditions as above (concave in x_i, and satisfies basic i-single crossing property in (x_i, t_i)), then conditions for theorem 1 are satisfied and we have i-directional monotone comparative statics. This can be generalized further to allow for heterogeneity across

25 The same result holds for minimum production quotas; constraints sets of the form $A(k) = \{(x_1, y) \mid \alpha_1 x_1 + \alpha_2 y \geq k\}$, where α_1 and α_2 are arbitrary positive constants. In this case as well, $k \leq k' \Rightarrow A(k) \sqsubseteq^{dso} A(k')$.

26 This would be consistent with weighting profits in different markets differently, such as when shareholders prefer profits in particular industries relatively more than in others.

industries, by using $\Pi(x;t) = \left(\sum_{i=1}^{N} \Pi_i(x_i, t_i)^{\sigma_i}\right)^{\frac{\gamma}{\bar{\sigma}}}$, with $0 < \sigma_i < 1$ for every i, $\bar{\sigma} = \frac{1}{N}\sum \sigma_i$ is the average of σ_i, and $\gamma > 0$.[27]

A main insight is that in an industry with strategic complements, monotone comparative statics continues to hold simultaneously with competitor actions and cross market capacity constraints, and in the presence of some classes of heterogeneous trade-offs across industries. Such results are not accessible with the standard existing results in the literature.

Example 4 (Emissions standards) Consider the emissions standards model in Montero (2002) and Bruneau (2004). A firm is producing an output $q \geq 0$ that causes pollution. It is subject to an emissions ceiling $e > 0$ and can produce more by engaging in costly abatement $a \geq 0$. The firm's payoff is given by $\pi(q, a; k) = p \cdot q - c(q) - kc(a)$. Here, revenue is $p \cdot q$, cost of output, $c(q)$ is assumed to be increasing and convex, as is cost of abatement, $c(a)$. The firm can consider technological progress $k \leq 1$, measured as a decrease in abatement cost to $kc(a)$. It is easy to check that Π is supermodular in (q, a), Π is concave, and technological innovation (decrease in k, or increase in $-k$) satisfies $\frac{\partial^2 \Pi}{\partial(-k)\partial q} = 0$ and $\frac{\partial^2 \Pi}{\partial(-k)\partial a} \geq 0$. Therefore, an increase in technological innovation, say $(-k) \leq (-k')$ implies that $\arg\max_A \Pi(q, a; k) \sqsubseteq_a^{dso} \arg\max_A \Pi(q, a; k')$, for arbitrary constraint set A. In particular, it holds for the emissions constraint set, $A(e) = \{(q, a) \mid q - a = e\}$. Moreover, at optimum, an increase in a leads to an increase in q, and therefore, *an increase in technological innovation increases both abatement and output*. This main result in Montero (2002) and Bruneau (2004) can be seen here from an easy calculation on the objective function.

Example 5 (Discrete labor supply) Recent models of labor supply frequently incorporate a discrete choice model, for example, Aaberge et al. (1995), van Soest (1995) and Hoynes (1996). In order to work with integer data, these models consider integer work-leisure choices. Let h denote hours worked and l denote hours of leisure. Given total hours available T, the constraint set is $B(T) = \{(h, l) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid h + l \leq T\}$. Using example 1-2, it is easy to check that $T \leq T' \Rightarrow B(T) \sqsubseteq^{dso} B(T')$. Preferences are given by $u(wh + I, l)$ where w is wage rate and I is non-labor income, both exogenously specified. Using our results, when preferences are supermodular and concave, both hours worked and leisure hours are increasing in the time constraint T, even in a discrete choice framework. In particular, for standard preferences such as Cobb-Douglas, CES, and their increasing transformations, this result holds. Moreover, consider the utility from Hoynes (1996), $u(wh + I, l) = \alpha_1 \ln(wh + I) + \alpha_2 \ln(l)$ with $\alpha_1, \alpha_2 > 0$. In this case, the basic h-single crossing property holds for parameters $(w, -I)$, because $\frac{\partial^2 u}{\partial(-I)\partial h} = \frac{\alpha_1 w}{(wh+I)^2} \geq 0$, $\frac{\partial^2 u}{\partial w \partial h} = \frac{\alpha_1 I}{(wh+I)^2} \geq 0$, $\frac{\partial^2 u}{\partial(-I)\partial l} = 0$, and $\frac{\partial^2 u}{\partial w \partial l} = 0$. Consequently, optimal labor supply increases when either wage rate goes up or non-labor income goes down. Notably, this result holds for discrete choices and for an arbitrary compact constraint set, and therefore, includes piece-wise linear and other

[27] The example shows that some standard formulations for considering trade-offs among profits in different industries are admissible here. Of course, additional cases can be considered as well, but no doubt, arbitrary profit functions will not work.

non-Walrasian budget sets common in applications. This indicates new directions for application of the results here.

Example 6 (Auctions with budget constraints) Another class of games with budget-type constraints is auctions with bidding constraints. These are common in practice (for example, ad auctions run by Google and Yahoo, Treasury auctions, and spectrum or electricity auctions), but less widely studied in the auction theory literature (for some examples, confer Rothkopf 1977; Palfrey 1980, and Dobzinski et al. 2012), partly because the problem with bidding constraints (or budget constraints) is harder to analyze. The results here can be applied in these cases as well.

Consider an auction of $N \geq 2$ indivisible objects. There are I bidders, and bidder i has exogenously specified valuations $(v_{i,1}, \ldots, v_{i,N})$ for the N objects, and can bid $(b_{i,1}, \ldots, b_{i,N})$ subject to the resource constraint

$$B^i(T_i) = \left\{b_i = (b_{i,1}, \ldots, b_{i,N}) \geq 0 \mid b_{i,1} + \cdots + b_{i,N} \leq T_i\right\}.$$

The probability that bidder i wins object n when bid profile for object n is $(b_{1,n}, \ldots, b_{I,n})$ is given by $F_{i,n}(b_{i,n}, b_{-i,n})$, where $\frac{\partial F_{i,n}}{\partial b_{i,n}} \geq 0$. The expected payoff to bidder i from winning object n is

$$u_{i,n}(b_{i,n}, b_{-i,n}, v_{i,n}) F_{i,n}(b_{i,n}, b_{-i,n}),$$

and as usual, the expected payoff from losing is normalized to 0. As usual, suppose utility for object n is increasing in $v_{i,n}$ ($\frac{\partial u_{i,n}}{\partial v_{i,n}} \geq 0$), and bids and valuations are complementary ($\frac{\partial^2 u_{i,n}}{\partial b_{i,n} \partial v_{i,n}} \geq 0$). To inquire into monotone comparative statics of optimal bids with respect to valuations, let

$$\mathcal{U}_{i,n}(b_{i,n}, b_{-i,n}, v_{i,n}) = u_{i,n}(b_{i,n}, b_{-i,n}, v_{i,n}) F_{i,n}(b_{i,n}, b_{-i,n})$$

denote bidder i's expected payoff from object n and suppose it is concave in $b_{i,n}$, for every n.

As in the example of multiple markets and multiple firms, we may consider a variety of aggregate payoffs, including additive payoffs,

$$\mathcal{U}_i(b_i, b_{-i}, v_i) = \sum_{n=1}^{N} \mathcal{U}_{i,n}(b_{i,n}, b_{-i,n}, v_{i,n}),$$

payoffs with Cobb-Douglas form,

$$\mathcal{U}_i(b_i, b_{-i}, v_i) = \times_{n=1}^{N} \mathcal{U}_{i,n}(b_{i,n}, b_{-i,n}, v_{i,n})^{\alpha_n},$$

where $\alpha_n > 0$, and payoffs with constant elasticity of substitution,

$$\mathcal{U}_i(b_i, b_{-i}, v_i) = \left(\sum_{n=1}^{N} \mathcal{U}_{i,n}(b_{i,n}, b_{-i,n}, v_{i,n})^{\sigma}\right)^{\frac{\gamma}{\sigma}},$$

with $0 < \sigma < 1$ and $\gamma > 0$, and others. In all these cases, it can be checked that $\mathcal{U}_i(b_i, b_{-i}, v_i)$ is quasisupermodular in b_i, satisfies n-concavity, and satisfies basic n-directional single crossing property for $(b_{i,n}, v_{i,n})$. From theorem 1, it follows that in auctions with bidding constraints, optimal bid for object n increases with simultaneous increases in its valuation and in bidding resources, and allowing for payoff linkages across objects.[28]

5 Conclusion

This paper presents an extension of the theory of monotone comparative statics in different directions in finite-dimensional Euclidean space. The new notions of i-single crossing property and basic i-single crossing property are similar in spirit to the single crossing property in the standard theory of monotone comparative statics, both are ordinal properties, and both can be naturally specialized to related cardinal and differential properties. The results here use more standard assumptions on the objective function, include parameters in the objective function, do not require the use of new binary relations or convex domains, and subsume results in Quah (2007). The results allow flexibility to explore comparative statics with respect to the constraint set, with respect to parameters in the objective function, or both. Moreover, the results here can be applied fruitfully to enhance the reach of existing results and to provide new applications.

Appendix A: Relation to Quah (2007)

Quah (2007) uses different techniques based on new binary relations, denoted ∇_i^λ and Δ_i^λ, and convex sets. Using these binary relations, he defines a new set order, termed $\mathcal{C}_i$-flexible set order, and a new notion of $\mathcal{C}_i$-quasisupermodular function. Some connections to these ideas are explored here.

Let X be a convex sublattice of $\mathbb{R}^N$ (that is, X is a sublattice that is also a convex set), and $i \in \{1, 2, \ldots, N\}$. For $a, b \in X$ and $\lambda \in [0, 1]$, let

$$a \Delta_i^\lambda b = \begin{cases} a & \text{if } a_i \leq b_i \\ \lambda b + (1-\lambda)(a \wedge b) & \text{if } a_i > b_i, \end{cases} \quad \text{and}$$

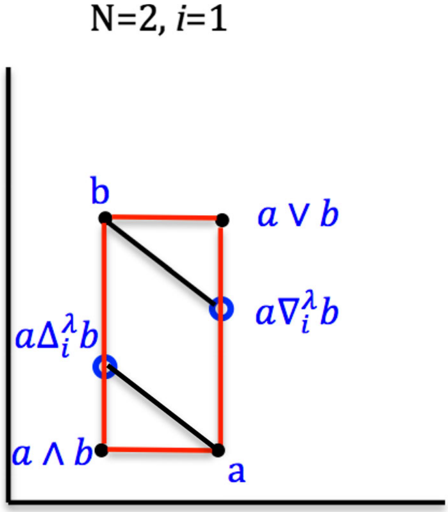
$$a \nabla_i^\lambda b = \begin{cases} b & \text{if } a_i \leq b_i \\ \lambda a + (1-\lambda)(a \vee b) & \text{if } a_i > b_i. \end{cases}$$

Figure 4 shows the graphical intuition.

When $a_i > b_i$, the set $\left\{a, a\Delta_i^\lambda b, a\nabla_i^\lambda b, b\right\}$ forms a "backward-bending" parallelogram, as compared to the standard lattice theory rectangle formed by the set $\{a, a \wedge b, a \vee b, b\}$. The shape of this parallelogram varies with λ, ranging from the

[28] Notably, no restrictions are placed on the effects of competitor bids on the payoffs of a given bidder. In particular, this result applies when bids across bidders may be strategic complements, strategic substitutes, or a mixture of the two. Moreover, the result allows for discrete bids for up to two objects, as described earlier.

Fig. 4 $\mathcal{C}_i$-Flexible set order



standard lattice theory rectangle when $\lambda = 0$ to the degenerate line segment formed by $\{a, b\}$ when $\lambda = 1$.

The binary operations Δ_i^λ, ∇_i^λ have some counter-intuitive properties when compared to the standard lattice operations $\wedge, \vee$. For example, *the relations* Δ_i^λ, ∇_i^λ *are non-commutative*: suppose $N = 2$, $i = 1$, consider $a = (1, 0)$, $b = (0, 1)$, and $\lambda = \frac{1}{2}$. Then $a\Delta_i^\lambda b = \frac{1}{2}b \neq b = b\Delta_i^\lambda a$, and $a\nabla_i^\lambda b = (1, \frac{1}{2}) \neq a = b\nabla_i^\lambda a$. Moreover, $a\Delta_i^\lambda b$ *and* $a\nabla_i^\lambda b$ *are not necessarily comparable in the underlying lattice order*: suppose $N = 2$, $i = 1$, and consider $a = (1, 1)$ and $b = (2, 0)$. Then for every $\lambda \in [0, 1]$, $a\Delta_i^\lambda b = a \not\leq b = a\nabla_i^\lambda b$. It is easy to see that additional classes of examples of these instances can be provided as well.

The binary relations Δ_i^λ, ∇_i^λ are used to define the $\mathcal{C}_i$-flexible set order and the notion of a $\mathcal{C}_i$-quasisupermodular function, as follows.

Let X be a convex sublattice of $\mathbb{R}^N$ and $i \in \{1, 2, \ldots, N\}$. For subsets A, B of X, A ***is lower than*** B ***in the*** $\mathcal{C}_i$***-flexible set order***, denoted $A \sqsubseteq_i^{\mathcal{C}} B$, if for every $a \in A$, $b \in B$, there is $\lambda \in [0, 1]$ such that $a\Delta_i^\lambda b \in A$ and $a\nabla_i^\lambda b \in B$. The $\mathcal{C}_i$-flexible set order is flexible in the sense that the choice of λ may vary for each $a \in A$ and $b \in B$, and therefore, the “backward bendedness” of the parallelogram may vary for each $a \in A$ and $b \in B$. On convex sublattices, the $\mathcal{C}_i$-flexible set order is the same as the i-directional set order, as shown next.

Proposition 4 *Let X be a convex sublattice of $\mathbb{R}^N$, $i \in \{1, 2, \ldots, N\}$, and A, B be subsets of X. The following are equivalent.*

(1) *A is lower than B in the $\mathcal{C}_i$-flexible set order ($A \sqsubseteq_i^{\mathcal{C}} B$).*
(2) *A is lower than B in the i-directional set order ($A \sqsubseteq_i^{dso} B$).*

Proof Suppose $A \sqsubseteq_i^{\mathcal{C}} B$. Fix $a \in A$, $b \in B$, and suppose $a_i > b_i$. Let $\lambda \in [0, 1]$ be such that $a\Delta_i^\lambda b \in A$ and $a\nabla_i^\lambda b \in B$. Let $t = 1 - \lambda \in [0, 1]$. Then $b - v =$

$b-t(b-a\wedge b) = (1-t)b+t(a\wedge b) = a\Delta_i^{1-t}b \in A$, and $a+v = a+t(a\vee b-a) = (1-t)a+t(a\vee b) = a\nabla_i^{1-t}b \in B$, as desired.

In the other direction, suppose $A \sqsubseteq_i^{dso} B$. Fix $a \in A$, $b \in B$. Suppose $a_i \leq b_i$. Then $a\Delta_i^{1-t}b = a \in A$ and $a\nabla_i^{1-t}b = b \in B$, as desired. Suppose $a_i > b_i$. Let $t \in [0, 1]$ be such that $v = t(b - a \wedge b) = t(a \vee b - a)$ satisfies $b - v \in A$ and $a + v \in B$. Then for $\lambda = 1 - t$, $a\Delta_i^{\lambda}b = (1 - t)b + t(a \wedge b) = b - t(b - a \wedge b) = b - v \in A$, and $a\nabla_i^{\lambda}b = (1 - t)a + t(a \vee b) = a + t(a \vee b - a) = a + v \in B$, as desired. □

The i-directional set order may be viewed as reformulating the $\mathcal{C}_i$-flexible set order to work more closely with monotone methods. In particular, i-directional set order does not invoke the binary relations Δ_i^{λ}, ∇_i^{λ}, it does not require convex sets, and it uses the standard properties of order and direction in $\mathbb{R}^N$.

Let X be a convex sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. The function f is ***$\mathcal{C}_i$-quasisupermodular***, if for every $a, b \in X$ and for every $\lambda \in [0, 1]$, $f(a) \geq (>)\ f(a\Delta_i^{\lambda}b) \Rightarrow f(a\nabla_i^{\lambda}b) \geq (>)\ f(b)$. One of the main results in Quah (2007) is the following: *for every $i \in \{1, \ldots, N\}$, $\arg\max_A f$ is increasing in A in the $\mathcal{C}_i$-flexible set order, if, and only if, f is $\mathcal{C}_i$-quasisupermodular.*

Notice that the property $\mathcal{C}_i$-quasisupermodular is symbolically similar to the notion of a quasisupermodular function. Its interpretation is more complex for two reasons: First, the use of the quantifier "for every $\lambda \in [0, 1]$" in the definition forces consideration of the whole line segment joining a and $a \vee b$ and the whole line segment joining $a \wedge b$ and b, and essentially forces consideration of convex sets, and second, the interpretive issues with using Δ_i^{λ}, ∇_i^{λ} carry over to this definition.

The use of the quantifier "for every $\lambda \in [0, 1]$" in this definition is required by the $\mathcal{C}_i$-flexible set order. This can be seen as follows. Suppose we consider weakening the definition of f is $\mathcal{C}_i$-quasisupermodular by requiring it to hold for only some collection of $\lambda \in [0, 1]$, as follows. Let X be a convex sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, $i \in \{1, 2, \ldots, N\}$, and Λ be a non-empty subset of $[0, 1]$. The function f is ***(i, Λ)-quasisupermodular***, if for every x, y in X, and every $\lambda \in \Lambda$, $f(x) \geq (>)\ f(x\Delta_i^{\lambda}y) \Rightarrow f(x\nabla_i^{\lambda}y) \geq (>)\ f(y)$. Notice that f is $\mathcal{C}_i$-quasisupermodular is a special case of this definition, when $\Lambda = [0, 1]$.

In order to characterize the type of monotone comparative statics possible with (i, Λ)-quasisupermodular functions, consider the following set order. Let X be a convex sublattice of $\mathbb{R}^N$, $i \in \{1, 2, \ldots, N\}$, and Λ be a non-empty subset of $[0, 1]$. For subsets A, B of X, A ***is (i, Λ)-lower than*** B, denoted $A \sqsubseteq_i^{\Lambda} B$, if for every $a \in A$, for every $b \in B$, there is $\lambda \in \Lambda$ such that $a\Delta_i^{\lambda}b \in A$ and $a\nabla_i^{\lambda}b \in B$. Notice that A is lower than B in the $\mathcal{C}_i$-flexible set order is a special case of this definition, when $\Lambda = [0, 1]$. Say that a function $f : X \to \mathbb{R}$ has ***(i, Λ)-increasing property***, if for every A, B subset of X, $A \sqsubseteq_i^{\Lambda} B \Longrightarrow \arg\max_A f \sqsubseteq_i^{\Lambda} \arg\max_B f$. We can prove the following result.

Proposition 5 *Let X be a convex sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, $i \in \{1, 2, \ldots, N\}$, and Λ be a non-empty subset of $[0, 1]$.*
f is (i, Λ)-quasisupermodular, if, and only if, f has (i, Λ)-increasing property.

Proof ($\Rightarrow$) Suppose f is (i, Λ)-quasisupermodular. Fix $A \sqsubseteq_i^{\Lambda} B$. Let $a \in \arg\max_A f$ and $b \in \arg\max_B f$. Notice that $A \sqsubseteq_i^{\lambda} B$ implies that there is $\lambda \in \Lambda$ such that

$a\Delta_i^\lambda b \in A$ and $a\nabla_i^\lambda b \in B$. Fix this λ. Thus $a \in \arg\max_A f \implies f(a) \geq f(a\Delta_i^\lambda b) \implies f(a\nabla_i^\lambda b) \geq f(b)$, where the last implication follows from (i, Λ)-quasisupermodularity of f. Moreover, as $b \in \arg\max_B f$, it follows that $f(a\nabla_i^\lambda b) = f(b)$, whence $a\nabla_i^\lambda b \in \arg\max_B f$. Furthermore, $f(a\nabla_i^\lambda b) = f(b) \implies f(a\nabla_i^\lambda b) \not> f(b) \implies f(a) \leq f(a\Delta_i^\lambda b)$, where the last implication follows from (i, Λ)-quasisupermodularity of f. As $a \in \arg\max_A f$, it follows that $f(a) = f(a\Delta_i^\lambda b)$, whence $a\Delta_i^\lambda b \in \arg\max_A f$, as desired.

($\Leftarrow$) Considering the contrapositive, suppose f is not (i, Λ)-quasisupermodular. Then there exists $\lambda \in \Lambda$, and there exist a, b in X, such that either (1) $f(a) \geq f(a\Delta_i^\lambda b)$ and $f(a\nabla_i^\lambda b) < f(b)$, or (2) $f(a) > f(a\Delta_i^\lambda b)$ and $f(a\nabla_i^\lambda b) \leq f(b)$. Notice that in either case, it must be that $a_i > b_i$. Therefore, $a\nabla_i^\lambda b \neq b$, $a\Delta_i^\lambda b \neq a$, and $a\Delta_i^\lambda b \neq a\nabla_i^\lambda b$. Let $C = \{a, a\Delta_i^\lambda b\}$ and $C' = \{b, a\nabla_i^\lambda b\}$. Then $C \sqsubseteq_i^\Lambda C'$. Suppose (1) is true. Then $a \in \arg\max_C f$ and $b = \arg\max_{C'} f$, but for every $\lambda' \in \Lambda$, $a\nabla_i^{\lambda'} b \notin \arg\max_{C'} f$, because $a_i > b_i$ implies that for every $\lambda' \in [0, 1]$, $a\nabla_i^{\lambda'} b \neq b$. Therefore, f does not have (i, Λ)-increasing property (for $C \sqsubseteq_i^\lambda C'$). Suppose (2) is true. Then $a = \arg\max_C f$ and $b \in \arg\max_{C'} f$, but for every $\lambda' \in \Lambda$, $a\Delta_i^{\lambda'} b \notin \arg\max_C f$, because $a_i > b_i$ implies that for every $\lambda' \in [0, 1]$, $a\Delta_i^{\lambda'} b \neq b$. Again, f does not have (i, Λ)-increasing property. □

The result in Quah (2007) is a special case of this result, when $\Lambda = [0, 1]$. The result here shows that if we want to weaken the notion of a $\mathcal{C}_i$-quasisupermodular function by requiring the condition to hold for fewer λ, then we must make the comparability of the set order more restrictive (that is, fewer sets can be ordered) by requiring less flexibility in the choice of λ as well. To say this differently, if we want a monotone comparative statics result applicable to a larger collection of constraint sets, we can expand the collection of sets that can be ordered by allowing the greatest flexibility in choosing λ, by setting $\Lambda = [0, 1]$. (This gives us the $\mathcal{C}_i$-flexible set order.) In this case, characterizing monotone comparative static requires imposing the strictest conditions on the objective function by requiring $\Lambda = [0, 1]$. In particular, for every a and b, we are forced to consider the whole line segment joining a and $a \vee b$ and the whole line segment joining $a \wedge b$ and b, and we are essentially forced to consider convex sets.

As mentioned in the text, corollaries to theorem 1 and the equivalence of i-directional set order and $\mathcal{C}_i$-flexible set order shows that on convex sublattices, a C_i-quasisupermodular function is equivalent to a function that is i-quasisupermodular and satisfies i-single crossing property on X. The following proposition shows this directly. For convenience, a part of the proof is written as the lemmas below.

Lemma 1 *For every $a, b \in \mathbb{R}^N$ with $a_i > b_i$, and for every $\lambda \in [0, 1]$, $a \wedge (a\Delta_i^\lambda b) = a \wedge b$.*

Proof Fix $a, b \in \mathbb{R}^N$ with $a_i > b_i$, and fix $\lambda \in [0, 1]$. Fix index $j = 1, \ldots, N$. As case 1, suppose $a_j \leq b_j$. Then $(a\Delta_i^\lambda b)_j = \lambda b_j + (1-\lambda)(a \wedge b)_j = \lambda b_j + (1-\lambda)a_j \geq a_j$, and therefore, $(a \wedge (a\Delta_i^\lambda b))_j = a_j = (a \wedge b)_j$. As case 2, suppose $a_j > b_j$. Then $(a\Delta_i^\lambda b)_j = \lambda b_j + (1-\lambda)(a \wedge b)_j = b_j = (a \wedge b)_j$, as desired. □

Lemma 2 *For every $a, b \in \mathbb{R}^N$ with $a_i > b_i$, and for every $\lambda \in (0, 1]$*

(1) $b = a\Delta_i^\lambda b + \frac{1-\lambda}{\lambda}(a\Delta_i^\lambda b - a \wedge b)$, *and*
(2) $a\nabla_i^\lambda b = a + \frac{1-\lambda}{\lambda}(a\Delta_i^\lambda b - a \wedge b)$.

Proof Fix $a, b \in \mathbb{R}^N$ with $a_i > b_i$, and fix $\lambda \in (0, 1]$. In this case, $a\Delta_i^\lambda b - a \wedge b = \lambda b + (1-\lambda)(a \wedge b) - a \wedge b = \lambda(b - a \wedge b)$, and therefore, $b - a\Delta_i^\lambda b = b - \lambda b - (1-\lambda)(a \wedge b) = (1-\lambda)(b - a \wedge b) = \frac{1-\lambda}{\lambda}(a\Delta_i^\lambda b - a \wedge b)$, showing (1). Similarly, $a \vee b - a\nabla_i^\lambda b = a \vee b - \lambda a - (1-\lambda)(a \vee b) = \lambda(a \vee b - a) = \lambda(b - a \wedge b)$, and therefore, $a\nabla_i^\lambda b - a = \lambda a + (1-\lambda)(a \vee b) - a = (1-\lambda)(a \vee b - a) = (1-\lambda)(b - a \wedge b) = \frac{1-\lambda}{\lambda}(a\Delta_i^\lambda b - a \wedge b)$, showing (2). □

Proposition 6 *Let X be a convex sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, \ldots, N\}$. The following are equivalent.*

1. *f is C_i-quasisupermodular*
2. *f is i-quasisupermodular and satisfies i-single crossing property on X*

Proof For (1) implies (2), to show i-single crossing property, fix x, y in X with $y_i < x_i$, fix $t \geq 0$, and let $v = t(y - x \wedge y)$. Let $z = y + v$ and $u = z - x \wedge z$. It is easy to check that $z - x \wedge z = z - x \wedge y$, and therefore, $u = z - x \wedge y = y + t(y - x \wedge y) - x \wedge y = (1+t)(y - x \wedge y)$. Notice also that $u = x \vee z - x$. Consequently, $v = t(y - x \wedge y) = \frac{t}{1+t}(1+t)(y - x \wedge y) = \frac{t}{1+t}u$. Let $\lambda = \frac{1}{1+t} \in (0, 1]$. Then $v = (1-\lambda)u$, and this implies that $y - x \wedge y = \lambda u$, because $z = y + v = y + (1-\lambda)u = y + u - \lambda u = y + z - x \wedge z - \lambda u$. Moreover, notice that $x\Delta_i^\lambda z = \lambda z + (1-\lambda)(x \wedge z) = x \wedge z + \lambda(z - x \wedge z) = x \wedge z + \lambda u = y$, and that $x\nabla_i^\lambda z = \lambda x + (1-\lambda)(x \vee z) = x \vee z + \lambda(x \vee z - x) = x \vee z - \lambda u = x \vee z + v - u = x \vee z + v - (x \vee z - x) = x + v$. Now suppose $f(x) \geq f(y) = f(x\Delta_i^\lambda z)$. Then $f(x + v) = f(x\nabla_i^\lambda z) \geq f(z) = f(y + v)$, where the inequality follows from C_i-quasisupermodularity. The case for strict inequality follows similarly. Thus f satisfies i-single crossing property. i-quasisupermodularity of f follows from C_i-quasisupermodularity for $\lambda = 0$.

For (2) implies (1), fix i, fix $a, b \in \mathbb{R}^N$, and fix $\lambda \in [0, 1]$. As case 1, suppose $a_i \leq b_i$. In this case, $a\Delta_i^\lambda b = a$, $a\nabla_i^\lambda b = b$, and therefore, $f(a) \geq f(a\Delta_i^\lambda b) \Rightarrow f(a\nabla_i^\lambda b) \geq f(b)$ holds trivially. Strict inequality holds vacuously. As case 2, suppose $a_i > b_i$. As subcase 1, suppose $\lambda \neq 0$. Let $x = a$, $y = a\Delta_i^\lambda b$, and $v = \frac{1-\lambda}{\lambda}\left(a\Delta_i^\lambda b - a \wedge (a\Delta_i^\lambda b)\right)$. Using lemma 1, $v = \frac{1-\lambda}{\lambda}(a\Delta_i^\lambda b - a \wedge b)$. Suppose $f(a) \geq f(a\Delta_i^\lambda b)$. That is, $f(x) \geq f(y)$. Then i-single crossing property implies $f(x+v) \geq f(y+v)$. By construction, $x+v = a + \frac{1-\lambda}{\lambda}(a\Delta_i^\lambda b - a \wedge b) = a\nabla_i^\lambda b$, where the last equality follows from lemma 2. Also, $y+v = a\Delta_i^\lambda b + \frac{1-\lambda}{\lambda}(a\Delta_i^\lambda b - a \wedge b) = b$, where the last equality follows from lemma 2. The case for strict inequality follows similarly. As subcase 2, suppose $\lambda = 0$. Then $a\Delta_i^\lambda b = a \wedge b$ and $a\nabla_i^\lambda b = a \vee b$, so the property follows from i-quasisupermodularity of f. □

Appendix B: Some proofs

One set of conditions under which sets can be ordered in the i-directional set order is as follows. Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$.

The function f is ***i-quasisubmodular on*** X, if for every $a, b \in X$ with $a_i > b_i$, $f(a) \leq (<) f(a \wedge b) \implies f(a \vee b) \leq (<) f(b)$. The function f satisfies ***dual i-single crossing property on*** X, if for every $a, b \in X$ with $a_i > b_i$, and for every $v \in \{s(b - a \wedge b) \mid s \in \mathbb{R}, s \geq 0\}$ such that $a + v, b + v \in X$, $f(a) \leq (<) f(b) \implies f(a + v) \leq (<) f(b + v)$.

Proposition 7 *Let X be a convex sublattice of* $\mathbb{R}^N$, $f : X \to \mathbb{R}$, *and* $i \in \{1, 2, \dots, N\}$. *If f is continuous, (weakly) increasing, i-quasisubmodular, and satisfies dual i-single crossing on X, then* $\tau \leq \tau' \implies \{x \mid f(x) \leq \tau\} \sqsubseteq_i^{dso} \{x \mid f(x) \leq \tau'\}$.

Proof Let $\tau \leq \tau'$, $A = \{x \mid f(x) \leq \tau\}$, $B = \{x \mid f(x) \leq \tau'\}$, and suppose $a \in A$, $b \in B$ with $a_i > b_i$. As case 1, suppose $f(b) \leq \tau$. In this case, let $v = 0$. Then $b - v = b \in A$ and $f(a) \leq \tau \leq \tau'$ implies that $a + v = a \in B$. As case 2, suppose $f(b) > \tau$. Then f is (weakly) increasing implies that $f(a \wedge b) \leq f(a) \leq \tau$. For $s \in [0, 1]$, consider $v(s) = s(b - a \wedge b)$. Then $s = 0$ implies $f(b - v(s)) > \tau$ and $s = 1$ implies $f(b - v(s)) \leq \tau$. By continuity, there is $\hat{s} \in (0, 1]$ such that $f(b - v(\hat{s})) = \tau$. Set $\hat{v} = \hat{s}(b - a \wedge b)$. Then $b - \hat{v} \in A$ and $f(b - \hat{v}) \geq f(a)$. As subcase 1, suppose $\hat{s} = 1$. Then $f(a \wedge b) = f(b - \hat{v}) \geq f(a)$ and i-quasisubmodularity implies that $f(b) \geq f(a \vee b)$, whence $a + 1(a \vee b - a) \in B$. As subcase 2, suppose $\hat{s} \in (0, 1)$. Applying dual i-single crossing to vectors to a and $b - \hat{v}$, with the directional vector $w = \frac{\hat{s}}{1-\hat{s}} \left[(b - \hat{v}) - a \wedge (b - \hat{v})\right]$ implies $f(b - \hat{v} + w) \geq f(a + w)$. But notice that $\hat{v} = \hat{s}(b - a \wedge b) = \hat{s}\left[(b - \hat{v}) - a \wedge b\right] + \hat{s}\hat{v} = \hat{s}\left[(b - \hat{v}) - a \wedge (b - \hat{v})\right] + \hat{s}\hat{v}$, and therefore, $\hat{v} = \frac{\hat{s}}{1-\hat{s}} \left[(b - \hat{v}) - a \wedge (b - \hat{v})\right] = w$. In other words, $f(b) \geq f(a + \hat{v})$, whence $a + \hat{v} \in B$. □

It is easy to check that for given prices $p \gg 0$, the function $\phi : X \to \mathbb{R}$, $\phi(x) = p \cdot x$ satisfies these conditions. This provides another proof that with respect to wealth w, Walrasian budgets sets are ordered in the i-directional set order.

To show the equivalence of i-increasing differences (u) on X and i-increasing differences (*) on X, consider first the following slight modification of i-increasing differences (u) on X. Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \dots, N\}$. f satisfies ***i-increasing differences*** (σu), if for every $b \in X, u \in \mathbb{R}^N$ with $u_i > 0$, for every $\sigma, s \geq 0$, such that $b + \sigma u, b + s(-u)_+, b + \sigma u + s(-u)_+ \in X$, $f(b + \sigma u) - f(b) \leq f(b + \sigma u + s(-u)_+) - f(b + s(-u)_+)$. Consider the following equivalence.

Lemma 3 *Let X be a sublattice of* $\mathbb{R}^N$, $f : X \to \mathbb{R}$, *and* $i \in \{1, 2, \dots, N\}$. *f satisfies i-increasing differences (u) on X, if, and only if, f satisfies i-increasing differences* (σu) *on X.*

Proof For sufficiency, fix $b \in X$, $u \in \mathbb{R}^N$ with $u_i > 0$, and fix $\sigma, s \geq 0$. If $\sigma = 0$, we are done, because left-hand side and right-hand side of the condition are both zero. Suppose $\sigma > 0$. Let $\hat{u} = \sigma u$ and $\hat{s} = \frac{s}{\sigma} \geq 0$. Then $\hat{u}_i > 0$ and $\hat{s}(-\hat{u}_+) = s(-u)_+$, and therefore,

$$\begin{aligned} f(b + \sigma u) - f(b) &= f(b + \hat{u}) - f(b) \\ &\leq f(b + \hat{u} + \hat{s}(-\hat{u})_+) - f(b + \hat{s}(-\hat{u})_+) \end{aligned}$$

$$= f(b + \sigma u + s(-u)_+) - f(b + s(-u)_+),$$

as desired. For necessity, let $\sigma = 1$. □

Now recall that f satisfies i-increasing differences (*) on X, if for every $b \in X$, $u \in \mathbb{R}^N$ with $u_i > 0$, for every $\sigma \geq 0$, $f(b+\sigma u+s(-u)_+) - f(b+s(-u)_+)$ is (weakly) increasing in s, (where we consider only points $b+\sigma u+s(-u)_+$, $b+s(-u)_+ \in X$). Consider the following equivalence.

Lemma 4 *Let X be a sublattice of $\mathbb{R}^N$, $f : X \to \mathbb{R}$, and $i \in \{1, 2, \ldots, N\}$. f satisfies i-increasing differences (σu) on X, if, and only if, f satisfies i-increasing differences (*) on X.*

Proof Suppose f satisfies i-increasing differences (σu) on X. To check for i-increasing differences (*) on X, fix $b \in X, u \in \mathbb{R}^N$ with $u_i > 0$, and $\sigma \geq 0$. Fix $s_1 \leq s_2$. If $\sigma = 0$, we are done, because the expression is 0 for all s. Suppose $\sigma > 0$. Let $\hat{b} = b + s_1(-u)_+$ and $\hat{s} = s_2 - s_1 \geq 0$. Then

$$\begin{aligned}
&f(b + \sigma u + s_1(-u)_+) - f(b + s_1(-u)_+)\\
&\quad = f(\hat{b} + \sigma u) - f(\hat{b})\\
&\quad \leq f(\hat{b} + \sigma u + \hat{s}(-u)_+) - f(\hat{b} + \hat{s}(-u)_+)\\
&\quad = f(b + \sigma u + s_1(-u)_+ + (s_2 - s_1)(-u)_+)\\
&\quad - f(b + s_1(-u)_+ + (s_2 - s_1)(-u)_+)\\
&\quad = f(b + \sigma u + s_2(-u)_+) - f(b + s_2(-u)_+),
\end{aligned}$$

as desired.

Suppose f satisfies i-increasing differences (*) on X. To check that f satisfies i-increasing differences (σu) on X, fix $b \in X$, $u \in \mathbb{R}^N$ with $u_i > 0$, and fix $\sigma, s \geq 0$. Let $s_1 = 0$. Then $s_1 \leq s$, and therefore, $f(b+\sigma u) - f(b) = f(b+\sigma u+s_1(-u)_+) - f(b + s_1(-u)_+) \leq f(b + \sigma u + s(-u)_+) - f(b + s(-u)_+)$, as desired. □

These lemmas imply the equivalence of i-increasing differences (u) on X and i-increasing differences (*) on X, as desired.

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