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RESEARCH ARTICLE

Formalization of information: knowledge and belief

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Abstract Billingsley (Probability and measure, Wiley, New Jersey, 1995) and Dubra and Echenique (Math Soc Sci 47(2):177–185, 2004) provide an example to show that the formalization of information by σ-algebras and by partitions need not be equivalent. Although Hervés-Beloso and Monteiro (Econ Theory 54(2):405–418, 2013) provide a method to generate a σ-algebra from a partition and another method for going in the opposite direction, we show that their two methods are in fact based on two *different* notions of information: (i) information as belief, (ii) information as knowledge. If information is conceived to allow for falsehoods, case (i) above, the equivalence between σ-algebras and partitions holds after applying the notion of posterior completion suggested by Brandenburger and Dekel (J Math Econ 16(3):237–245, 1987). If information is conceived not to allow for falsehoods, case (ii) above, the equivalence holds only for measurable partitions and countably generated σ-algebras.

Keywords Information · σ-Algebras · S5 knowledge · KD45 belief · Formalization of information

JEL Classification C60 · C70

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1 Introduction

In any model that deals with a decision maker (henceforth DM) facing uncertainty, the DM's information is often described by either a signal (equivalently, a random variable), a partition or a σ-algebra. Specifically, one signal is more informative than another if it is sufficient in Blackwell's sense for another; one partition is more informative than another if it is finer; a σ-algebra is more informative than another if it is larger.[1] A natural question is whether all these three orderings can be equivalently used to represent information. In other words, it is to ask whether there is a mapping from one category of representation to another while preserving the ordering in the two categories that are being used. The answer to this question had been understood to be positive. Nevertheless, as we shall see below, the understanding is far from complete.

Billingsley (1995) raises concerns that partitions and σ-algebras may not always be equivalently used by presenting a simple but powerful example: a unit interval is given as the state space equipped with the Lebesgue measure. A partition that consists of every singleton indicates that the DM knows exactly in which state she lies. On the contrary, the smallest σ-algebra generated by the partition implies that the DM is totally ignorant, for it contains countable or co-countable sets that are of Lebesgue measure zero.[2] In addition, Dubra and Echenique (2004) highlight Billingsley's concern by embedding his example in the context of a decision problem. They consider another partition that contains only two cells. This partition is obviously less informative than the partition in Billingsley's example. However, if one compares the expected utility values conditional on the smallest σ-algebras generated by those partitions, the value based on the two-cell partition is larger. That is, the σ-algebra generated by the two-cell partition is more informative.

In response to these cautionary warnings, Hervés-Beloso and Monteiro (2013) (henceforth HM) argue that one may disregard them. By taking a partition as a primitive representation of information, they introduce a notion of an *informed set* which corresponds to a (possibly uncountable) union of partition cells. The collection of all informed sets is indeed a σ-algebra. If one generates σ-algebras in this way, a strictly finer partition always yields a larger σ-algebra. Arguably, the collection of informed sets represents the informational content of a given partition. To establish the equivalence between partitions and σ-algebras, they also suggest another method of deriving a partition from a given σ-algebra. Given a measure space equipped with a strongly Blackwell σ-algebra, HM suggest to form a partition by collecting atoms of a σ-algebra if it is countably generated.[3] If not, they suggest to consider a countably

[1] For a pair of partitions, the strictly finer partition distinguishes more elements, implying that a DM can say more accurately about the true state (the state in which she lies). For a pair of σ-algebras, the larger one contains more sets. For larger number of sets, a DM is able to say whether it contains the true state or not, thus having more information.

[2] By the smallest σ-algebra generated by the partition, we mean that the σ-algebra contains all the complements and the countable unions of partition cells.

[3] A countably generated σ-algebra is the smallest σ-algebra generated by a collection of countably many subsets of the state space.

generated σ-algebra that differs from the given σ-algebra by null sets.[4] This implies, although HM do not so explicitly argue, that the informational content of a σ-algebra is captured by the corresponding countably generated σ-algebra.

In this paper, our primary goal is to show that HM leave unsettled the following question, "What is the information (equivalently, the informational content) preserved when one generates a σ-algebra from a partition or when one goes in the opposite direction?" HM claim that it is the collection of informed sets, and they interpret the notion of an informed set to denote the set of which occurrence (or non-occurrence) a DM knows. This naturally leads one to ask a question about the difference between what one merely knows and what one is informed of. Unfortunately, however, HM are silent on this question. In addition, the informational content, as HM claim, is also captured by the countably generated σ-algebra that differs from a σ-algebra by null sets. This implies that if both a partition and a σ-algebra contain the same informational content, the collection of informed sets of the partition must be countably generated. We present a counterexample in which this is not the case (Example 5). Furthermore, the collection of informed sets of a partition may contain non-measurable sets, because the informed sets do not depend on a given measurable space. We show that this indeed happens in Billingsley's example (Example 4). This poses a technical impossibility of defining a probability measure on the non-measurable set when computing the expected utility value as in Dubra and Echenique (2004), not to mention a conceptual difficulty of how to understand that a DM is informed about a set lying outside the event space.[5] More importantly, HM's treatment provides a contradictory answer about whether a probability measure conveys any informational content or not. As noted, informed sets of a partition are invariant to any choice of a measure space and a measure defined on it. This suggests that a probability measure does not convey any information. Contradictorily, a probability measure conveys information if one considers the informational content embodied in a σ-algebra, as it is unique up to null sets.

The secondary goal of this paper is to tackle these issues and to establish an equivalence relationship between a partition and a σ-algebra in representing information. Our innovation is to bring out with especial salience the two distinct notions of information, knowledge and belief, that are well recognized among researchers working in epistemic logic and game theory.[6] The distinction lies in whether information is conceived to be factual or not. To elaborate, if one insists that information cannot be false in order to distinguish it from a rumor, then he conceives information to arise from knowledge. On the contrary, if one allows for the possibility that information may turn out to be false, then he conceives information to arise from belief.

[4] A null set is a set to which a DM ascribes zero probability. HM refers to it as a negligible set of states.

[5] The existence of non-measurable sets can be addressed by Theorem 4 and the following Remark 3 in HM. However, the notion of an informed set, as it is defined in HM, fails to accommodate this: the collection of informed sets in Billingsley's example, according to HM, is the power set even when the underlying σ-algebra is strongly Blackwell (See Example 4 in HM). In fact, we propose the notion of an informed event to accommodate Theorem 4 and Remark 3 in HM.

[6] See, for example, Aumann (1999a, b), Maschler et al. (2013), and Meyer (2003).

The advantage of bringing out these two notions of information is that each notion, either knowledge or belief, is formally defined as an operator from a measurable space (or, equivalently, an event space) to itself that satisfies a certain set of axioms (Definitions 4, 5). One can thus see easily whether a mathematical object such as a partition and a σ-algebra qualifies for being a formalization of information (as knowledge/belief), by inspecting the relationship between a knowledge/belief operator to the mathematical object of one's interest. By taking advantage of the two notions, we resolve the issues that HM leave open. Firstly, we show that the notion of an informed event imposes a counterfactual restriction on that of knowledge/belief. To be specific, we define a K-informed event for information as knowledge, and a B-informed event for information as belief. An informed event requires that if one knows/believes whether an event occurs or not at one state, then he must know/believe it even in a hypothetical situation that he lies at other states (Example 3). Secondly, we show that the collection of K-informed events is the restriction of the collection of informed sets (defined by HM) to a measurable space, thereby resolving the issue regarding the presence of non-measurable set (Lemma 3). This is immediate from the definition of knowledge/belief being an operator from a measurable space to itself.

Turning to the remaining issues, we show that if one conceives information as knowledge, measurable partitions and countably generated σ-algebras can be used interchangeably to formalize information (Theorem 1). This implies that the preserved informational contents are the K-informed events of measurable partitions. Moreover, it also reveals that probability does not convey any information, for the K-informed event is invariant to a specific choice of a probability measure.

A further question is whether we need restrict the use of partitions or σ-algebras only to the case where partitions are measurable or σ-algebras are countably generated. We argue that if one conceives information as belief, we do not need such a restriction. By adopting the technique of *posterior completion*[7] proposed by Brandenburger and Dekel (1987), we show that if the posterior completion of a σ-algebra is larger then the posterior completion of a partition is strictly finer, and vice versa. Then, what is the informational content in this case? We argue that the informational content is indeed the collection of B-informed events, and it depends on a specific choice of a probability measure. Specifically, a proper regular conditional probability (either directly from a σ-algebra or from the smallest σ-algebra generated by a partition) captures the notion of belief. More importantly, the collection of B-informed events is the posterior completion of a given σ-algebra. Since B-informed events are defined in relation to a given probability measure, probability conveys information.

The paper is structured as follows: we present preliminary definitions including the notions of knowledge and belief in Sect. 2. In Sect. 3, under the conception of information as knowledge, we establish an equivalence between measurable partitions and countably generated σ-algebras in formalizing information. Moreover, we discuss the issues regarding the notion of informed sets as formalized by HM. Section 4

[7] The posterior completion of a σ-algebra is to create the smallest σ-algebra by adding events that are either measure zero or one with a proper regular conditional probability measure, into a given σ-algebra. The posterior completion of a partition is to add in those events to the partition.

consists of an equivalence result under the conception of information as belief. Then, we conclude in Sect. 5.

2 Preliminaries

Partitions and σ-algebras Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω is a non-empty set of states endowed with a σ-algebra $\mathcal{F}$, so-called the event space. Measurable sets of the σ-algebra $\mathcal{F}$ are called events. We assume that Ω is a complete separable metric space, and the event space $\mathcal{F}$ is a strongly Blackwell σ-algebra.[8] The complement of an event E is denoted by $\neg E$.

Definition 1 Let X and Y be partially ordered sets (posets) with the partial orderings $\succeq^X$ and $\succeq^Y$. A mapping $\Phi : X \to Y$ is an order isomorphism if Φ is bijective and preserves order in the following sense: $x \succeq^X x' \iff \Phi(x) \succeq^Y \Phi(x')$. If such an order isomorphism exists, X and Y are said to be order isomorphic.

Definition 2 (*Partition*) Let $(\Omega, \mathcal{F})$ be given. A collection of non-empty events is called a *partition* and denoted by Π if it satisfies the following:

(1) $\cup\{E|E \in \Pi\} = \Omega$;
(2) If $E, F \in \Pi$ and $E \neq F$, then $E \cap F = \emptyset$.

Note that we define a partition to be a collection of events (or, equivalently, measurable sets). Let Π_ω denote an element of Π containing a state ω, and it is unique. For two partitions Π and Π', we say that Π is *finer* than Π', denoted by $\Pi \succeq^P \Pi'$, if for each $\omega \in \Omega$, $\Pi_\omega \subseteq \Pi'_\omega$. Let P be a collection of all partitions of Ω. Then, $\succeq^P$ is a partial ordering on P and $(P, \succeq^P)$ is a partially ordered set (poset).

Definition 3 (*Sub-σ-algebra*) Let $(\Omega, \mathcal{F})$ be given. A sub-σ-algebra $\mathcal{G}$ is a sub-collection of events satisfying the following two properties:

(1) Closed under complements: for any $E \in \mathcal{G}$, $\neg E \in \mathcal{G}$.
(2) Closed under countable unions: for any countable number of events $\{E_i\}_{i \in I}$ with $E_i \in \mathcal{G}$, $\cup_{i \in I} E_i \in \mathcal{G}$.

For a σ-algebra $\mathcal{G}$ and a state $\omega \in \Omega$, an atom $\mathscr{A}(\omega, \mathcal{G}) = \cap\{G \in \mathcal{G}|\omega \in G\}$ is the smallest set containing ω in a σ-algebra $\mathcal{G}$. Whenever $\mathcal{G}$ is obvious, we simply denote it by $\mathscr{A}_\omega$.

Let Σ be a collection of all sub-σ-algebras of Ω. A sub-σ-algebra $\mathcal{G}$ is larger than $\mathcal{H}$ if for every $E \in \mathcal{H}$, $E \in \mathcal{G}$. This naturally defines a partial ordering $\succeq^\sigma$ on Σ such that for two sub-σ-algebras $\mathcal{G}$ and $\mathcal{H}$, $\mathcal{G} \succeq^\sigma \mathcal{H}$ if $\mathcal{G}$ is larger than $\mathcal{H}$. Then, $(\Sigma, \succeq^\sigma)$ is a poset.

For the two posets $(P, \succeq^P)$ and $(\Sigma, \succeq^\sigma)$, define a mapping $F : (P, \succeq^P) \to (\Sigma, \succeq^\sigma)$ such that for $\Pi \in P$, $F(\Pi)$ is the smallest σ-algebra generated by the partition cells of Π. Then, as the following example from Billingsley (1995) shows, F is not an (order) isomorphism.

[8] A σ-algebra is a strongly Blackwell σ-algebra if it is separable and every two countably generated sub-σ-algebras with the same atom coincide.

Example 1 (Billingsley) Let $\Omega = [0, 1] \subset \mathbb{R}$ endowed with a Borel σ-algebra $\mathcal{F}$. Let $\Pi = \{\{\omega\} | \omega \in \Omega\}$ and $\Pi' = \{\left[0, \frac{1}{2}\right), \left[\frac{1}{2}, 1\right]\}$. Then, $F(\Pi) = \{E \in \mathcal{F} |$ either E or $\neg E$ is countable$\}$ and $F(\Pi') = \{\emptyset, \left[0, \frac{1}{2}\right), \left[\frac{1}{2}, 1\right], \Omega\}$. Clearly, Π is finer than Π' ($\Pi \succeq^P \Pi'$). However, neither σ-algebra is larger than the other: neither $F(\Pi) \succeq^\sigma F(\Pi')$ nor $F(\Pi') \succeq^\sigma F(\Pi)$.

Belief and Knowledge The following definitions are standard in the literature on epistemic logic and game theory. For example, see Aumann (1999a, b); Maschler et al. (2013), and Meyer (2003).

Definition 4 (*Belief*) Let $(\Omega, \mathcal{F})$ be given. An operator $B : \mathcal{F} \longrightarrow \mathcal{F}$ is said to be a *belief* if B satisfies the following axioms:

A1 Conjunction: For any countable index set I and events $\{E_i\}_{i \in I}$ with $\cap_{i \in I} E_i \in \mathcal{F}$, $\cap_{i \in I} B(E_i) = B(\cap_{i \in I} E_i)$.
A2 Consistency: $B(E) \cap B(\neg E) = \emptyset$.
A3 Positive introspection: $B(E) \subseteq B(B(E))$ for $E \in \mathcal{F}$.
A4 Negative introspection: $\neg B(E) \subseteq B(\neg B(E))$ for $E \in \mathcal{F}$.

For $\omega \in \Omega$ and $E \in \mathcal{F}$, $\omega \in B(E)$ is read as "A DM *believes* an event E at a state ω." Therefore, for an event E, $B(E)$ is an event that whenever it occurs, the DM believes that the event E occurs. In this sense, $B(E)$ is the event that is an evidence based on which the DM believes E.

Definition 5 (*Knowledge*) Let $(\Omega, \mathcal{F})$ be given. An operator $K : \mathcal{F} \longrightarrow \mathcal{F}$ is said to be *knowledge* if it satisfies the axioms of a belief operator and the following additional axiom:

A5 Non-delusion: $K(E) \subseteq E$ for $E \in \mathcal{F}$.

Note that a knowledge operator K is also a belief operator, but the converse does not hold in general. In what follows, we shall use B to denote a belief operator and K a knowledge operator. Similarly to the case of belief, we say that the DM *knows* at ω that the event E occurs, or simply that the DM knows E at ω if $\omega \in K(E)$.

Any belief operator B satisfies the following properties:

A6 Necessitation: $B(\emptyset) = \emptyset$.
A7 Monotonicity: $E \subseteq F$ implies $B(E) \subseteq B(F)$.

The proof is easy, so we omit it.[9] Given a belief operator, one can completely describe what the DM believes at each state, or his *doxastic* status. Similarly, a knowledge operator specifies what the DM knows at each state, or his *epistemic* status. If one chooses a different belief (or knowledge) operator, it indicates a different doxastic (or epistemic) status as it is illustrated in the following example.

Example 2 Let $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{F} = 2^\Omega$. Consider two knowledge operators, K and K' such that $K(\{\omega\}) = \{\omega\}$ for $\omega \in \Omega$, and $K'(\{\omega\}) = \emptyset$. Let ω be the true

[9] Interested readers may see, for example, Bacharach (1985).

state. For any event E with $\omega \in E$, $\omega \in K(E)$ but $\omega \notin K'(E)$ unless $E = \Omega$. The knowledge operator K thus implies that a DM knows all the events that actually occur at the true state. On the contrary, K' indicates that the DM does not know any event that occurs, except for that the state space Ω itself occurs.

Note that the notion of belief and thus of knowledge rely on the event space $\mathcal{F}$. Although the definitions given in this paper are standard in the literature on epistemic logic and game theory, this reliance may raise an issue about why some sets of states, if they lie outside the event space, are precluded from being the subjects of belief and knowledge. This issue becomes trivial if the event space is given as the powerset. Hence, we shall focus on the case where the event space is strictly smaller than the powerset. Then, a natural question arises. What is the meaning of an event if it does not merely mean a set of states? Before answering this, one cannot understand why the set of states being an event is essential in defining the notion of belief and thus of knowledge. Unfortunately, however, there is no consensus about why some sets of states are not events. Savage (1972) thus insists the event space to be the powerset, but for a technical need to define a countably additive probability measure, the event space is required to be smaller as in Arrow (1966). Shafer (1986) interprets this restriction as *complexity of describing states, thus of comparing acts*. Villegas (1964), implicit though, takes this point by taking events to be a primitive of uncertainty. Taking Shafer's point of view, we interpret the event space to be the collection of sets of states which the DM is able to recognize.[10] Accordingly, sets of states lying outside the event space are not recognizable to the DM. As the DM cannot believe/know those that he cannot recognize, we may preclude those sets of states from being the subjects of belief and thus of knowledge.

Now, we define an informed event.

Definition 6 (*Informed event*) For a belief operator $B : \mathcal{F} \longrightarrow \mathcal{F}$, an event $E \in \mathcal{F}$ is an *B-informed event* if $B(E) \cup B(\neg E) = \Omega$. Similarly, for a knowledge operator K, an event $E \in \mathcal{F}$ is said to be *K-informed event*. A DM is said to be *B-informed (K-informed, resp.) about an event E at* ω if E is an B-informed (K-informed, resp.) event and $\omega \in B(E)$ ($\omega \in K(E)$, resp).

The above definition draws a distinction between what one knows/believes and what one is informed about. Although he knows/believes the event, he may not be informed about it. For him to be informed, he must know either the event occurs or not at any state. This requires that the DM has *counterfactual* knowledge/belief about the event. To illustrate this possibilities, consider a variant of Example 2.3 in Halpern (1999).

Example 3 Bob is in a room with the light on. The door is painted either red or blue, and he can tell which color. However, he might not have distinguished the colors, had the room been dark. Formally, there are four states, $\{(red, off), (blue, off), (red, on), (blue, on)\}$, where (red, off) denotes a state in which the door is red and the light is off, and the other states can be similarly interpreted.

[10] This interpretation is similar to the view in Heifetz et al. (2006). They consider events to be "those that can be "known" or be the object of awareness." For more discussion about the conception of an event, see Al-Najjar (2009).

Let *RED*, *BLUE*, *ON*, and *OFF* be the events that the door is red, the door is blue, the light is on, and the light is off. Let K be the knowledge operator describing Bob's knowledge. Then, $K(ON) = ON$, $K(OFF) = OFF$, $K(RED) = \{(red, on)\}$, and $K(BLUE) = \{(blue, on)\}$. Suppose that only the event RED is of an agent's interest, and the realized state is (red, on). As a consequence, Bob *knows* that the event *RED* occurs. Were the realized state to be (red, off), however, he would have not known that *RED* occurs, nor does $BLUE = \neg RED$ occur. For $K(RED) \cup K(\neg RED) = ON \neq \Omega$, *RED* is not an informed event. Therefore, Bob is *not* informed of the event *RED*.

In this example, an event ON is a K-informed event. At the state (red, on), an agent knows that the light is on. In addition, he would know whether the light is on or off, even in his imagination that any other state might have occurred.

The following lemma shows that a K-informed event is sufficient for a DM to know itself. In this sense, a K-informed event represents information.

Lemma 1 *Let E be a K-informed event. Then, E is self-evident*[11]*: $E = K(E)$.*

Proof Suppose that E is a K-informed event, i.e., $K(E) \cup K(\neg E) = \Omega$. By **A5**, $K(E) \subseteq E$, so it suffices to show that $E \subseteq K(E)$. By **A2**, $K(E) \cap K(\neg E) = \emptyset$ and thus $\neg K(E) = K(\neg E)$. Again by **A5**, $\neg K(E) = K(\neg E) \subseteq \neg E$. Thus, $E \subseteq K(E)$. □

By definition of knowledge and belief, it is easy to see that a K-informed event is a B-informed event, but not every B-informed event is a K-informed event. Moreover, a B-informed event is not necessarily self-evident.

3 Representation of information as knowledge

We first present a well-known result on the relationship between a partition and a knowledge operator.

Lemma 2 *For a partition $\Pi \in P$, define $K_\Pi(E) = \{\omega | \Pi(\omega) \subseteq E\}$ for each $E \in \mathcal{F}$. Then, K_Π satisfies **A1–A5**. For an operator $K : \mathcal{F} \to \mathcal{F}$ satisfying **A1–A5**, define a partition $\Pi_K = \{\Pi_K(\omega) | \omega \in \Omega\}$, where $\Pi_K(\omega) = \cap\{E \in \mathcal{F} | \omega \in K(E)\}$. Then, $\Pi = \Pi_{K_\Pi}$.*

For the proof, see Aumann (1999a). According to the above lemma, a K-informed event can be defined with respect to a partition in the following way: E is a K-informed event with respect to a partition Π if $E = K_\Pi(E)$. By adapting the notion of a K-informed event to a partition, we can compare our notion of a K-informed event directly with HM's notion of an informed set. For comparison, we present HM's notion of an informed set.

Definition 7 (*Informed set in HM*) A set $E \subseteq \Omega$ is an informed set defined by a partition Π if for every $F \in \Pi$, either $F \subseteq E$ or $F \subseteq \neg E$. The collection of informed sets of Π is denoted by $\mathcal{I}_\Pi$.

[11] This term originates in Aumann (1999a). Whenever a self-evident event occurs, it informs the DM of its occurrence. The self-evident event, therefore, *is* the knowledge about itself.

The definition of an informed set by HM is related to ours by the following lemma. Let $\mathcal{F}_\Pi$ denote the collection of K-informed events adapted to a partition Π.

Lemma 3 *Let $(\Omega, \mathcal{F})$ be given. For a partition Π, let $\mathcal{I}_\Pi$ denote a collection of its informed sets defined by HM, and let $\mathcal{F}_\Pi$ denote a collection of its K-informed events. Then, $\mathcal{F}_\Pi = \mathcal{I}_\Pi \cap \mathcal{F}$. Moreover, $\mathcal{F}_\Pi$ is a sub-σ-algebra of $\mathcal{F}$.*

The collection of informed sets by HM does not have to be a sub-σ-algebra. That is, there may exist an informed set that is non-measurable.

Example 4 Let $\Omega = [0, 1]$ equipped with a Borel σ-algebra, and let μ be the Borel measure defined on it. Let $\Pi = \{\{\omega\}|\omega \in \Omega\}$ be a partition that contains all singletons. Then, the collection of its informed sets $\mathcal{I}_\Pi$ is the powerset. Obviously, this is larger than the Borel σ-algebra and contains a well-known non-measurable set, so-called Vitali set. See Royden (1988) for its definition.

Now, we investigate the relationship between a knowledge operator and a σ-algebra. From the discussion on partitions, one can easily see that a knowledge operator defines a σ-algebra. What is not clear is whether a σ-algebra may define a knowledge operator. For our purpose, we need the following definition.

Definition 8 (*Countably generated σ-algebra*) A sub-σ-algebra $\mathcal{G}$ is *countably generated* if there is a collection of countably many events $\mathcal{U} = \{E_i | i \in \mathbb{N}\}$ such that $\mathcal{G}$ is the smallest σ-algebra containing $\mathcal{U}$.

We show that a countably generated σ-algebra also represents information as knowledge.

Lemma 4 *Let $\mathcal{G}$ be a countably generated sub-σ-algebra. Define for an event $E \in \mathcal{F}$,*

$$K(E) = \cup\{G \in \mathcal{G} | G \subseteq E\}.$$

Then, K is indeed a knowledge operator. Moreover, every event in $\mathcal{G}$ is a K-informed event, i.e., $K(G) = G$ for every $G \in \mathcal{G}$.

Proof To show that K is a knowledge operator, it suffices to show **A1**,**A4** and **A5**, because they implies the rest Bacharach (1985). For **A1**, let $(E_i)_{i\in I}$ be given for a countable index set I. Then, $\cap_{i\in I} K(E_i) = \cup\{\cap_{i\in I} G_i \in \mathcal{G} | G_i \subseteq E_i, \forall i \in I\} = \cup\{\cap_{i\in I} G_i \in \mathcal{G} | \cap_{i\in I} G_i \subseteq \cap_{i\in I} E_i\} = K(\cap_{i\in I} E_i)$. For **A4**, since a countably generated σ-algebra $\mathcal{G}$ is a sub-σ-algebra of a strongly Blackwell σ-algebra $\mathcal{F}$, it is closed under complements and arbitrary unions, and thus $\neg K(E) \in \mathcal{G}$ holds. Then, $K(\neg K(E)) = \cup\{G \in \mathcal{G} | G \subseteq \neg K(E)\} = \neg K(E)$. Lastly, **A5** and the last claim that $K(G) = G$ for $G \in \mathcal{G}$ trivially follow from the definition of K. □

As both partitions and countably generated σ-algebras represent information as knowledge, one may wonder whether they can be always equivalently used. Unfortunately, however, this is not true.

Example 5 Let $\Omega = [0, 1]$ endowed with a Borel σ-algebra $\mathcal{F}$. Let μ be the Borel measure. Define a mapping $\phi : [0, 1] \to [0, 1]$ such that for $\omega \in [0, 1]$, $\phi(\omega) = \omega + \alpha$ if $\omega + \alpha \leq 1$ and $\phi(\omega) = \omega + \alpha - 1$ if $\omega + \alpha > 1$, where α is an irrational number. Let $\omega \sim \omega'$ be an equivalence relation on $[0, 1]$ so that $\omega \sim \omega'$ if and only if $\phi^n(\omega) = \omega'$ for some $n \in \mathbb{N}$. Then, $\Pi(\omega) = \{\omega' | \omega' \sim \omega\}$ is countable and dense in $[0, 1]$. Moreover, the collection of these subsets $\Pi = \{\Pi(\omega) | \omega \in [0, 1]\}$ is a partition of Ω. The informed events of this partition are well known to be ϕ-invariant measurable subsets of Ω and they have either measure 0 or measure 1 (Cornfeld et al. 2012).[12] Then, the collection of informed events $\mathcal{F}_\Pi$ contains an atom of measure 1, which cannot be an element of Π, and thus, it is not countably generated. Moreover, a partition Π' generated by $\mathcal{F}_\Pi$ is not the same as the partition Π.

The above example illustrates that if the collection of K-informed events from a partition is not countably generated, the partition generated by such a σ-algebra does not preserve K-informed events when one goes from a σ-algebra to a partition. Therefore, we restrict our attention to partitions whose collections of K-informed events are countably generated σ-algebras.

Definition 9 A partition Π is said to be *measurable* if $\mathcal{F}_\Pi$ is countably generated.

Let Σ^c be a sub-collection of Σ such that it contains all countably generated sub-σ-algebras. We naturally endow Σ^c with the partial ordering $\succeq^\sigma$ restricted to Σ^c. With a slight abuse of notations, write it also as $\succeq^\sigma$. Then, $(\Sigma^c, \succeq^\sigma)$ is a poset. Let P^M denote a collection of all measurable partitions of Ω, endowed with a partial ordering $\succeq^P$ restricted to P^M. Then, $(P^M, \succeq^P)$ is a poset. Now, we have our first main result as follows:

Theorem 1 *The collection of measurable partitions* $(P^M, \succeq^P)$ *and the collection of countably generated sub-σ-algebras* $(\Sigma^c, \succeq^\sigma)$ *are order isomorphic: Define* $\Phi : (P^M, \succeq^P) \to (\Sigma^c, \succeq^\sigma)$ *such that for* $\Pi \in P$, $\Phi(\Pi) = \mathcal{F}_\Pi$ *is a collection of informed events. Define* $\Psi : (\Sigma^c, \succeq^\sigma) \to (P^M, \succeq^P)$ *such that for a countably generated sub-σ-algebra* $\mathcal{G} \in \Sigma^c$, $\Psi(\mathcal{G}) = \{\mathscr{A}(\omega, \mathcal{G}) | \omega \in \Omega\}$ *is a partition that contains atoms of* $\mathcal{G}$. *Then, the following properties hold.*

(1) Φ *is injective and order-preserving.*
(2) Ψ *is injective and order-preserving.*
(3) $\Phi \circ \Psi = I_{\Sigma^c}$ *and* $\Psi \circ \Phi = I_{P^M}$, *where* I_{Σ^c} *and* I_{P^M} *are the identity functions defined on* Σ^c *and* P^M, *respectively.*

Moreover, the informational content of a measurable partition Π *or a countably generated sub-σ-algebra* $\mathcal{G}$ *is the collection of K-informed events, and a K-informed set is defined by a knowledge operator K deriving from* Π *or* $\mathcal{G}$.

Remark 1 Note that an atom $\mathscr{A}(\omega, \mathcal{G})$ of a countably generated sub-σ-algebra is an event (a measurable set) because a countably generated sub-σ-algebra of a strongly

[12] The collection of informed sets suggested by HM consists of ϕ-invariant subsets of Ω. The collection includes non-measurable subsets, and the collection of informed events excludes those non-measurable subsets as it is obvious from Lemma 3.

Blackwell σ-algebra $\mathcal{F}$ is closed under arbitrary unions as long as it is measurable with respect to a larger σ-algebra. See Remark 3 of HM.

For comparison, we restate the result of HM in the following.

Lemma 5 *Let* $(P, \succeq^P)$, $(\Sigma, \succeq^\sigma)$, *and* $(\Sigma^c, \succeq^\sigma)$ *be given. Define* $\Phi : (P, \succeq^P) \to (\Sigma, \succeq^\sigma)$ *such that for* $\Pi \in P$, $\Phi(\Pi) = \mathcal{F}_\Pi$ *is a collection of informed events. Define* $\Psi : (\Sigma^c, \succeq^\sigma) \to (P, \succeq^P)$ *such that for a countably generated sub-*σ*-algebra* $\mathcal{G}$, $\Psi(\mathcal{G}) = \{\mathscr{A}(\omega, \mathcal{G}) | \omega \in \Omega\}$ *is a partition that contains atoms of* $\mathcal{G}$. *Then, the following holds.*

(1) Φ *is injective and order-preserving.*
(2) Ψ *is injective and order-preserving.*
(3) *For* $\mathcal{G} \in \Sigma^c$, $(\Phi \circ \Psi)(\mathcal{G}) = \mathcal{G}$, *i.e.,* $\Phi \circ \Psi = I_{\Sigma^c}$, *where* I_{Σ^c} *is the identity function defined on* Σ^c.

For proof of Theorem 1, see HM.

Remark 2 Note that the codomain of Φ is Σ, not Σ^c. Due to the existence of non-measurable partition, as we show in Example 5, $\Psi \circ \Phi = I_P$ does not hold. That is, Φ cannot have Ψ as its inverse, thus $(P, \succeq^P)$ and $(\Sigma, \succeq^\sigma)$ are not order isomorphic. The proof of Theorem 1 follows naturally from the above lemma and the definition of a measurable partition.

We are concluding this section by showing how our result addresses the problem identified in Billingsley's example.

Example 6 Recall that in Billingsley's example, $\Omega = [0, 1]$ endowed with a Borel σ-algebra $\mathcal{F}$. Let Π be the partition that contains every singleton. Then, the collection of K-informed events corresponding to Π consists of every event in $\mathcal{F}$. As the measurable space $(\Omega, \mathcal{F})$ is assumed to be a complete separable metric space, $\mathcal{F}$ is countably generated. Therefore, the partition Π' generated from $\mathcal{F}$ by collecting all of its atoms is indeed the same as Π.

4 Representation of information as belief

In this section, we fix $(\Omega, \mathcal{F}, \mu)$, and we additionally assume that $\mathcal{F}$ is a Borel σ-algebra. We first argue that the generical equivalence of σ-algebras as it is defined in HM indeed represents information as belief, not as knowledge. For this purpose, we present some definitions.

Definition 10 (*Generical Equivalence of* σ*-algebra*) Any two sub-σ-algebras $\mathcal{G}$ and $\mathcal{H}$ are *generically equivalent* with respect to a probability measure μ if

(1) for every $G \in \mathcal{G}$, there is $H \in \mathcal{H}$ such that $\mu(G \triangle H) = 0$, and
(2) for every $H \in \mathcal{H}$, there is $G \in \mathcal{G}$ such that $\mu(G \triangle H) = 0$.

Definition 11 (*Proper Regular Conditional Probability*) Let $(\Omega, \mathcal{F}, \mu)$ and let $\mathcal{G}$ be a sub-σ-algebra. Then, a *regular conditional probability* is a function $Q : \mathcal{F} \times \Omega \to [0, 1]$ satisfying the following:

(1) for each $\omega \in \Omega$, $Q(\cdot, \omega)$ is a probability measure on $\mathcal{F}$.
(2) for each $E \in \mathcal{F}$, $Q(E, \cdot)$ is a version of $p(E|\mathcal{G})$ such that $p(E|\mathcal{G})$ is $\mathcal{G}$-measurable and integrable, and $\int_G p(F|\mathcal{G})d\mu = \mu(F \cap G)$ for all $G \in \mathcal{G}$.

Moreover, the regular conditional probability Q is said to be *proper* if $Q(E, \omega) = \mathbb{1}_E(\omega)$ for each $E \in \mathcal{G}$, where $\mathbb{1}_E(\omega) = 1$ if $\omega \in E$, and 0 otherwise.

By our assumption on the measurable space $(\Omega, \mathcal{F})$, a proper regular conditional probability exists Blackwell and Ryll-Nardzewski (1963).[13] Now, we show that one can define a belief operator by a proper regular conditional probability.

Lemma 6 *Let $\mathcal{G}$ be a sub-σ-algebra, and let $Q(E, \omega)$ be a proper regular conditional probability derived from the probability space $(\Omega, \mathcal{F}, \mu)$ and $\mathcal{G}$. Define an operator $B : \mathcal{F} \to \mathcal{F}$ such that for each event $E \in \mathcal{F}$,*

$$B(E) = \{\omega \in \Omega | Q_\omega(E) = 1\}.$$

*Then, B satisfies **A1–A4**. That is, B is a belief operator.*

For the proof, see Brandenburger and Dekel (1987). In the above lemma, $\mathcal{G}$ can be any σ-algebra which, for example, can be the smallest σ-algebra generated by a partition. Therefore, one can always define a belief operator regardless of whether one starts from a partition or from a σ-algebra. Similarly to the case of knowledge, we consider a collection of all B-informed events and denote it by $\mathcal{F}_Q$

In general, B does not satisfy **A5**, i.e., $B(E) \subseteq E$ does not necessarily hold. Therefore, B is not a knowledge operator. Moreover, note that for each $\omega \in \Omega$, Q_ω is not a complete measure on $\mathcal{G}$ as the following example illustrates.

Example 7 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = 2^\Omega$, and a sub-σ-algebra $\mathcal{G} = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3\}, \Omega\}$. The probability measure μ is given as $\mu(\{\omega_1\}) = \mu(\{\omega_3\}) = 0.5$. Let $E = \{\omega_2\}$ and $F = \{\omega_1, \omega_2\}$. The posterior beliefs for E and F at ω_3 can be calculated as $Q(F, \omega_3) = Q(E, \omega_3) = 0$. On the measurable space $(\Omega, \mathcal{G})$, Q_{ω_3} is not a complete measure, for $E \notin \mathcal{G}$.

Motivated by this observation, Brandenburger and Dekel (1987) propose the following:

Definition 12 (*Posterior Completion*) The posterior completion of a σ-algebra $\mathcal{G}$ is the σ-algebra $\hat{\mathcal{G}}$ generated by $\mathcal{G}$ and the class of sets $\{G \in \mathcal{G} | Q(G, \omega) = 0$ for every $\omega \in \Omega\}$. That is, $\hat{\mathcal{G}} = \{G \in \mathcal{G} | Q(G, \omega) = 0$ or 1 for every $\omega \in \Omega\}$ and it is said to be the posterior-completed σ-algebra.

Although the definition takes a sub-σ-algebra as primitive, one can take a partition as primitive as well by the following procedure: for a given partition Π, generate the

[13] This reveals why we need to restrict $\mathcal{F}$ to be a Borel σ-algebra, instead of being a strongly Blackwell σ-algebra in this subsection. If $\mathcal{F}$ is not a Borel σ-algebra, a proper regular conditional probability may not exist. See Shortt (1984).

smallest σ-algebra containing the partition cells, say $\mathcal{H}$, and then apply the procedure described in the above definition to obtain the posterior-completed σ-algebra $\widehat{\mathcal{H}}$. Then, the posterior-completed partition $\hat{\Pi}$ is the collection of the atoms of $\widehat{\mathcal{H}}$. As a matter of fact, the posterior completion of a partition is to add in B-informed events. All these imply that the posterior-completed σ-algebra is indeed a collection of all B-informed events.

Lemma 7 *Let $\mathcal{G}$ be a sub-σ-algebra, and let B be the resulting belief operator (by Lemma 6). The posterior-completed σ-algebra of $\mathcal{G}$ is indeed a collection of B-informed events:*

$$\hat{\mathcal{G}} = \{E \in \mathcal{F} | B(E) \cup B(\neg E) = \Omega\}.$$

By definition of the posterior-completed σ-algebra, the proof is obvious. In Example 7, the posterior completion leads to the powerset.

Define a binary relation $\sim$ such that for all two sub-σ-algebras $\mathcal{G}$ and $\mathcal{H}$, $\mathcal{G} \sim \mathcal{H}$ if $\hat{\mathcal{G}} = \hat{\mathcal{H}}$. It is not hard to see that this relation is an equivalence relation. That is, the two sub-σ-algebras are considered to be equivalent if their posterior-completed σ-algebras are identical. Now, we connect the notion of generical equivalence to the notion of a posterior completion.

Lemma 8 *Any sub-σ-algebra is generically equivalent to its posterior completion with respect to the proper regular conditional probability measure Q.*

Proof Let $\mathcal{G}$ be a sub-σ-algebra, and let $\hat{\mathcal{G}}$ be its posterior-completed σ-algebra with respect to a proper regular conditional probability Q. Clearly, $\mathcal{G} \subseteq \hat{\mathcal{G}}$. Take any event $E \in \hat{\mathcal{G}}$. If $E \in \mathcal{G}$, it is trivial. Suppose that $E \notin \mathcal{G}$. Then, for any $\omega \in E$, either $Q(E, \omega) = 0$ or 1. If $Q(E, \omega) = 0$, trivially there exists an empty set in $\mathcal{G}$ satisfying $Q(E \triangle \emptyset, \omega) = 0$. Otherwise if $Q(E, \omega) = 1$, there exists an event $F \in \mathcal{G}$ such that $E \subset F$ and thus $Q(F, \omega) = 1$. Hence, $Q(E \triangle F, \omega) = Q(F \setminus E, \omega) = 0$. □

We are concluding this section by presenting our second main result that establishes an equivalence between partitions and σ-algebras for representing information as belief. Let $P^{pc} = P/\sim$ denote a collection of all posterior completion of partitions of Ω, endowed with a partial ordering $\succeq^P$ restricted to P^{pc}.[14] Then, $(P^{pc}, \succeq^P)$ is a poset. Similarly, let $\Sigma^{pc} = \Sigma/\sim$ denote a collection of all posterior completion of sub-σ-algebras, endowed with a partial ordering $\succeq^\sigma$ restricted to Σ^{pc}. Then, $(\Sigma^{pc}, \succeq^\sigma)$ is a poset.

Theorem 2 *The collection of all posterior-completed partitions $(P^{pc}, \succeq^P)$ and the collection of all posterior-completed sub-σ-algebras $(\Sigma^{pc}, \succeq^\sigma)$ are order isomorphic: Define $\Phi : (P^{pc}, \succeq^P) \to (\Sigma^{pc}, \succeq^\sigma)$ such that for $\Pi \in P$, $\Phi(\Pi) = \mathcal{F}_\Pi$ is a collection of B-informed events. Define $\Psi : (\Sigma^{pc}, \succeq^\sigma) \to (P^{pc}, \succeq^P)$ such that for a posterior-completed sub-σ-algebra $\mathcal{G} \in \Sigma^{pc}$, $\Psi(\mathcal{G}) = \{\mathscr{A}(\omega, \mathcal{G}) | \omega \in \Omega\}$ is a partition that contains atoms of $\mathcal{G}$. Then, the following properties hold.*

[14] The equivalence relation $\sim$ between any two partitions Π and Π' is defined so that the smallest σ-algebras generated by these partitions, denoted by $\sigma(\Pi)$ and $\sigma(\Pi')$, have the same posterior-completed σ-algebra, i.e., $\sigma(\Pi) \sim \sigma(\Pi')$.

(1) *Φ is injective and order-preserving.*
(2) *Ψ is injective and order-preserving.*
(3) *$\Phi \circ \Psi = I_{\Sigma^{pc}}$ and $\Psi \circ \Phi = I_{P^{pc}}$, where $I_{\Sigma^{pc}}$ and $I_{P^{pc}}$ are the identity functions defined on Σ^{pc} and P^{pc}, respectively.*

Moreover, the informational content of a posterior-completed partition Π or a posterior-completed sub-σ-algebra $\mathcal{G}$ is the collection of B-informed events, and a B-informed set is defined by a belief operator B deriving from Π or $\mathcal{G}$ through a proper regular conditional probability.

Proof (1) is trivial, for the posterior-completed σ-algebra is the collection of all B-informed events of the posterior-completed partition. For (2), suppose that $\mathcal{G}$ and $\mathcal{G}'$ are two different posterior-completed σ-algebras such that $\mathcal{G} \subseteq \mathcal{G}'$. The corresponding partitions are $\Pi = \{\mathscr{A}(\omega, \mathcal{G}) | \omega \in \Omega\}$ and $\Pi' = \{\mathscr{A}(\omega, \mathcal{G}') | \omega \in \Omega\}$. Take any $\omega \in \Omega$. Then, $\mathscr{A}(\omega, \mathcal{G}') = \cap\{G \in \mathcal{G}' | \omega \in G\} = \cap\{G \in \mathcal{G} \cup \mathcal{H} | \omega \in G\} \subseteq \cap\{G \in \mathcal{G} | \omega \in G\} = \mathscr{A}(\omega, \mathcal{G})$. As to (3), it is easy to see that two different posterior-completed partitions cannot yield the same σ-algebra. Therefore, it suffices to show that two different posterior-completed σ-algebras generate two different partitions. Suppose that $\mathcal{G}$ and $\mathcal{G}'$ are two different posterior-completed σ-algebras. Assume without loss of generality that there exists an event $E \in \mathcal{G}$ but $E \notin \mathcal{G}'$. Suppose to the contrary that the corresponding partitions are the same, i.e., $\Pi = \{\mathscr{A}(\omega, \mathcal{G}) | \omega \in \Omega\} = \{\mathscr{A}(\omega, \mathcal{G}') | \omega \in \Omega\}$. Since Π is the posterior-completed partition, there exists $\omega' \in \Omega$ and $\Pi'(\omega') \subseteq E$ such that $Q'(\Pi'(\omega'), \omega') = 1$ where Q' is the proper regular conditional probability measure defined by $\mathcal{G}'$ together with μ. Then, $Q'(E, \omega') = 1$ because $\Pi'(\omega') \subseteq E$. This implies that $E \in \mathcal{G}'$, for $\mathcal{G}'$ contains every event F such that $Q'(F, \omega') = 1$. This contradicts to the assumption that $E \notin \mathcal{G}'$. □

The above theorem shows that after completing each σ-algebra $\mathcal{G}$ with respect to a proper regular conditional probability measure Q (defined jointly by $\mathcal{G}$ and μ), the σ-algebra $\mathcal{G}$ uniquely determines a partition Π.

Our result, which is based on the technique of posterior completion, provides a different result from HM regarding what is a partition that preserves the informational content of the sub-σ-algebra in Billingsley's example.

Example 8 Consider the following σ-algebra $\mathcal{G}$ in Billingsley's example:

$$\mathcal{G} = \{E \in \mathcal{F} | \text{ either } E \text{ or } \neg E \text{ is countable}\}.$$

The posterior completion of $\mathcal{G}$ is thus $\mathcal{F}$ which is the Borel σ-algebra. The partition generated from this posterior-completed σ-algebra $\mathcal{F}$ is the partition that contains every singleton. This is, in fact, the partition that generates $\mathcal{G}$.

Remark 3 Recall that in HM, the partition claimed to have the same informational content as $\mathcal{G}$ is the coarsest partition $\Pi' = \{\Omega\}$. Notice that $\mathcal{G}$ is the smallest σ-algebra generated by the finest partition $\Pi = \{\{\omega\} | \omega \in \Omega\}$. As $\mathcal{G}$ contains every singleton, a DM can distinguish each state from the other. This is the information that $\mathcal{G}$ inherits from the partition Π. However, HM's treatment of $\mathcal{G}$ ignores this information, while focusing solely on the information provided by the uniform probability distribution.

On the other hand, our treatment requires the informational content of $\mathcal{G}$ to come from both the partition Π *and* the uniform probability distribution conditioned on Π, as one usually defines a conditional probability. The information contained in Π is not lost, thus implying that the DM is fully informed of which state occurs. Hence, the informational content of $\mathcal{G}$ must be equal to the underlying event space, which is the Borel σ-algebra $\mathcal{F}$.

5 Conclusion

In this paper, we establish an equivalent relationship between partitions and σ-algebras as formalizations of information and equip the notion of an informational content with a precise and intuitive meaning by viewing it through the two different but related notions of knowledge and belief. Although both a partition and a σ-algebra have been prevalently used to formally represent information, there has only been a vague understanding about the relationship between the two. However, Billingsley (1995) and Dubra and Echenique (2004) raise a concern about the use of σ-algebra by coming up with an example in which a partition and the σ-algebra generated by it fail to contain the same informational content.

Hervés-Beloso and Monteiro (2013) engage this example and elaborate on the meaning of information. They provide a notion of an informed set and suggest the two alternative methods: one for generating a σ-algebra from a partition and the other for going in the opposite direction. However, we find out that their suggestion still leaves the meaning of information ambiguous. When it comes to a partition, the information content captured by the notion of an informed set depends neither on a given measurable space nor on a probability measure. On the other hand, for a given σ-algebra, the informational content, in general, relies on a specific choice of a probability measure. Even when information content is captured by a countably generated σ-algebra, HM are silent about whether or not it is a collection of all informed sets for some partition.

By separating the notion of information into the two notions of knowledge and belief, we elaborate on the meaning of information in relation to a probability measure. The two notions are distinct regarding whether the concept of information is required to satisfy the truthfulness or not. If one allows for falsity, the notion one works with is that of belief. We show that a proper regular conditional probability, and the posterior completion of a partition/a σ-algebra correspond to this conception of information. Specifically, the presence of null events captures the possible falsity of information. Based on the conception of information as belief, we show that partitions and σ-algebras can be equivalently used after applying the technique of the posterior completion proposed by Brandenburger and Dekel (1987). The idea behind posterior completion is to add in null events to a partition (or a σ-algebra) to generate a new partition (a new σ-algebra) that allows a DM to incorporate the possibility of falsity in his information. On the other hand, if the concept of information is based on knowledge, information must be independent of one's belief (which is captured by a probability measure). In this case, we show that only measurable partitions and countably generated σ-algebras can be equivalently used.

We conclude that although the distinction between knowledge and belief matters for the equivalence between partitions and σ-algebras when formalizing information either by a partition or by a σ-algebra, one can safely assume information as belief in a practical sense. In almost all economic models, a partition or a σ-algebra is equipped with a probability measure to formalize information of a DM. Therefore, the only thing one needs to make sure is to apply posterior completion before using a partition or a σ-algebra to analyze the problem in his hand.

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