Econ Theory (2018) 65:819–853
https://doi.org/10.1007/s00199-017-1040-5

RESEARCH ARTICLE

Deficit, monetization, and economic growth: a case for multiplicity and indeterminacy

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Received: 12 June 2016 / Accepted: 23 February 2017 / Published online: 9 March 2017

Abstract This paper develops an original analysis of deficit monetization in a growth model with transaction costs, in which economic growth interacts with productive public expenditures. This interaction generates two positive balanced growth paths (BGP) in the long run: a high BGP and a low BGP. The transitional dynamics show that multiplicity cannot be rejected if transaction costs affect both consumption and investment expenditures, with possible indeterminacy of the high BGP. Importantly, deficit monetization is shown to reduce the parameter space producing indeterminacy.

Keywords Endogenous growth · Deficit · Monetization · Indeterminacy · Public debt

JEL Classification E41 · E52 · E62 · H62 · H63

Electronic supplementary material The online version of this article (doi:10.1007/s00199-017-1040-5) contains supplementary material, which is available to authorized users.

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1 Introduction

In recent years, the Great Recession shaped the conduct of monetary and fiscal policies in an unprecedented way. On the fiscal side, governments of developed countries launched massive debt-financed spending programs that might have generated potentially explosive debt paths (see Chen and Imrohoroglu 2015, for the United States). On the monetary side, many central banks implemented "unconventional" monetary policies, and bought an unprecedented amount of public (and private) debt. Consequently, the age of central banks "independence" (from fiscal policy) and of monetary policy isolationism seems to be over (see, e.g., Taylor 2012). Regarding monetary policy, the post-2008 era raises two key issues: (i) how to conduct an effective policy in the presence of a zero lower bond on the nominal interest rate? (ii) What are the possible consequences of the debt monetization programs adopted by central banks? Although these questions are not unrelated, our paper focuses on the second issue and particularly on the effect of public deficits and debt monetization on economic growth, from both short- and long-run perspectives. We address this question in a theoretical endogenous growth setup.

On a theoretical level, assessing the effect of monetizing public debt and deficits on inflation is a long-standing question, which has been posed since the seminal "*unpleasant monetarist arithmetics*"of Sargent and Wallace (1981). However, to the best of our knowledge, there is no work that addresses the question of the impact of deficit monetization on economic growth, despite of its relevance both at present and from a historical perspective (see, e.g., Rousseau and Stroup 2011).

To correct for this caveat, we build an endogenous growth model with permanent public indebtedness. To give a role to public expenditures, we model endogenous growth based on the canonical model of Barro (1990) with public spending entering the production function as a flow of productive services. In this setup, we introduce several innovations. First, to address the question of public debt and deficit monetization, we consider a general budget constraint for the Government, in which public expenditures can be financed by taxes, public debt or money emission. This creates a richer environment to study government finance, compared to the balanced-budget rule used by Barro (1990). Second, contrary to the usual modeling of an exogenous money supply, we suppose that money creation is proportional to fiscal deficits. This allows us to analyze the impact of deficit monetization, which in the long run corresponds to monetizing a share of public debt. Third, to introduce money, we resort to a very general setup, namely a generalized transaction costs specification based on the fact that resources are used up in the process of exchange, along the lines of Brock (1974, 1990), Jovanovic (1982) or Kimbrough (1986).[1]

Our findings are threefold. First, regarding the balanced growth path (hereafter BGP), our model exhibits two steady states (a high BGP and a low BGP). This multiplicity comes from the interaction between economic growth and public expenditure.

[1] This specification is more general than typical "cash-in-advance" models because it allows for interest-elastic money demand. Furthermore, it is also more general than "money-in-the-utility-function" (MIUF) approaches because money demand is generated by the need for a liquid asset to finance either consumption only or both consumption and investment.

On the one hand, economic growth positively depends on public spending that raises the marginal productivity of private capital. On the other hand, public spending is positively related to economic growth in the Government's budget constraint, because growth reduces the debt burden in the long run. Thus, a high rate of economic growth mitigates the detrimental impact of the debt burden and boosts productive expenditure, which, in turn, supports economic growth. In contrast, a low rate of economic growth leads to a high public debt ratio that crowds-out productive public spending and lowers growth. Consequently, two perfect-foresight BGPs emerge in the long run.

Second, regarding the impact of deficits and monetization in the long run, we show that the low BGP depends positively on deficits and negatively on monetization. Along the high BGP, however, public deficits increase economic growth only if they are sufficiently monetized, while the direct impact of monetization depends on the interest-elasticity of money demand. In support of our theoretical findings, a calibration shows the existence of an optimal, welfare-maximizing, degree of monetization in the long run. This calibration closely reproduces a number of stationary features of the US economy.

Third, regarding transitional dynamics, the results change dramatically depending on what type of expenditures are subject to transaction costs. In the special case with transaction costs on consumption only, the high BGP is saddle-point stable and the low BGP is unstable, and thus multiplicity can be removed. However, in the general case with transaction costs on both consumption and investment, the low BGP becomes saddle-point stable, and the high BGP becomes locally undetermined or saddle-point stable, depending on the parameters. Therefore, multiplicity can no longer be excluded: according to the initial public debt ratio, both steady states are reachable. If the initial public debt ratio is "high", the economy is condemned to remain in the neighborhood of a poverty trap, with economic growth approaching zero. If, conversely, the initial public debt ratio is "low", the economy will converge toward the high BGP, but the exact transition path may be undetermined. We show in particular that "high" levels of monetization are beneficial to the determinacy of the high BGP.

Our model can be seen as unifying two strands of literature. On the one hand, it extends and challenges prior results concerning the impact of public debt on long-run economic growth. For example, in endogenous growth settings with wasteful public expenditures, Saint-Paul (1992) and Futagami and Shibata (1998) show that higher debts and deficits impede economic growth. These findings are extended by Minea and Villieu (2010, 2012) in endogenous growth models with productive public expenditures. The mechanism that drives the harmful effect of fiscal deficits on long-run economic growth comes from the impact of the debt burden, which always outweighs the financing allowed by deficits. The present paper shows that these results can be reversed if deficits are monetized: a sufficiently high dose of monetization would allow the economy to overcome the crowding-out effect of public debt in the long run.

On the other hand, an important strand of literature explores money as a source of indeterminacy in endogenous growth models. It is well known that indeterminacy can arise when the central bank follows an exogenous money growth rule (see Michener and Ravikumar 1998). In "cash-in-advance" (hereafter, CIA) endogenous growth models, as first developed by Wang and Yip (1992), Palivos et al. (1993), and Palivos and Yip (1995), several mechanisms may also give rise to multiplicity and/or indeterminacy. For example, in an endogenous growth model with transaction costs and endogenous labor supply, Itaya and Mino (2003) show that labor externalities can produce indeterminacy. In Suen and Yip (2005), indeterminacy comes from a strong intertemporal substitution effect on capital accumulation. Moreover, Bosi and Magris (2003) and Bosi et al. (2010) show that indeterminacy and multiplicity can occur in discrete-time CIA models, when the cash requirement affects only a part of consumption. Finally, Bosi and Dufourt (2008) and Chen and Guo (2008) highlight that the form of the CIA constraint, and specifically the extent to which it affects investment, is a key factor in generating indeterminacy.

Our model extends these works in several dimensions. First, we introduce a role for government spending, through productive public expenditures, and we relax a key assumption of this literature (the balanced-budget rule). Second, by accounting for the possibility of deficit monetization, we go beyond the hypothesis that the money supply is exogenous and study indeterminacy in the context of a "passive" monetary authority. In this context, we show that the degree of monetization can be used as a selection device among different convergent paths, as indeterminacy can be removed at high degrees of monetization. Third, our results are insensitive to the specific form of transaction costs or the CIA constraint (provided that it affects investment expenditure). Along these lines, the last Section shows that our results still hold in the presence of microfoundations of the transaction cost function.

The paper is structured as follows. Section 2 presents the model. Section 3 describes the long-run solution. Section 4 studies the effect of deficit and monetization along the BGPs. Section 5 discusses transitional dynamics and the possibility of indeterminacy. Section 6 establishes the microfoundations of the transaction costs, and Sect. 7 concludes the paper.

2 The model

We consider a continuous-time endogenous growth model describing a closed economy populated by a private sector, and fiscal and monetary authorities.

2.1 The private sector

The private sector consists of a producer-consumer infinitely lived representative agent with perfect foresight, who maximizes the present value of a discounted sum of instantaneous utility functions based on consumption $c_t > 0$. With $\rho > 0$ the discount rate and $S := -u_{cc}c_t/u_c > 0$ (with $u_c := \mathrm{d}u(c_t)/\mathrm{d}c_t$) being the consumption elasticity of substitution, Households' welfare is

$$U = \int_0^\infty u(c_t)\exp(-\rho t)\mathrm{d}t, \quad u(c_t) = \begin{cases} \frac{S}{S-1}\left\{(c_t)^{\frac{S-1}{S}} - 1\right\}, & \text{if } S \neq 1 \\ \log(c_t), & \text{if } S = 1. \end{cases} \tag{1}$$

For lifetime utility U to be bounded, it must be true that $(S-1)\gamma_c < S\rho$, where $\gamma_x := (1/x)(\mathrm{d}x/\mathrm{d}t)$ defines the long-run growth rate of any variable x.[2]

Output is produced using a constant returns-to-scale technology at the private level but with a public good externality, namely $y_t = Ak_t^\alpha (L_t g_t)^{1-\alpha}$, where k_t and g_t stand for private capital and productive public expenditure, respectively. The population L_t will be normalized to unity, so that all variables are per capita, and we obtain

$$y_t = Ak_t^\alpha g_t^{1-\alpha}. \tag{2}$$

The elasticity of output to private capital is $\alpha \in (0, 1)$. Following Barro (1990) and the long-lasting growth literature inspired by the empirical work of Aschauer (1989) and Munnell (1990), we assume that public expenditure provides "productive services", with an elasticity $1-\alpha$.

To motivate a demand for real balances, the model must account for imperfections in the process of exchange. The long-lasting literature aiming at introducing money in general equilibrium resulted in two alternative reduced forms of money demand, namely money in the utility function (MIUF) and cash-in-advance (CIA) models. In some sense, the former approach may be viewed as more general because it gives rise to an interest-elastic demand for money, while typical CIA models lead to a strict quantitative equation with a constant (exogenous) velocity of money, as soon as the nominal interest rate is positive. Furthermore, the CIA specification is a special case of MIUF, when money and consumption are strict complements in utility (Asako 1983).[3]

However, the MIUF approach loses its generality once the CIA constraint affects both investment and consumption goods. Indeed, it would be quite unusual to introduce investment in the utility function. On this basis, the CIA version might be seen as more general. Moreover, CIA specifications have proven sensitive to the type of goods subject to the money constraint. Along this line, Stockman (1981) first shows that money is not superneutral in the long run when the CIA constraint affects both investment and consumption, a result extended to endogenous growth setups by Palivos and Yip (1995). In addition, as we stated in the Introduction, multiplicity and indeterminacy dramatically depend on the form of the CIA constraint.

This paper provides a more general approach that encompasses the above cases. Standard money demand depends both on the nominal interest rate (say, R_t) and on expenditures (say, e_t) that are constrained by the use of money, namely $m_t = m(e_t, R_t)$. To obtain such a specification, we develop a transaction costs approach to the demand for money, based on the fact that resources are used up in the process of exchange, as suggested by numerous works (Brock 1974, 1990; Jovanovic 1982; Feenstra 1986;

[2] The standard transversality condition $(S-1)\gamma_c < S\rho$ is necessary to bound Households' welfare, as shown in particular by Kamihigashi (2002). With an infinite horizon, this condition corresponds to the no-Ponzi game constraint $\gamma^* < r^*$ (with γ^* and r^* denoting BGP values of economic growth and the real
Footnote 2 continued
interest rate, respectively), which means that public debt must be repaid in the long run. This condition does not preclude the possibility that $\gamma_t > r_t$ in the short run.

[3] In an important paper, Feenstra (1986) proves the formal equivalence between the MIUF approach and a large class of models with transaction costs. Nevertheless, this "functional equivalence" does not apply to models with transaction costs on investment, such as that developed in this paper.

Kimbrough 1986). To this end, we assume that some expenditures (e_t), to be defined below, are subject to a transaction cost

$$T\left(e_t, m_t\right) = \psi\left(e_t/m_t\right) e_t, \tag{3}$$

that satisfies the following standard assumptions: (i) $\psi(\cdot)$ is continuous, increasing, non-negative and twice continuously differentiable, and (ii) $s\psi''(s) + 2\psi'(s) > 0$, for all $s \geq 0$. Assumption (i) means that the transaction cost is smooth and assumption (ii) is a necessary and sufficient condition ensuring that money demand is decreasing in the nominal interest rate (see Appendix A).[4]

This transaction technology supposes that a fraction $\psi(\cdot) \in (0, 1)$ of expenditures is wasted in the process of exchange. This fraction depends negatively on real balances, as money provides liquidity services. Indeed, money does not provide direct utility but frees up resources spent on transactions. Along these lines, the velocity of money (e_t/m_t) will play a significant role in equilibrium, contrary to the basic CIA specification with an exogenous velocity of money. In particular, by defining the set of expenditures as $e_t = \phi^c c_t + \phi^k(\dot{k}_t + \delta k_t)$, this transaction technology allows us to study two special cases: transaction costs on consumption only ($\phi^c > 0$ and $\phi^k = 0$) or on consumption and investment ($\phi^c > 0$ and $\phi^k > 0$). The impact of technical progress (that lowers the cost of credit) on the velocity of money can then be analyzed through changes in the coefficient ϕ^c or ϕ^k (for interesting qualitative and quantitative analyses of the US velocity of money, see, e.g., Ireland 1994; Benk et al. 2010).

The formal equivalence between the transaction cost function (3) and a CIA specification can be established by using an isoelastic specification, namely

$$\psi\left(e_t/m_t\right) := \frac{\omega\mu}{1+\mu}\left(\frac{e_t}{m_t}\right)^{1/\mu}, \tag{4}$$

with $\mu \geq 0$ and ω a positive scale parameter ensuring small transaction costs. In particular, we obtain the CIA constraint $e_t = m_t$ as a special case when $\mu \to 0$.[5] On the contrary, if $\mu > 0$, we obtain an interest-elastic money demand. Equation (4) will be used in Sects. 4 and 5 below, but alternative specifications of the general form (3) will be examined below. In particular, Sect. 6 will develop a financial intermediation approach of the demand for money, based on the work of Gillman and Kejak (2011), which provides microfoundations to transaction costs. In this respect, the transaction technology (3) can reflect intermediation costs in financial markets.

Taking into account imperfections in the process of exchange, Households are subject to the following budget constraint (we define $\dot{x}_t := \mathrm{d}x_t/\mathrm{d}t, \forall x_t$)

$$\dot{k}_t + \dot{b}_t + \dot{m}_t = r_t b_t + (1-\tau)\, y_t - c_t - \delta k_t - \pi_t m_t - T\left(e_t, m_t\right) + \varrho_t. \tag{5}$$

[4] This assumption is weaker than the more common specification of strict convexity of the transaction cost function. See, e.g., Schmitt-Grohé and Uribe (2004), for similar assumptions.

[5] Writing $m_t = \left[\frac{\omega e_t}{(1+1/\mu)\psi(\cdot)}\right]^{\mu} e_t$, we use $\lim_{\mu\to 0}\left(\frac{1}{1+1/\mu}\right)^{\mu} \to 1$, and, since $\lim_{\mu\to 0}\left(\frac{\omega}{\psi(\cdot)}\right)^{\mu} \to 1$, we obtain $m_t \to e_t$.

Notations are standard. τ is a flat tax rate on output and Households use their net income $((1-\tau)\, y_t)$ to consume (c_t) and invest $(\dot{k}_t + \delta k_t$, with δ the rate of private capital depreciation). They can buy Government bonds (b_t), which return the real interest rate r_t, and hold money. All variables are defined in real terms (i.e., nominal variables are deflated by the price level P_t), and $\pi_t m_t$ is the depreciation of real money holdings due to inflation $(\pi := \dot{P}_t / P_t)$. In addition, since goods are used-up in transacting, Households' budget constraint must contain the transaction cost term $T(\cdot)$. Finally, to close the model, ϱ_t is a lump-sum transfer that equals, in equilibrium, the value of transaction costs $T(\cdot)$ levied on Households.[6]

2.2 Monetary and fiscal authorities

The Government provides productive public expenditures, levies income taxes, and borrows from Households. He also collects the seignorage on real balances.[7] Fiscal deficit (d_t), can be financed either by issuing debt $(\dot{b}_t)$ or money $(\dot{M}_t / P_t)$, with M_t the (nominal) money supply. Hence, the following budget constraint, in real terms

$$\dot{b}_t + \frac{\dot{M}_t}{P_t} = r_t b_t + g_t - \tau y_t =: d_t. \tag{6}$$

Some points deserve attention. First, the budget constraint (6) is an extension of Barro (1990) and Minea and Villieu (2012). Indeed, in his seminal paper, Barro (1990) only considers balanced-budget rules (hereafter BBR, i.e., $g_t = \tau y_t$), while Minea and Villieu (2012) deal with public debt, but without money $(\dot{b}_t = r_t b_t + g_t - \tau y_t)$. In the present model, by using public debt and seigniorage, the Government can make productive expenditure eventually higher than fiscal revenues (τy_t). This new feature is crucial, since, as we will see, the monetization of deficits will make it possible to go beyond Barro's BGP, in contrast with Minea and Villieu (2012).

Second, it must be emphasized that, to obtain an endogenous growth solution, productive public expenditure must be endogenous in the Government's budget constraint. Effectively, with exogenous public spending, the production function (2) exhibits decreasing return of capital, without any possible growth in the long run. By assuming the BBR $g_t = \tau y_t$, Barro (1990) takes (endogenous) public expenditure to be proportional to income, leading, in equilibrium, to a "Ak"-type production function $(y_t = A^{1/\alpha} \tau^{(1-\alpha)/\alpha} k_t)$. In his model, the BGP is exclusively determined by the Keynes–Ramsey relationship and there are no transitional dynamics. By introducing the dynamics of public debt and monetization, our model strongly departs from such an "Ak" specification. Effectively, on the one hand, the BGP will depend on the interaction between two relations, giving rise to multiplicity and, on the other hand,

[6] One can consider that transaction costs are levied by the banking sector, which rebates its profits to Households (see Sect. 6).

[7] In our model, high-powered money is the only form of money, so that the Central Bank collects the seignorage and transfers it to the Government. Developments related to the presence of a banking sector will be introduced below.

our model exhibits non-trivial transitional dynamics. Both features highly contrast with simple Barro-type endogenous growth models.

Third, at this stage the model is not closed, because there is one free variable in the Government budget constraint (6). To close the model, the Government must fix either the public spending or the public debt path. Since, for an endogenous growth solution to emerge, public expenditure cannot be chosen as the instrument, we take the deficit (d_t)—that determines the public debt path—as the instrument. This characterizes a great number of countries that adopted deficit rules. To this end, the simplest way to proceed would be to assume that the Government fixes the deficit-to-output ratio, namely $d_t = \theta y_t$. Here, we adopt a slightly different approach, by specifying a gradual adjustment path of the deficit-to-output ratio to a long-run target (θ). Let $d_{yt} := d_t/y_t$ be the deficit-to-GDP ratio and $\theta := d^*/y^*$ its long-run target (with stars denoting steady-state values). The Government makes the deficit ratio evolve according to

$$\dot{d}_{yt} = -\xi \left(d_{yt} - \theta\right). \tag{7}$$

Thus, the fiscal policy instruments are the flat tax rate (τ), the targeted deficit-to-output ratio in the long run (θ), and the speed of adjustment of the current deficit to this target (ξ). A low value of the last parameter describes a "gradualist" strategy (i.e., the speed of adjustment of the deficit ratio is small), and a high value represents a "shock therapy" strategy, which gives rise to a faster reduction in the deficit ratio.

Furthermore, the monetary authorities must determine the deficit share that they will monetize. For simplicity, we assume that a fraction $\eta \in [0, 1]$ of the deficit is monetized at each instant,[8] namely, anticipating on money equilibrium (that requires the real value of money supply to be equal to the demand for real balances, i.e., $M_t/P_t = m_t$)

$$\frac{\dot{M}_t}{P_t} = \dot{m}_t + \pi_t m_t = \eta d_t. \tag{8}$$

It follows that the Government must cover the remaining part of deficit by issuing public debt

$$\dot{b}_t = (1-\eta) d_t. \tag{9}$$

2.3 Equilibrium

The solution of the Households' program (see Appendix A) conducts to

$$\frac{\dot{c}_t}{c_t} = S\left[r_t - \rho - \frac{\phi^c f'(R_t)\dot{R}_t}{1+\phi^c f(R_t)}\right], \tag{10}$$

$$\frac{(1-\tau)\,\alpha A\,(g_t/k_t)^{1-\alpha}}{1+\phi^k f(R_t)} - \delta = r_t - \frac{\phi^k f'(R_t)\dot{R}_t}{1+\phi^k f(R_t)}, \tag{11}$$

8 We could introduce an exogenous trend in the money supply, without any change in the qualitative results (see Sect. 4.3 below).

where the interest factor $f(R_t)$ measures the cost of financing expenditure (with $f'(R_t) > 0$, as shown in Appendix A). This measure depends on the nominal interest rate, i.e., the "inflation tax", as defined by Phelps (1973).

Equation (10) is the well-known Keynes–Ramsey rule obtained in optimal growth models. Without transaction costs, this rule is the usual one $(\dot{c}_t/c_t = S(r_t - \rho))$. With transaction costs on consumption goods $(\phi^c > 0)$, the consumption path is affected by the nominal interest rate, which represents a part of the effective cost of consumption. Thus, in periods with increasing (resp. decreasing) nominal interest rates, the growth rate of consumption will be lower (reps. higher) than under the usual Keynes–Ramsey rule. This explains the presence of the last term in Eq. (10).

Equation (11) defines the real return to capital. Without transaction costs on capital goods $\left(\phi^k = 0\right)$, this return is simply the real interest rate (the rate of return on Government bonds), namely: $(1-\tau)\,\alpha A\,(g_t/k_t)^{1-\alpha} - \delta = r_t$. With transaction costs on investment $\left(\phi^k > 0\right)$, the return to capital is lower, as it must be deflated by the financing cost $\left(1 + \phi^k f(R_t)\right)$, as shown by the first term on the LHS of (11). In addition, the nominal interest rate introduces a wedge between the return on bonds and the return on capital: with a growing nominal interest rate, the return on capital will be lower, as shown by the second term on the RHS of (11). Indeed, Households expect a greater inflation tax on capital goods in the future, while real bonds (that are not subject to transaction costs) are immunized against the inflation tax. This corresponds to the Stockman (1981) effect with a CIA constraint on investment goods.

To obtain endogenous growth solutions, we transform variables into long-run stationary ratios. To this end, all variables that grow in steady state will be deflated by the capital stock, namely $x_k := x_t/k_t$ (and we henceforth remove time indexes). Thus, the path of the capital stock is obtained from the goods market equilibrium

$$\frac{\dot{k}}{k} = y_k - c_k - g_k - \delta, \tag{12}$$

with the production function defined as

$$y_k = A g_k^{1-\alpha}. \tag{13}$$

Under the usual regularity assumptions (i) and (ii), the transaction technology (3) gives rise to a unique money demand-to-capital ratio (see Appendix A)

$$m_k = e_k \Psi(R), \tag{14}$$

where $\Psi'(\cdot) \leq 0$, and, using (12),

$$e_k = \phi^k \left(A g_k^{1-\alpha} - g_k\right) + \left(\phi^c - \phi^k\right) c_k. \tag{15}$$

The precise form of function $\Psi(\cdot)$ depends on the specification of transaction costs, through the function $T(\cdot)$. In general, the money demand (14) is interest-elastic (except in the special CIA case in which $\Psi'(\cdot) = 0$) and corresponds to the desired specification $m_t = m(e_t, R_t)$.

The deficit-to-capital ratio comes from the Government's budget constraint (6)

$$d_k = rb_k + g_k - \tau y_k, \tag{16}$$

and the behavior of the monetary and fiscal authorities (8) and (9) leads to

$$\frac{\dot{m}}{m} = \eta\left(\frac{d_k}{m_k}\right) - \pi, \tag{17}$$

$$\frac{\dot{b}}{b} = (1-\eta)\left(\frac{d_k}{b_k}\right). \tag{18}$$

Assuming Fisher's equation $R = r + \pi$, relations (10)–(18), together with the deficit rule (7), fully characterize the equilibrium of the model.

3 The long-run endogenous growth solution

We define a BGP as a path on which consumption, capital, public spending, money, output, public debt, and deficit grow at the same (endogenous) rate ($\gamma^* = \dot{c}/c = \dot{k}/k = \dot{g}/g = \dot{m}/m = \dot{y}/y = \dot{b}/b = \dot{d}/d$), while the real ($r^*$) and the nominal ($R^*$) interest rates (and, as a consequence, the inflation rate π^*) are constant. Thus, in steady state, the real interest rate is defined by the marginal product of capital, amended to accommodate the financing cost of investment

$$r^* = \frac{(1-\tau)\,\alpha A g_k^{*\,1-\alpha}}{1+\phi^k f(R^*)} - \delta, \tag{19}$$

and the rate of economic growth is simply

$$\gamma^* = S\left(r^* - \rho\right). \tag{20}$$

For notational convenience, we define: $\varepsilon(\gamma^*) := r^*/\gamma^* = S^{-1} + \rho/\gamma^*$, with $\varepsilon(\gamma^*) > 1$ since the standard transversality condition ensures that $r^* > \gamma^*$.

In addition, as $d_k^* = \theta y_k^*$ in steady state, we obtain, by (18)

$$(1-\eta)\,\theta A g_k^{*\,1-\alpha} = \gamma^* b_k^*, \tag{21}$$

and, from the definition of the deficit in the Government's budget constraint (6)

$$r^* b_k^* = (\theta+\tau)\,A g_k^{*\,1-\alpha} - g_k^*. \tag{22}$$

Section 3.2 computes the long-run solution of the model, and Sect. 4 will examine comparative statics with respect to changes in deficit and monetization parameters. To provide some intuition for our results, the following Subsection considers the effect of such changes, for a given long-run growth rate.

3.1 A preliminary analysis

Let us first consider a fixed growth rate and examine the impact of deficits and monetization.

Proposition 1 (Deficits and monetization in the steady state) *For a given rate of long-run economic growth* (γ^*):

(i) *any increase in the degree of deficit monetization increases the public-expenditure-to-capital ratio in the long run*;

(ii) *any increase in the deficit target reduces the public-expenditure-to-capital ratio in the long run if monetization is small (namely* $\eta < \bar{\eta}$*), but increases it if monetization is large* ($\eta > \bar{\eta}$)*, where:* $\bar{\eta} := 1 - 1/\varepsilon(\gamma^*) \in (0, 1)$.

Proof By (20), (21) and (22), the public-spending-to-capital ratio is, in steady state

$$g_k^* = \left[(\theta + \tau) A - (1 - \eta) \theta A \varepsilon \left(\gamma^*\right)\right]^{1/\alpha}, \tag{23}$$

where $\left.\frac{\partial g_k^*}{\partial \eta}\right|_{\gamma^*} > 0$ and, $\left.\frac{\partial g_k^*}{\partial \theta}\right|_{\gamma^*} \geq 0 \Leftrightarrow 1 \geq (1 - \eta)\, \varepsilon(\gamma^*) \Leftrightarrow \eta \geq \bar{\eta}$. □

From (23) without deficit ($\theta = 0$), we find the solution of Barro (1990), namely: $g_k^* = (\tau A)^{1/\alpha} =: g_k^B$. With deficit but no monetization ($\theta > 0$ and $\eta = 0$), we obtain: $g_k^* = [\tau A - \theta A(\varepsilon(\gamma^*) - 1)]^{1/\alpha} < g_k^B$. Since the standard transversality condition ensures $\varepsilon(\gamma^*) > 1$, the public spending ratio is lower than under a BBR, as described by Minea and Villieu (2012). The basic mechanism driving this crowding-out effect is the following. On the one hand, deficits generate a permanent flow of new resources ($\dot{b}$). On the other hand, public debt generates a permanent flow of new unproductive expenditures (the debt burden rb). In steady state, the standard transversality condition ($r^* > \gamma^* = \dot{b}/b$) means that the latter prevails over the former ($rb > \dot{b}$), so that any rule that authorizes permanent deficits involves net costs for public finance in the long run, irrespective of the precise nature of this rule.

Proposition 1 shows that this configuration radically changes if deficit monetization is authorized. Indeed, the debt burden can now be accommodated by money creation. Thus, if monetization is high enough, an increase in deficit can generate additional resources for productive public expenditures, in contrast with Minea and Villieu (2012). Intuitively, this is because the new resources provided by the deficit are devoted to productive public spending, while the (additional) interest burden is financed by issuing new money. Suppose, for example, that the deficit is fully monetized [$\eta = 1$ in (23)]; compared to the BBR used in Barro (1990), taxes are now supplemented by the deficit, and productive expenditures are larger in steady state: $g_k^* = [(\theta + \tau) A]^{1/\alpha} > g_k^B$ if $\theta > 0$.

More generally, Proposition 1 shows that the monetization of fiscal deficits (i) alleviates their harmful effect on productive expenditures, and (ii) can even, if large enough, reverse this effect. Since economic growth positively depends on public expenditures, the impact of deficits on long-run growth will be likely to depend on the degree of monetization. However, these results are only preliminary, because, in equilibrium, γ^* is an endogenous function of parameters (including θ and η). In the following Subsection, we fully characterize the long-run solution of the model.

3.2 The steady state

The long-run solution of the model is computed in Appendix C. Endogenous growth solutions are obtained at the intersection of two relations between γ^* and g_k^*.

The first relation is directly linked to the Government's budget constraint (GBC) and comes from Eqs. (19), (20) and (23)

$$\gamma^* = \frac{S\rho(1-\eta)\theta A}{S[(\theta+\tau)A - g_k^{*\alpha}] - (1-\eta)\theta A} =: \mathcal{G}(g_k^*). \tag{24}$$

Relation (24) depicts a positive association between public expenditure and long-run economic growth. Effectively, in the Government budget constraint, higher public expenditures increase fiscal deficits, while higher economic growth, by weakening the debt burden (as a share of GDP), reduces deficit. Hence, we have the positive association displayed in Eq. (24).

The second relation comes from the determination of real interest rate (19) and Keynes–Ramsey rule (20)

$$\gamma^* = S\left\{\frac{(1-\tau)\,\alpha A {g_k^*}^{1-\alpha}}{1+\phi^k f(R^*)} - \rho - \delta\right\}, \tag{25}$$

with $R^* := R\left(\gamma^*, g_k^*\right)$ (see Appendix C).

Equation (25) represents the tradeoff between holding bonds and capital in households' portfolio, and can be called *bond market clearing* (BMK) relation. In this relation, the positive association between public expenditure and economic growth is based on the fact that public expenditures are productive, thus enhancing the return to private capital and the incentive to invest in the long run. This relation also illustrates the link between government expenditure (the numerator in the first term) and the inflation tax (the interest factor in the denominator), when investment is subject to a cash requirement.

The steady states are found at the cross-points of GBC (24) and BMK (25) curves. The latter describes an implicit function between γ^* and g_k^*, which can be rewritten as

$$g_k^* =: \mathcal{F}\left(\gamma^*\right), \tag{26}$$

where $\mathcal{F} \in C^\infty(\mathbb{R}_+^*)$ is an increasing strictly convex function (see Appendix C). Therefore, using the GBC relation, the steady-state solutions can be computed as

$$\gamma^* = \mathcal{G}\left(\mathcal{F}\left(\gamma^*\right)\right). \tag{27}$$

In general, the model exhibits multiplicity: there are two solutions that verify (27). To provide some intuition about this multiplicity, let us first study a special case without deficits or money.

Definition 1 (*Steady-state solutions without public deficit or money*) Without public deficit or money ($\theta = \omega = 0$) the model generates two solutions: a no-growth solution (that we call the "Solow" solution $\gamma^S = 0$) and a positive growth solution (that we call the "Barro" solution $\gamma^B > 0$).

The "Solow" solution is obtained by setting $\theta = 0$ in (24). Consequently, the rate of economic growth becomes $\gamma^S = 0$, and, from (25), the real interest rate is $r^S = \rho$. The public spending ratio is obtained by (19) (notice that $Q^* = 0$ if $\omega = 0$), namely: $g_k^S = [(\rho + \delta)/\alpha A(1-\tau)]^{1/(1-\alpha)}$. The pair (g_k^S, γ^S) characterizes point S in Fig. 1 below. However, there is another long-run solution, which corresponds to the growth rate of Barro (1990), if $g_k^* = g_k^B = (\tau A)^{1/\alpha}$ in (24). This solution gives rise to a "0/0" case of indeterminacy, but the rate of economic growth can easily be computed from (25)

$$\gamma^B = S\left[\alpha A\,(1-\tau)\,(A\tau)^{(1-\alpha)/\alpha} - \rho - \delta\right]. \tag{28}$$

The Barro solution corresponds to a zero stock of public debt in the steady state $\left(b_k^B = 0\right)$ and is depicted by point B in Fig. 1 below.

The intuition is as follows. Barro (1990) assumes a BBR with zero public debt at any instant (including the initial time $t = 0$), which excludes multiplicity by removing the possibility of a no-growth solution. In our model, in contrast, the case $\theta = 0$ corresponds to a BBR at any time, but public debt can be positive at date $t = 0$, which generates multiplicity.[9] Indeed, a no-growth solution appears when public debt at time $t = 0$ is so high that the debt burden captures most public resources, which in turn does not allow economic growth to emerge due to the lack of productive spending. This critical debt ratio is such that $b_k^S = (g_k^S)^{1-\alpha}[\tau A - (g_k^S)^\alpha]/\rho > 0$. If $b_{k0} = b_k^S$, growth cannot appear in the steady state, and the economy is locked into a poverty trap, namely a no-growth BPG ($\gamma^S = 0$) where public debt remains at its initial level. On the contrary, if public debt is initially zero, the economy can grow at the positive endogenous rate $\gamma^B > 0$, as productive public expenditures are not crowded out by the debt burden.

The question of how the economy converges to these BGPs will be addressed in Sect. 5 below. Let us now turn our attention to the general long-run solutions of the model in the presence of deficit ($\theta > 0$) and money ($\omega > 0$).

Proposition 2 (Multiplicity of BGPs) *For* $\theta, \omega > 0$ *and* $\eta \in (0, 1)$*, two and only two BGPs characterize the long-run solution of the model: a high BGP* (γ^{*h}) *and a low BGP* (γ^{*l})*, where* $0 < \gamma^{*l} < \gamma^{*h}$.

Proof See Appendix C.

Relations (24) and (25) are depicted in Fig. 1, where point H characterizes the high BGP, while point L denotes the low BGP.

In our setup, multiplicity comes from the interaction between Households' intertemporal behavior (BMK) and the Government's budget constraint (GBC). On the one hand, economic growth positively depends on the public-expenditures-to-capital ratio, which increases the marginal productivity of private capital in the bond market clearing condition (25). On the other hand, in the Government's budget constraint (24),

[9] Notice that, with $\theta = 0$, public debt must be constant but not necessarily zero in the long run.

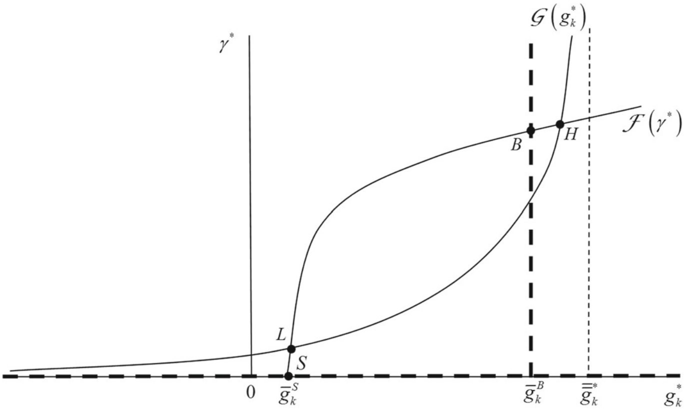

Fig. 1 Multiplicity of BGPs

economic growth allows reducing the public debt burden in the long run (in steady state $b_k^* = (1-\eta)\,\theta y_k^*/\gamma^*$), thus promoting productive public spending.

This interaction generates multiplicity: for the same set of parameters, a high BGP (H) and a low BGP (L) coexist. Indeed, a high rate of growth, by reducing the debt burden, allows increasing public expenditure, which further enhances growth, while a low rate of growth magnifies the crowding-out effect of debt on productive public spending, which, in turn, decreases growth. Consequently, there are two perfect-foresight BGPs, one associated with low public debt and high expected economic growth and the other associated with high expected public debt and low growth.

4 The long-run effect of deficit monetization

From the present, to save notation, we define $Q := f(R)$, and, to obtain explicit results and emphasize the role of the interest-elasticity of money demand, we use the transaction cost specification (4). The relationship between the nominal interest rate R and the transaction cost factor Q is thus: $Q = [(1+\mu)\,\omega^{\mu} R]^{1/(1+\mu)}$, with $Q = R$ in the special CIA case ($\mu \to 0$). With such a specification, the money demand (14) is written as (see Appendix A)

$$m = e(Q/\omega)^{-\mu}, \tag{29}$$

with μ being the (absolute value of the) elasticity of money demand to the interest factor Q.

We first compute analytically the effect of deficit monetization in the long run (Sect. 4.1) before illustrating our results within a calibrated economy similar to that of the US (Sect. 4.2), and examining the case of a zero interest rate (Sect. 4.3).

4.1 Analytical results

The following Proposition establishes the effect of public deficits in the long run.

Proposition 3 (The effect of deficit in the steady state)

(i) *Along the low BGP* (γ^{*l})*, any upward shift in the deficit target* (θ) *increases economic growth, irrespective of the degree of monetization.*
(ii) *Along the high BGP* (γ^{*h})*, there is a critical level of the degree of monetization* η *(say,* $\bar{\eta}^h$*), such that any upward shift in the deficit target* (θ) *reduces growth if monetization is small* ($\eta < \bar{\eta}^h$) *but increases it if monetization is large* ($\eta > \bar{\eta}^h$).

See Appendix D.
From (27), we can define the following implicit function

$$\mathcal{H}\left(\gamma^*\right) := \mathcal{G}(\mathcal{F}\left(\gamma^*\right)) - \gamma^* \ = 0. \tag{30}$$

Using the implicit function theorem, the effect of the deficit ratio on the BGPs can be obtained as

$$\left.\frac{\mathrm{d}\gamma^*}{\mathrm{d}\theta}\right|_{\gamma^*=\gamma^{*i}} = -\left.\frac{\partial_\theta \mathcal{H}\left(\gamma^*,\theta\right)}{\partial_\gamma \mathcal{H}\left(\gamma^*,\theta\right)}\right|_{\gamma^*=\gamma^{*i}}, \quad i \in \{h,l\},$$

where, for $(\theta, \omega) \to (0, 0)$,

$$\begin{cases} \partial_\gamma \mathcal{H}\left(\gamma^*,\theta\right)\big|_{\gamma^*=\gamma^{*l}} \to \partial_\gamma \mathcal{H}\left(\gamma^*,\theta\right)\big|_{\gamma^*=\gamma^S} > 0 \\ \partial_\gamma \mathcal{H}\left(\gamma^*,\theta\right)\big|_{\gamma^*=\gamma^{*h}} \to \partial_\gamma \mathcal{H}\left(\gamma^*,\theta\right)\big|_{\gamma^*=\gamma^B} < 0 \end{cases}. \tag{31}$$

In addition, we have $\mathrm{Sign}\{\partial_\theta \mathcal{H}\left(\gamma^*,\theta\right)|_{\gamma^*=\gamma^{*i}}\} = \mathrm{Sign}\{\eta - \bar{\eta}^i\}$, $i \in \{h,l\}$, where, defining $\upsilon := \phi_k/\phi_c$ and $x\left(\gamma^i\right) := 1 - \tau - (1-\upsilon)\left(\gamma^i + \delta\right) A^{\frac{-1}{\alpha}} \tau^{\frac{\alpha-1}{\alpha}}$,

$$\bar{\eta}^i = \frac{(1-\alpha)\, x\left(\gamma^i\right)\left[\varepsilon\left(\gamma^i\right) - 1\right]}{(1-\alpha)\, x\left(\gamma^i\right) \varepsilon\left(\gamma^i\right) - \alpha\tau\upsilon\,(1+\mu)}. \tag{32}$$

In the neighborhood of the Solow BGP ($\gamma^S = 0$), $\varepsilon(\gamma^S) \to +\infty$ and $\bar{\eta}^S \to 1$, such that, as $\partial_\gamma \mathcal{H}(\gamma^*,\theta)|_{\gamma^*=\gamma^S} > 0$ in (31): $\mathrm{Sign}\{\frac{d\gamma^*}{d\theta}|_{\gamma^*=\gamma^S}\} = \mathrm{Sign}\{1-\eta\} \geq 0$, for any $\eta \leq 1$, which proves point (i). In the neighborhood of the Barro BGP, $\bar{\eta}^h \to \bar{\eta}^B$, with $\partial_\upsilon \bar{\eta}^B \geq 0$, $\partial_\mu \bar{\eta}^B \leq 0$, and where $\bar{\eta}^B$ is defined in (32) for $i = B$, and, since $\partial_\gamma H(\gamma^*,\theta)|_{\gamma^*=\gamma^B} < 0$ in (31), $\mathrm{Sign}\{\frac{d\gamma^*}{d\theta}|_{\gamma^*=\gamma^B}\} = \mathrm{Sign}\{\eta - \bar{\eta}^B\}$, which proves point (ii).[10] □

From Proposition 3, along the high BGP, the deficit ratio positively impacts economic growth if money demand is relatively inelastic and transaction costs on investment are relatively low. Effectively, any increase in the interest-elasticity of money demand (μ) or in transaction costs on investment (υ) moves up the borderline $\bar{\eta}^B$ and reduces the parameter space in which the deficit has a positive impact (see Fig. 2a).

[10] Equation (32) provides an explicit value for $\bar{\eta}^B$ because γ^B is independent of η.

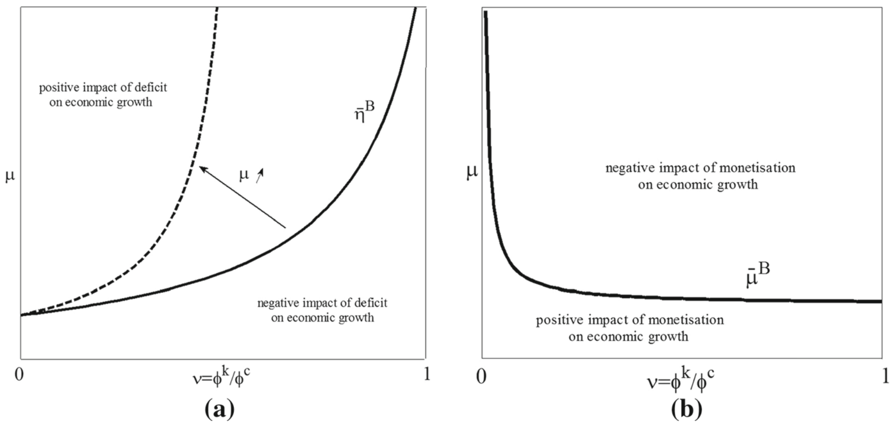

Fig. 2 Comparative statics along the high BGP. **a** Increase in the deficit ratio. **b** Increase in monetization

These results exemplify the interplay between public deficits and the inflation tax. Indeed, public deficits exert both a positive effect on growth, by financing additional productive public spending, and a negative effect, by increasing the debt burden. The latter effect can be mitigated by monetizing deficits, but this also entails costs, as monetization increases the inflation tax (the nominal interest rate and, in turn, transaction costs are increased). If private investment is highly subject to transaction costs or if money demand responds strongly to the interest rate, the gain from monetizing deficits is fairly low, as the inflation tax has a strong harmful impact on growth. Thus, the effect of the deficit on growth will be positive if $\eta > \bar{\eta}^B$ and negative otherwise (see Fig. 2a).

Along the low BGP, the marginal return on productive spending is sufficiently strong that the direct impact of the deficit outweighs the effect of the inflation tax, independent of the parameters. Let us now analyze the direct effect of monetization on the BGPs.

Proposition 4 (The effect of monetization in the steady state)

(i) *Along the low BGP (γ^{*l}), any upward shift in the rate of deficit monetization reduces economic growth.*
(ii) *Along the high BGP (γ^{*h}), there is a critical level of the interest-elasticity of money demand μ (say, $\bar{\mu}^h$) such that any upward shift in the rate of deficit monetization increases economic growth if the interest-elasticity of money demand is small $\left(\mu < \bar{\mu}^h\right)$ but decreases it if this elasticity is large $\left(\mu > \bar{\mu}^h\right)$.*

See Appendix D.
Using the implicit function theorem in (30), the effect of monetization on the BGPs can be obtained as

$$\left.\frac{\mathrm{d}\gamma^*}{\mathrm{d}\eta}\right|_{\gamma^*=\gamma^{*i}} = -\left.\frac{\partial_\eta \mathcal{H}(\gamma^*, \eta)}{\partial_\gamma \mathcal{H}(\gamma^*, \eta)}\right|_{\gamma^*=\gamma^{*i}}, \quad i \in \{h, l\},$$

where, for $(\theta, \omega) \to (0, 0)$, $\text{Sign}\{\partial_\eta \mathcal{H}(\gamma^*, \eta)|_{\gamma^*=\gamma^{*i}}\} = \text{Sign}\{\bar{\mu}^i - \mu\}, i \in \{h, l\}$, with

$$\bar{\mu}^i = \frac{(1-\alpha)x(\gamma^{*i})\varepsilon(\gamma^{*i})}{\alpha\tau\upsilon} - 1. \tag{33}$$

In the neighborhood of the Solow BGP ($\gamma^S = 0$), $\varepsilon(\gamma^S) \to +\infty$ and $\bar{\mu}^S \to +\infty$, such that $\partial_\eta \mathcal{H}(\gamma^*, \eta)\big|_{\gamma^*=\gamma^S} < 0$ and, by (31), $\frac{\mathrm{d}\gamma^*}{\mathrm{d}\eta}|_{\gamma^*=\gamma^S} < 0$, for $\mu > 0$, which proves point (i). In the neighborhood of the Barro BGP, $\bar{\mu}^h \to \bar{\mu}^B$, with $\partial_\upsilon \bar{\mu}^B \leq 0$, where $\bar{\mu}^B$ is defined in (33) for $i = B$, and $\text{Sign}\{\frac{d\gamma^*}{d\eta}|_{\gamma^*=\gamma^B}\} = \text{Sign}\{\bar{\mu}^B - \mu\}$, which proves point (ii).[11] □

Following Proposition 4, along the high BGP monetization has a positive effect on growth if money demand is relatively inelastic. This is notably the case in the special CIA case in which $\mu \to 0$. If money demand is very elastic to the interest rate $\left(\mu > \bar{\mu}^B\right)$, however, monetization impedes economic growth because it creates inflation and increases transaction costs in the long run. The effect of the inflation tax on economic growth is related to the importance of transaction costs on investment (υ). If investment is not subject to transaction costs $\left(\upsilon = 0 \Rightarrow \mu^B \to +\infty\right)$, monetization is always beneficial to long-run economic growth, as the debt burden is alleviated without any cost on private investment. On the contrary, as soon as $\upsilon > 0$, a trade-off appears between the benefits of debt burden alleviation due to monetization and the increased transaction costs on private capital accumulation due to the associated inflation tax. As Fig. 2b shows, the former effect prevails if $\mu < \bar{\mu}^B$, and vice versa.

The low BGP is not subject to such a threshold effect because monetization is detrimental to economic growth, independent of the parameters. Effectively, along the low BGP, output is so low that any increase in monetization leads to an increase in the public-debt-to-GDP ratio: the detrimental impact of the inflation tax on production outweighs the benefit of debt burden monetization, even for very low levels of the interest-elasticity of money demand.

4.2 A numerical illustration

Although all of our results are established *analytically*, it is interesting to assess whether their magnitude is consistent with the long-run properties of a developed economy such as the US. Moreover, our analytical results are obtained for "small" values of the deficit ratio (formally $\theta \to 0$), and it must be verified that these results are robust in the case of larger values. To this end, we present some simulations from a calibration of our model.

Our numerical results are based on reasonable values for parameters (see Table 1). We choose a typical discount rate $\rho = 0.02$ to match long-run historical data for the risk-free real interest rate. The consumption elasticity of substitution (inverse of the risk-aversion coefficient) is, as a rule, fixed at $S = 1$. Regarding the technology, we set $A = 0.5$ to obtain a realistic rate of economic growth, and the capital share in the production function is $\alpha = 0.7$, as in Gomes et al. (2013), close to the value

[11] Equation (33) provides an explicit value for $\bar{\mu}^B$, because γ^B is independent of μ.

Table 1 Baseline calibration (high BGP)

Parameters		
Households		
S	1	Intertemporal elasticity of substitution
ρ	0.02	Discount rate
ω	0.01	Scale parameter for transaction costs
$\mu/(1+\mu)$	0.5	Elasticity of the demand for money
ϕ^c	1	Cash requirement for consumption
ϕ^k	0.2	Cash requirement for investment
Technology		
A	0.5	Productivity parameter
α	0.7	Capital share in the production function
δ	0.05	Depreciation rate
Government		
τ	0.4	Tax rate on income
θ	0.025	Long-run deficit ratio (target value)
η	0.25	Monetized share of deficit

	Target values		
	Model	Data	Source
Long-run economic growth	0.034	0.033	Bureau of Economic Analysis, 1950–2015
Long-run inflation rate	0.0354	0.0358	Bureau of Labor Statistics, 1950–2015
After-tax return of capital	0.054	0.0516	Gomme et al. (2011)
Investment-to-capital ratio	0.084	0.088	Gillman and Kejak (2011)
Velocity of money (e/m)	3.73	4.53	BEA and Federal Reserve Bank, 1959–2015
Public debt to GDP	0.55	0.572	Bureau of Economic Analysis, 1950–2015
Public deficit to GDP	0.025	0.025	US Office of Management and Budget, 1950–2015

(0.715) used by Gomme et al. (2011). Such a capital share allows us to reproduce the empirical results of Munnell (1990) on the elasticity of output to productive public spending ($1-\alpha=0.3$). The depreciation rate of capital is set at $\delta=0.05$, which roughly corresponds to the average value of depreciation rates used in Gomme and Rupert (2007).

Regarding the Government's behavior, the income tax rate is $\tau=0.4$, according to, e.g., Trabandt and Uhlig (2011) and Gomes et al. (2013), and we fix the deficit ratio at its long-run average value in the US data, namely $\theta=0.025$ from 1950 to 2015. Regarding the monetary features of the model, the interest rate elasticity of the demand for money ($\mu/(1+\mu)$) is set at the usual value of 1/2 (in absolute value), that is $\mu = 1$, consistent with Baumol (1952)'s well-known "square root rule", and $\omega = 0.01$, to ensure low transaction costs (in our baseline calibration, transaction costs $T(e, m)$ represent a share of 0.63% of GDP). According to the literature, the coefficient of consumption in the CIA constraint is set at $\phi^c = 1$, and we choose $\phi^k = 0.2$ to reproduce the fact that a considerable share of investment goods are credit-financed. Finally, the part of deficit that is monetized is fixed $\eta = 0.25$ in the baseline calibration, but this parameter will be considered over the range (0, 1) in our simulations below.

Despite the highly stylized nature of our model, the baseline calibration allows us to replicate some salient facts characterizing the US economy. Effectively, the high BGP (which we consider to describe the long-run features of the US economy) is characterized by a 3.4% long-run rate of economic growth (3.3% in the data) and a 3.54% inflation rate (3.58% in the data).[12] The calibration well reproduces the after-tax rate of return to capital (5.4%; while the average mean over the period 1954–2008 is 5.16% in Gomme et al. 2011), and the investment-to-capital ratio (0.084) is comparable to the estimate (0.088) used in Gillman and Kejak (2011). Furthermore, the steady-state debt-to-output ratio (55%) is consistent with US data (57.2% on average during the period 1950–2015), and the computed velocity of money (3.73) roughly corresponds to observations.[13]

From this baseline calibration, we undertake some simulations to quantitatively assess the effect of changes in the degree of monetization (η). Figure 3 shows that, for the parameters in Table 1, the high BGP positively depends on monetization. In conformity with Proposition 4, this is the case because the elasticity of money demand is less than its critical value, i.e., $\mu < \bar{\mu}^h$. Indeed, with respect to our baseline calibration, $\bar{\mu}^h = 1.82 > 1$. Quite naturally, Fig. 3 also shows that the long-run inflation rate positively depends on monetization. For small degrees of monetization, the BGP is characterized by deflation because money emissions are not large enough to balance the rate of growth of money demand (in an endogenous setup the demand for real balances increases at the same rate as the economy in steady state).[14]

To obtain normative results, we also compute the effect of monetization on long-run welfare.[15] On a BGP, Households' welfare (1) writes

[12] With the parameters in Table 1, the corresponding values of economic growth and inflation for the low BGP are 0.15 and 2.05%, respectively. In the calibration, one period corresponds to one year.

[13] The apparent M1 velocity y/m was 6.86 during the period 1959–2014. To obtain the corresponding e/m that we use in the model, we compute $e/m = (e/y)(y/m)$, with $e/y = c/y + \phi^k(\dot{k} + \delta k)$. Since, in the long run (1950–2015, BEA) the consumption-to-GDP ratio has been 63.11% and the private gross investment-to-GDP ratio 17.24%, with $\phi^k = 0.2$, we obtain $e/y = 0.66$, that is $e/m = 0.66 \times 6.86 = 4.53$.

[14] Long-run deflation can easily be avoided by introducing an exogenous positive trend in money supply.

[15] We focus on steady-state welfare effects, namely we compare different BGPs associated with different values of monetization. In other words, we are not interested in the transition from one steady state to another, and we do not study transitional dynamics following a change in parameters; thus, we perform comparative statics among different BGPs. Indeed, as we will show in the following Section, transitional dynamics may give rise to indeterminacy of the high BGP, thereby making it impossible to assess welfare effects on the transition path. For an analysis of Ramsey fiscal policy in an endogenous growth setup, see, e.g., Park (2009).

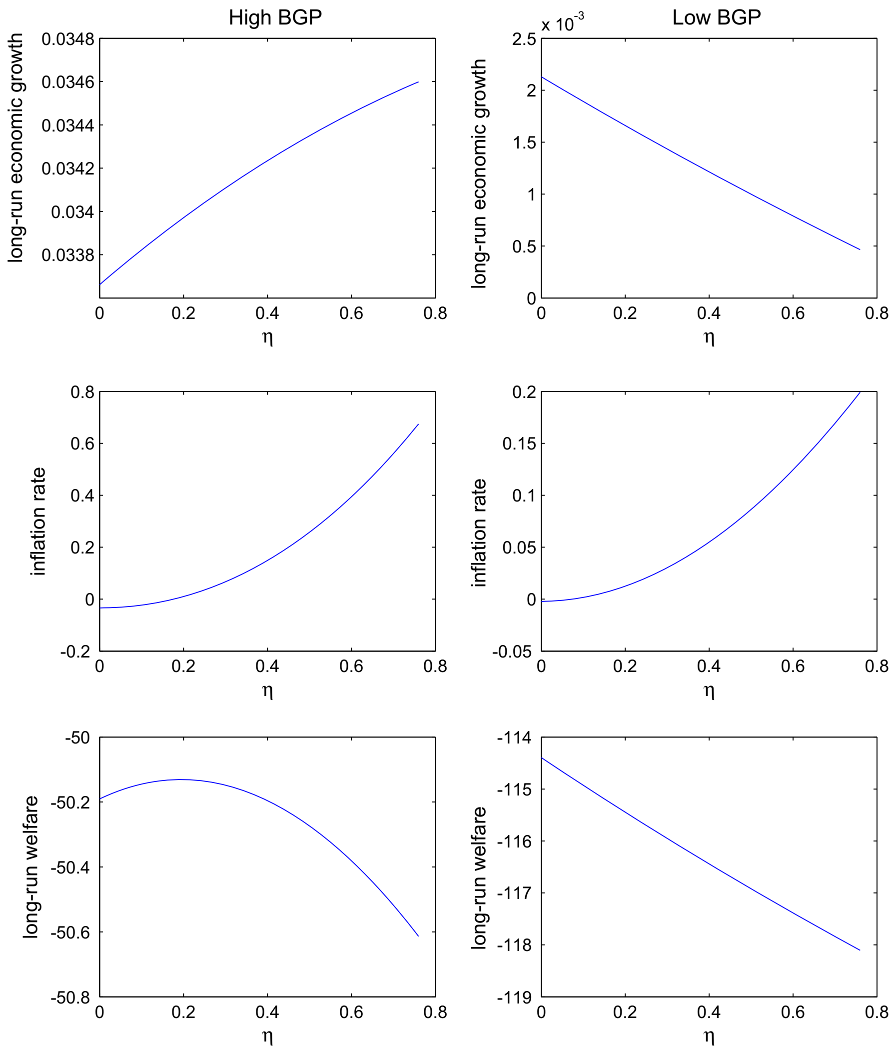

Fig. 3 Effect of monetization on the BGP

$$U = \begin{cases} \left\{\rho\left[\log\left(c^*\right) + \log\left(K_0\right)\right] + \gamma^*\right\}/\rho^2 & \text{if } S = 1, \\ \frac{S}{S-1}\left[\frac{\left(c^* K_0\right)^{\frac{S-1}{S}}}{\rho - \gamma^*\left(\frac{S-1}{S}\right)} - \frac{1}{\rho}\right] & \text{if } S \neq 1, \end{cases}$$

where K_0 is the initial capital stock (that we normalize to one in our simulations). Since a change in the degree of monetization impacts economic growth and consumption in steady state, namely $c_k^* = y_k^* - \gamma^* - g_k^* - \delta$, its effect on welfare might differ from its effect on growth.

Figure 3 shows that this is effectively the case. For our benchmark calibration, there is an optimal degree of monetization degree (approximatively 20% of the deficit) that maximizes the Household's welfare on the high BGP. This degree of monetization is the one that optimally trades off the gain in future consumption due to the increased rate of economic growth and the loss in current consumption that has to be incurred to generate future growth. Specifically, this loss comes from the higher inflation brought about by the rise in monetization, which causes an increase in the inflation tax on consumption (and investment) owing to transaction costs. In our baseline calibration, the optimal degree of monetization corresponds to a long-run inflation rate of 1% and economic growth of 3.4%.

4.3 The case of a zero nominal interest rate[16]

In the last decade, there has been much discussion of the case of a zero nominal interest rate, in line with the so-called "zero lower bound" that constraints conventional monetary policies. As is well known, a zero interest rate corresponds to the Friedman (1969) rule and leads to the policy prescription that the central bank should seek a deflation rate equal to the real interest rate in the long run. This prescription comes up against policies aiming at avoiding the zero lower bound, which require a positive rate of inflation. Avoiding the zero lower bound is motivated by business cycle considerations, while the Friedman rule is motivated by long-run efficiency. Since our model is deterministic, there is no room for stabilization issues.[17] Furthermore, the nominal interest rate is an equilibrium price and not a policy instrument for the Central Bank. Therefore, the case of a zero nominal interest rate can be addressed to assess the long-run costs of transacting.[18]

To this end, we introduce, beyond money issued to monetize public deficit in relation (8), an exogenous rate of growth of the money supply (η_0), controlled by the Central Bank, namely $\dot{M}/M = \eta_0 + \eta(d_k/m_k) =: \Theta$. The money equilibrium (17) becomes $\dot{m}/m = \Theta - R + r$, and, in the long run ($\dot{m}/m = \gamma^*$), the zero nominal interest rate can be achieved by a policy that deflates money supply at a rate $\hat{\Theta} = \gamma^* - r^*$.

16 We thank an anonymous referee for suggesting that we analyze this issue.

17 In fact, our model may be understood as depicting "the day after", once conventional monetary policies are ineffective and the Central Bank conducts an "unconventional" monetary policy, in the form of monetizing public deficits. For an interesting analysis of unconventional monetary policy and the liquidity trap, see Giraud and Pottier (2016).

18 Formally, our model can account for a zero interest rate by introducing a slight change in the transaction technology (4). To this end, assume that money demand (as a ratio of expenditure) is satiated for a positive level $\bar{m}$ such that $\psi\left(e_t/m_t\right) := \omega\mu\left(e_t/m_t - 1/\bar{m}\right)^{1/\mu}/(1+\mu)$. Thus, with a zero nominal interest rate, money demand is finite ($m_t/e_t = \bar{m}$) and the transaction cost is zero. Similar specifications have been used by Kimbrough (1986), Rebelo and Vegh (1995) and Schmitt-Grohé and Uribe (2004), among others. This corresponds, in MIUF models, to the assumption that there is a positive satiation level of money holdings such that the marginal utility of money is zero, following Friedman (1969). Thanks to this specification, money demand is well determined with a zero nominal interest rate, and along the BGP, economic growth and welfare are independent of the satiation parameter $\bar{m}$. Thus, for $R_t = 0$, the equilibrium of our economy corresponds to that of a pure credit economy. Such a demand for money, the microfoundations of which are established in Sect. 6, leaves our results (including the properties of transitional dynamics, analyzed in the following Section) qualitatively unchanged.

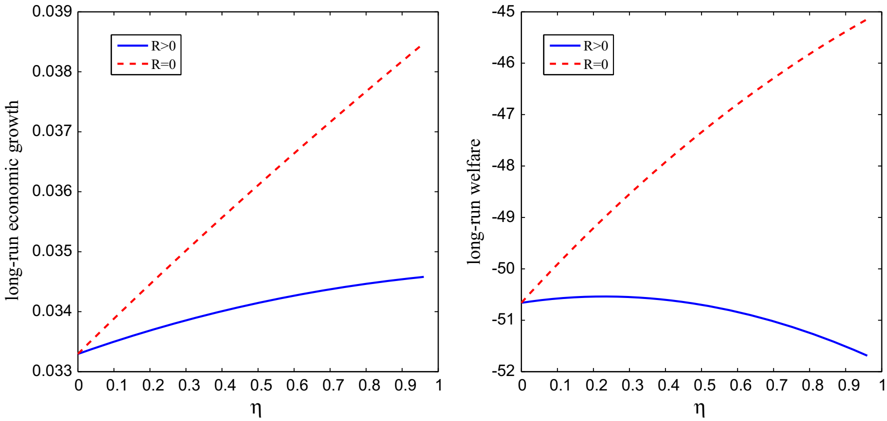

Fig. 4 A zero versus positive interest rate (high BGP)

This value corresponds, in our context, to the Friedman rule.[19] Effectively, in such a situation, the private cost of holding money equals the social cost of producing it (zero), as transaction costs are canceled.

Figure 4 describes the link between the endogenous component of the money growth rate (η) and economic growth and welfare. The continuous curve corresponds to Fig. 3 above with positive nominal interest rate, while the dashed curve depicts the optimal rule $\Theta = \hat{\Theta}$. In the latter case, as the degree of monetization increases, the interest rate can be kept at zero because the exogenous component of money creation adjusts to the endogenous component. As illustrated in Fig. 4, with a zero interest rate, long-run growth and welfare are higher than they are under positive rates, for any degree of monetization, thanks to the absence of transaction costs. For our benchmark calibration, long-run welfare now positively depends on monetization, as a money expansion no longer generates increases in the inflation tax. Therefore, in Fig. 4, the difference between the dashed and solid lines measures the long-run growth and welfare losses associated with the increasing transaction costs when deficit monetization increases. For full monetization, these losses are, respectively, 0.4 points of permanent economic growth (3.45% per annum against 3.85%) and 12% of welfare. If we consider the degree of monetization that maximizes long-run welfare in the case of positive interest rates (20%) these losses translate to 0.1 points of economic growth and 3% of long-run welfare.

Consistent with a realistic calibration, our results show that the effect of monetization on long-run economic growth and welfare is quantitatively important, even for moderate levels of inflation, as monetization affects both the financing cost of private

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[19] Indeed, without endogenous growth ($\gamma^* = 0$ and $r^* = \rho$), the optimal policy is to deflate the money stock at a rate equal to the subjective discount rate ($\hat{\Theta} = -\rho$). In our model, since a part of money creation is endogenous, to obtain this optimal rule, the exogenous component of money creation must offset the endogenous component, namely $\hat{\eta}_0 = \hat{\Theta} - \eta(d_k/m_k)$. Thus, to keep the nominal interest rate at zero, monetary authorities must set the discretionary component η_0 at the level $\hat{\eta}_0$ consistent with the unique optimal rate of money growth $\hat{\Theta}$. Obviously, this unique optimal policy can be reached with different degrees of monetization (η), as shows Fig. 4.

capital and the burden of public debt. This creates interactions that generate growth and welfare effects in the long run, and a possible indeterminacy of the transition path in the short run, as shown in the following Section devoted to transitional dynamics.

5 Transitional dynamics

Outside the steady state, the model gives rise to a five-variable reduced form, which can be solved recursively (see Appendix B), namely, for $\phi^k \zeta(g_k) \neq 0$

$$\begin{cases} \text{(a)}\ \dot{d}_y = -\xi \left(d_y - \theta\right) \\ \text{(b)}\ \dot{b}_k = (1-\eta)\, d_y A g_k^{1-\alpha} - \gamma_k b_k \\ \text{(c)}\ \dot{Q} = \frac{1}{\phi^k}\left[(r+\delta)\left(1+\phi^k Q\right) - (1-\tau)\,\alpha A g_k^{1-\alpha}\right] \\ \text{(d)}\ \dot{c}_k = S\left[r - \rho - \phi^c \dot{Q}/(1+\phi^c Q)\right] c_k - \gamma_k c_k \\ \text{(e)}\ \dot{g}_k = \frac{1}{\phi^k \zeta(g_k)}\left\{\eta d_y A g_k^{1-\alpha}\left(\frac{Q}{\omega}\right)^{\mu} + \left(r - R - \gamma_k + \frac{\mu \dot{Q}}{Q}\right) e_k - \left(\phi^c - \phi^k\right)\dot{c}_k\right\} \end{cases} \tag{34}$$

where $R = \frac{\omega}{1+\mu}\left(\frac{Q}{\omega}\right)^{1+\mu}$, and

$$\gamma_k = A g_k^{1-\alpha} - g_k - c_k - \delta =: \gamma(c_k, g_k), \tag{35}$$

$$e_k = \left(\phi^c - \phi^k\right) c_k + \phi^k \left(A g_k^{1-\alpha} - g_k\right) =: e\left(c_k, g_k\right), \tag{36}$$

$$r = \left(\left(d_y + \tau\right) A g_k^{1-\alpha} - g_k\right) / b_k =: r\left(d_y, g_k, b_k\right), \tag{37}$$

$$\zeta\left(g_k\right) := \frac{\partial}{\partial g_k}\left(A g_k^{1-\alpha} - g_k\right) = (1-\alpha)\, A g_k^{-\alpha} - 1. \tag{38}$$

The first equation of the reduced-form (34) is the evolution of the deficit ratio (7). The second equation describes the law of motion of the public debt ratio, relative to the share of the deficit that is not monetized [we use (9) together with the definition of the public debt ratio b_k]. Relation (34c) comes from the trade-off between the return to private capital and indexed bonds (11), which, in turn, provides the law of motion of the nominal interest factor (Q) that reflects the role of the inflation tax. The fourth equation in system (34) is the Keynes–Ramsey rule that governs consumption behavior, and relation (34e) establishes the law of motion of productive public spending. Indeed, as we previously emphasized, public spending is endogenous in our setup. In the money equilibrium, the rate of growth of the real money supply ratio ($\dot{M}/M - \dot{K}/K - \pi = \eta d/m - R + r - \gamma_k$) must coincide with the rate of growth of the real money demand ratio ($\dot{m}_k/m_k = \dot{e}_k/e_k - \mu \dot{Q}/Q$). Since, in goods market equilibrium, the evolution of Households' expenditure depends on that of productive public spending that determines production capacities ($\dot{e}_k = \phi^k \zeta\left(g_k\right) \dot{g}_k + \left(\phi^c - \phi^k\right) \dot{c}_k$), this provides the (endogenous) law of motion of public expenditure in equilibrium.

To analyze the stability of the BGPs we have to specify the variables that can jump in the reduced-form (34). First, the initial deficit-to-output ratio $\left(d_{y0}\right)$ cannot jump because it is defined by the smooth adjustment dynamics (7). Second, the debt-to-output ratio $b_{k0} = b_0/k_0$ cannot freely jump.[20] Consequently, there are only 3 free jumpable variables and 2 predetermined variables in system (34).

5.1 A special case: transaction costs on consumption only

In the special case in which the transaction technology does not affect investment goods $\left(\phi^k = 0\right)$, Eqs. (34c) and (34d) are not defined, and the reduced form becomes a four-variable one, namely (see Appendix B)

$$\begin{cases} \text{(a)} \quad \dot{d}_y = -\xi \left(d_y - \theta\right) \\ \text{(b)} \quad \dot{b}_k = (1-\eta)\, d_y A g_k^{1-\alpha} - \gamma_k b_k \\ \text{(c)} \quad \dot{Q} = \frac{Q(1+\phi^c Q)}{\mu+(\mu+S)\phi^c Q} \left[\frac{\omega}{1+\mu} \left(\frac{Q}{\omega}\right)^{1+\mu} - \frac{\eta d_y A g_k^{1-\alpha}}{\phi^c c_k} \left(\frac{Q}{\omega}\right)^{\mu} - (1-S)\, r - \rho S \right] \\ \text{(d)} \quad \dot{c}_k = S\left[r - \rho - \phi^c \dot{Q}/(1+\phi^c Q)\right] c_k - \gamma_k c_k \end{cases} \tag{39}$$

The crucial difference relative to (34) is that the public spending ratio is no longer part of the reduced form but is obtained by means of (37), which rewrites as

$$g_k := g\left(d_y, b_k\right), \tag{40}$$

while the real interest rate is simply, by (34c),

$$r = (1-\tau)\, \alpha A g_k^{1-\alpha} - \delta =: r\left(g_k\right). \tag{41}$$

The linearization of (39) in the neighborhood of BGPs provides the following system

$$\begin{pmatrix} \dot{d}_y \\ \dot{b}_k \\ \dot{Q} \\ \dot{c}_k \end{pmatrix} = \mathbf{J}_1^i \begin{pmatrix} d_y - d_y^{*i} \\ b_k - b_k^{*i} \\ Q - Q^{*i} \\ c_k - c_k^{*i} \end{pmatrix}, \quad i \in \{h, l\}, \tag{42}$$

where $\mathbf{J}_1^i$ stands for the Jacobian matrix in the neighborhood of BGP $i = h, l$. According to the Blanchard-Kahn conditions, the steady state is (saddle path) stable and well determined if $\mathbf{J}_1^i$ contains exactly 2 positive and 2 negative eigenvalues (with one eigenvalue equal to $-\xi$).

[20] Effectively, the initial stock of capital (k_0) is predetermined and cannot jump. The real stock of debt ($b_0 := B_0/P_0$) is also predetermined if public debt is indexed. In this case, the nominal quantity of public debt (B_0) jumps every time there is a jump in the price level (P_0): $\Delta B_0 = b_0 \Delta P_0$, such that the real debt (b_0) does not jump. If public debt is non-indexed, it is the nominal stock of public debt (B_0) that is predetermined, and the real stock of debt jumps every time there is a jump in the price level: $\Delta b_0/b_0 = -\Delta P_0/P_0$. In this case, the real stock of debt can jump, but its jump is constrained by the jumps in the other variables of the reduced-form (34). In accordance with money neutrality, the stock of nominal money is predetermined, and a jump in the price level is feasible only if there is a jump in the real-balances-to-capital ratio ($m_{k0} = M_0/K_0 P_0$). Since, by (14), m_{k0} depends on the variables in the reduced-form (34), its jumps depends on the jumps of Q_0, c_{k0}, g_{k0} and b_{k0}. Consequently, even if b_{k0} is a jumpable variable, there are only 3 degrees of freedom in the set of initial jumps.

Proposition 5 (Stability) *For small values of the deficit ratio*

(i) *the low BGP is over-determined (unstable), and*
(ii) *the high BGP is well determined (saddle-path stable).*

Proof See Appendix E.
Appendix E shows that in the neighborhood of the Barro BGP, $\mathbf{J}_1^B$ contains two positive and two negative eigenvalues, while in the neighborhood of the Solow BGP, $\mathbf{J}_1^S$ contains one negative and three positive eigenvalues. By continuity, these properties are verified for positive (and low) values of the deficit ratio. Therefore, the high BGP is well determined, while the low BGP is unstable. □

Thanks to Proposition 5, we can exclude multiplicity on the basis of local dynamics: the high BGP is the only relevant equilibrium path when investment is not subject to transaction costs. This is no longer the case in the general version of the model, as we show in the following.

5.2 Transaction costs on consumption and investment

With transaction costs on consumption and investment, the linearization of (34) in the neighborhood of the BGPs provides the following system

$$\begin{pmatrix} \dot{d}_y \\ \dot{b}_k \\ \dot{Q} \\ \dot{c}_k \\ \dot{g}_k \end{pmatrix} = \mathbf{J}_2^i \begin{pmatrix} d_y - d_y^{*i} \\ b_k - b_k^{*i} \\ Q - Q^{*i} \\ c_k - c_k^{*i} \\ g_k - g_k^{*i} \end{pmatrix}, \quad i \in \{h, l\}, \tag{43}$$

where $\mathbf{J}_2^i$ stands for the Jacobian matrix in the neighborhood of BGP $i = h, l$. According to the Blanchard-Kahn conditions, the determination of the BGP is ensured if $\mathbf{J}_2^i$ contains exactly 3 positive and 2 negative eigenvalues (with one eigenvalue equal to $-\xi$).

Proposition 6 (Multiplicity and indeterminacy) *For small values of the deficit ratio*

(i) *the low BGP is well determined (saddle-path stable), and*
(ii) *the high BGP is locally undetermined or well determined (saddle-path stable), depending on parameters.*

Proof See Appendix F.
The inspection of the reduced-form (34) reveals that the dynamics fundamentally shift with the value of the term $\zeta\left(g_k^*\right)$. Effectively, the system is not defined for $\zeta\left(g_k^*\right) = 0$, and the determinant of the Jacobian matrix $\mathbf{J}_2^i$ changes sign whenever $\zeta\left(g_k^{*i}\right)$ changes sign. Yet, to be fully determined, the BGP must be associated with exactly 2 negative eigenvalues. As the determinant of the Jacobian matrix is the product of the 5 eigenvalues, such a configuration is possible only if it is positive, i.e., BGP determinacy cannot be ensured when the determinant is negative. On the one hand,

Appendix F shows that, in the neighborhood of the Solow BGP (i.e., $g_k \to g_k^S$), we have $\zeta(g_k) > 0$ for any g_k, and $\mathbf{J}_2^S$ has exactly 2 negative and 3 positive eigenvalues; thus the Solow BGP is saddle path. By continuity, this must be true for positive (but small) values of the deficit ratio. This proves point (i). On the other hand, in the neighborhood of the Barro BGP (i.e., $g_k \to g_k^B$), on the contrary, $\zeta(g_k)$ changes sign depending on parameters and $\mathbf{J}_2^B$ has 2 negative and 3 positive eigenvalues if $\zeta(g_k) < 0$, but 3 negative and 2 positive eigenvalues if $\zeta(g_k) > 0$. By continuity, this property is verified for positive (but low) values of the deficit ratio. In the former case the high BGP is well determined, while in the latter it exhibits local indeterminacy. This proves point (ii). □

In the neighborhood of the low BGP, the public spending ratio is very small (see Fig. 1); hence, $\zeta\left(g_k^{*l}\right) > 0$, for any g_k^{*l}, and the BGP is well determined. This is not true in the neighborhood of the high BGP. The high BGP is locally determined only if $\zeta\left(g_k^{*h}\right) < 0$, namely if $g_k^{*h} > \hat{g}_k^* := A^{1/\alpha}(1-\alpha)^{1/\alpha}$. By (23), this criterion amounts to

$$\tau > 1 - \alpha - \theta\left[1 - (1-\eta)\,\varepsilon\left(\gamma^{*h}\right)\right]. \tag{44}$$

Relation (44) shows that determinacy of the high BGP is the more likely to occur (i) the higher is the degree of monetization (η) and (ii) the higher is the tax rate (τ).[21] The effect of the deficit ratio, meanwhile, depends on the degree of monetization. Effectively, an increase in the deficit ratio supports determinacy of the high BGP only if monetization is high enough, namely if $\eta > \bar{\bar{\eta}}$, where $\bar{\bar{\eta}}$ is implicitly defined by $\bar{\bar{\eta}} = 1 - 1/\varepsilon(\gamma^h\left(\bar{\bar{\eta}}\right))$. In the other case ($\eta < \bar{\bar{\eta}}$), any increase in the deficit ratio results in high-BGP indeterminacy.

As Fig. 5 shows, determinacy is ensured above the curve defined by (44), which pivots downward as monetization increases.[22] Without deficit ($\theta = 0$), to ensure the determinacy of the high BGP, the tax rate must be higher than the Barro (1990) optimal rate, namely $\tau^B := 1-\alpha$. This is no longer the case with public deficit, as determinacy can be ensured with tax rates lower than τ^B, especially if monetization is high. The higher monetization is, the more likely the BGP is determined, for a given tax rate and deficit ratio.

Proposition 6 shows that multiplicity cannot be removed on the basis of the local dynamics of the two BGPs. On the contrary, if transaction costs affect consumption and investment, both steady states are reachable, depending on the initial level of the public debt ratio. If the initial public debt ratio is "high", the economy will be attracted by the low-growth BGP, which can be regarded as a poverty trap, with economic growth approaching zero. If the initial public debt ratio is "low", on the contrary, the economy will converge toward the high BGP, but the exact path that the economy will follow during the transition may be subject to sunspot equilibria, i.e., the existence of a continuum of equilibrium paths converging toward the high BGP, starting from the same initial values of the state variables. In such a case, as Fig. 5 suggests, deficit

[21] Considering that the direct effect prevails, as θ is small.

[22] As an illustration, Fig. 5 is drawn by using our baseline calibration (Table 1). Of course, Proposition 6 and Eq. (44) are established analytically and do not depend on the parameters' values.

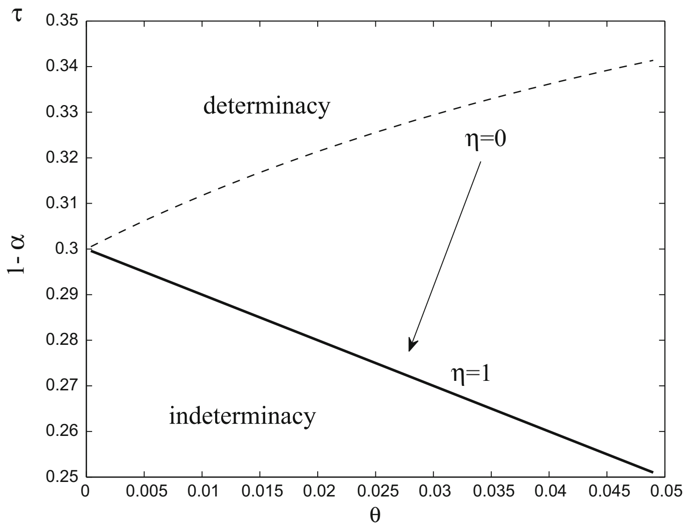

Fig. 5 Determinacy and indeterminacy of the high BGP

monetization can be used as a selection device to solve indeterminacy and obtain a unique transition path.

5.3 Discussion

Our general result is that the high BGP is in some sense "more stable" than the low BGP. In essence, this property comes from the Government's budget constraint, in which the dynamics of the public debt ratio are driven by the difference between the debt burden and economic growth. As usual in the analysis of Government's budget constraint, a sufficiently high economic growth rate allows circumventing the inherent unstable dynamics of public debt, thus stabilizing the public debt ratio. This is the case in the neighborhood of the high BGP. On the contrary, along the low BGP, economic growth is very low and cannot stabilize the evolution of the public debt ratio. This explains why, in the special case with transaction costs on consumption only, the high BGP is saddle-point stable, while the low BGP is unstable.

In the general version of the model, the same reasoning applies, but the reduced-form (43) has one additional equation, based on the evolution of a jumpable variable (the public spending ratio g_k), which can create indeterminacy. Indeed, the nature of the dynamics of $\dot{g}_k$ fundamentally shifts with respect to $\zeta\,(g_k)$. The intuitive explanation of this shift is the following. The term $\zeta\,(g_k) = d\,(y_k - g_k)/dg_k$ is the response of the difference between output and public spending following an increase in public spending, or, in other words, the net impact of an additional unit of productive public expenditure on the goods market equilibrium. Thus, any rise in g_k increases (decreases) private demand if $\zeta(g_k) > 0$ ($\zeta(g_k) < 0$). Since money is used in transactions, money demand positively (if $\zeta(g_k) > 0$) or negatively (if $\zeta(g_k) < 0$) depends on productive public expenditure.

Yet, in the money market equilibrium, money emissions are defined, in real terms, by the difference between the monetization of the public deficit and the depreciation of real balances due to inflation ($\dot{M}/P = \eta d - \pi m$). Suppose that there is an upward jump in public spending from the high BGP, such that $g_k > g_k^*$, *ceteris paribus*. As a result of the excess demand in goods equilibrium, the inflation rate jumps up, and the depreciation of the money stock (πm) exceeds deficit monetization (ηd). Then, the real stock of money decreases ($\eta d < \pi m \Rightarrow \dot{M}/P < 0$). In equilibrium, money demand must decline, and thus, private demand must decrease ($\dot{e}_k < 0$), which implies: $\dot{g}_k < 0$ if $\zeta(g_k) > 0$, or $\dot{g}_k > 0$ if $\zeta(g_k) < 0$. In the first case, the law of motion of g_k is stable, leading to the indeterminacy of the BGP (recall that g_k is a jumpable variable), while in the latter, the law of motion of g_k is unstable, leading to the determinacy of the BGP.

Hence, for configurations of parameters satisfying (44), the public spending ratio becomes so large that the derivative $\zeta\ (g_k)$ becomes negative. As a result of this novel source of instability, the high BGP loses its undesirable property of being stable and undetermined, and becomes saddle-path stable.

Our indeterminacy result it quite general, compared to the literature. In existing studies, multiplicity and indeterminacy arise from increasing returns in the production function,[23] the value of the intertemporal elasticity of substitution in the utility function,[24] or the timing or the fraction of transactions that are subject to the cash requirement. In particular, in two-sector endogenous growth models, using a discrete-time approach, Bosi et al. (2010) show that indeterminacy crucially depends on the timing of (intra-period) monetary arrangements and on the specification of preferences. In contrast, our indeterminacy result is not sensitive to the consumption elasticity of substitution or to the form of the utility function. It is not more sensitive to the timing of monetary payments because, in continuous time, any intra-period mechanism disappears. Moreover, in our setting, indeterminacy does not depend on the interest-elasticity of money demand: indeterminacy arises if investment goods are subject to transaction costs, but not if consumption only is affected, independent of the interest-elasticity of money demand (and in particular in the special CIA case with a zero elasticity).[25]

This feature outlines the motivation for introducing a general transaction cost technology that includes capital goods in growth models, as pioneered by Palivos and Yip (1995). In this respect, our analysis emphasizes an original source of multiplicity and indeterminacy, namely the interaction between deficit monetization and the form of the money demand, and especially how money demand reacts to changes in public expenditures in goods market equilibrium.

[23] Jha et al. (2002), e.g., find that the technology is a key determinant of the stability of the equilibrium.

[24] In one-sector "Ak"-type endogenous growth models, such as Suen and Yip (2005) and Chen and Guo (2008), local indeterminacy is due to the presence of an intertemporal substitution effect, the strength of which positively depends on the intertemporal elasticity of substitution in consumption.

[25] Compared to Bosi and Magris (2003) and Bosi et al. (2005), who show the importance of having a partial CIA constraint on consumption goods (namely $\phi^c < 1$ or $\phi^c = 1$), or Chen and Guo (2008) and Bosi and Dufourt (2008), our indeterminacy result does not depend on the exact fraction of investment expenditures that are subject to transaction costs (provided that it is strictly positive: $\phi^k > 0$).

6 Alternative specifications and microfoundations of transaction costs

As shown in the preceding Section, the interaction between deficit monetization and the inflation tax on the return to capital due to transaction costs on investment plays a crucial role in producing indeterminacy. Thus far, we have established our results by using a very general specification of the transaction technology. However, this specification was not explicitly related to the behavior of the financial sector. This Section addresses this issue by building microfoundations of the transaction technology through the introduction of a banking sector. To this end, we follow the pioneering work of Gillman and Kejak (2005, 2011), who develop an original approach to the microfoundations of money demand by introducing a financial intermediary sector.

Much of the literature regarding transaction costs is based on Baumol (1952) and Tobin (1956) and supposes that the use of money allows economizing some broker fees or "shoe-leather costs" (see, e.g., Jovanovic 1982; Romer 1986, among others). In this Section, we adopt a different approach and, using a simplified version of the Gillman and Kejak financial microfoundations of money demand, we obtain an observational equivalence between the production cost of banking activities and the transaction cost incurred by Households.

To prove this equivalence, we add a financial block to our model. As in Gillman and Kejak (2011), the financial sector includes a "mutual bank", owned by Households, that serves as financial intermediary.[26] Yet, the banking sector makes it possible to finance expenditures using credit bank, and the velocity of money becomes endogenous from the Households' choice between using credit versus money. Now, Households consume and invest using either real money (m_t) or credit (q_t), hence the exchange constraint

$$m_t + q_t \geq e_t. \tag{45}$$

In addition, private expenditures (e_t) are generated by deposits (denoted in real units by d_t) held at the financial intermediary, namely, $d_t = e_t := \phi^c c_t + \phi^k i_t$. Households must pay a fee on credit services ($p_{qt}q_t$), where p_{qt} denotes the real price per unit of credit. Thus, Households' budget constraint (5) can be rewritten as

$$\dot{k}_t + \dot{b}_t + \dot{m}_t = r_t b_t + (1-\tau)\, y_t - c_t - \delta k_t - \pi_t m_t - p_{qt} q_t + \varrho_t, \tag{46}$$

where ϱ_t is a lump-sum transfer that represents financial intermediary's profit, which is rebated to Households.[27]

The financial intermediary collects the fee for credit services ($p_{qt}q_t$) and incurs a cost c per unit of deposit. The cost function of the intermediary deserves attention. The cost of funds is one of the most important input costs for a financial institution, and a large part of banks' funding cost is the cost of refinancing that notably reflects any expenses or financial costs generated by reserve requirements. Indeed, the financial intermediary collects a share of deposits to build up reserves (either due to legal

[26] Details of the analysis are found in Gillman and Kejak (2011), from which we adopt notations. We simply describe here how our model can be amended to incorporate a financial sector.

[27] This specification is directly linked to footnote 6.

restrictions or to avoid liquidity risk). Presumably, the cost of refinancing is increasing with the amount of credit that the bank offers (the bank uses primarily the payment facilities from the Central Bank, then resorts to the interbank market where it faces increasing risk or liquidity premia). In this way, as the credit-to-deposit ratio increases, the reserve-to-deposit ratio goes down and, to restore reserves, the intermediary seeks to refinance and suffers from rising costs. Therefore, the higher the credit-to-deposit ratio, the higher the cost of obtaining liquidity for the bank. Hence, we assume a cost function $c := c(q_t/d_t)$, with $c'(\cdot) > 0$, and $c''(\cdot) > 0$,[28] and the intermediary maximizes its profit $\Pi_{Qt} := p_{qt}q_t - c\,(q_t/d_t)\,d_t$, subject to a standard liquidity constraint $d_t \geq m_t$ and a solvency restriction

$$q_t + m_t = d_t. \tag{47}$$

Using the following normalized variables $q_t^* := q_t/d_t$, and $\Pi_{Qt}^* := \Pi_{Qt}/d_t$, the intermediary's program becomes

$$\max_{q_t^*} \Pi_{Qt}^* := p_{qt}q_t^* - c\left(q_t^*\right), \tag{48}$$

where both the solvency and liquidity constraints are satisfied in equilibrium.

The solution of Households' and financial intermediary's program is now, omitting time indexes (see Appendix G)

$$\frac{\dot{c}}{c} = S\left[r - \rho - \left(\frac{\phi^c \dot{R}}{1 + \phi^c R}\right)\right], \tag{49}$$

$$\frac{(1-\tau)\alpha A\,(g/k)^{1-\alpha}}{1 + \phi^k R} - \delta = r - \left(\frac{\phi^k \dot{R}}{1 + \phi^k R}\right), \tag{50}$$

$$m = e\left[1 - c_1^{-1}(R)\right], \text{ where } c_1(\cdot) := c'(\cdot). \tag{51}$$

As can be seen, the laws of motion of the nominal interest rate (50) and the growth rate of the consumption ratio (49) are particular cases of the general model (10) and (11), where the cost of financing expenditures is simply $f(R) = R$. Indeed, in this Section, the cost of using money for Households simply relates to the cost of credit (p_q) that corresponds, in equilibrium, to the marginal cost of money R.

Additionally, relation (51) establishes financial microfoundations of the demand for money. Money demand is proportional to private expenditure and depends on the marginal cost of deposits $(c_1^{-1}(R))$. Since $c(\cdot)$ is an increasing convex function, money demand decreases with respect to R. Effectively, the higher the nominal interest rate, the higher the share of expenditure that is financed with credit, leading to an increase in money velocity (e/m) in accordance with the solvency restriction.

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[28] Compared with Gillman and Kejak (2011), we disregard complications associated with the fact that the production function of credit services can require inputs as a share of private capital or labor. Instead, we assume that the cost to build up reserves depends solely on the credit-to-deposit ratio. This allows us to keep the model simple and focus on the microfoundations of the transaction technology.

It must be emphasized that Eq. (51) is a special case of the general specification (14) used in the preceding Sections. Indeed, money demand can be written as

$$m = e\Psi(R), \text{ with } \Psi(R) = 1 - c_1^{-1}(R).$$

As shown in Appendix A, the function $\Psi(\cdot)$ exclusively depends on the share of expenditure devoted to transactions $\psi(\cdot)$, as

$$m = e\Psi(R) \Leftrightarrow (e_t/m)^2\psi'(e/m) = R.$$

Therefore, the loss of expenditures due to the exchange process (that we defined in Sect. 2) can be directly linked to the ousting of a share of deposits for building up reserves by the financial intermediary.

Remarkably, our financial specification allows us to obtain an explicit formal equivalence between the microfoundations of money demand and the transaction technology. Effectively, the transaction cost function $T(\cdot)$ can be identified as[29]

$$T(e, m) = e\left[c_0 + \int_1^{e/m} \frac{1}{s^2} c_1\left(\frac{s-1}{s}\right) \mathrm{d}s\right], \tag{52}$$

with $c(0) =: c_0$ being a constant of integration.

Equation (52) establishes, at a very general level, the observational equivalence between any convex cost function $c(\cdot)$ and any technology function $T(\cdot, \cdot)$.

Specifically, by defining the following cost function $c(x) = (\omega\mu/(1+\mu))(1-x)^{-1/\mu}$, the associated transaction technology can be computed as $\psi(e/m) = (\omega\mu/(1+\mu))(e/m)^{1/\mu}$, giving rise to the simple money demand with constant interest-elasticity that we have used in the paper, namely $m = e\Psi(R)$, with (as defined in Appendix A): $\Psi(R) = \bar{\omega}R^{-\mu/(1+\mu)}$. This specification corresponds to the usual one derived from standard MIUF or CIA models (see, e.g., Walsh 2010) and translates to the log–log demand for money defended by Lucas (2000), following Meltzer (1963) and the tradition of the inventory-theoretic approach of Baumol (1952) and Tobin (1956), namely

$$\ln(m/e) = \ln(\bar{\omega}) - \frac{\mu}{1+\mu}\ln(R), \tag{53}$$

where $\mu/(1+\mu)$ is the constant interest-elasticity of money demand.

By defining another cost function, such that:

$$c(x) = \frac{\lambda\sigma}{1+\sigma}(\bar{m} + x - 1)^{(1+\sigma)/\sigma},$$

with $\sigma > 1$, the corresponding transaction technology would be $\psi(e/m) = (\lambda\sigma/(1+\sigma))(\bar{m} - (m/e))^{(1+\sigma)/\sigma}$, giving rise to the following money demand:

[29] Indeed, by defining $x := m/e$, we have $\Psi(R) = x$, $\psi'(1/x) = x^2R$, and $\Psi(R) = 1 - c_1^{-1}(R)$. Consequently, we find that $\psi'(1/x) = x^2\Psi^{-1}(x) = x^2c_1(1-x)$; hence, $\psi'(s) = c_1((s-1)/s)/s^2$.

$m/e = \bar{m}\,(1 - (R/\lambda)^{\sigma})$. Taking logarithms, we have, for small interest rates

$$\ln(m_t/e_t) \simeq \ln(\bar{m}) - (R/\lambda)^{\sigma}. \quad (54)$$

Such a specification implies a variable interest-elasticity of money demand ($\sigma(R/\lambda)^{\sigma}/(1-(R/\lambda)^{\sigma})$), as in Rebelo and Vegh (1995) and Gillman and Kejak (2005), for example, rather than the constant one implied by typical inventory-theoretic models. Furthermore, with $\sigma = 1$, we obtain the money demand of Cagan (1956), namely: $\ln(m/e) \simeq \ln(\bar{m}) - (R/\lambda)$, with $1/\lambda$ being the interest semi-elasticity of money demand.[30] Therefore, our general specification (52) covers a wide range of money demand functions.

Clearly, our results concerning the determinacy of the BGPs in Sect. 5 are insensitive to money demand specification (53) or (54), as the sign of the determinant in the reduced-form (43) only depends on the sign of $\zeta\left(g_k^*\right)$. Consequently, our analytic results are robust to a very general set of functional specifications based on explicit microfoundations of money demand and transaction costs. These findings underline the generality of our argument that deficit monetization can be a reliable policy tool for supporting economic growth and reducing indeterminacy.

7 Conclusion

In a growth setup, the interaction between money and public debt can generate multiplicity and indeterminacy. In our model, multiplicity refers to the coexistence of two achievable BGPs in the long run: a high BGP and a low BGP. Indeterminacy refers to the transition path toward the high BGP, which is locally indeterminate for a large range of parameters.

Overall, from an economic policy standpoint, our results provide two new motivations for monetizing deficits.

On the one hand, along the high BGP, monetization avoids (or limits) the crowding-out effect of public debt on productive public expenditure in the long run. Usually, monetization is defended for providing seigniorage revenues, or because inflation surprises can reduce the cost of capital. Yet, seigniorage revenues are fairly small, and inflation surprises cannot be perpetuated in rational expectation equilibria; thus, our motivation for monetizing deficits to increase public spending might be stronger. Indeed, our model highlights a composition effect in public finance, namely the substitution of a non-interest-bearing asset (money) for public debt in the Government's budget constraint. This change in the composition of government liabilities generates a less distortionary way of finance for productive public expenditure. Nevertheless, monetization also produces distortions, by increasing transaction costs and the associated inflation tax. Hence, its positive effect on growth only holds if the interest-elasticity of

30 As discussed in Lucas (2000) and Ireland (2009), such a specification describes a very different behavior of money demand at low interest rates, relative to the log–log formulation. As the nominal interest rate approaches zero, the latter implies that real balances become arbitrarily large, while the former implies that real balances reach the finite satiation point $\bar{m}$, as in footnote 18 above. This may have strong implications for the optimum quantity of money and the welfare cost of inflation (Wolman 1997).

money demand is sufficiently low. Therefore, monetization can be viewed as a policy support, rather than the ultimate tool for promoting long-run economic growth.

On the other hand, with transaction costs on investment, transitional dynamics in the neighborhood of the high BGP crucially depend on the degree of monetization. For "small" monetization rates, the low BGP is locally determined (saddle path), but the high BGP becomes locally undetermined. However, for sufficiently high monetization rates, both BGPs are characterized by the saddle-path property and are locally determined. Thus, a large dose of monetization might allow avoiding, whenever present, BGP indeterminacy.

Our findings are consistent with several significant results in the literature, illustrating the importance of the transaction technology in generating long-run multiplicity and/or indeterminacy of perfect-foresight equilibria. Yet, we provide an original mechanism according to which deficit monetization could be used as a selection device to solve indeterminacy and obtain a unique transition path. Furthermore, this mechanism is established for a very general, microfounded, approach to transaction costs. In this way, our setup provides useful insights to explore the role of monetary policy as a tool for macroeconomic stabilization.

Certainly, the issue of deficit monetization deserves future research. One strand of work could explore how endogenous taxes, in addition to monetization, impact the deficit-growth relationship, and act as a potential second source of multiplicity or indeterminacy. Moreover, one could examine in greater detail the type of public spending financed by deficit monetization, e.g. by considering public capital, differentiating between productive and unproductive public expenditure, or accounting for public-financed human capital (in the spirit of Bond et al. 1996), possibly in a two-sector model. Finally, our setup provides an appropriate environment for studying the dynamic strategic interaction between monetary and fiscal policies in a context of growing public debt.

Acknowledgements We are extremely grateful to Nicholas Yannelis (Editor) and to two anonymous referees, for extensive comments on a previous version of the paper. We benefited from many comments from C. Aubin, T. Azomahou, E. Bond, J.-B. Chatelain, J.-L. Combes, M. Copaciu, C. Cornand, F. Dufourt, C. Litan, V. Midrigan, M. Mutascu, T. Palivos, C. Pintilescu, T. Seegmuller, A. Venditti, M. Voia, P. Wang, T. Xepapadeas, and C. Yip, and especially from F. Magris and D. Onori. We thank participants to conferences GDRE (Clermont-Ferrand and Nice), GEBA (Iasi), Journées LAGV (Marseille), and to the seminar at LEO (Orléans). Usual disclaimers apply.

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