Econ Theory (2018) 65:751–780
https://doi.org/10.1007/s00199-017-1038-z

RESEARCH ARTICLE

Mid-auction information acquisition

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Received: 11 September 2015 / Accepted: 17 February 2017 / Published online: 3 March 2017

Abstract An English auction is studied in which bidders can acquire information during the bidding process, allowing for heterogeneity both in ex-ante private information and the cost of information acquisition. The best response has a simple characterization where the optimal information acquisition time is unaffected by the other bidders' strategies. We prove the existence of an equilibrium in a novel way by characterizing it as a fixed point in the space of bid distributions rather than the space of bid functions. Furthermore, we show that when bidders have homogeneous ex-ante private information about valuations: (1) The English auction generates more revenue than the Vickrey auction when the number of bidders is sufficiently large; and (2) the English auction is more efficient than the Vickrey auction when the information acquisition cost are relatively small. We present numerical simulations that show that these effects can be large. Our findings provide an additional explanation for the popularity of the English auction, even in settings where the bidders' valuations are independent.

Keywords Dynamic auctions · Information acquisition · English auctions

JEL Classification D44 · D83

1 Introduction

This paper studies a continuously ascending price independent private values auction where bidders are allowed to acquire further information about the value of a good during the game.

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Given the fundamental role that asymmetric information plays in auction theory, developing ways to make information acquisition endogenous is a very natural agenda for research; we contribute to the literature by examining mid-auction information acquisition. In some contexts, the bidding process is slow enough that bidders may have the opportunity to invest in due diligence during the auction; for example, some of the spectrum auction took up to 80 days, and this time frame makes a mid-auction information acquisition strategy practically feasible. Alternatively, the mid-auction information acquisition process may be interpreted more broadly to represent the cognitive process of mentally focusing to obtain a clearer assessment of one's own willingness to pay for the object being auctioned; under this interpretation, mid-auction information acquisition may play a role in virtually any dynamic auction.

This paper presents a model of an English auction with continuously ascending prices. We characterize the optimal information acquisition strategy for bidders that are heterogeneous about their information acquisition cost, their valuation for the good, and possibly their ex-ante information about their valuation. An insight from the analysis is that the optimal timing of the information acquisition depends on the bidder's own information, but not on the beliefs about other players strategies. There is a one-to-one mapping between the time of information acquisition and the bidder's personal cost of information that does not depend on the equilibrium behavior of other players.

We provide a proof of existence of an equilibrium in a novel way by applying Schauder's fixed-point theorem on the set of bid distributions, rather than bid functions. This method of proof can be helpful in models with multidimensional types and single dimensional action spaces, since it reduces the dimensionality of the space under consideration.

We also study how the expected revenue of this auction compares with a one-shot auction where bidders are prevented from acquiring information during the auction. We present a theorem that states that the revenue of the English auction is larger provided that the number of bidders is sufficiently large and there is no heterogeneity on the bidders ex-ante information about valuations.

We also study efficiency gains. The analysis is involved because with endogenous information acquisition the two auctions generate different allocations and different levels of information acquisition costs. We show that for moderate levels of information acquisition costs, the English auction is more efficient.

We also present simulations that show that gains in revenue and efficiency can be large in some circumstances (up to 6 and 1%, respectively).

Our findings suggest that mid-auction information acquisition may be an additional reason why English auctions seem to be so much more popular than their one-shot counterpart, the Vickrey auction (Compte and Jehiel 2007). The literature on auctions provides several alternative explanations of this fact. Milgrom and Weber (1982) have shown that under affiliation a dynamic English auction generates more revenue than an auction under one-shot, sealed-bid rules. An English auction may also be preferred because it is more immune to manipulation by the auctioneer: In a Vickrey auction, a dishonest auctioneer can profit by introducing a fake bid between the winner and the second highest bid. Any attempt of doing so in an English auction would involve the risk of not selling the good.

An alternative explanation for why dynamic auctions might be useful is related to the fact that bidders often have to invest in acquiring information about the value of the item being traded. In circumstances where this information acquisition is important, dynamic auctions may be preferable because they may lead to more aggressive bidding and/or participation, and therefore, to higher final selling prices.

Engelbrecht-Wiggans (1988) has attempted to formalize this insight by working with two-stage auctions. However, the analysis of two-stage auctions is quite difficult, and hence, the existing results have been limited to very restrictive functional assumptions. Similarly, Compte and Jehiel (2007) investigate the hypothesis in a context of an ascending auction with discrete bids and valuations distributed over a finite support. They establish that expected revenue in the dynamic auction is larger than in the one-shot auction in their setting, but do not provide a full characterization of optimal bidding. Our aim in this paper is to propose a more flexible and tractable way to capture this economic intuition.

The present analysis does not impose unusual restrictions on the valuation distribution. Furthermore, the information structure allows bidders to have different initial signals of their valuation and different privately known costs of acquiring information. In addition, the framework accounts for the possibility that some bidders already know their valuation at the outset of the auction, or conversely, that some bidders may not be able to acquire any information at all. We permit these possibilities by allowing bidders to be heterogeneous along three dimensions: their valuation for the item, their ex-ante information about the valuation, and the information acquisition costs.

A methodological contribution of this paper is that it proposes a novel way to characterize and prove the existence of an equilibrium in private value auctions. In this class of games, the payoff-relevant aspect of the equilibrium strategy profile of the other players for any given bidder is the distribution of the highest bid (or the lowest bid, in the case of procurement). This property allows us to characterize a symmetric equilibrium as a fixed point in the space of highest bid distributions, rather than in the space of strategies. This characterization is useful, as the former space is mathematically less complex: In the current model, the space of highest bid distributions is the space of increasing, right-continuous functions from a compact interval into [0, 1], and each strategy is a pair of mappings from two or three dimensions into the information acquisition time and the bid. Using this shift in perspective allows us to prove that an equilibrium exists in a novel way, using the Schauder fixed-point theorem.

The rest of the paper is structured as follows: The next section discusses the relationship of this paper with the literature on information acquisition in auctions. Section 3 presents the setup that is used throughout the paper. The problem of characterizing the bidder's best response is investigated in Sects. 4, and 5 contains a proof that an equilibrium for this game exists. Section 6 presents a simplified version of the game where bidders have homogeneous ex-ante private information about valuations. In this setting, we compare the equilibrium of the dynamic auction with the one-shot sealed-bid auction. We find that the dynamic auction provides larger expected revenues when the number of bidders is sufficiently large and is more efficient when information acquisition costs are moderate. Section 7 provides results of numerical simulations that show that the revenue gain may be substantial even with as few as two bidders. Section 8 provides some concluding remarks.

2 Connections with the literature

In most of the literature on information acquisition in auctions, the analysis is restricted to ex-ante information acquisition. Matthews (1984) and Persico (2000) study models where bidders can purchase information out of a continuum of alternative degrees of informativeness. These authors resolve the non-trivial problem of ranking distributions in terms of informativeness in different ways. Due to the simple structure of the information acquisition problem that is examined in this paper, this ranking is immediate here. On the other hand, the present model allows for information acquisition at any moment in a continuous time game, whereas these papers consider only information acquisition prior to the auction.

In addition to Engelbrecht-Wiggans (1988), two other papers, namely Compte and Jehiel (2007) and Rasmusen (2006), have independently investigated the theme of mid-auction information acquisition in ascending auctions. Compte and Jehiel (2007) consider an English auction where a set of bidders begin the auction informed and another set begin uninformed, having a choice to learn their valuations during the game. They also study a symmetric model where any bidder has an exogenous probability q of being uninformed.[1] Unlike in this paper, in Compte and Jehiel (2007) there is no heterogeneity in the information acquisition costs or in the prior beliefs among the bidders. Conversely, they allow bidders to observe the rivals' drop-out points; here, we limit the analysis to an auction format where this information is not available. Another major difference is that their setting is discrete (both in terms of actions and distributions) and our setting is continuous.

Finally, subsequent work by Gretschko and Wambach (2014) and Miettinen (2013) study mid-auction information acquisition in Dutch auctions. Gretschko and Wambach (2014) investigate Dutch auctions in the same setting that is considered here and employ the analytical tools I propose in this paper to characterize and prove the existence of an equilibrium and obtain a revenue ranking: They show that if the cost of information acquisition is low, the descending price auction generates less revenue than the first-price auction. Their work is complementary to this paper: Taken together, the papers establish a theory of mid-auction information acquisition in all four classical auction formats under the same setting.

Miettinen (2013) characterizes the equilibrium for a situation where there is no heterogeneity in either the information acquisition cost or the prior information about valuations, and an exogenous fraction of the bidders have already acquired information before the auction begins. In the equilibrium described by Miettinen, bidders randomize their information acquisition point over a region of prices over which the informed bidders do not bid. He also shows that in this setting, as the number of bidders becomes large, the Dutch auction produces more revenue than the first-price auction.

[1] In a related paper, Compte and Jehiel (2004) study an environment where one bidder obtains better information as time proceeds, but this work is less closely related to the current paper than Compte and Jehiel (2007) since information acquisition is exogenous. Rasmusen (2006) study information acquisition in a more restricted setting with two bidders and a specific distributional assumption for the value of the good.

3 Setup

We study an auction of a single good among n bidders with symmetric and independent private valuations. Valuations are i.i.d. random variables $v_1, \ldots, v_n$. The distribution function of v_i is F_v, and this function is absolutely continuous with support $[0, \bar{v}]$.

The auction rules are also conventional: Our model for the English ascending price auction is a blind Japanese button auction, where the price p begins at a low level (which is assumed without loss of generality to be 0) and increases continuously. The bidders should decide at which price to drop out. The auction ends when only one bidder is left, and he or she pays the price at which the last of the other bidders dropped out. If all the remaining bidders drop out at the same time, the winner is selected at random, i.e., each of these bidders has an equal probability of winning. The auction is “blind”: During the auction, a bidder only observes the price clock and does not observe any action of the other players (neither if and when they drop out nor if and when they acquire information).

Each bidder can have two possible levels of information about her own valuation: They observe an informative signal w_i. At any point during the auction, she can learn v_i instantaneously at a cost c_i. There is heterogeneity on the information acquisition cost: the costs c_i are i.i.d. F_c and are private information of each bidder.

We assume that the conditional expectation of v_i is strictly increasing in w_i; without further loss of generality, we set $w_i = E[v_i|w_i]$. Bidders may be heterogeneous or homogeneous regarding their ex-ante prior information about valuations. In Sects. 4 and 5, we assume the conditional expectations w_i are i.i.d. F_w. In Sect. 6, we consider the simpler case were bidders have homogeneous prior beliefs about their valuations, in which case F_w is degenerate.

Both c_i and w_i are known by player i in the beginning of the game, but are not directly observed by anyone else. It is common knowledge that all w_i are i.i.d. F_w and the c_i are i.i.d. F_c. The distribution of v_i conditional on w_i is represented by $F_{v|w}$, and a similar notation is used for the other conditional distribution functions. Except for w_i and v_i, all other variables are assumed to be independent.[2]

It is convenient to impose some further assumptions on the distributions of c_i, w_i and v_i. They have bounded intervals as supports; the support of v_i is $[0, \bar{v}]$ and the support for c_i is $[0, \bar{c}]$. The distribution of c_i has an atom at 0: this feature accommodates bidders that already have all the information at the outset of the auction in the same framework. Conversely, if the maximum information acquisition cost $\bar{c}$ is very high, the framework accommodates bidders that cannot acquire information. Additionally, the presence of an atom at 0 is used in proving the existence of an equilibrium in Sect. 5. Apart from the atom at 0, the distributions of c_i and v_i are assumed to have densities that are bounded above and away from zero everywhere in the support.

When there is heterogeneity in w_i (Sects. 4, 5), we assume F_w is absolutely continuous, with density bounded above and away from zero everywhere in the support. We also assume that the maximum of the support of w_i is strictly below $\bar{v}$. Bounds on

[2] Independence across different bidders is an important simplifying assumption. Independence between c_i and w_i or v_i is not important, and the analysis would not change much without it.

densities are not required to characterize the equilibrium in Sect. 4, but will be used to prove existence of equilibrium in Sect. 5 and to study its properties in Sect. 6.

During the auction, the only information about the behavior of other players that a bidder observes is whether all of them have dropped out or not, i.e., if the auction has ended or not. We conjecture that the assumption on the unobservability of drop-out points does not qualitatively affect the analysis (even though it greatly simplifies part of it). Notice that due to the independent private values assumption, knowledge of another player's drop-out point/valuation does not change i's estimate of her own v_i, and hence, a linkage effect as in Milgrom and Weber (1982) is not expected to exist.

Assuming information acquisition is unobservable is possibly not innocuous, as direct observation of the information acquisition point can in principle generate additional strategic effects that have not been accounted for in the present model.

4 The individual bidder's problem

We begin by studying the individual bidder's problem taking as given the behavior of the other bidders. We fix a specific player i and whenever it is clear from the context we drop the subscript i in the notation and relabel $v_i = v$, etc.

It is convenient to summarize the other bidders' behavior in a reduced-form fashion. Let the random variable y represent the price at which the last of the other bidders drops out if player i were to stay in the game forever. The price y is the price bidder i will pay for the item if she wins.

The price y is a function of (c_{-i}, w_{-i}, v_{-i}), a random variable that is independent of the bidder's private information on c_i and w_i (and v_i). Let the distribution of y be F_y. During the auction, no information about y is obtained during the auction except that y is greater than the current price.

The possible strategies of the bidder can be split into two groups: either she decides never to acquire information and place a bid that depends only on w_i, or she decides to wait until a price $\hat{p}$ is reached and acquire information at this point. Under this formulation, $\hat{p} = 0$ represents immediate information acquisition and $\hat{p} = \infty$ represents never acquiring information and never dropping out of the auction. Thus, all pure strategies are represented by a tuple of four choices as a function of the bidder private information: a binary decision about acquiring information; a choice of information acquisition timing $\hat{p}$; a choice of when to drop out if information is not acquired; and a choice of when to drop out if information is acquired, as a function of the observed value. Figure 1 presents a pure strategy graphically.

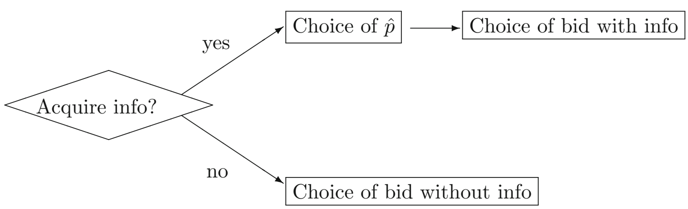

Fig. 1 A schematic representation of a bidder's strategy

The bidding decision in this game, as in standard English auctions, is strategically simple: It is always weakly dominant to stay in the auction as long as the conditional expectation of the value of the item is above the current price and drop out otherwise. If the bidder never acquires information, she will drop out at w. If she acquires information, she will drop out at v, or immediately, if she finds that $v < \hat{p}$.

The novel aspects of the analysis are the determination of the optimal information acquisition point $\hat{p}$ and the decision whether to acquire information or not. Sections 4.1 and 4.2 present each in turn.

4.1 Optimal timing of information acquisition

The task in this section is to find the function $\hat{p}(w, c, F_y)$, which is defined as the optimal information acquisition point for a bidder that observes a signal w of her valuation, faces a cost c to acquire information, and expects the other bidders to behave in such a way that the highest price at which any of them stays in the auction is distributed according to F_y.

We have the following characterization:

Proposition 1 *Let $\hat{p}$ satisfy*

$$c = \int_0^{\hat{p}} (\hat{p} - v)\, \mathrm{d}F_{v|w}(v) = \int_0^{\hat{p}} F_{v|w}(v)\, \mathrm{d}v.$$

Then $\hat{p}$ is an optimal information acquisition point.

Proof First, note that since $F_{v|w}$ is absolutely continuous, integrating by parts one obtains $\int_0^t (t - v)\, \mathrm{d}F_{v|w}(v) = \int_0^t F_{v|w}(v)\, \mathrm{d}v$ for any $t \in [0, \bar{v}]$. This establishes the second equality in the formula. Note that $\int_0^t (t - v)\, \mathrm{d}F_{v|w}(v)$ is strictly increasing in t.

If the distribution of the highest bid among the other players is F_y, then the expected utility of a bidder that decides to wait until p to acquire information and afterward acts optimally is:[3]

$$U(p) = \int_0^p E\left[v - y|w\right] \mathrm{d}F_y(y) + \int_p^{\bar{v}} \left(E\left[\max\{v - y, 0\}|w\right] - c\right) \mathrm{d}F_y(y);$$

if $y < p$, the bidder receives $v - y$; if $y \geq p$, she pays c and learns v before the end of the auction, having the option to quit if $v < y$.

For $p < p'$,

[3] I am indebted to an anonymous referee for suggesting a proof based on that function.

$$
\begin{aligned}
U(p') - U(p) &= \int_p^{p'} \{E[v - y|w] - E[\max\{v - y, 0\}|w] + c\}\, \mathrm{d}F_y(y) \\
&= \int_p^{p'} \left\{ c + \int_0^{\bar{v}} (v - y)\mathrm{d}F_{v|w}(v) - \int_y^{\bar{v}} (v - y)\mathrm{d}F_{v|w}(v) \right\} \mathrm{d}F_y(y) \\
&= \int_p^{p'} \left\{ c - \int_0^{y} (y - v)\mathrm{d}F_{v|w}(v) \right\} \mathrm{d}F_y(y) \\
&= \int_p^{p'} \left\{ c - \int_0^{y} F_{v|w}(v)\, \mathrm{d}v \right\} \mathrm{d}F_y(y).
\end{aligned}
$$

The integrand is 0 when $y = \hat{p}$, negative if $y > \hat{p}$ and positive if $y < \hat{p}$. Therefore, $U(p) - U(\hat{p}) \leq 0$ for all p, and the inequality is strict if there is a positive probability that y falls between p and $\hat{p}$. □

A remarkable property of the optimal information acquisition point is that it does not depend on the behavior of other players; the proof above did not require any assumption about the distribution of y. It is also the unique optimum, provided there is a positive probability of bidding in any neighborhood of $\hat{p}$.

To see why the optimal information acquisition point is independent of the behavior of other players, consider the trade-off involved in deciding to delay the information acquisition over a given period. Due to the independent private values assumption, actions of other players during this period do not provide any information about the bidder's own valuation, but may affect expected payoffs since they may lead to the auction ending during this period. If the auction does not end, it makes no difference delaying the information acquisition. If it ends, then it makes a difference: Deciding to delay the time to acquire the bidder has the benefit of saving the acquisition cost c, while running the risk of later learning that the item is not worth the price paid. Notice, however, that both the expected benefit and the expected cost are proportional to the probability of the auction ending; hence, the latter does not affect which one is greater.

From Formula 1, one immediately obtains that $\hat{p}$ is a strictly increasing function of c, and that if $c > 0$, $\hat{p} > 0$: A bidder always finds it optimal to wait to acquire information, unless the information acquisition cost is zero.

One can also use the formula to investigate comparative statics regarding w. For example, if w is good news about v (that is, if $w > w'$, then $F_{v|w}$ first-order stochastically dominates $F_{v|w'}$; Milgrom 1981), then $\hat{p}$ is increasing in w.

4.2 Whether to acquire information: an option trade

To investigate whether a bidder should acquire information, we compare the expected utility of acquiring information with the expected utility of participating in the auction without learning v. The former was calculated in Sect. 4.1 to be

$$
U(\hat{p}) = \int_0^{\hat{p}} E[v - y|w]\, \mathrm{d}F_y(y) + \int_{\hat{p}}^{\bar{v}} (E[\max\{v - y, 0\}|w] - c)\, \mathrm{d}F_y(y);
$$

the latter is $\check{U} = \int_0^w E[v - y|w]\,\mathrm{d}F_y(y)$, since conditional on y the utility of the bidder is $E[v - y|w]$ if $y < w$ and 0 otherwise.

It is convenient to write the net expected benefit of acquiring information as the expectation of a function of y:

$$U(\hat{p}) - \check{U} = \int r(y; c, w)\mathrm{d}F_y(y)$$

where

$$r(y; c, w) = \begin{cases} (w - y) - (w - y) = 0 & \text{if } y < \hat{p} \\ \int_y^{\bar{v}} (v - y)\mathrm{d}F_{v|w} - c - \int_0^{\bar{v}} (v - y)\mathrm{d}F_{v|w} = \int_0^y F_{v|w} - c & \text{if } y \in [\hat{p}, w) \\ \int_y^{\bar{v}} (v - y)\mathrm{d}F_{v|w} - c = \int_0^y F_{v|w} - c + w - y & \text{if } y > w \end{cases}$$

or more compactly

$$\begin{aligned} r(y; c, w) &= \max\left\{\int_0^y F_{v|w}(s)\mathrm{d}s - c, 0\right\} - \max\{y - w, 0\} \\ &= \max\left\{\int_{\hat{p}}^y F_{v|w}(s)\mathrm{d}s, 0\right\} - \max\{y - w, 0\} \end{aligned}$$

The decision whether to acquire information is an option trade on the underlying asset y. The expression $\int r\mathrm{d}F_y$ is the expected profit of an option strategy composed by selling a call option on y at the strike price w and buying a call option on $\int_0^y F_{v|w}(s)\mathrm{d}s$ with price c. Figure 2 shows the shape of the r function: It is an asymmetric spread that pays if y is close to w, has a negative value if y is too high, and has a value of 0 if y is too low. This shape suggests that information acquisition depends negatively on the variance of y with respect to $v|w$.

For comparison purposes, Fig. 2 also depicts the shape of the option r_0 that would determine the information acquisition decision of a bidder in a one-shot auction. In that case, the expected gain from learning about v is $\int r_0(y; c, w)\mathrm{d}F_y(y)$, where

$$r_0(y; c, w) = \int_0^y F_{v|w}(s)\mathrm{d}s - c - \max\{y - w, 0\}.$$

Comparing r and r_0, one sees that the possibility of acquiring information during the auction provides a value to the bidder: If $y < \hat{p}$, she can save on the information acquisition cost.

Figure 3 represents schematically a bidder's best response in this game. The strategies of the opponents only affect the decision whether to acquire information (this decision should be affirmative if $\int r(t, c, w)\mathrm{d}F_y > 0$). After this decision has been made, the best response prescribes behavior that is unaffected by the rivals' strategies: If information is going to be acquired, it should happen when the auction price reaches the price $\hat{p}$ that solves $c = \int_0^{\hat{p}} (\hat{p} - v)\mathrm{d}F_{v|w}$, and the bidder will either drop out immediately (if $v < \hat{p}$) or stay in the auction until the price reaches v. If

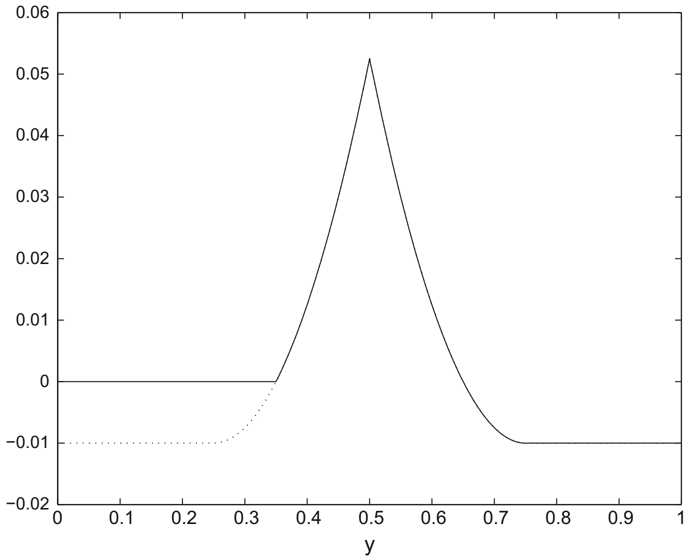

Fig. 2 A Graph of the r function (the *solid line*) and the r_0 function (the *dotted line*), for the case where $v|w \sim U[0.25, 0.75]$ and $c = 0.01$

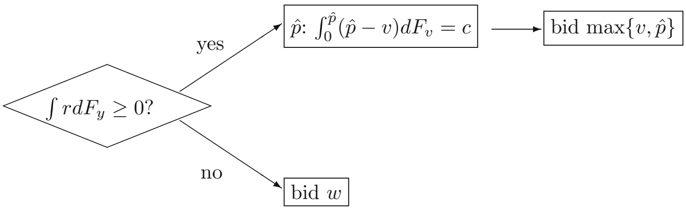

Fig. 3 A schematic representation of a best response strategy

$\int r(t, c, w)\mathrm{d}F_y < 0$, then the bidder never acquires information and drops out when the price reaches $w = E[v|w]$.

5 Existence of an equilibrium

Section 4 described the single best response strategy against any profile of the opponents' strategies that lead to an absolutely continuous distribution of drop-out points. However, this is not sufficient to guarantee that a symmetric equilibrium exists; we must still show there exists a distribution of drop-out points that leads bidders to bid in a way that generates itself.

This section provides an existence result for the case where there is a positive probability that bidders have zero information acquisition cost (Proposition 3). Readers uninterested in the technical aspects of the proof may skip to Sect. 6.

There is an extensive literature on the existence of equilibria in auction games. They can be divided into lattice-theoretic methods (e.g., Topkis 1979; Milgrom and Shannon

1994; Athey 2001; McAdams 2003) and topological methods that allow for specific types of discontinuity (e.g., Dasgupta and Maskin 1986; Reny 1999; Simon and Zame 1990; Jackson and Swinkels 2005). These methods are not directly applicable to our model, as the first group requires monotonicity properties that are not valid in our setting and the second group focuses on discontinuity problems that do not arise here because our assumptions make a bidder's expected profit continuous. The issue here is whether one can find a set of types A that is willing to acquire information given that types in A are acquiring information. We shall proceed directly by defining an operator over this set and applying a fixed-point theorem from functional analysis.

Let $\mathcal{F}$ be the set of all absolutely continuous distributions over $[0, \bar{v}]$, and let $\mathcal{A}$ be the collection of all measurable subsets of types (c, w).

We define two mappings between these spaces. $T : \mathcal{A} \to \mathcal{F}$ gives the distribution of y_i that would arise if a bidder was acquiring information if her type was in A, i.e.,

$$\begin{aligned} T(A)(x) &= \Pr[A] \Pr\left[\max\{\hat{p}, v\} \le x | A\right] + (1 - \Pr[A]) \Pr\left[w \le x | A^c\right] \\ &= \int_A \Pr\left[\max\{\hat{p}, v\} \le x | c, w\right] \mathrm{d}F_{c,w} + \int_{A^c} \Pr\left[w \le x | c, w\right] \mathrm{d}F_{c,w}. \end{aligned}$$

Notice that $T(A) \in \mathcal{F}$, as it is a mixture of absolutely continuous distributions. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ under the sup norm.[4]

Define $R : \overline{\mathcal{F}} \to \mathcal{A}$ as follows:

$$R(F) = \left\{ (c, w) \mid \int r(t, c, w) \mathrm{d}F^{n-1}(t) \ge 0 \right\},$$

where $r(t, c, w) = \max\left\{\int_0^t F_{v|w}(s)\mathrm{d}s - c, 0\right\} - \max\{t - w, 0\}$. For an absolutely continuous distribution F, $R(F)$ selects the best response A to it. Notice that because any distribution function is of finite variation and r is continuous with respect to y, the integral $\int r \mathrm{d}F^{n-1}$ is well defined, and for a sequence $F_k \to F$, $\lim_{k\to\infty} \int r \mathrm{d}F_k^{n-1} = \int r \mathrm{d}F^{n-1}$ by Helly's second theorem (e.g., Natanson 1961).

Using this notation, the object that we need to find to obtain a symmetric equilibrium is a distribution $F^* \in \mathcal{F}$ such that the information acquisition decisions that are consistent with it do generate it; that is, we need to find a fixed point

$$F^* = T\left(R(F^*)\right).$$

It is convenient at this point to impose bounds on the densities of w, c and v:

Assumption 1 (*density bounds*) Assume that the distributions of w and v are absolutely continuous, the distribution of c is absolutely continuous except possibly for an atom π at 0 and there are positive constants M_c, M_w, m_v and M_v such that $f_w \le M_w$, $m_v \le f_v \le M_v$, and $f_c \le M_c$ everywhere.

[4] $\overline{\mathcal{F}} \backslash \mathcal{F}$ contains continuous distribution functions with a singular part.

A set is *relatively compact* if it is a subset of a compact set, and a continuous application is a *compact map* if its image is relatively compact. We seek to apply the Schauder fixed point theorem, which states that a compact map from a closed, convex subset of a normed linear space onto itself has a fixed point (e.g., Brown 1993). The closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ is a closed, convex subset of the normed linear space of functions from $[0, \overline{v}]$ to $[0, 1]$. It remains to verify that $T \circ R$ is a compact map, i.e., a continuous map with a relatively compact image.

Lemma 1 *$T(\mathcal{A})$ is relatively compact.*

Proof In the Appendix. □

The continuity of the $T \circ R$ operator depends on the following condition:

Lemma 2 *If $F \in \mathcal{F}$ is such that* $\Pr\left\{(c, w) \mid \int r \mathrm{d}F^{n-1} = 0\right\}$ *is zero, then $T \circ R$ is continuous at F.*

Proof In the Appendix. □

An equilibrium would necessarily exist if one could restrict the analysis to distributions where $\Pr\left[\left\{(c, w) \mid \int r \mathrm{d}F^{n-1} = 0\right\}\right] = 0$. Unfortunately, this condition is not necessarily valid for distributions that concentrate mass in low values: because $r = 0$ for sufficiently low values of y (see Fig. 2), against these distributions a positive mass of types will be indifferent between acquiring information or not and Lemma 2 cannot be applied.

It is not hard to impose assumptions that avoid this technical problem. For example, suppose that with some positive probability π bidders start the game already knowing v. This assumption is equivalent to having an atom π in the distribution of c at 0, since bidders that start knowing v behave in exactly the same way as bidders with zero cost.

This assumption is sufficient for existence. For $\pi > 0$, define $\mathcal{F}_\pi = \{(1 - \pi)F + \pi F_v \mid F \in \overline{\mathcal{F}}\}$, where F_v is the (unconditional) distribution of v. The next proposition shows that we can restrict our attention to this set, and that $T \circ R$ is continuous there.

Proposition 2 *Suppose $F_c(0) = \pi > 0$. Then:*

1. *$T \circ R(\overline{\mathcal{F}}) \subset \mathcal{F}_\pi$, so if a fixed point exists, it will be in $\mathcal{F}_\pi$.*
2. *$T \circ R$ is continuous in $\mathcal{F}_\pi$.*

Proof In the Appendix. □

Then, we can state an existence result:

Proposition 3 *If $F_c(0) = \pi > 0$, a symmetric equilibrium exists.*

Proof After Proposition 2, it only remains to show that there exists a fixed point for the $T \circ R$ map in $\mathcal{F}$.

The set $\mathcal{F}_\pi$ is convex and closed. By Lemma 1 and Proposition 2, the restriction of $T \circ R$ on $\mathcal{F}_\pi$ is a compact map, and by Proposition 2 its image is in $\mathcal{F}_\pi$. Hence, the Schauder theorem applies, and therefore, a fixed point F^* exists in $\mathcal{F}_\pi$. As F^* is in the image of $T \circ R$, it also belongs to $\mathcal{F}$. □

6 Comparison with the Vickrey auction

In this section, we compare the welfare properties of an auction with mid-game information acquisition with the one-shot Vickrey auction: In particular, we want to investigate if the alternative of acquiring information during the game leads to more aggressive bidding and larger expected revenues for the seller.

In a Vickrey auction, bidders simultaneously post sealed bids. The winner of the auction is the bidder who placed the highest bid, and he or she pays the price of the second highest bid. Since all bidding occurs simultaneously, all information acquisition must necessarily happen before the auction. We continue to assume that players do not directly observe the information acquisition of other players. Therefore, the Vickrey auction is strategically equivalent to the dynamic auction we have been studying, with the added restriction that bidders can only acquire information at the start of the game.

To simplify the analysis, in this section we restrict attention to the case where all bidders share the same prior distribution regarding v (that is, w is the same for all bidders). The analysis of the individual bidder best response in Sect. 4 is unchanged in this case.[5]

We shall also assume in this section that the cost distribution has no atom at zero. In the case of homogeneous ex-ante private information, existence of an equilibrium is guaranteed with or without an atom, and ruling it out simplifies the exposition of the revenue comparison result.

The characterization of the equilibrium is simpler in this case: because the expected net gain from information acquisition is decreasing in the information acquisition cost, we know that the set of types that acquire information in equilibrium is $A = [0, c^*] \times \{w\}$; the set of types that acquire information is summarized by c^*, the highest cost type that acquires information in equilibrium.

Let $\alpha = F_c(c^*)$ be the ex-ante probability that a bidder acquires information in equilibrium. To characterize the equilibrium, it is sufficient to characterize α, since it holds a one-to-one relationship to c^*.

More precisely, once we know the equilibrium value of α the equilibrium of the game can be described as follows: the set of types that acquire information have cost below $c^* = F_c^{-1}(\alpha)$; they acquire information when the price reaches the value $\hat{p}(c)$ that solves $\int_0^{\hat{p}} F_v(v)\mathrm{d}v = c$ and they drop out at $\max\{v, \hat{p}\}$. Bidders with cost above c^* do not acquire information and drop out at w.

The following proposition provides an equation that characterizes α, and therefore, the equilibrium of this game.

Proposition 4 *For* $a \in [0, 1]$, *let*

$$Er(a; n) = -\int_{F_{\hat{p}}^{-1}(a)}^{w} a^{n-1} F_v(t)^n \mathrm{d}t + \int_{w}^{\bar{v}} [1 - a(1 - F_v(t))]^{n-1} (1 - F_v(t))\, \mathrm{d}t$$
$$+ \int_{F_{\hat{p}}^{-1}(a)}^{\bar{v}} F_v(t)\mathrm{d}t + w - \bar{v}.$$

[5] Optimal bidding will lead to a tie with positive probability at w, since at that price all bidders that do not buy information will simultaneously drop out. The specifics of the tie breaking rule are not important here, since all winning bidders have an expected utility of zero.

Then the equilibrium value of α *satisfies*

$$Er(\alpha, n) = 0.$$

Proof Let c^* be the highest cost type that acquires information in equilibrium, and let $\bar{p}$ be his or her optimal information acquisition time. He or she must be indifferent between acquiring information or not. Therefore, it must be that

$$\begin{aligned}
0 &= \int r(y; c^*, w)\mathrm{d}F_y(y) \\
&= \int_{\bar{p}}^{\bar{v}} \left(\int_{\bar{p}}^{y} F_v(t)\mathrm{d}t \right) \mathrm{d}F_y(y) - \int_{w}^{\bar{v}} (y - w)\mathrm{d}F_y(y) \\
&= \int_{\bar{p}}^{\bar{v}} F_v(t)\mathrm{d}t F_y(\bar{v}) - 0 - \int_{\bar{p}}^{\bar{v}} F_y(s)F_v(s)\mathrm{d}s \\
&\quad - \left((\bar{v} - w)F_y(\bar{v}) - 0 - \int_{w}^{\bar{v}} F_y(s)\mathrm{d}s \right) \\
&= \int_{\bar{p}}^{w} \left(1 - F_y(s)\right) F_v(s)\mathrm{d}s + w - \bar{v} + \int_{w}^{\bar{v}} F_y(s)\mathrm{d}s,
\end{aligned}$$

applying integration by parts and observing that $F_y(\bar{v}) = 1$.

Since this is the last bidder to acquire information, his or her decision is contingent on the behavior of other players during the period between $\bar{p}$ and $\bar{v}$, when all player have either already acquired information and decided to drop out at price v or decided to not acquire information and drop out at w. Therefore, the equilibrium distribution of the highest bid is easy to characterize over this range:

$$F_y(y) = \begin{cases} (\alpha F_v(y))^{n-1} & \text{if } y \in (\bar{p}, w) \\ (\alpha F_v(y) + (1 - \alpha))^{n-1} & \text{if } y \geq w \end{cases}$$

(For $y < \bar{p}$ the distribution is affected by information acquisition points, but does not affect the calculation of c^*.) Substituting $F_y(y)$ and $\bar{p} = F_{\hat{p}}^{-1}(\alpha)$ in the former expression yields the desired expression.[6] □

When $a = 0$, $Er(a; n) > 0$; in equilibrium the probability of information acquisition is always positive, since the information acquisition cost can be arbitrarily small. If $Er(a; n)$ is positive when $a = 1$, in equilibrium all bidders acquire information; this is a possibility if the highest possible information acquisition cost $\bar{c}$ is small. Otherwise, since $Er(a; n)$ is strictly decreasing in a, there is a unique value of α that satisfies equation $Er(\alpha; n) = 0$.

In the one-shot Vickrey auction, mid-auction information acquisition is not allowed, and this has two effects on the bid distribution: The distribution of bids below $\bar{p}$

[6] Since F_c and $\hat{p}(c)$ are one-to-one, the distribution of $\hat{p}$ is also one-to-one. We can write $\bar{p}$ as an increasing function $F_{\hat{p}}^{-1}(\alpha)$ of α.

changes, but also the set of types that acquire information shifts. Since $r \geq r_0$, we know that holding other player's behavior constant, the value of information is higher with mid-auction information acquisition, and more types acquire information. The next proposition shows that this is still true when we compare equilibria.

Proposition 5 *Let α_0, α be the equilibrium fraction of types that acquire information in the one-shot Vickrey auction and in the English auction, respectively. Then $\alpha_0 \leq \alpha$.*

Proof We prove the result by providing a characterization of α_0, analogous to the one we have for α. In a one-shot auction, the highest type c_0^* that acquires information is determined by the following equation:

$$\begin{aligned} 0 &= \int r_0(y)\mathrm{d}F_y \\ &= r_0(\bar{v}) - \int_0^w F_y(y)r_0'(y)\mathrm{d}y - \int_w^{\bar{v}} F_y(y)r_0'(y)\mathrm{d}y \\ &= \int_0^{\bar{v}} F_v(s)\mathrm{d}s - c_0^* - \bar{v} + w - \int_0^w F_y(y)F_v(y)\mathrm{d}y + \int_w^{\bar{v}} F_y(y)\left(1 - F_v(y)\right)\mathrm{d}y \end{aligned}$$

To explicitly state the effect of α_0 in this equation, as before we write $F_y(y) = (\alpha_0 F_v(y))^{n-1}$ for $y < w$ and $F_y(y) = (1 - \alpha_0\,(1 - F_v(y)))^{n-1}$ for $y > w$ and let $F_{\bar{p}}^{-1}(\alpha_0)$ satisfy $\int_0^{F_{\bar{p}}^{-1}(\alpha_0)} F_v(s)\mathrm{d}s = c_0^*$.

We then have the following equation characterizing the equilibrium of the one-shot auction:

$$\begin{aligned} Er_0(\alpha_0; n) &= -\int_0^w \alpha_0^{n-1}F_v(y)^{n-1}F_v(y)\mathrm{d}y + \int_w^{\bar{v}} \left[1 - \alpha_0\,(1 - F_v)\right]^{n-1}(1 - F_v)\mathrm{d}y \\ &\quad + \int_{F_{\bar{p}}^{-1}(\alpha_0)}^{\bar{v}} F_v(y)\mathrm{d}y + w - \bar{v} \\ &= 0 \end{aligned}$$

Comparing the expressions we find that for any $a > 0$, $Er(a; n) > Er_0(a; n)$, since the difference is $\int_0^{F_{\bar{p}}(\alpha)} \alpha^{n-1}F_v(y)^{n-1}F_v(y)\mathrm{d}y > 0$. Thus, $\alpha_0 \leq \alpha$. □

We now turn to the investigation of the asymptotic properties of the equilibria when the number of bidders goes to infinity. This is needed because our revenue comparison result applies to auctions with many bidders.

The next proposition shows that as the number of bidders grows, each bidder acquires less and less information, but the expected number of bidders that become informed grows higher.

Proposition 6 *As the number of bidders n grows to infinity, the equilibrium fraction of bidder types that acquire information in both games converge to zero at a rate slower than $1/n$: as $n \to \infty$, $\alpha, \alpha_0 \to 0$, but $n\alpha, n\alpha_0 \to \infty$.*

Proof The equilibrium fraction of bidders that acquire information in the one-shot auction $\alpha_0(n)$ solves the equation $Er_0(\alpha_0(n); n) = 0$, where

$$Er_0(a; n) = \int_w^{\bar{v}} (1 - a(1 - F_v(y)))^{n-1} (1 - F_v(y))\,dy - \int_0^w a^{n-1} F_v(y)^n dy - F_c^{-1}(a).$$

$Er_0(a; n)$ is a decreasing function of a. Since $Er_0(0; n) = \int_0^w F_v(y)dy > 0$, we know that $\alpha_0(n) > 0$, for all n. For any $a > 0$, $\lim_{n\to\infty} Er_0(a; n) = -F_c^{-1}(a) < 0$. Since $\alpha_0(n) \in [0, 1]$, a compact set, the sequence $\{\alpha_0(n)\}$ converges to 0, since no subsequence can converge anywhere else.

Suppose now that, counter to the claim, for all n, $n\alpha_0 \le M$, for a finite bound M. Then $(1 - \alpha_0(1 - F_v(y)))^{n-1} \ge (1 - M/n(1 - F_v(y)))^{n-1} \to e^{-M(1-F_v(y))} > 0$. Since $\alpha_0 \to 0$, that would imply that $\liminf Er_0(\alpha_0) \ge e^{-M(1-F_v(y))}$, a contradiction since $Er_0(\alpha_0) = 0$ by definition. We have thus proved the claim for the one-shot auction.

For the dynamic auction, the equilibrium fraction of bidder that acquire information $\alpha(n)$ is the solution to $Er(\alpha(n); n) = 0$, where

$$Er(a; n) = \int_w^{\bar{v}} (1 - a(1 - F_v(y)))^{n-1} (1 - F_v(y))\,dy - \int_{F_{\hat{p}}(a)}^w a^{n-1} F_v(y)^n dy - F_c^{-1}(a).$$

For $a > 0$, $\lim_{n\to\infty} Er(a; n) = -F_c^{-1}(a) < 0$, and $\alpha(n) \to 0$, by the same argument used for α_0. Since $\alpha_0(n) \le \alpha(n)$, α_n cannot converge faster than $1/n$, either. □

It is important to notice that the previous result is valid for the case where there is no atom at zero in the cost distribution. If a $F_c(0) = \pi > 0$, then α_0, $\alpha \to \pi$ and trivially $n\alpha_0$ and $n\alpha_0 \to \infty$.

The fact that the expected number of bidders that acquire information in equilibrium goes to infinity means that information acquisition is economically meaningful even when the number of bidders is large. For example, it implies that the expected revenue of both auctions converge to $\bar{v}$, rather than w.

6.1 Revenue

The next result is the main proposition of this section:

Proposition 7 *There exists n^* such that if the number of bidders is above n^*, the expected revenue in the English auction is larger than the one-shot Vickrey auction.*

Proof In the equilibrium of the English auction, with probability α a bidder drops out at $\max\{\hat{p}, v\}$ and with probability $1 - \alpha$ she drops out at w. Consider the random variable $X(a)$ where

$$X = \begin{cases} v & \text{prob. } a \\ w & \text{prob. } 1 - a \end{cases}$$

Since the distribution of $X(\alpha)$ is first-order stochastically dominated by the bid distribution, the expected revenue in the English auction is larger than $E\left[X(\alpha)_{(2:n)}\right]$, the expectation of the second order statistic of $X(\alpha)$.

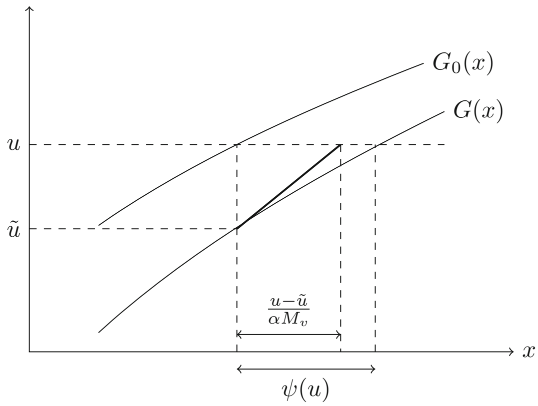

Fig. 4 A lower bound for $\psi(u) = G^{-1}(u) - G_0^{-1}(u)$

The equilibrium bid in the one-shot auction is $X(\alpha_0)$, that is, v with probability α_0 and w with probability $1 - \alpha_0$. In order to prove the result, we need to show that for sufficiently high n, $E\left[X(\alpha)_{(2:n)}\right] \geq E\left[X(\alpha_0)_{(2:n)}\right]$.

Ganuza and Penalva (2010) have shown that, if a random variable is greater than another in the convex order, then for sufficiently high n, the expectation of the second order statistic of the former is greater then the latter (Theorem 5 (ii); see also Board (2009), Proposition 2). It is easy to prove that $X(\alpha)$ is greater than $X(\alpha_0)$ in the convex order since $\alpha \geq \alpha_0$; however, our situation here is more delicate since α and α_0 depend on the number of bidders as well.

To complete the proof, instead of relying on this argument we explicitly compute a lower bound for the difference between expected revenues in the two games. As in the proof in Ganuza and Penalva (2010), we begin by expressing this difference as an integral in terms of quantiles:

$$E\left[X(\alpha)_{(2)}\right] - E\left[X(\alpha_0)_{(2)}\right] = \Delta ER = n(n-1)\int_0^1 \psi(u)u^{n-2}(1-u)\mathrm{d}u$$

where $\psi(u) = G^{-1}(u) - G_0^{-1}(u)$ and G and G_0 are the distributions of $X(\alpha)$ and $X(\alpha_0)$.

We obtain a lower bound for $\psi(u)$ as follows. For $u > 1-\alpha_0(1 - F_v(w))$, let $\tilde{u}$ be the quantile such that $G^{-1}(\tilde{u}) = G^{-1}(u)$ (see Fig. 4). Since $G(x) = 1-\alpha(1 - F_v(x))$ and $G_0(x) = 1 - \alpha_0(1 - F_v(x))$ in this range, $\tilde{u} = 1 - \frac{\alpha}{\alpha_0}(1 - u)$.

The function G^{-1} is differentiable between $\tilde{u}$ and u and the derivative is greater than $1/(\alpha M_v)$, where M_v is an upper bound for f_v. Therefore, $G^{-1}(u) \geq G^{-1}(\tilde{u}) + \frac{1}{\alpha M_v}(u - \tilde{u})$. Using $G^{-1}(\tilde{u}) = G_0^{-1}(u)$, we obtain a bound for $\psi(u)$:

$$\psi(u) \geq \frac{\alpha - \alpha_0}{M_v \alpha\, \alpha_0}(1 - u), \quad u > 1 - \alpha_0(1 - F_v(w)).$$

An analogous calculation when $u < \alpha F_v(w)$ leads to a similar lower bound for $\psi(u)$ in this range:

$$\psi(u) \geq \frac{\alpha - \alpha_0}{m_v \alpha\, \alpha_0} u, \quad u < \alpha F_v(w).$$

Finally, we know that for $u \in [\alpha F_v(w), 1 - \alpha_0 (1 - F_v(w))]$ we have $\psi(u) \geq 0$, since $G_0^{-1}(u) = w$ and $G^{-1}(u) \geq w$ in this range.

Therefore, we obtain a lower bound for $\Delta ER = E\left[X(\alpha)_{(2)}\right] - E\left[X(\alpha_0)_{(2)}\right]$:

$$\begin{aligned}
\Delta ER &= n(n-1) \int_0^1 \psi(u) u^{n-2}(1-u)\mathrm{d}u \\
&\geq n(n-1) \int_0^{\alpha F_v(w)} \frac{\alpha - \alpha_0}{m_v \alpha\, \alpha_0} u u^{n-2}(1-u)\mathrm{d}u \\
&\quad + n(n-1) \int_{1-\alpha_0(1-F_v(w))}^1 \frac{\alpha - \alpha_0}{M_v \alpha\, \alpha_0}(1-u)u^{n-2}(1-u)\mathrm{d}u \\
&= \frac{\alpha - \alpha_0}{(n+1)\alpha\alpha_0} \left[\frac{1}{M_v} - \frac{P}{m_v} (\alpha F_v(w))^n - \frac{Q}{M_v} (1 - \alpha_0 (1 - F_v(w)))^{n-1} \right]
\end{aligned}$$

where $P = (n-1)(n+1-n\alpha F_v(w))$ and $Q = n(n+1) - 2(n+1)(n-1)(1 - \alpha_0 (1 - F_v(w)) + n(n-1)(1 - \alpha_0 (1 - F_v(w)))^2$ are polynomials in n. Since the term outside the bracket is positive, the sign of ΔER is the sign of the term in brackets. From Proposition 6, we know that as $n \to \infty$, $(\alpha F_v(w))^n \to 0$ and $(1 - \alpha_0 (1 - F_v(w)))^{n-1} \to 0$ exponentially, and therefore, the term in brackets converges to $1/M_v > 0$. We conclude that for sufficiently high n, the expected revenue of the dynamic auction is larger than in the one-shot auction. □

It is surprising and perhaps counter-intuitive that Proposition 7 applies for auctions with many bidders. When the auction has many bidders, both the one-shot and the ascending auction generate revenue near the maximum of the valuation distribution; one should expect that the difference in expected revenue should go down as the number of bidder grows (as the numerical simulations in Sect. 7 illustrate). Why then assuming n is large is needed for the result?

The reason is that the key effect of more information acquisition is to make the bid distribution more spread out (technically, larger in the convex order). That, unfortunately, does not always translate to a larger revenue, because the expectation of the second order statistic does not necessarily go up when the underlying distribution spreads out when n is small (in fact, it goes *down* when $n = 2$).[7] Assuming that n is large allows us to guarantee that, as bidding becomes more aggressive at the top of the bid distribution, expected revenue goes up. Board (2009) and Ganuza and Penalva (2010) provide additional discussion of this issue.

[7] The simulations in Sect. 7 show that even with $n = 2$ the revenue of the English auction tend to be larger, since bidders bid aggressively at the beginning of the auction as well.

6.2 Efficiency

When comparing the dynamic and one-shot auctions, it is natural to ask also about how social welfare changes. Welfare may differ for two different reasons: Because the allocation of the item may change, and because the amount of resources spent in information acquisition may change.

When an auction rule change leads bidders to acquire more information, it becomes easier for the mechanism to select the bidder with the highest value, and therefore, allocative efficiency improves; but at the same time, there is more spending in information acquisition costs. These two effects tend to operate in opposite directions, and it is difficult to obtain general statements about which effect dominates.

A definite ranking on *net* allocative welfare can be obtained for the case where information acquisition costs are low enough so that all types prefer to eventually acquire information in equilibrium, that is, $\alpha = \alpha_0 = 1$.[8]

The following proposition establishes that in that case, the ascending auction is always more efficient that the one-shot auction: The expected cost savings are always larger than the loss in allocative efficiency.

Proposition 8 *Suppose the cost distribution is such that all bidders prefer to acquire information at the equilibria of the ascending auctions and the one-shot auction. Then, the ascending auction is more efficient: the expected valuation of the winner minus the expected sum of the information acquisition costs is larger in the ascending auction.*

Proof In an equilibrium of the ascending auction where all bidders eventually decide to acquire information, each bidder i stays in the auction until her information acquisition point $\hat{p}_i$ is reached, drops out immediately if $v_i < \hat{p}_i$ or continues until the price reaches v_i. Therefore, player i stays in the auction until the price reaches $b_i = \max\{v_i, \hat{p}_i\}$. If we order all values of b_i, the auction ends when the player with the second highest value for b_i is reached, and the winner is the bidder with the highest value for b_i.

Let $[1] = \text{argmax}_i b_i$ be the index of the winner, and [2] be the index of the bidder with the second highest value of b_i. The value of the allocation of the ascending auction is $v_{[1]}$, which is equal or less than $v_{(1)}$, the value of the allocation in the one-shot auction.

The expected difference in (gross) allocative efficiency is

$$\Delta AE = E\left[v_{[1]} - v_{(1)}\right] < 0;$$

on the other hand, the ascending auction allows some information acquisition cost savings. In the one-shot auction, all bidders pay their costs c_i upfront. In the ascending auction, all bidders except the winner eventually pay the cost as well, but the winner sometimes does not pay. More precisely, the cost saving happens when the auction ends before the price reaches the information acquisition point of the winner, that is, when $b_{[2]} < \hat{p}_{[1]}$. The expected difference in information acquisition costs is

[8] I am indebted to an anonymous referee for suggesting this line of investigation.

$$\Delta CS = E\left[c_{[1]}\mathbb{I}\left\{b_{[2]} < \hat{p}_{[1]}\right\}\right] > 0.$$

Our objective is to show that the net efficiency gain $\Delta CS + \Delta AE$ is positive.

We begin by characterizing the distribution of $v_{[1]}$ and $\hat{p}_{[1]}$. In an auction without heterogeneity in w, $\hat{p}_i$ is a function of c_i and is therefore independent of v_j, for all i, j. We have

$$\begin{aligned} P\left(\hat{p}_{[1]} \in [s, s+\epsilon), v_{[1]} \in [t, t+\epsilon)\right) \simeq\; & nP\left(\hat{p}_1 \in [s, s+\epsilon)\right) P\left(v_1 \in [t, t+\epsilon)\right) \\ & \times P\left(\hat{p}_2, \ldots, \hat{p}_n < \max\{s, t\}\right) \\ & \times P\left(v_1, \ldots, v_n < \max\{s, t\}\right); \end{aligned}$$

either $\hat{p}_{[1]}$ or $v_{[1]}$ is a maximum, but (usually) not both. Taking limits, we obtain an expression for the joint density:

$$f_{\hat{p}_{[1]}, v_{[1]}}(s, t) = nF_{\hat{p}}\left(\max\{s, t\}\right)^{n-1} F_v\left(\max\{s, t\}\right)^{n-1} f_{\hat{p}}(s) f_v(t).$$

The probability of the auction ending before the winner acquires information, conditional on $\hat{p}_{[1]}$ and $v_{[1]}$, can be derived as follows:

$$\begin{aligned} P(b_{[2]} < s | \hat{p}_{[1]} = s, v_{[1]} = t) = P(&\hat{p}_2, \ldots, \hat{p}_n, v_2, \ldots, v_n < s | \hat{p}_1 = s, v_1 = t, \\ &\hat{p}_2, \ldots, \hat{p}_n, v_2, \ldots, v_n < \max\{s, t\}) \\ = &\frac{F_{\hat{p}}(s)^{n-1} F_v(s)^{n-1}}{F_{\hat{p}}\left(\max\{s, t\}\right)^{n-1} F_v\left(\max\{s, t\}\right)^{n-1}} \end{aligned}$$

We are now in a position to explicitly compute the expected cost saving. Let $c_{[1]} = C(\hat{p}_{[1]}) = \int_0^{\hat{p}_{[1]}} F_v(t)\mathrm{d}t$; then

$$\begin{aligned} \Delta CS &= E\left[C(\hat{p}_{[1]})\mathbb{I}\left\{b_{[2]} < \hat{p}_{[1]}\right\}\right] \\ &= \int_0^{\bar{p}} \int_0^{\bar{v}} C(s) P\left(b_{[2]} < s | \hat{p}_{[1]} = s, v_{[1]} = t\right) f_{\hat{p}_{[1]}, v_{[1]}}(s, t)\mathrm{d}t\, \mathrm{d}s \\ &= \int_0^{\bar{p}} \int_0^{\bar{v}} C(s) \frac{F_{\hat{p}}(s)^{n-1} F_v(s)^{n-1}}{F_{\hat{p}}\left(\max\{s, t\}\right)^{n-1} F_v\left(\max\{s, t\}\right)^{n-1}} \\ &\quad \times nF_{\hat{p}}\left(\max\{s, t\}\right)^{n-1} F_v\left(\max\{s, t\}\right)^{n-1} f_{\hat{p}}(s) f_v(t)\mathrm{d}t\, \mathrm{d}s \\ &= \int_0^{\bar{p}} \int_0^{\bar{v}} C(s) F_{\hat{p}}(s)^{n-1} F_v(s)^{n-1} n f_{\hat{p}}(s) f_v(t)\mathrm{d}t\, \mathrm{d}s \\ &= \int_0^{\bar{p}} C(s) F_{\hat{p}}(s)^{n-1} F_v(s)^{n-1} n f_{\hat{p}}(s)\mathrm{d}s. \end{aligned}$$

The marginal density of $v_{[1]}$ is

$$
\begin{aligned}
f_{v_{[1]}}(t) &= \int_0^{\bar{p}} f_{\hat{p}_{[1]},v_{[1]}}(s,t)\mathrm{d}s \\
&= \int_0^{\bar{p}} nF_{\hat{p}}\left(\max\{s,t\}\right)^{n-1} F_v\left(\max\{s,t\}\right)^{n-1} f_{\hat{p}}(s) f_v(t)\mathrm{d}s \\
&= \int_0^{t} nF_{\hat{p}}(t)^{n-1}F_v(t)^{n-1}f_{\hat{p}}(s)f_v(t)\mathrm{d}s \\
&\quad + \int_t^{\bar{p}} nF_{\hat{p}}(s)^{n-1}F_v(s)^{n-1}f_{\hat{p}}(s)f_v(t)\mathrm{d}s \\
&= nF_{\hat{p}}(t)^n F_v(t)^{n-1} f_v(t) \\
&\quad + f_v(t)\int_t^{\bar{p}} nF_{\hat{p}}(s)^{n-1}F_v(s)^{n-1}f_{\hat{p}}(s)\mathrm{d}s
\end{aligned}
$$

in the interesting region $t \in [0,\bar{p})$. In the region $t \in [\bar{p},\bar{v}]$, $f_{v_{[1]}}(t) = nF_v(t)^{n-1}f_v(t) = f_{v_{(1)}}(t)$.

The difference in allocative efficiency is

$$
\begin{aligned}
\Delta AE &= \int_0^{\bar{v}} t\left(f_{v_{[1]}}(t) - f_{v_{(1)}}(t)\right)\mathrm{d}t = \int_0^{\bar{p}} x\left(f_{v_{[1]}}(t) - f_{v_{(1)}}(t)\right)\mathrm{d}t \\
&= \int_0^{\bar{p}} t\left\{\left(F_{\hat{p}}(t)^n - 1\right)nF_v(t)^{n-1}f_v(t)\right. \\
&\quad \left. + f_v(t)\int_t^{\bar{p}} nF_{\hat{p}}(s)^{n-1}F_v(s)^{n-1}f_{\hat{p}}(s)\mathrm{d}s\right\}\mathrm{d}t \\
&= 0 - 0 - \int_0^{\bar{p}} F_v(t)^n \frac{\partial}{\partial t}\left[t(F_{\hat{p}}(t)^n - 1)\right]\mathrm{d}t \\
&\quad + \int_0^{\bar{p}} tf_v(t)\int_t^{\bar{p}} nF_{\hat{p}}(s)^{n-1}F_v(s)^{n-1}f_{\hat{p}}(s)\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^{\bar{p}} F_v(t)^n\left(1 - F_{\hat{p}}(t)^n\right)\mathrm{d}t - \int_0^{\bar{p}} tF_v(t)^n nF_{\hat{p}}^{n-1}f_{\hat{p}}(t)\mathrm{d}t \\
&\quad + \int_0^{\bar{p}}\int_0^{s} tf_v(t)nF_{\hat{p}}(s)^{n-1}F_v(s)^{n-1}f_{\hat{p}}(s)\mathrm{d}t\,\mathrm{d}s
\end{aligned}
$$

using integration by parts from the second to the third line, and changing the order of integration in the last line.

Also applying integration by parts, $C(s) - sF(s) = \int_0^s F_v(t)\mathrm{d}t - sF(s) = -\int_0^s tf_v(t)\mathrm{d}t$. We can conclude that

$$
\begin{aligned}
\Delta EE &= \Delta CS + \Delta AE \\
&= \int_0^{\bar{p}} (C(s) - sF_v(s))F_v(s)^{n-1}nF_{\hat{p}}^{n-1}f_{\hat{p}}(s)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{\bar{p}} F_v(t)^n \left(1 - F_{\hat{p}}(t)^n\right) \mathrm{d}t \\
&+\int_0^{\bar{p}} \int_0^{s} t f_v(t) n F_{\hat{p}}(s)^{n-1} F_v(s)^{n-1} f_{\hat{p}}(s) \mathrm{d}t\, \mathrm{d}s \\
=& -\int_0^{\bar{p}} \left(\int_0^{s} t f_v(t) \mathrm{d}t\right) F_v(s)^{n-1} n F_{\hat{p}}^{n-1} f_{\hat{p}}(s) \mathrm{d}s \\
&+\int_0^{\bar{p}} F_v(t)^{n-1} \left(1 - F_{\hat{p}}(t)^n\right) \mathrm{d}t \\
&+\int_0^{\bar{p}} \int_0^{s} t f_v(t) n F_{\hat{p}}(s)^{n-1} F_v(s)^{n-1} f_{\hat{p}}(s) \mathrm{d}t\, \mathrm{d}s \\
=& \int_0^{\bar{p}} F_v(t)^n \left(1 - F_{\hat{p}}(t)^n\right) \mathrm{d}t > 0
\end{aligned}
$$

□

I am indebted to an anonymous referee for pointing out that Proposition 8 is a consequence of the expression for the optimal information acquisition point presented in Proposition 1. The argument is as follows: Efficiency differs only in situations where one of the bidders does not acquire information—if all bidders acquire information, the final allocation and the information acquisition costs are the same in both auctions.

Consider then a situation where the winner i does not acquire information in the English auction. The cost saving is c_i. Let j be the index with the highest value besides the winner. The loss in efficiency is $v_j - v_i$ if this expression is positive or 0 otherwise. Since the winner does not know her own value, v_i could be less than v_j or even $\hat{p}_i$; but we know that $v_j \leq \hat{p}_i$, and therefore, from Proposition 1,

$$
\int_0^{v_j} (v_j - v_i) \mathrm{d}F_v(v_i) \leq \int_0^{\hat{p}_i} (\hat{p}_i - v_i) \mathrm{d}F_v(v_i) = c_i.
$$

Therefore, the expected efficiency loss (once one takes the expectation conditional on the identity of the winner) is less than the expected saving in information acquisition cost.

7 Numerical simulations

This section presents explicit computations of expected revenue and social surplus for some choices of distributions for c, w, and v. This exercise establishes some quantitative meaning to the comparative statics finding that the dynamic auction is superior to the one-shot procedure in terms of expected revenue and efficiency.

Table 1 Expected revenue for the seller

$v\|w \sim$:	n	If info was free (true)	If info was free (simul.)	One-shot	Dynamic	% gain
$U[0, 2w]$	2	0.2593	0.2599	0.2702	0.2870	6.20
	3	0.4410	0.4420	0.4382	0.4524	3.25
	4	0.5750	0.5762	0.5619	0.5703	1.50
$U[w, w+1]$	2	0.7667	0.7655	0.7880	0.8006	1.60
	3	1	0.9999	0.9852	0.9953	1.03
	4	1.1274	1.1278	1.0930	1.1039	1.00
$U[w, 1]$	2	0.6296	0.6283	0.6452	0.6535	1.28
	3	0.7795	0.7790	0.7638	0.7650	0.16
	4	0.8465	0.8461	0.8131	0.8135	0.05

In the simulations we present, we assume $w \sim U[0, 1]$ and $c \sim U[0, 0.05]$. We present results for three alternatives for the distribution of $v|w$, for n between 2 and 4.[9]

The three alternatives for the distribution of $v|w$ were $U[0, 2w]$, $U[w, w+1]$ and $U[w, 1]$. The reason for these choices was to look at distributions where the variance increases, stays constant, and decreases with w. This is of interest because according to the discussion of Sect. 4.2, the impact of w through variance is a potentially important determinant of information acquisition.

Table 1 presents the computed expected revenue of the seller under each circumstance. In order to provide a benchmark, the first column shows what would be the revenue if information was costless to all bidders (i.e., if every bidder would drop out at v).[10] The second and third columns show the expected revenue in the one-shot and the dynamic auctions. Finally, the last column shows the percentage difference of revenue between the dynamic and the one-shot auction.[11]

[9] The models have also been simulated for all values for n up to 10. Results are the same, although profit differentials across auction rules become smaller as n grows high.

[10] A counterintuitive finding is that sometimes the dynamic auction is more profitable than if information was for free. This can only occur however for $n = 2$. The logic is the following: suppose c is extremely high, so that nobody effectively buys information. In this case the revenue is the expected value of the second order statistic of a sample of $E[v|w_i]$, rather than of v_i. With many bidders, the latter is larger than the former, but not when the number of bidders is 2: in this case, $E[\min\{E[v|w_1], E[v|w_2]\}] > E[\min\{v_1, v_2\}]$. (I thank Paul Milgrom for pointing me that).

[11] The equilibrium was computed by searching for a fixed point in the set A of (w, c)-types by direct iteration of the $R \circ T$ operator. A is represented numerically as a subsample of a quasi-Monte Carlo sequence.
In order to provide some evidence of the accuracy of the simulations, 8 sequence lenghts from 8000 to 20000 points have been used. In all cases, the different simulations lead to results within an interval of width 0.0008 or less. Based on this crude accuracy analysis, the numbers reported in the following tables are accurate up to the third digit. (Percentage gain figures are less accurate, with results disagreeing up to 0.19. But in all cases that inaccuracy is not enough to revert the reported signs).

Table 2 Ex-ante expected payoff to each bidder

$v\|w\sim$:	n	If info was free (true)	If info was free (simul.)	One-shot	Dynamic	% gain
$U[0, 2w]$	2	0.2407	0.2413	0.2187	0.2140	−2.18
	3	0.1500	0.1502	0.1345	0.1309	−2.65
	4	0.1052	0.1055	0.0931	0.0914	−1.82
$U[w, w+1]$	2	0.2333	0.2343	0.2018	0.1994	−1.17
	3	0.1167	0.1170	0.1019	0.1008	−1.07
	4	0.0742	0.0744	0.0655	0.0642	−2.00
$U[w, 1]$	2	0.1204	0.1208	0.0998	0.0961	−3.74
	3	0.0454	0.0455	0.0399	0.0396	−0.73
	4	0.0231	0.0231	0.0232	0.0231	−0.34

Table 3 Expected efficiency

$v\|w\sim$:	n	If info was free (true)	If info was free (simul.)	One-shot	Dynamic	% gain
$U[0, 2w]$	2	0.7407	0.7426	0.7077	0.7149	1.02
	3	0.8906	0.8827	0.8417	0.8453	0.43
	4	0.9958	0.9981	0.9344	0.9361	0.18
$U[w, w+1]$	2	1.2333	1.2340	1.1916	1.1994	0.66
	3	1.35	1.3509	1.2910	1.2979	0.53
	4	1.4242	1.4253	1.3549	1.3605	0.41
$U[w, 1]$	2	0.8704	0.8699	0.8448	0.8456	0.10
	3	0.9158	0.9154	0.8835	0.8839	0.04
	4	0.9389	0.9384	0.9059	0.9060	0.01

In percentage terms, the increased revenue of a dynamic procedure ranges from 0 to 6%—arguably, an economically significant figure. In all cases, the gain is positive.

It is interesting to note that as n grows high, the gain becomes small, both in absolute and percentage terms. This observation, coupled with the asymptotic comparison result, suggests that the expected revenue is generally larger with the dynamic procedure.

Table 2 shows the ex-ante expected payoff of an individual bidder under each rule for all settings, i.e., the expected profit average over all (c, w)-types. The expected payoff under the dynamic procedure is lower than in the one-shot auction. So dynamic auctions seem to benefit the seller partially at the expense of the bidders.

Table 3 presents expected efficiency (net of information acquisition costs) under each auction. Once again the dynamic auction rule is preferable under this criterion, although the gains in efficiency are smaller than those in seller revenue.

Figures 5, 6, and 7 exhibit how the sets of types that acquire information (top panels) and the distributions of the individual drop-out points (bottom panels) are under each

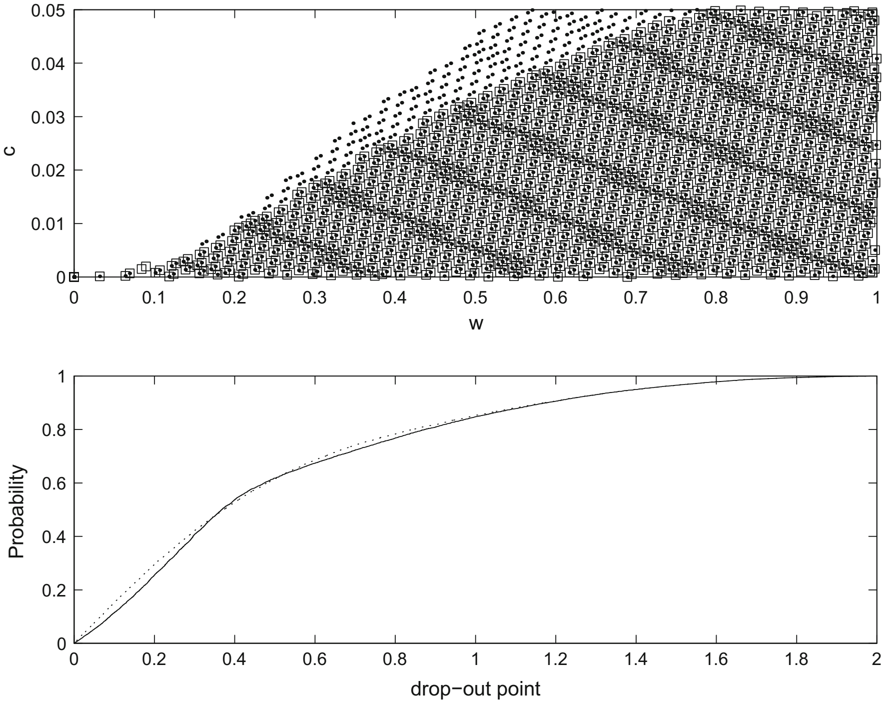

Fig. 5 Information acquisition sets and drop-out point distributions when $v|w \sim U[0, 2w]$. *Top panel*: shape of the equilibrium R_0 (*squares*) and R (*dots*) sets. *Bottom panel*: distribution function of the bidder's drop-out point at the dynamic (*solid line*) and one-shot (*dotted line*) auctions. All graphs assume $n = 2$, $w \sim U[0, 1]$, and $c \sim U[0, 0.05]$

alternative. For convenience, only equilibria with $n = 2$ are depicted. Equilibria with more bidders have smaller information acquisition regions, but the shape of these regions and of the drop-out distributions are qualitatively similar.

As the top panels show, the information acquisition regions are indeed monotone in c, but not necessarily so in w. A more optimistic signal about the good's valuation can make the bidder more (as in the first specification) or less (as in the second one) eager to acquire information, depending on how this news affect the dispersion of her valuation *vis-à-vis* the auction price.

The bottom panels show the distribution function of a bidder's drop-out price, in the dynamic (solid line) and the one-shot (dotted line) auctions. In all cases, bidding is more aggressive in the dynamic game, and almost in a first-order stochastic dominance sense.

8 Concluding remarks

This paper has investigated a model of an auction where bidders have the freedom to improve the information they have about their valuations at any point in time. The model accommodates heterogeneity both in the prior information each bidder

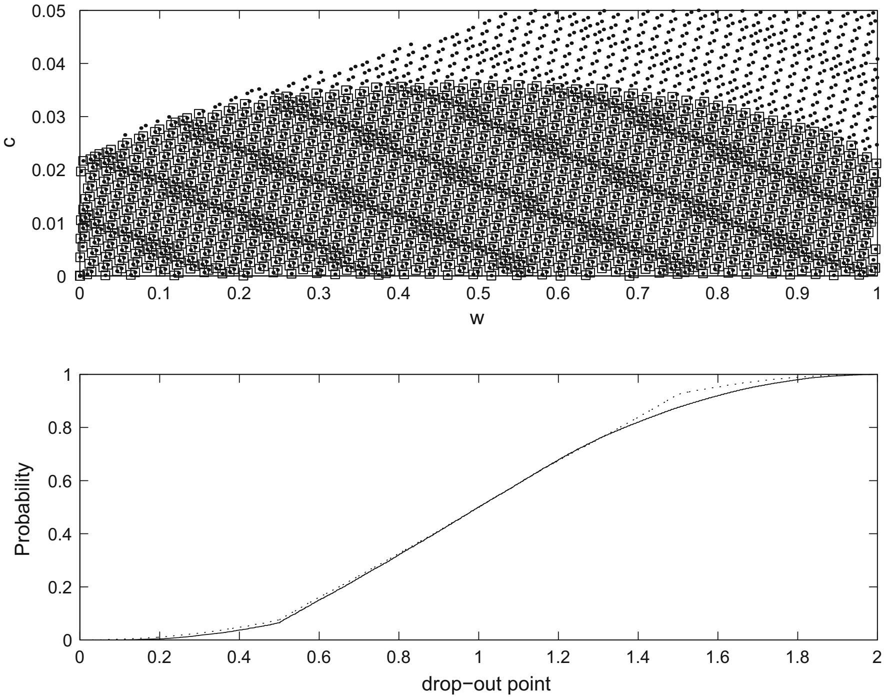

Fig. 6 Information acquisition sets and drop-out point distributions when $v|w \sim U[w, w+1]$. *Top panel*: shape of the equilibrium R_0 (*squares*) and R (*dots*) sets. *Bottom panel*: distribution function of the bidder's drop-out point at the dynamic (*solid line*) and one-shot (*dotted line*) auctions. All graphs assume $n = 2$, $w \sim U[0, 1]$, and $c \sim U[0, 0.05]$

possesses and in the information acquisition costs. Because these costs can be zero or very high, the model allows for treating the cases of bidders who cannot acquire information or already have done so in the same framework.

In spite of this generality, we found a remarkably simple formula to determine the optimal moment to acquire information. Furthermore, the formula determines a time that is independent of the strategies followed by the other players.

Another virtue of the model is that it accommodates the analysis of a one-shot version of the auction. We have compared the expected revenue and the expected efficiency of the equilibrium of the English auction with the one-shot auction. We have proved that the expected revenue of the seller is larger in the dynamic auction when the number of bidders is sufficiently large. We also show that the expected efficiency of the ascending auction is larger, in the case where all types acquire information. These results provide an additional explanations for the prevalence of the English auction in practice, even under independent private values.

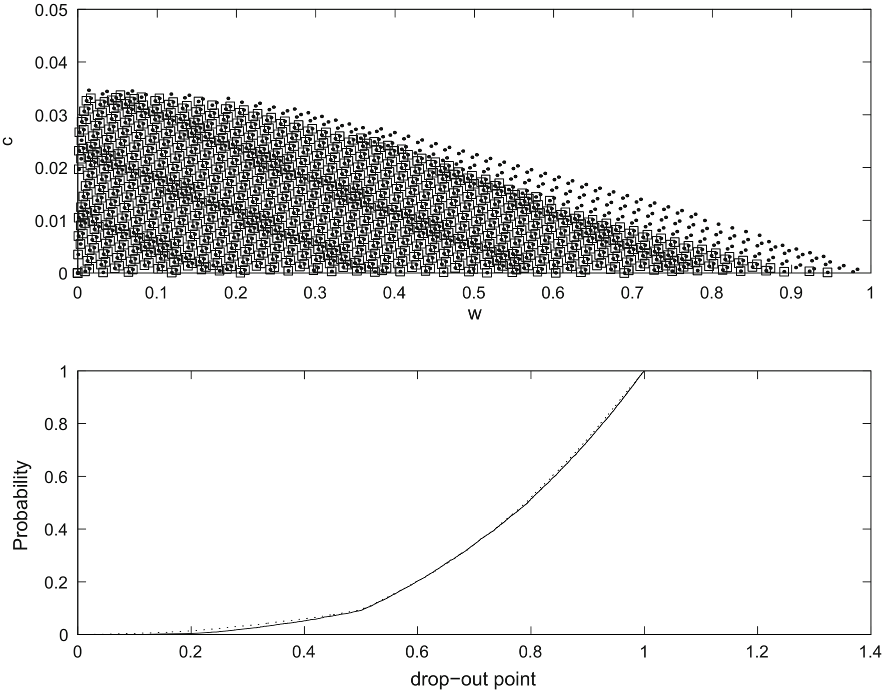

Fig. 7 Information acquisition sets and drop-out point distributions when $v|w \sim U[w, 1]$. *Top panel*: shape of the equilibrium R_0 (*squares*) and R (*dots*) sets. *Bottom panel*: distribution function of the bidder's drop-out point at the dynamic (*solid line*) and one-shot (*dotted line*) auctions. All graphs assume $n = 2$, $w \sim U[0, 1]$, and $c \sim U[0, 0.05]$

Acknowledgements This paper is an updated version of chapter 2 of my Ph.D. dissertation. I thank Paul Milgrom, Ilan Kremer and two anonymous referees for most helpful suggestions, and Vinicius Carrasco for encouragement. I acknowledge the support of a Melvin and Joan Lane Stanford Graduate Fellowship. All errors are mine.

9 Appendix: Proofs

Proof of Lemma 1 According to the Ascoli–Arzelà theorem, if X is any compact metric space, and $S \subset C(X)$ is equicontinuous and bounded, then S is relatively compact.

Take $X = [0, \bar{v}]$ and $S = T(\mathcal{A})$ in the Ascoli–Arzelà theorem statement. Because all distributions are bounded in the sup norm, it only remains to verify equicontinuity.

Take $\epsilon > 0$. Let $\delta < \epsilon / (M_c + M_v + M_w)$. Let F be a distribution in $T(\mathcal{A})$.

So, for any $x \in [0, \bar{v}]$, one can write $0 \leq F(x+\delta) - F(x) = \Pr[y_1 \in [x, x+\delta]] \leq \Pr[v \in [x, x+\delta]] + \Pr\left[\hat{p} \in [x, x+\delta]\right] + \Pr[w \in [x, x+\delta]]$.

Next, we observe that, using the Jacobian rule and the definition of $\hat{p}$, we have that $f_{\hat{p}}(s) = \int F_{v|w}(s) f_c \left(\int_0^s F_{v|w}(t)\mathrm{d}t\right) f_w(w)\mathrm{d}w$. Inasmuch as $f_c \leq M_c$ and $F_{v|w} \leq 1$, we obtain $f_{\hat{p}} \leq M_c$. Furthermore, $f_w \leq M_w$. It then follows that, for sufficiently small δ,

$$0 \leq F(x+\delta) - F(x) \leq (M_c + M_v + M_w)\,\delta < \epsilon.$$

For the case where $x = 0$, notice that $v \geq \hat{p}(0, w) = 0$. Therefore, the atom in the $\hat{p}$ distribution is irrelevant, since we can write $0 \leq F(\delta) - F(0) \leq \Pr[v \in [x, x+\delta]] + \Pr[w \in [x, x+\delta]] < (M_v + M_w)\delta < \epsilon$. Consequently, $T(\mathcal{A})$ is equicontinuous. □

Proof of Lemma 2 Take $F_t \to F$ uniformly. Then $\int r\mathrm{d}F_t^{n-1} \to \int r\mathrm{d}F^{n-1}$, by the Helly Second theorem. We can write

$$\begin{aligned} &T \circ R(F)(x) \\ &\quad = \int \mathbb{I}\{w \leq x\} + \mathbb{I}\left\{\int r\mathrm{d}F^{n-1} \geq 0\right\} \left(\Pr\left[\max\{\hat{p}, v\} \leq x | c, w\right] - \mathbb{I}\{\check{p} \leq x\}\right) \mathrm{d}F_{(c,w)}, \end{aligned}$$

where $\mathbb{I}\{\}$ denotes the indicator function. Applying Cauchy–Schwarz and the fact that indicator functions are bounded by 1, we obtain that

$$\begin{aligned} &(T \circ R(F_k)(x) - T \circ R(F)(x))^2 \\ &= \left(\int \left(\mathbb{I}\left\{\int r\mathrm{d}F_k^{n-1} \geq 0\right\} - \mathbb{I}\left\{\int r\mathrm{d}F^{n-1} \geq 0\right\}\right)\right. \\ &\quad \times \left.\left(\Pr\left[\max\{\hat{p}, v\} \leq x | c, w\right] - \mathbb{I}\{w \leq x\}\right) \mathrm{d}F_{(c,w)}\right)^2 \\ &\leq \int \left(\mathbb{I}\left\{\int r\mathrm{d}F_k^{n-1} \geq 0\right\} - \mathbb{I}\left\{\int r\mathrm{d}F^{n-1} \geq 0\right\}\right)^2 \mathrm{d}F_{(c,w)} \\ &\quad \times \int \left(\Pr\left[\max\{\hat{p}, v\} \leq x | c, w\right] - \mathbb{I}\{w \leq x\}\right)^2 \mathrm{d}F_{(c,w)} \\ &\leq \int \left(\mathbb{I}\left\{\int r\mathrm{d}F_k^{n-1} \geq 0\right\} - \mathbb{I}\left\{\int r\mathrm{d}F^{n-1} \geq 0\right\}\right)^2. \end{aligned}$$

For any point outside $\{(c, w) \mid \int r\mathrm{d}F^{n-1} = 0\}$, $\mathbb{I}\left\{\int r\mathrm{d}F_k^{n-1} \geq 0\right\}$ converges pointwise to $\mathbb{I}\left\{\int r\mathrm{d}F_k^{n-1} \geq 0\right\}$. Therefore, the limit of the integral of the last expression is of a function that is zero almost everywhere. In addition, because the last expression does not depend on x, convergence is uniform and continuity is established. □

Proof of Proposition 2 For the first statement, notice that $r(y, 0, w) \geq 0$. Thus, because we are resolving any indifference in favor of acquiring information, for any $A \subset R(\overline{\mathcal{F}})$, $\{c = 0\} \subset A$ (meaning that all types with $c = 0$ are in A). Additionally, $\hat{p}(0, w) = 0$. As a result, for such A, separating the types where $c = 0$, we obtain

$$\begin{aligned} T(A)(x) = &\int_{A^c \cap \{c>0\}} \mathbb{I}\{w \leq x\}\mathrm{d}F_{c,w} + \int_{A \cap \{c>0\}} \Pr[\max\{\hat{p}, v\} \leq x | c, w]\mathrm{d}F_{c,w} \\ &+ \int_{\{c=0\}} \Pr\left[\max \hat{p}, v \leq x | c, w\right] \mathrm{d}F_{c,w} \end{aligned}$$

$$
\begin{aligned}
&= (1-\pi)\left[\int_{A^c\cap\{c>0\}} \mathbb{I}\{w \le x\}\mathrm{d}F_{c|c>0,w}\right.\\
&\quad \left. + \int_{A\cap\{c>0\}} \Pr\left[\max\{\hat{p}, v\} \le x|c, w\right]\mathrm{d}F_{c|c>0,w}\right]\\
&\quad +\pi \int_{\{c=0\}} \Pr[v \le x|w]\mathrm{d}F_{c,w}\\
&= (1-\pi)F(x) + \pi \int F_{v|w}(x)\mathrm{d}F_w\\
&= (1-\pi)F(x) + \pi F_v(x),
\end{aligned}
$$

by the law of iterated expectations. Here, F is the distribution defined as the term between square brackets.

For the second statement, by Lemma 2, it is enough to verify that the measure of $\{\int r\mathrm{d}F^{n-1} = 0\}$ is zero. From the Envelope theorem, $\frac{\partial}{\partial c}\int r\mathrm{d}F^{n-1} = 1 - F^{n-1}(\hat{p})$. For all distributions in $\mathcal{F}_\pi$, and any $x < \bar{v}$, $F(x) < 1-\pi F_v(x) < 1$. Thus, $\int r\mathrm{d}F^{n-1}$ is strictly increasing in c everywhere, and for each w, there is at most one $c > 0$ such that $(c, w) \in \left\{\int r\mathrm{d}F^{n-1} = 0\right\}$. Furthermore, this c can never be zero, because if F is in $\mathcal{F}_\pi$, there is a positive probability of y_i that occurs in any interval in the support $[0, \bar{v}]$. We conclude that the integral $\int r(t, 0, w)\mathrm{d}F^{n-1}$ is positive for these types. □

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