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RESEARCH ARTICLE

General economic equilibrium with financial markets and retainability

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Abstract A theory of economic equilibrium for incomplete financial markets in general real assets is developed in a new formulation with currency-denominated prices. The "goods" are not only commodities, and they can influence utility through retention as an alternative to consumption. Perfect foresight is relinquished in a rolling horizon approach to markets which lets agents pursue long-term interests without being sure of future prices. The framework is that of an economy operating in a fiat currency that agents find attractive to retain, in balance with other needs. The attractiveness comes from Keynesian considerations about uncertainty which until now have not been brought in. The existence of equilibrium is established directly—not just generically—and moreover under weaker assumptions on endowments than before, except that utility functions are taken to be concave. Agents do not need to start out with, or end up with, positive amounts of everything. With a single currency denominating the units of account in all states, price indeterminacy is avoided and all contracts issued in the financial markets can be interpreted as "real contracts." Derivative instruments and collateralized contracts based on money prices are thereby encompassed for the first time. Transaction costs on sales of contracts, generated endogenously, lead

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to bid–ask spreads and in particular to a gap between interest rates for lending and borrowing money.

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1 Introduction

Concerns about the future have an undoubted influence on the decisions of economic agents in making plans that balance immediate and later needs while taking advantage of opportunities and protecting against hazards. Nonetheless, it has not been easy to capture this convincingly in the theory of economic equilibrium with incomplete financial markets (GEI), especially when so-called real assets are bought and sold and more than a merely generic assurance of the existence of equilibrium is sought. Anticipated prices of goods must be an important ingredient, but their informational status and connection to uncertain future supply and demand have posed difficulties along with the serious question of whether they can be denominated in a currency that persists beyond the present to fulfill its Keynesian[1] role as a store of value and a hedge against the unforeseen. Apart from elaborate schemes to force the explicit use of money in transactions through cash-in-advance constraints, the GEI models available so far have not provided for that. They have mostly expressed prices in unscaled "units of account" which are not linked to any currency and can only reflect relative values within a single state of the economy. Financial instruments involving payments or options tied to a real currency have thus been left by the wayside along with even the lending and borrowing of money between agents. Another awkward point has been the almost universal reliance of GEI theory on requiring agents to be endowed with positive amounts of every good and often to have preferences that forbid anything but positive amounts in equilibrium as well.

In this paper, we push beyond such limitations by developing a model that, while sharing core features with previous ones, differs sharply in other ways. It allows the agents to hold null quantities of some goods both initially and in equilibrium while moving significantly past tradition in capturing an agent's preferences.[2] It expands sources of value beyond consumption while mitigating the drawbacks of finite-horizon

[1] Keynes wrote that the "desire to hold money as a store of wealth is a barometer of the degree of our distrust of our own calculations and conventions concerning the future…" Keynes (1978b), and that money is "above all a subtle device for linking present and future" Keynes (1978a). More can be read about his views and their present-day relevance in recent publications of Skidelsky (2008, 2009).

[2] The original version of this paper, with the same model, results, and equilibrium proof as here, was circulated and made website-available in May 2010 under the title "General economic equilibrium with incomplete markets and money"; the results in a streamlined case were published in Jofre et al. (2011). The paper's title was subsequently changed to "General economic equilibrium with financial markets and money" and finally, in 2013, to the present title. The content in these versions has remained the same except for evolving attempts at better explaining the ideas and their consequences.

modeling (with its inherent "doomsday"), thereby enabling the incorporation of a "Keynesian" currency and much more. In a two-stage format, the model generalizes "goods" beyond their traditional range and allows agents to get utility by retaining them instead of consuming them. Retention, concerning more than customary "durable" goods and affecting utility not only at time 0, the present, but also at time 1, the modeled future, helps to bridge from a past before time 0 to an unmodeled future after time 1. It serves also as a vehicle for an agent's longer-term interests and worries. In line with this, the model departs from the perfect foresight approach of Radner and others by adopting a different attitude toward prices at time 1, thereby letting retention of goods at time 1 have a different quality than at time 0. The prices in future spot markets do not then reflect with certainty the eventual buying power with money and that gives an additional reason for agents to build attitudes toward holding onto currency into their preferences.

The items we treat as *goods* (for want of a better term!) are "generalized goods": more than just commodities. They can be *anything tradable in markets that enters each state of the economy in fixed supply within the agents' holdings*. A good retained in the present might automatically change into something else in the future. Some goods may deteriorate in passage, but some may persist unaltered. Among the latter can be "monetary goods," identifiable as being readily exchangeable for other goods as a resource backup which agents are attracted to retain, at least in some proportion (as dictated by their utility functions), and can freely do so. Fiat money, acting as the accepted currency in a given economy,[3] fits this description in particular. Quantities of currency, limited in overall supply, normally do, after trading, pass from present to future unchanged in the holdings of agents (cash and interest-free accounts) in proportions dictated by preferences.

Our generalized conception of goods also includes "investment goods" which morph into future-sensitive returns of other goods, maybe monetary.[4] Investment goods are distinguished in our model from two-party "contracts" which sellers can issue to buyers at time 0. The supplies of such contracts are not fixed in advance, thus failing the test for generalized goods.

Although our assumptions on utility will guarantee that *any* "monetary good" could serve as the price numéraire in all markets, we focus on the case of an economy that indeed functions in a single currency. We make that particular "good," referred to as *money* for convenience, be the numéraire for units of account. No financial distinction is needed then between real contracts for the delivery of goods and nominal contracts for the delivery of units of account, which can be recast as money delivery contracts. Moreover, the indeterminate scaling of units of account associated with the nominal contract equilibrium of Cass (1984) and Werner (1985), as highlighted by Geanakoplos (1990) and Magill and Shafer (1991), is obviated in a simpler manner than the layered transaction structures proposed by Magill and Quinzii (1992) and Dubey and Geanakoplos (2003) with cash-in-advance constraints that moreover limit the total value of transactions in a given state of the economy to the total supply of money in

[3] For a review of the economic importance of fiat money, see (Krugman and Wells 2009, Chapter 14).

[4] More about "investment goods" will come later. Observe that this is not akin to production, because the utility effects permitted for retention would not be appropriate for inputs to production.

that state—which our model avoids. The capability of relating economic value across different states, in assessments of inflation or deflation in particular, is thereby secured. Our money framework makes it possible—for the first time in a GEI model—to allow the delivered amounts of goods in a contract (instead of just their market value) to depend on the money-scaled prices in the future states, as in financial instruments like put/call options or loans backed by collateral. Furthermore, it broadens foundations for the existence of equilibrium by significantly relaxing the assumptions typically imposed on the agents' endowments. They only need to be able, without trading, to survive (1) *individually* without exhausting their initial *money* and (2) *collectively* in all states with an *aggregate* surplus of each other good. That improvement, relying also on innovations in handling budget constraints, lets boundaries of the agents' survival sets[5] come realistically into play.

Along with these developments in the theory of equilibrium, we have a secondary aim of promoting *variational analysis* as a methodology which can have many uses in economics. The familiar paradigm of *classical analysis* is that of fundamentally reducing a model, at some point, to solving n equations in n unknowns, whether linear or nonlinear, and under strong assumptions of smoothness and interiority. This is seen everywhere in economics, especially in looking toward stability and computation. Beyond that, *convex analysis* Rockafellar (1970) has long been put to work, but the much larger body of mathematics that now surrounds it in variational analysis Rockafellar and Wets (1997) has not yet attracted as much attention from economists as it might. Variational geometry provides an alternative to the reliance on differential geometry, which has influenced a large body of work, starting with Debreu (1970), but emphasizes results that can only be claimed to hold *generically*. A strong advantage of variational analysis is the capability that it affords for working at the boundaries of the agents' survival sets as well as, eventually, handling the effects of other one-sided constraints coming from externalities. In an economy with a rich range of goods, agents should not be obliged to hold positive quantities of all of them.

The contribution we make in that direction is to formulate equilibrium as a *variational inequality problem*.[6] In fact, variational inequality problems are the new paradigm beyond "n equations in n unknowns." They are supported by extensive theory that includes generalizations of the classical implicit function theorem, as available for instance in the book Dontchev and Rockafellar (2009).[7] Passage to a variational inequality formulation also articulates an extended form of equilibrium that includes "money rates" along with the agents' decisions.[8] It furthermore opens the door to a

[5] Agents are then no longer required to enter each state with a positive quantity of every good, as typically has been assumed to facilitate proving the existence of equilibrium.

[6] For existence of solutions, this amounts to a advantageously structured setup for locating a fixed point.

[7] We aim at applying that theory in later work to investigate the degree to which an equilibrium, as formulated here, may be "locally unique" and nicely behaved with respect to shifts in parameters such as the agents' endowments. For one-stage nonfinancial models, that program has already been initiated with surprising results in our paper (Jofre et al. 2013), drawing on recent work of Dontchev and Rockafellar (2012).

[8] Our existence development centers, in effect, on this full combination of equilibrium elements as the targeted "fixed point." Condensing it to a customary type of fixed point argument in "price space" alone would be extremely unwieldy and result in lost information.

range of methodology, hitherto unexploited in GEI, which has potential for getting beyond past obstacles in stability analysis and computation.
Summary of the main new features of the model.

- Retention of goods, directly affecting an agent's utility, is introduced as a fundamental source of value beyond consumption. Time periods and markets are reinterpreted in this light to help capture at least roughly, even in a two-stage framework, ongoing economic activity that emerges from the past and continues to an indefinite future.
- The concept of goods is generalized accordingly to cover even financial elements, including currency, which can be desired for retention apart from consumption. A currency can serve then as the numéraire which links prices across all states and times.
- Markets in financial assets are reconstituted so as to handle "investment goods" in one way and two-party "contracts' in another, thereby letting the former to be open to retention.
- The goods deliveries promised in contracts can depend on the currency-denominated future prices of those goods. Instruments like call/put options and collateralized loans are covered then without the scaling ambiguities of the "units of account" in previous approaches.
- The retainability of money underpins the existence of equilibrium even if agents have little or no interest in most of the goods in the markets and without the typical assumption that agents are individually endowed with more of every good than their survival would require.
- Market-derived costs associated with issuing contracts induce bid–ask spreads in contract prices and in particular a gap between interest rates in the borrowing and lending of money.
- Connections between marginal utility and money lead to imputed probabilities and discount rates. Refined rules of asset evaluation utilizing them furnish new insights into agents' behavior.

In Sect. 2, we present the basics of the model, formulate equilibrium, explain some of its immediate characteristics, and state our existence theorem. A discussion of the key features, especially the treatment of money, and connections with other models is undertaken then in Sect. 3. In Sect. 4, we introduce the money rates and employ them in a saddle point characterization of optimality in the agents' utility problems. This is followed by results that tie the money rates to the marginal utility of money and develop corresponding discounting rules for the pricing of contracts. Section 5 develops the variational inequality representation of equilibrium that our proof of existence is organized around. The "Appendix" lays out the proof itself in truncation steps with novel features such as the application of duality theory in convex optimization in order to derive bounds on the money rates. Such effort is vital in the face of our extremely relaxed assumptions on survival, and in working directly with money-denominated prices instead of relative prices in a simplex format.

2 Goods, contracts, and equilibrium

The model has a single present state $s = 0$ at time $t = 0$ but a possible multiplicity of future states $s = 1, \dots, S$ at time $t = 1$. Agents $i = 1, \dots, I$ are endowed in these states with nonnegative vectors $e_i(s)$ of goods $l = 0, 1, \dots, L$,[9,10] and can plan for trading those goods in markets as governed by price vectors $p(s)$ with component prices $p_l(s)$. Good 0 will play a special role as the money in the economy.[11] Because good 0 will end up scaling all the units of account, we only deal with *money-denominated* prices and accordingly take

$$p_l(s) \geq 0 \quad \text{for } l = 1, \dots, L, \quad \text{but } p_0(s) = 1 \text{ in all states } s. \tag{1}$$

However, we do not see the spot markets in future goods as having the same economic character as the present "real" markets. Instead we see them as informational "stagings" or "rehearsals" for markets which will take place only later, as will be explained in Sect. 3.

The trading of agent i aims at achieving an optimal balance over time and uncertainty between *consumption*, represented by goods vectors $c_i(s)$, and *retention*, represented by goods vectors $w_i(s)$; it will be convenient to speak of $w_i(s)$ as the *wealth* of agent i at the end of the period in state s. The preferences of agent i among these vectors are ordered by their overall *utility*

$$u_i(w_i, c_i), \quad w_i = (w_i(0), w_i(1), \dots, w_i(S)), \quad c_i = (c_i(0), c_i(1), \dots, c_i(S)), \tag{2}$$

for a *concave* function $u_i : (I\!R^{1+L})^{2(1+S)} \to [-\infty, \infty)$ which is *nondecreasing* (but not necessarily always increasing) in all arguments and *upper semicontinuous*.[12] The set

$$U_i = \left\{ (w_i, c_i) \,\middle|\, u_i(w_i, c_i) > -\infty \right\}, \quad \text{with} \quad \emptyset \neq U_i \subset \left(I\!R^{1+L}_+ \right)^{2(1+S)}, \tag{3}$$

which is convex because u_i is concave, is the *survival set* for agent i. The upper semicontinuity of u_i corresponds to the level sets of the form $\left\{ (w_i, c_i) \,\middle|\, u_i(w_i, c_i) \geq \alpha \right\}$ for finite α being closed, but the set U_i need not be closed.[13] However, we assume that u_i is actually *continuous relative to* U_i. Note that neither differentiability nor *strict*

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9 The endowments in state $s = 0$ serve also as a repository for resources transmitted from the past.

10 Nonnegativity in goods can be relaxed by an obvious trick, which could be important eventually. The goods quantities here could be reinterpreted as distances above negative but naturally generated lower bounds for the "true" goods in the economy, with negativity standing for "debts" carried over from past obligations.

11 The amounts of this money good component in the endowment vectors $e_i(s)$ might be provided or manipulated by a government, but that will not be pursued here.

12 Assuming concavity in place of quasi-concavity is highly beneficial to our later analysis. Although this is more restrictive than usual, our utilities are in other ways much less restrictive than usual.

13 Utility might tend to $-\infty$ at a point in the boundary of U_i is approached, or even if it stays bounded it could jump to $-\infty$ as the boundary is crossed.

concavity is asked of u_i. Agent i might, in reality, have *zero interest in most of the goods*, and quickly dwindling interest in others.[14]

Through U_i, which can be far more general than the orthant usually seen, or even a shifted orthant produced by lower bounds on some goods, intertemporal constraints on consumption and retention may even be reflected. Many details of utility structure could of course be plumbed for consequences, such as an expectational form $u_i(w_i, c_i) = \sum_{s=1}^{S} \beta_{is} u_{is}(w_i(0), c_i(0), w_i(s), c_i(s))$ with β_{is} being the probability assigned by agent i to the future state s. Even better, $u_i(w_i, c_i)$, could be the minimum of such expectations with respect to a collection B_i of alternative probability assignment vectors β_i so as to reflect "ambiguity" in the assessments of the agent. Such ambiguity has been a topic for many economists, for instance in Maccheroni et al. (2006), Strzalecki (2011) and very recently He and Yannelis (2015), and it ties in well with the theory of "measures of risk" in finance, cf. Rockafellar and Uryasev (2013). [15]

The role of retention. The wealth $w_i(0)$ at time 0 is passed on to state s at time 1 as $A_i(s)w_i(0)$, where $A_i(s)$ is a matrix with nonnegative entries. We say that

$$\text{a good } l \text{ can } \textit{freely be saved} \text{ if column } l \text{ of } A_i(s) \text{ has 1 in row } l, \text{ but otherwise 0.}$$

Instead a good could gain or diminish in quantity, or evolve in this manner to something else, even a boundle of other goods, about which more will be said below.[16] Some goods may be more suitable for consumption or for retention, and that could be accommodated with additional complexity of notation, but it is easier mathematically to allow a dual role for every good and let utility functions sort out what happens. For goods suitable for both, there is an advantage also because a quantity can be split into the two different modes. An agent could plan at time 0 to consume some wine in a state s at time 1 while retaining some more for the unmodeled future after time 1. Agents expect that future to come into view when time 1 is reached.

Attractiveness of goods. In tackling the problem differently, we proceed as follows. A good l will be called *attractive for consumption* to agent i in state s if any increase in the l component of the vector $c_i(s)$ results in higher utility, or on the other hand, *attractive for retention* if this holds for $w_i(s)$. It is *always* attractive for consumption, or as the case may be, retention, if this holds for every state $s = 0, 1, \ldots, S$. The attractiveness of a good serves as a specific source of insatiability in an agent's utility. For the money in our model, this will be a key property.

[14] An equilibrium model that does not account for this must, in our opinion, be seen as seriously falling short. Such lack of interest can extend to either consumption or retention or both, depending on the "good."

[15] It should be noted, in connection with our assumption of concavity of u_i instead of the more common quasi-concavity, that this is followed also by researchers working nowadays with "ambiguity," since expectations of quasi-concave functions can hardly ever themselves be just quasi-concave.

[16] This transformation might seem like elementary "home production" in which inputs within $w_i(0)$ lead to output bundles $A_i(s)w_i(0)$, but an important distinction needs to be underscored. Retaining $w_i(0)$ may boost u_i, but in production the benefits have always been connected with outputs. Agents have never been portrayed as getting utility directly from the quantities of goods they may devote to inputs, and therefore our retention vectors $w_i(0)$ cannot rightly be interpreted as production inputs.

Financial support assumption. Good 0 is always attractive to every agent i for retention and can freely be saved.

Financial markets. We distinguish between financial instruments that *deliver from exogenous sources* (outside the agents' holdings) and, on the other hand, assets in the form of *freshly written two-party agreements between agents*, where one party promises delivery and the other receives it. The instruments belonging to the first category are handled as *investment goods*. The term *contracts* is reserved here for instruments in the second category, even though various instruments in the first category might sometimes be echoes of contracts written in the past.

The contracts in our model are not themselves "goods," even in our broadened conception. Rather they are general "real assets" which deliver in goods, and they can be held in either "long" positions (buyers) or "short" positions (sellers). However, it is important to understand that because their deliveries leave total supplies unaffected, they come down just to transferring to the recipient's budget the *market value* of the promised goods in *money* units calculated from the money-denominated prices, at which point the recipient can buy those goods, but may have other preferences. The total quantities of goods promised for delivery are liberated then from having to bow to the total supplies.[17] Since essentially only *values* have to be delivered by a two-party contract in the end, instead of physical piles of cash, *the restricted supply of such cash is no impediment to the scale of transactions in the economic model*.

Investment goods are initially present in fixed amounts within the agents' endowments. They are exchangeable in markets (even in fractions) and are open to being transferred from time 0 to time 1 in the retention process described above. An investment good in the form of a bond could, for instance, be converted in passage to a money payment plus a truncated bond. An investment good in the form of owning a share of stock could turn into future-state-dependent dividends with ownership persisting.[18] Yet another type of investment good could be the right to a portion of some production stream, which in the future would yield bundles of products.

The possible presence of investment goods alongside of two-party contracts in the financial market enables coverage, for example, of equities in production firms in the special version admitted by Geanakoplos et al. (1990).[19]

Contract structure and transaction costs. For two-party contracts, we permit the deliveries to *possibly depend on the market prices in the state of delivery*. In that way, we

[17] In comparison, the nominal asset models of Cass (1984) and Werner (1985) make payments directly in "units of account," which may seem akin to money but are unbounded in availability. Such units are valid only in a single state and, unless anchored to a currency as here, are replaced to different, unrelated units in the other states.

[18] Investment goods, as envisioned, may live on after transition. An agent can desire to retain them at time 1 even though the model has no time 2, because of anticipations of the future built into utility.

[19] The agents in Geanakoplos et al. (1990) have perfect foresight into the production decisions of the firms (however carried out). From a modeling perspective, there is no difference between that and simply assuming the outputs in each future state are known in advance. The exclusion of short-selling in Geanakoplos et al. (1990) further reinforces this interpretation of their equities as investment goods in our sense instead of two-party contracts.

cover various "options," as explained later with examples. This is a new capability based on being able to operate with prices tied to a single numéraire—"money."[20]

For tractability, we consider as usual that the contracts can only be of finitely many unit types, indexed by $k = 0, 1, \ldots, K$, with contract 0 having a special role to be described shortly. Agents can buy or sell them at prices $q_k \geq 0$ comprising a price vector q. Contract k promises delivery of the goods vectors $D_k(s, p(s)) \geq 0$ in the future states s, at least one of which, for some s, has *a positive component in some good that is attractive to some agent*, regardless of the particular prices. These vectors are the columns of the matrix

$$D(s, p(s)) \in I\!R_+^{(1+L)\times(1+K)} \quad \text{for} \quad s = 1, \ldots, S.$$

Contract 0 delivers one unit of good 0 in every future state $s = 1, \ldots, S$. *Buying this contract at price q_0 corresponds to lending q_0 units of money at time 0 in return for surely receiving one unit of money at time 1. Selling corresponds similarly to borrowing.*

The contracts can be bought or sold in any amounts (not necessarily integral), and in this way, agent i can put together a portfolio of long and short positions represented by vectors z_i^+ and z_i^- in $I\!R_+^K$ at market cost $q[z_i^+ - z_i^-]$.[21] Simultaneously buying and selling a contract k is not excluded, but is hindered by another provision. Namely, we suppose that

any agent, in selling a unit of k, must use up a goods vector $D_k(0, p(0))$

which satisfies independently of $p(0)$ a bound $D_k(0, p(0)) \geq D_k^*(0) \geq 0$ for a vector $D_k^*(0)$ having *a positive component for at least one good that is attractive to some agent* initially. This could refer of course to money (fees, taxes?), but also to service goods (professional labor?) connected for instance with confirming the reliability of a seller's delivery promises. In terms of the matrix $D(0, p(0))$ with columns $D_k(0, p(0))$, we have

$$\tau_k(p(0)) = p(0)D_k(0, p(0)) = \text{the transaction cost in } money \text{ units for selling contract } k.$$

The portfolio (z_i^+, z_i^-) thus consumes the goods vector $D(0, p(0))z_i^-$ and incurs in money units the total transaction cost[22] $\sum_{k=1}^K \tau_k(p(0))z_{ik}^- = p(0)D(0, p(0))z_i^-$.

The matrices $D(s, p(s))$ for $s = 0, 1, \ldots, S$ are assumed to depend *continuously*, if at all, on the money-denominated prices. We furthermore allow for the possibility that some agents might not be "qualified" to take on the delivery obligations associated

20 Tying deliveries to future "units of account" in the established framework of GEI cannot have the same effect because of inherent ambiguities in scaling.

21 For the sake of the algebra in our formulas, we consistently regard $p(s)$ and q as row vectors, in contrast to $w_i(s)$, $c_i(s)$, z_i^+ and z_i^-, which we regard as column vectors.

22 Note that this cost is essentially budgetary and does not, in itself, force actual money to change hands out of the limited supply of good 0 at time 0.

with certain contracts. Specifically, we introduce for each agent i an index set and a corresponding subset of $I\!R_+^{1+K}$,

$$K_i = \{k \in [1, K] \mid \text{ agent } i \text{ must not sell contract } k\},$$
$$I\!R_+^{1+K}(i) = \{z_i^- = (z_{i0}^-, z_{i1}^-, \ldots, z_{iK}^-) \mid z_{ik}^- = 0 \text{ for } k \in K_i\}, \tag{4}$$

and the *obligation constraint*

$$z_i^- \in I\!R_+^{1+K}(i). \tag{5}$$

No constraint is imposed on the purchases in z_i^+. Of course K_i might be empty as a particular case, and then $I\!R_+^{1+K}(i) = I\!R_+^{1+K}$. Note however that we have required $0 \notin K_i$, so that all agents are at least able to get buying power at time 0 by promising to pay money at time 1 in a manner independent of the future state s. This is intended as additional confirmation of the special monetary role of good 0. On the other hand, we suppose that

$$\bigcap\nolimits_i K_i = \emptyset \tag{6}$$

so as to insure that for every contract k there is as least one agent i able to sell it.

Excess demands. To handle the effects of the agents' decisions, we introduce notation for the excess demands they induce relative to the agents' endowments $e_i(s)$, namely

$$\begin{aligned} &d_i(0, p(0)) = w_i(0) + c_i(0) + D(0, p(0))z_i^- - e_i(0),\\ &d_i(s, p(s)) = w_i(s) + c_i(s) - D(s, p(s))\left[z_i^+ - z_i^-\right] - e_i(s)\\ &\quad -A_i(s)w_i(0) \text{ for } s > 0, \quad \text{with goods components } d_{il}(s, p(s)),\\ &\quad d_{il}(0, p(0)), \quad \text{for } l = 0, 1, \ldots, L. \end{aligned} \tag{7}$$

We speak of

$$\begin{aligned} (p, q) = (p(0), p(1), \ldots, p(S), q) \in \left(I\!R_+^{1+L}\right)^{1+S}\\ \times I\!R_+^{1+K} \text{ with } p_0(s) = 1 \text{ for all } s \end{aligned} \tag{8}$$

as a *price system* for the economy. Such a money-denominated price system affords a picture like that seen in the literature on nominal assets. However, there is a fundamental difference with that literature, because here the total quantity of money in any state s is bounded in our model by the endowments in that state plus the amounts of money that agents may have saved or otherwise transmitted from the past.[23]

[23] In nominal asset models like those in which Cass (1984) and Werner (1985) demonstrated the existence of equilibrium, with the units of account in state s considered to be denominated by the money in state s, there are no such bounds. In effect, each state has its own special money, and the supply of that money is infinite.

Utility optimization. The optimization problem $\mathcal{P}_i(p, q)$ faced by agent i, with respect to a price system (p, q), is to choose

$$(w_i, c_i) \in U_i, \qquad z_i^+ \in I\!R_+^{1+K}, \quad z_i^- \in I\!R_+^{1+K}(i), \tag{9}$$

to maximize $u_i(w_i, c_i)$ subject to budget constraints which are expressed through (7) by

$$p(0)\, d_i(0, p(0)) + q\left[z_i^+ - z_i^-\right] \leq 0, \quad p(s)\, d_i(s, p(s)) \leq 0 \quad \text{for } s > 0. \tag{10}$$

The inequality form of the budget constraints is not necessary, because equality will prevail in the end, but is convenient for our eventual appeal to Lagrange multipliers for these constraints.

Definition of equilibrium. A price system (p, q) furnishes an equilibrium if the problems $\mathcal{P}_i(p, q)$ have solutions for which the excess demands (7) satisfy the market-clearing conditions

$$\sum_i d_{il}(s, p(s)) \begin{Bmatrix} = 0 & \text{if } p_l(s) > 0 \\ \leq 0 & \text{if } p_l(s) = 0 \end{Bmatrix} \text{ for goods } l = 1, \ldots, L \text{ in all states } s,$$

$$\sum_i d_{i0}(s, p(s)) = 0 \quad \text{for good } l = 0 \text{ in all states } s,$$

$$\sum_i z_{ik}^+ = \sum_i z_{ik}^- \quad \text{for all contracts } k = 0, 1, \ldots, K.$$

The alternatives for the excess demands on the goods $l \neq 0$ reflect free disposal when the price is 0. That has no role for good $l = 0$, for which the price is fixed at 1.

Basic observations about equilibrium. In any equilibrium under the assumptions that have been made, necessarily

(a) $p_l(s) > 0$ if good l is attractive in state s to some agent i for consumption or retention,
(b) every contract k delivers a value $p(s)\, D_k(s, p(s)) > 0$ in at least one future state s,
(c) every contract k has transaction cost $\tau_k(p(0)) = p(0)\, D_k(0, p(0)) > 0$,
(d) every contract k has price $q_k > 0$,
(e) all budget constraints for all agents i are tight:

$$p(0)\, d_i(0, p(0)) + q[z_i^+ - z_i^-] = 0 \text{ and for } s > 0 \text{ also } \ p(s)\, d_i(s, p(s)) = 0,$$

(f) $\sum_i z_{ik}^+ - \sum_i z_{ik}^- = 0$, and no agent i can have both $z_{ik}^+ > 0$ and $z_{ik}^- > 0$ for any contract k.

Property (a) holds because utility would soar to infinity in problem $\mathcal{P}_i(p, q)$ if this good could be obtained cost-free in state s by agent i. Properties (b) and (c) then follow from the attractiveness assumptions on the delivery matrices $D(s, p(s))$. The attractiveness of money precludes optimality from occurring with slackness in the budget constraints, hence (e). From (b) we must have (d) to keep buyers from wanting arbitrarily large quantities of contract k. Through (c), on the other hand, any (z_i^+, z_i^-) portfolio with $z_i^- \neq 0$ must engender a positive transaction cost. This acts as a disincentive to taking short positions and keeps agents from taking long and short positions simultaneously in a contract k or from promising superfluous deliveries, thus giving us (f).

Ample survivability assumption. The agents have available to them particular choices

$$(\hat{w}_i, \hat{c}_i) \in U_i \quad \text{in combination with } \left(\hat{z}_i^+, \hat{z}_i^-\right) = (0, 0), \tag{11}$$

which, in extension of the notation in (7), result in excess demands that satisfy, for $s = 0, 1, \ldots, S$,

(a) $\hat{d}_{il}(s) \leq 0$ for $l = 0, 1, \ldots, L$, with $\hat{d}_{i0}(0) < 0$,
(b) $\sum_i \hat{d}_{il}(s) < 0$ for $l = 1, \ldots, L$.

It should be noticed that the possible dependence on prices in the general notation of excess demands has been suppressed from these conditions *because that dependence falls away when there is no contract activity* as in (11). Another observation, important for our subsequent use of ample survivability, is that, because money can freely be saved, *ample survivability could equivalently be formulated with $\hat{d}_{i0}(s) < 0$ for all s in (a)*, not just $s = 0$.

Theorem 1 (existence of an equilibrium). *Under the financial support assumption and the ample survivability assumption, along with the stipulated conditions on the utility functions u_i and the delivery matrices $D(s, p(s))$, an equilibrium exists.*

In other words, an equilibrium with money-denominated prices is sure to exist if the agents, without any trading in the markets, can survive with *individual* surpluses of money initially and a *collective* surplus of every other good in every state. (A surplus can be arbitrarily tiny.)

The importance of price-dependent deliveries. The effect of admitting price-dependent delivery matrices $D(s, p(s))$ is to enlarge the scope of contracts to include examples of financial instruments like the following.

Example 1 Call and put options. Let $l \neq 0$ designate some good (maybe an investment good), and consider a contract that, in each future state s, delivers money in the amount

$$\alpha(s) \max\left\{0, \kappa - p_l(s)\right\}.$$

This would be a *call* option-type contract; a *put* option-type contract would deliver

$$\alpha(s) \max\left\{0, p_l(s) - \kappa\right\}.$$

Instead of money as good 0, an option such as in Example 1 could deliver a quantity of l itself or something more complicated. The variants are obviously extensive and might even have $\kappa(s)$ in place of just κ, or for that matter, strike prices in several goods simultaneously.

Other forms of options fitting into our price-dependent framework could connect up with collateral that counters potential default.

Example 2 Loans with collateral. Consider, with respect to money amounts $m(s)$ for $s = 1, \ldots, S$ and a nonzero goods vector g (collateral), a contract that delivers in future state s

$$\begin{cases} m(s) & \text{if } m(s) \leq p(s)g, \\ g & \text{if } m(s) > p(s)g. \end{cases}$$

The second case corresponds to defaulting on the debt owed when the value of the associated collateral is "under water" with respect to it. The buyer then gets the collateral instead.[24]

Collateral and default are major topics which cannot be discussed here in detail, with so much else on the agenda, however the ability of our model to cover them, at least in part, is evident from Example 2. Default was modeled with penalties by Dubey et al. (2005), but restricted participation of agents as sellers of contracts has also played a role in the subject. A more elaborate approach, tuned to collateral proposals of Geanakoplos and Zame (2002a, b), has been developed by Seghir and Torres-Martinez (2011).[25] However, elementary obligation constraints like ours in (5) do not have a place in their model.

3 Discussion of the key issues

The transaction costs introduced in the issuance of two-party contracts have a role in the existence of equilibrium because they act through budgets to bound contract totals,[26] but they also generate a bid–ask spread for each contract. As will be explained in Sect. 4, this leads to refined rules of asset pricing in terms of market-imputed probabilities of future states, which can differ among agents and therefore help explain why some take long positions while others take short positions. In the case of Contract 0, the transaction cost leads to an endogenously generated difference between the interest rates for borrowing and lending.

Transaction costs tied *endogenously* like ours to the consumption of resources are not new. They have earlier been employed by others, but in different patterns and never with the amounts consumed depending on the current market prices for those goods. Laitenberger (1996) imposed them on both buyers and sellers at time 1, permitting asymmetry. Arrow and Hahn (1999) later had them at time 0 like us, but shared equally by buyers and sellers. In other related work, Préchac (1996) relied on *exogenous* costs of brokers who arrange transactions.

[24] This could be broadened in many ways, for instance in replacing $m(s)$ by a (generalized) goods vector $h(s)$, the market value of which is to be compared with that of g. On the other hand, g could become $g(s)$.

[25] Sales of contracts are bounded in Seghir and Torres-Martinez (2011) by the amounts of collateral an agent acquires. The rules for that enter exogenously, like our sets K_i.

[26] That does not exclude the well observed phenomenon that promised delivery amounts of a commodity like copper may far exceed the available supply. Some of the promises, in aggregate, can cancel out others.

Although Theorem 1 targets a two-stage equilibrium with a wide array of features beyond the one-stage equilibrium of Arrow and Debreu (1954), comparisons can be made with the historical attempts in that classical context to soften assumptions of strong survivability. The farthest advances can be attributed to Florig (2001a, b),[27] but already in (Arrow and Debreu 1954, Theorem 2) there are elements akin to our ample survivability that can be gleaned from the arguments, if not actually evident on the surface. A fundamental difference, of course, is the absence of anything like money in Arrow and Debreu (1954), but another is the focus there on including production. That led to such complications that the specialization of their conditions to pure exchange is hard to decipher with clarity.[28]

A retention scheme with some resemblance to ours has been employed by Seghir and Torres-Martinez (2011) in a context of "collateral ideas" inspired by unpublished suggestions of Geanakoplos and Zame (2002a, b). But in that work, adhering to perfect foresight and not trying to deal with "money," there is a fixed categorization of goods into being "durable" or not, and no trade-off between consumption or retention is available to the agents. Aside from that line of research and a contribution of Dubey and Geanakoplos (2003) (see below), perfect foresight models stemming from Radner (1972) have not allowed for retention. Indeed, they have made no attempt to reflect the possibility (not to speak of the reality) of continuing activity outside of their time framework and have restricted goods to being consumed in single time periods with no possibility of carry-over to a future period.

Such models have had difficulties with the value of "money," in particular, and that did not begin with them; a long history of struggles over money is recounted by Duffie (1990). A cogent summary, with various references, has been offered more recently by Walsh in his book on monetary theory (Walsh 2010, Chapter 2). The core problem is outlined very clearly by Geanakoplos (1990, Sect. 6). Fiat money, such as paper currency, might be treated as a special kind of good, but how then would it have any value? In common opinion, for a good to be positively priced at some time, some agent must want to consume it then. However, "consuming" money would be akin to removing quantities of it from the economy and that lacks plausibility as the natural platform from which money could function as a numéraire. Even if money could be assigned value in a model as a means of facilitating exchange, that value would drop away "at the end of time" and, in the propagation of perceptions back to the present, it would come out as worthless from the start.

Dubey and Geanakoplos (2003) have put together a model that is perhaps closest to ours in spirit, although very different in structure. They deal explicitly with fiat money and allow it to be retained. They also allow broadly for retention of "perishable goods" within a scheme of home production.[29] However, *no utility accrues from*

[27] Florig's results are so extremely subtle and complex in their statements that it is very hard to see how to apply them effectively to specific situations.

[28] We have held back here from trying, within our framework, to include intertemporal production carried out by a "firm" because of the serious additional modeling challenges it would entail. For current ideas on meeting those challenges, see Britz et al. (2015).

[29] It would be easy to add general "home production," i.e., production with a single agent-owner, to our model, but we have refrained from doing so in order not to obscure the main ideas in our approach.

their retention in contrast to ours, and money is therefore unable to acquire value in Keynesian response to uncertainty. In compensation, they introduce cash-in-advance constraints, requiring that purchases must be made through direct trades with cash. But cash received cannot be utilized until the following period except through government intervention with "short-term loans" within a period.[30] This approach is anyway inconsistent in our view with the modeling of markets by "periods" in which much can happen but only the *net results* are recorded. In such modeling, money should be able to move around, in particular. so that quantities involved in transactions can be netted out. The total value of transactions ought not then be bounded by the money supply within a given period.

Other contributions to the existence of equilibrium with incomplete markets, in line with ours in omitting production (unless merely confined in inputs and outputs to the separate states, which succumbs to easier treatment) and not insisting on every good being attractive to every agent, have not been as broadly based. Chae (1988)succeeded in obtaining it always if the contracts were restricted to paying multiples of some numéraire good, or basket of goods: *numéraire* assets. Laitenberger (1996) got it always for general real assets like Radner's by likewise replacing exogenous bounds on sales by transaction costs generated endogenously much like ours, but without "money." Developments building on Cass (1984) and Werner (1985) and centering on nominal assets only, thus having cope with the corresponding indeterminacy of equilibrium underscored by Magill and Shafer (1991, Sect. 3) as well as Geanakoplos (1990), have stopped at *generic* existence in a context of differential topology; see Duffie and Shafer (1985).

A commodity like gold might instead be the "monetary good" taken as numéraire, but then the disparate "units of account" in existing GEI theory would not be linked to a currency of the kind that financial markets typically do operate in. Gold as "commodity money" has been explored by Dubey et al. (2003) in the absence of future uncertainty and with attention only to efficiency of equilibrium, not its existence. In the article (Geanakoplos and Polemarchakis 1986) a special "money-like" good serves as numéraire and is the basis of all delivery contracts, but it is a commodity locked into consumption without possibilities for retention, and the assumptions deployed for establishing equilibrium are severe.[31] Numéraire assets appear in the work of Ageloni and Cornet (2006) but likewise in terms of a consumption good which is perishable and cannot be carried over; see also the papers of Aouani and Cornet (2009, 2011).

Keynesian underpinnings. Market models with gold or some other "consumption good" as numéraire can never embrace the simple economic truth that ordinary *money*, alone, provides the scale in which agents really compare values. Our ability here to deal directly with that relies on our approach to preferences which insists that agents

[30] Their two-party contracts deliver only in money, and deliveries must be carried out in cash. That could raise serious questions about supplies. Anyway, their equilibrium proof depends on very strong assumptions like *every* good being attractive to *every* agent for consumption, together with a complicated and indirect "gains to trade hypothesis."

[31] Furthermore the financial market has the peculiarity that profits made from selling holdings in other goods cannot be used to buy contracts.

always find money attractive for retention. That last point is crucial, of course, and we must now give our reasons for it.

The attractiveness of fiat money can partly be viewed as cultural, i.e., tied to social agreement, tradition, taxation, and confidence that it can store value—aspects suitably built into an agent's utility.[32,33] To these reasons for attractiveness, we add Keynesian considerations. In harmony with our aim of enabling a two-stage model to capture *ongoing* activity within a longer time line, an agent's utility for money can *reflect beliefs about future buying power, coming from past experience and shared expectations, however imprecise, just as it can incorporate "untested" assessments of uncertainties lying ahead*. The same can be said about the utility of retaining goods like land and other enduring resources. The support provided by good 0 in our economy will be elaborated now in terms of agents being surely able to borrow or lend money.

Buying a unit of contract 0 is tantamount to putting a unit of money aside in an interest-bearing savings account. We contrast this with just retaining a unit of money, not in such an account. The utility benefits of lending through contract 0, which arrive in the future, must be balanced with that retention utility in equilibrium. This feature, closely connected with the discounting of future income, enters the endogenous determination of the interest rate in equilibrium in our model. An appreciation of the difference between lending out money in this sense at time 0 and simply retaining it until time 1 is therefore all-important for a correct understanding of our approach.[34]

We take retention of money as referring to deposits in secure accounts that do not correspond to "investments" yielding interest.[35] Depositing money in an interest-bearing account, with the level of interest determined endogenously in equilibrium, is regarded instead as a version of *lending*. It relinquishes access, permitting the money to be used temporarily by another party, and it earns a reward as compensation. This is contrasted in our model with retaining the money and not getting that compensation.

The agents retaining money in various amounts do so for reasons related to their preferences, i.e., their utility. Among those reasons (despite imperfect mirroring due to gross discretization) are *liquidity* in the sense of unhindered access, and *distrust* over unexpected events in the interim. These are Keynesian financial motivations that were also deemed essential by Hicks (1935) in his thoughtful and enlightening essay, which still today makes good reading.[36]

[32] The so-called money illusion provides strong evidence. People commonly express preferences that involve money without facing up to the effects of inflation or deflation. This may better be interpreted as indicating that money enters preferences in other ways than just what it can be anticipated to be able to buy.

[33] Although in microeconomics an agent's utility has generally not been applied to money holdings, this has been commonplace in macroeconomics, starting with Sidrauski (1967).

[34] Inevitably, any model in this subject is highly distilled from the world in its discretization of time and uncertainty and must be assessed mainly for its usefulness in furnishing basic insights (as with mathematical models everywhere).

[35] This may be combined with holding on to a currency's notes and coins, which do, of course, pass from present to future unchanged in the possession of various agents.

[36] Hicks also saw "frictions" of dealing with a savings account as a disincentive relative to just retaining money. Here we will only impose transaction costs on borrowers, since that suffices with minimum complications, but inflicting them also on lenders/depositors would be easy.

Once this dichotomy between lending and retaining is accepted, the next question is how a balance between them would be set in an equilibrium. Our model's answer focuses on the utility that agents associate with either action and specifically on "money rates" which we introduce to convert money values at equilibrium into utility values. These rates fit into a study of the *marginal* utility of money that will be undertaken for its effects on preferences.

Reconsideration of future states and prices. More must be said now about the modeling of future states and prices. This is a complex issue of long standing; cf. Arrow (1974) for his penetrating remarks. The fundamental purpose of equilibrium theory is to determine how markets *in the present* can be brought into balance between supply and demand by *present* prices for goods and contracts. But market activity in the present is concerned in part with plans that agents make for the future, and those plans cannot help but be influenced by what agents anticipate about prices in the future. How can that influence be modeled "realistically" at the level of a relatively simple abstraction?

In one approach, that of "temporary equilibrium" cf. Grandmont (2006), agents act on beliefs and anticipations, but the different prices they individually come up with might have little connection with eventual supply and demand. Financial markets are modeled tenuously, and hedging opportunities are weak. In the alternative approach to equilibrium with incomplete markets that goes back to Radner (1972), the prices do make future markets clear, but the question of how these prices might be known in the present is not answered satisfactorily. The idea is roughly that the agents possess the ability to *guess* them correctly, with perfect foresight and in universal agreement.

The notion of a Walrasian "broker" (or "auctioneer") has proved valuable for modeling how the price of a good in some market can bring about a balance between supply and demand. Instead of agents having to bargain separately with each other until agreement is reached, the market functions as a sort of information exchange on supplies and demands which is coordinated by the broker entity. In Radner's scheme, such "brokers" operate in every future state as well as in the present, and agents are supposed to surmise the results of their operations *in advance*.

As we see it, though, the agents' perceptions about future prices must come from *information developed in the present*, if they are to make sense. Therefore, the "brokers" concerned with those prices must be understood as *acting in the present*. Agents, in trying to come up with optimal decisions, seek information about what the market situation might turn out to be in a given future state in connection with a buy/sell component of a plan. One can think of this as giving rise to a rough information process modeled by (abstract) "brokers" who, in response to accumulated requests of the sort, report back on what may be *previewed* about total supply and demand for a given good and state. That way, some basis for surmising market-clearing prices would be created, even though the true prices would only be known later.[37]

Rolling horizon. We adopt this interpretation here as more plausible than perfect foresight in explaining the role of projected future prices. However, it carries with it the caveat that spot markets *operating in the future* are not claimed to be represented in

[37] It must be stressed that the advances in our results would persist even if our future prices were interpreted as perfect forecasts. What would slip, however, would be the capability of building on Keynes' ideas about money to ensure it has value, which has never before been attempted in a GEI model.

the equilibrium. Everything really revolves around how *present* prices may respond to the agents' communal anticipations of the future, however imperfect. Although our model is formally two-stage, it is closer in this respect to being a one-stage model which partly tries to account for the future through *tentative* future exchanges of goods in combination with whatever hedges against uncertainty might be available through investments. Nevertheless, it has a *rolling horizon* quality: When the next stage of future truly arrives, the agents face a renewed version with a new time 0 and a new time 1.

We see this as offering a workable compromise in how to look at the future, but at the same time take it as advising that models directly including multiple future stages might not be soundly appropriate. It is hard to think of the markets in the fully branched multiplicity of uncertain states in all those stages as being assisted effectively in price projection by "broker entities" *in the present*. The conclusion then is that, for *microeconomic* theory as here,[38] a two-stage (or few-stage) approach to equilibrium may be better than one purporting to take a prolonged future into account. But to bring this now full circle, a two-stage approach can hardly be convincing if burdened by doomsday effects. That is a major motivation for our insistence on the retention of goods, instead of just consumption, as potentially having utility.

Guarding against the unforeseen. This attitude toward framing the future also opens the way for Keynes' important ideas about *unrepresented* eventualities to enter the picture. In modeling future states by information processes in the present, we implicitly tie them to eventualities that agents can contemplate *in common* without necessarily agreeing on their likelihood. Individual agents could have worries not shared by others, and for those concerns, there would be no market support or feedback. Our version of equilibrium accordingly makes no pretense to offer more than a partial glimpse of the future and thus *confronts the agents with significant irreducible uncertainties*. Those could well induce them to postpone some transactions and hang on to money in the manner suggested by Keynes,[39] which our model enables them to do. As underscored by Arrow (1974), markets for coping with the future are bound to be sparse. This can be understood as saying that agents will never be able to allay all concerns through markets and will have to take some precautions outside of them. Being able to retain money, and other goods as well independently of the available markets, can help.

4 Money versus utility

In understanding the implications of equilibrium in our model with money-denominated prices, an important role will be played by "money rates" which, at optimality in the utility problem of an agent i, convert money values in the states $s = 0, 1, \dots, S$ into utility values. Such rates, arising from Lagrange multipliers associated with budget constraints, are essential in appreciating how *an agent's decisions*

[38] In macroeconomic theory, a long-range future is fundamental, and even the notion of perfect foresight relative to rational expectations Muth (1961) is different.

[39] "The possession of actual money lulls our disquietude; and the premium we require to make us part with money is a measure of the degree of our disquietude" (Keynes 1978b). Note that this quote also supports the distinction between money held and money lent as an investment, which is fundamental to our treatment.

are actually based on an individual attitude toward future probabilities and discounting. The transaction costs we have introduced will be shown to force differences in those attitudes. This is an aspect of the model which deserves to be brought out carefully and examined for its effects. Lagrange multipliers are familiar in equilibrium studies, but their application to the agents' problems $\mathcal{P}_i(p, q)$, with concave instead of merely quasi-concave utilities, produces a saddle point characterization of optimality which has not been exploited for the valuable additional insights that accrue.

When we come to the variational representation of equilibrium in the next section, it will be important even to enlarge the format of the equilibrium to include the money rates as elements partnered with the market prices and the plans of the agents. In preparation for that later development, but helpful already below, we introduce *complementary slackness* notation which facilitates a more compact description of equilibrium. In terms of

$$\beta \in N_+(\alpha) \iff \alpha \geq 0, \quad \beta \leq 0, \quad \alpha\beta = 0, \tag{12}$$

a price system (p, q) furnishes an if and only if the problems $\mathcal{P}_i(p, q)$ have solutions for which the excess demands (7) satisfy

$$\sum_i d_{i0}(s, p(s)) = 0 \quad \text{and} \quad \sum_i d_{il}(s, p(s)) \in N_+(p_l(s))$$
$$\text{for } l = 1, \ldots, L \text{ in all states } s, \tag{13}$$

$$\sum_i z^+_{ik} - \sum_i z^-_{ik} \in N_+(q_k) \quad \text{for } k = 0, 1, \ldots, K. \tag{14}$$

In (14), the market clearing for contracts has been relaxed, but harmlessly. Agents are allowed to promise deliveries that no one will pay for, i.e., by issuing contracts k having price $q_k = 0$ (with unwanted deliveries assigned to disposal). However, that will not happen in equilibrium, as will be confirmed shortly.

Definition (*money rates*) The multiplier $\lambda_i(s)$ for the budget constraint of agent i in state s will be called the *money rate* of agent i in that state.

It rescales the money to units of (present) utility which can be balanced against preferences in the utility function u_i. The balance is expressed by the Lagrangian

$$\begin{aligned}
L_i(w_i, c_i, z^+_i, z^-_i; \lambda_i) &= u_i(w_i, c_i) \\
&\quad + \lambda_i(0)\Big(p(0)\,[\,e_i(0) - w_i(0) - c_i(0) - D(0, p(0))z^-_i\,] - q\,\left[z^+_i - z^-_i\right]\Big) \\
&\quad + \sum_{s>0} \lambda_i(s)p(s)\,\left[e_i(s) + A_i(s)w_i(0) + D(s, p(s))\left(z^+_i - z^-_i\right) - w_i(s) - c_i(s)\right] \\
&= u_i(w_i, c_i) - \lambda_i(0)p(0)d_i(0, p(0)) - \sum_{s>0} \lambda_i(s)p(s)d_i(s, p(s)) \\
&\quad - \lambda_i(0)q[z^+_i - z^-_i],
\end{aligned} \tag{15}$$

where the excess demand notation in (7) has been brought in along with the vector notation $\lambda_i = (\lambda_i(0), \lambda_i(1), \ldots, \lambda_i(S))$. For given λ_i (and prices), the maximization of L_i with respect to the other variables stands for an auxiliary problem in which agent i acts without paying attention to budgets, looking instead only at an aggregated utility based on investments, consumption and retention, both present and future. The Lagrangian expression (15) can also be organized as

$$\begin{aligned} L_i(w_i, c_i, z_i^+, z_i^-; \lambda_i) &= u_i(w_i, c_i) + \sum_s \lambda_i(s)p(s)e_i(s) - \sum_s \lambda_i(s)p(s)\,c_i(s) \\ &- \left[\lambda_i(0)p(0) - \sum_{s>0}\lambda_i(s)p(s)A_i(s)\right] w_i(0) - \sum_{s>0}\lambda_i(s)p(s)\,w_i(s) \\ &- \left[\lambda_i(0)q - \sum_{s>0}\lambda_i(s)p(s)D(s,p(s))\right] z_i^+ \\ &+ \left[\lambda_i(0)[q - p(0)D(0,p(0))] - \sum_{s>0}\lambda_i(s)p(s)D(s,p(s))\right] z_i^-. \end{aligned} \tag{16}$$

Theorem 2 (Saddle point characterization of an agent's optimality) *The decision elements* (9) *for agent i in problem $\mathcal{P}_i(p,q)$ are optimal if and only if, for some money rate vector λ_i, they provide a saddle point of the Lagrangian* (15)–(16) *with respect to maximization in the elements*

$$(w_i, c_i) \in U_i, \quad z_i^+ \in I\!R_+^K, \quad z_i^- \in I\!R_+^K(i), \tag{17}$$

and minimization with respect to $\lambda_i \geq 0$. Having a saddle point corresponds in this way to the following set of conditions:

(A) (w_i, c_i) *maximizes over U_i the expression*

$$\begin{aligned} &u_i(w_i, c_i) - \left[\lambda_i(0)p(0) - \sum_{s>0}\lambda_i(s)p(s)A_i(s)\right] w_i(0) \\ &- \sum_{s>0}\lambda_i(s)p(s)\,w_i(s) - \sum_s \lambda_i(s)p(s)\,c_i(s), \end{aligned}$$

(B)

$$\begin{cases} \sum_{s>0}\lambda_i(s)p(s)D_k(s,p(s)) - \lambda_i(0)q_k \in N_+(z_{ik}^+) \quad \text{for all } k, \\ \lambda_i(0)[q_k - p(0)D_k(0,p(0))] - \sum_{s>0}\lambda_i(s)p(s)D_k(s,p(s)) \\ \quad \in N_+\left(z_{ik}^-\right) \quad \text{for } k \notin K_i, \end{cases}$$

(C) $\lambda_i(s) > 0$ *for $s = 0, 1, \ldots, S$, and in the notation* (7), *also*

$$p(0)\,d_i(0,p(0)) + q[z_i^+ - z_i^-] = 0, \quad p(s)\,d_i(s,p(s)) = 0 \quad \text{for } s > 0.$$

The proof Theorem 2 goes as follows. Having a saddle point is always sufficient for optimality in a setting of convexity like this, and it is necessary under the Slater condition,[40] namely that the budget constraints can be satisfied with strict inequality; cf. (Rockafellar 1970, Sect. 28). That holds because of the strict inequalities for good 0 in (a) of our ample survivability assumption. The maximization half of the saddle point condition breaks down into the separate conditions in (A) and (B). Through (A) and the insatiability of utility, it requires all the money rates to be positive. The rest of (C) corresponds then to the minimization half of the saddle point condition.

Basic properties of money rates. Any money rate vector $\lambda_i = (\lambda_i(0), \lambda_i(1), \ldots, \lambda_i(S))$ in the saddle point condition of Theorem 2 satisfies

$$\lambda_i(s) > 0, \quad \lambda_i(0) - \sum_{s>0} \lambda_i(s) > 0, \tag{18}$$

so that in letting

$$\rho_i(s) = \frac{\lambda_i(s)}{\lambda_0(s)} \quad \text{for } s = 1, \ldots, S, \quad \rho_i = \sum_{s>0} \rho_i(s), \quad \pi_i(s) = \frac{\rho_i(s)}{\rho_i}, \tag{19}$$

one has $\lambda_i(0) - \sum_{s>0} \lambda_i(s) = [1 - \rho_i]\lambda_i(0)$ with

$$0 < \rho_i < 1 \quad \rho_i(s) = \rho_i \pi_i(s), \quad \pi_i(s) > 0, \quad \pi_i(1) + \cdots + \pi_i(S) = 1. \tag{20}$$

The positivity of $\lambda_i(s)$ in (18) for $s > 0$ is revealed by the maximization in (A) of Theorem 2 through the attractiveness assumed for the retention of money such states. The second inequality in (18) arises the same way from the more special maximization with respect to retention of money in state 0 and the definition of good 0 being "freely saved."

Definition (*discount rates and imputed probabilities*). The factor $\rho_i(s)$ is the *discount rate* of agent i for money in state s, whereas ρ_i is the *overall discount rate* of agent i for future money. The fraction $\pi_i(s)$ will be called the *imputed probability* of the future state s for agent i.

Similar discounts and probabilities are invoked (in different notation) in the Geanakoplos 1990 introduction to GEI theory (Geanakoplos 1990, Theorem 3) in order to shed more light on the prices of assets. We use them analogously in the theorems that follow, but our orientation is directly toward money and, in the case of contracts k, must account also for the influence of

$$\begin{aligned}\tau_k(p(0)) &= p(0)D_k(0, p(0)) \\ &= \text{the positive transaction cost in money for selling contract } k.\end{aligned} \tag{21}$$

[40] Optimality in the case of merely *quasi-concave* utility can't be characterized by a saddle point.

(The positivity of $\tau_k(p(0))$ was noted earlier, right after the definition of equilibrium.) A new issue, which is very important for us in relation to the Keynesian aspects of our approach, is understanding the trade-offs between contracts and retention in transmitting wealth to the future.

Theorem 3 (discounting effects in retention). *For goods $l \neq 0$ other than money, present and future prices are constrained by retention to satisfy*

$$\lambda_i(0)p_l(0) \;\geq\; \sum_{s>0} \lambda_i(s)p_l(s)A_i(s) \text{for the goods } l > 0, \tag{22}$$

which can be expressed in discounted expectation form as the martingale-like inequality

$$p_l(0) \geq \rho_i \sum_{s>0} \pi_i(s)p_l(s)A_i(s). \tag{23}$$

These relations must hold with strict inequality if good l is attractive to agent i for retention in state 0. On the other hand, they must hold as equations if the quantity of good l retained by agent i in state 0 is at a level at which the marginal utility for a decrease in that quantity is 0. Moreover, in an equilibrium these properties of goods prices must hold for all agents i simultaneously.

The inequalities in Theorem 3 are again, like (18) which corresponds to $l = 0$, immediate from the maximization in (A) of Theorem 2 through the monotonicity of u_i and the possible attractiveness of good l. The equation case takes advantage of our assumption that u_i is concave: A concave function has one-sided derivatives (possibly different) for increases and decreases in any variable, in particular. Because $\lambda_i(0) > 0$ from (18), the maximization in (A) would be thrown off in the circumstance described unless the difference in (22) were 0.

This result has special importance for the modeling of one-sided financial instruments like pre-existing bonds as investment goods by way of our retention features. Perhaps an agent might get some enjoyment, however slight, from holding a bond, in which case strict inequality would be seen in (23), but there is no compulsion toward that in our framework. In the absence of such attractiveness, the price of a bond would obey (23) as a martingale equation. Bear in mind, though, that the imputed probabilities are those of agent i only. The combined effect from all the agents, as indicated in the final part of the theorem, would be more crucial.

Theorem 4 (discounting effects on contract prices). *In line with the transaction costs in (21), the prices q_k of the contracts k must satisfy the martingale-like inequalities*

$$q_k - \tau_k(p(0)) \;\leq\; \rho_i \sum_{s>0} \pi_i(s)p(s)\, D_k(s, p(s)) \;\leq\; q_k. \tag{24}$$

Equality holds on the right when $z^+_{ik} > 0$, i.e., agent i buys some of contract k, but equality holds on the left when $z^-_{ik} > 0$, i.e., agent i sells some of contract k. Thus, *agent i will not buy any of contract k if strict inequality holds on the right and will not sell any of contract k if strict inequality holds on the left. Moreover, in any equilibrium these properties of contract prices must hold for all agents i simultaneously.*

The relations in Theorem 4 re-express the complementary slackness conditions in (B) of Theorem 2 through the discount rates and imputed probabilities we have introduced. Observe that the position of the price q_k in the strict interval between the bounds in (24) partitions the agents i into three distinct categories: agents who potentially could buy, agents who potentially could sell, and agents who definitely would refrain from either buying or selling contract k. This is similar to the analysis of Arrow and Hahn (1999) of the influence of transaction costs.

The case of contract $k = 0$, which delivers one unit of the money good 0 in every future state s, deserves particular attention because of what it tells about the overall discount rates of the agents. Recall here our assumption that all agents can buy and sell contract 0; cf. (4).

Corollary 1 (discounting effects on lending and borrowing). *For contract 0, with transaction cost $\tau_0(p(0)) > 0$, Theorem 4 requires that*

$$q_0 - \tau_0(p(0)) \;\leq\; \rho_i \;\leq\; q_0. \tag{25}$$

Equality holds on the right when $z^+_{i0} > 0$, i.e., agent i lends some money, but equality holds on the left when $z^-_{i0} > 0$, i.e., agent i borrows some money. Thus, agent i will not consider lending money unless $\rho_i = q_0$ and will not consider borrowing money unless $\rho_i = q_0 - \tau_0(p(0))$. If $q_0 > \rho_i > q_0 - \tau_0(p(0))$, agent i will definitely neither lend nor borrow. In particular, an agent i having $\rho_i < q_0$ would prefer passing money to the future through retention rather than lending.

These effects can alternatively be seen from the perspective of the corresponding endogenously determined interest rates:

$$\frac{1}{q_0} - 1 = \text{ the interest rate received by lenders,}$$
$$\frac{1}{q_0 - \tau_0(p(0))} - 1 = \text{the interest rate paid by borrowers.}$$

The final assertion of Corollary 1 pins down a Keynesian reason why an agent would prefer holding onto money instead of investing it. A discount rate ρ_i lower than the market rate q_0 could signal a distrust of the market's appraisal of the uncertainties being faced.

Corollary 2 (existence of borrowers). *If in an equilibrium there are any agents at all who borrow money, then*

$$q_0 = \max_{i=1,\dots,I} \rho_i, \quad q_0 - \tau_0(p(0)) = \min_{i=1,\dots,I} \rho_i, \quad q_0 > \tau_0(p(0)).$$

Without the transaction cost $\tau_0(p(0)) > 0$ for borrowing money, everything would simplify, of course. Equality would hold throughout in (25), and every agent would have the same discount rate $\rho = q_0$. Even then, the imputed probabilities $\pi_i(s)$ of the agents would not have to agree, although other relations in Theorem 4 and earlier in Theorem 3 could place severe limitations on the extent of their disagreement. But in our equilibrium, we do have a positive transaction cost $\tau_0(p(0))$, so that, unless no agent at all lends or borrows, some difference in their discount rates is inevitable in any equilibrium, as reflected in Corollary 2.

As the concluding contribution in this section, we characterize the money rates of agent i as describing the marginal utility of having slightly more or less money in the present or future. The money already available, and possible changes to it, are captured by the vectors

$$e_{i0} = (e_{i0}(0), e_{i0}(1), \ldots, e_{i0}(S)), \quad \Delta e_{i0} = (\Delta e_{i0}(0), \Delta e_{i0}(1), \ldots, \Delta e_{i0}(S)).$$

The corresponding effects on maximum utility, with prices and the endowments in all other goods fixed, are described by the (utility) *value function*

$$v_i(\Delta e_{i0}) = \begin{cases} \text{maximum utility in problem } \mathcal{P}_i(p,q) \\ \text{when } e_{i0} \text{ is replaced by } e_{i0} + \Delta e_{i0}. \end{cases}$$

Here, $v_i(0)$ equals the maximum utility in the unmodified problem $\mathcal{P}_i(p,q)$. The *marginal utility* with respect to a shift in money endowments in the direction of Δe_{i0} is accordingly defined to be

$$dv_i(\Delta e_{i0}) = \lim_{\varepsilon \searrow 0} \frac{v_i(\varepsilon \Delta e_{i0}) - v_i(0)}{\varepsilon}.$$

Theorem 5 (money rates and marginal utility). *Under ample survivability, the value function v_i is concave with $v_i(0)$ finite, and if agent i actually has positive endowments of money in all states,*[41] *v_i is surely finite on some neighborhood of 0. Then the marginal value function dv_i is not only well defined and finite but also expressed by*

$$dv_i(\Delta e_{i0}) = \min_{\lambda_i \in \Lambda_i} \left\{ \lambda_i(0)\Delta e_{i0}(0) + \lambda_i(1)\Delta e_{i0}(1) + \cdots + \lambda_i(S)\Delta e_{i0}(S) \right\},$$

where Λ_i is the set of all money rate vectors λ_i associated with optimality in Theorem 2, this being a nonempty, compact convex set.

If there is only one such money rate vector $\lambda_i \in \Lambda_i$, then v_i is differentiable at 0 with this λ_i as its gradient vector. In that case,

$$\lambda_i(s) = \frac{\partial v_i}{\partial e_{i0}(s)}(0) = \text{the marginal utility of money in state } s.$$

[41] This assumption, for convenience, really loses no generality because ample survivability requires a positive endowment of money initially, and a tiny amount of could freely be saved.

This result, although new from the economic standpoint, at least in characterizing marginal utility in optimality with rigor, is a standard consequence of the saddle point representation of optimality in Theorem 2. The role of Lagrange multipliers in expressing the one-sided derivatives of value functions in optimization problems with convexity is fully laid out in Rockafellar and Wets (1997, Chapter 11H).

5 Variational formulation

Variational inequalities have not previously been employed in the theory of economic equilibrium, apart from some publications in mathematics concerning *one-stage* models, cf. the book of Facchinei and Pang (2003) and our own papers (Jofre et al. 2005, 2007), and their references. A brief summary of the facts and concepts is desirable therefore before proceeding with this methodology, which is so central to our approach.

We start with subgradient sets and normal cones of convex analysis. A proper convex function on $I\!R^n$ is a convex function $f : I\!R^n \to (-\infty, \infty]$ for which the set

$$\operatorname{dom} f = \{x \mid f(x) < \infty\}, \quad \text{the effective domain of } f,$$

is nonempty. The convexity of f corresponds to the convexity of the set

$$\operatorname{epi} f = \{(x, \alpha) \in I\!R^n \times I\!R \mid f(x) < \alpha\}, \quad \text{the epigraph of } f.$$

Lower semicontinuity of f corresponds to the epigraph being closed. The set of *subgradients* of f at a point x is defined by

$$y \in \partial f(x) \iff f(x') \geq f(x) + y[x' - x] \quad \text{for all } x'. \tag{26}$$

It is a closed convex set, surely empty if $x \notin \operatorname{dom} f$, and reduces to a single element y if and only if f is differentiable at x, with y then being the gradient $\nabla f(x)$. The set of pairs satisfying (26) is regarded in general as the graph of a "set-valued" mapping ∂f from $I\!R^n$ to $I\!R^n$.

An important special case occurs when f is the indicator of a nonempty, closed, convex set $C \subset I\!R^n$, namely with $\delta_C(x) = 0$ if $x \in C$ but $\delta_C(x) = \infty$ if $x \notin C$. Then $\operatorname{dom} f = C$, and the subgradient mapping ∂f reduces to the normal cone mapping N_C associated with C, for which

$$y \in N_C(x) \iff x \in C \quad \text{and} \quad y \cdot [x' - x] \leq 0 \quad \text{for all} \quad x' \in C. \tag{27}$$

Having $x \in \operatorname{int} C$ is equivalent to having $N_C(x)$ consist of just $y = 0$.
Variational inequalities. In terms of a vector function $F : I\!R^n \to I\!R^n$,

(a) a variational inequality of *functional* type has the form

$$-F(x) \in \partial f(x)$$

for a proper, lower semicontinuous, convex function f on $I\!R^n$;

(b) a variational inequality of *geometric* type has the form

$$-F(x) \in N_C(x)$$

for a nonempty, closed, convex set C in $I\!R^n$.

These conditions first gained popularity in infinite-dimensional engineering applications involving partial differential operators. The "inequality" in their name comes from the possibility of writing them through (26) or (27) as systems of inequalities on $F(x)$. They are "variational" that way through interpretation of the inequalities, but more profoundly by their connection to the classical framework of the inverse theorem or implicit function theorem, in which a problem gets embedded parametrically in family of similar problems which indicate how it may vary.

A variational inequality of either type is actually a sort of "generalized equation." Indeed, one simply gets $F(x) = 0$, the vector version of n real equations in n real unknowns, when $f = \delta_C$ with $C = I\!R^n$, i.e., when $f \equiv 0$. Variational inequalities of geometric type go beyond simple equations by insisting that certain normality relations hold on the boundary of a constraining set C. A remarkable feature is that, when $F \in \mathcal{C}^1$, the set

$$G = \left\{ (v, x) \in I\!R^n \times I\!R^n \;\middle|\; v - F(x) \in \partial f(x) \right\} \tag{28}$$

is a "Lipschitz manifold" of dimension n in $I\!R^n \times I\!R^n$. Solutions to $-F(x) \in \partial f(x)$ are obtained in principle by intersecting G with the subspace $v = 0$ in $I\!R^n \times I\!R^n$. This reveals a function-like quality of the graph G which is comparable to the case of an equation $F(x) = 0$, where G is the graph of the (generally set-valued) inverse of F. In that setting, the existence and potential uniqueness of a solution x and how it may vary when v shifts away from 0, are tied to the classical inverse function theorem and a full rank condition on Jacobian matrix for F at x. Broader parameterizations $F(v, x) = 0$ bring up the classical implicit function theorem. Those theorems have heavily been employed for a long time in economics and other areas, to handle specific issues but also in judging whether a problem may be reasonably posed, e.g. in having "the number of equations equal to the number of unknowns." In fact, though, that picture has solidly been extended in more recent times from equations to variational inequalities, with derivatives of F replaced by generalized one-sided derivatives. This is explained in much detail in Dontchev and Rockafellar (2009); other useful background can be found in Rockafellar and Wets (1998, Chapter 12).

The main thing to appreciate is that, by managing to arrive at a variational inequality formulation of some problem, one is not just exercising a preference, but enabling passage into a rich universe of analysis beyond classical calculus.

For computing solutions, most of the attention has been paid so far to the class of variational inequalities that are *monotone* in the sense that[42] $[F(x') - F(x)]\cdot[x' - x] \geq$

[42] This sense of monotonicity, with a long history and literature in the mathematics of optimization and partial differential equations, takes the opposite sign from the one often associated with this term in economics.

0 for all x, x'. Such monotonicity guarantees in particular that the set of solutions, if not a singleton, is at least a closed, convex set. In economics, of course, a multiplicity of isolated equilibria is normally expected instead in the absence of uniqueness. There is no surprise, then, in the fact that the equilibrium variational inequality about to be developed here will *not* be monotone.

To capture an equilibrium in our present context and analyze it by this methodology, we need to take advantage of the way that variational inequalities can be built up in "modules."

Modular formulation of variational inequalities. A family of conditions

$$F_j(x_1, \dots, x_r) \in \partial f_j(x_j) \quad \text{for } j = 1, \dots, r \quad \text{with} \quad x_j \in I\!R^{n_j}, \tag{29}$$

in which each f_j is a proper, lower semicontinuous, convex function on $I\!R^{n_j}$ and each F_j is a function from $I\!R^{n_1} \times \cdots \times I\!R^{n_r} \to I\!R^{n_j}$, is equivalent to the (functional-type) variational inequality $-F(x) \in \partial f(x)$ for the proper, lower semicontinuous, convex function

$$f(x) = f(x_1, \dots, x_r) = f_1(x_1) + \cdots + f_r(x_r) \tag{30}$$

and the function $F : I\!R^{n_1} \times \cdots \times I\!R^{n_r} \to I\!R^{n_1} \times \cdots \times I\!R^{n_r}$ given by

$$F(x) = F(x_1, \dots, x_r) = \Big(F_1(x_1, \dots, x_r), \dots, F_r(x_1, \dots, x_r)\Big). \tag{31}$$

A crucial aspect in this formulation is that every real variable among the components of $x = (x_1, \dots, x_r) \in I\!R^n$ with $n = n_1 + \cdots + n_r$ has a place finally in $-F(x) \in \partial f(x)$. Of course some of the functions f_j could be indicators δ_{C_j}, so that the composite variational inequality could incorporate normality conditions $F_j(x_1, \dots, x_r) \in N_{C_j}(x_j)$ on x_j which also depend perhaps on other vectors $x_{j'}$ with $j' \neq j$. Anyway, the domain will in general be the product of the effective domains $\operatorname{dom} f_i$ of the functions f_i.

In our equilibrium model, there will be functional modules coming from the utility functions u_i, but primarily we will have geometric modules coming from complementary slackness relations in the N_+ notation of (12) through the fact that

$$N_+ = N_C \ \text{ for } \ C = [0, \infty) \subset I\!R^1. \tag{32}$$

Another feature is motivated by the significance of the money rates $\lambda_i(s)$ in providing discount rates and imputed probabilities. This suggests that an equilibrium ought to incorporate these money rates directly, as the dual variables in the agent's optimization problems.

Definition (*enhanced equilibrium*). The specification of an equilibrium price system (p, q), together with elements w_i, c_i, z_i^+, z_i^-, solving the associated problems $\mathcal{P}_i(p, q)$ and the money rate vectors λ_i that combine with them in optimality, as in Theorem 2, will be called an *enhanced* equilibrium.

Enhanced equilibrium that includes Lagrange multipliers for budget constraints was previously treated in our paper (Jofre et al. 2007) in the context of a one-stage model. In following that pattern here, we have the additional feature of money interpretations but also incentives coming from variational analysis. By these means, we achieve a variational inequality representation which is in prime condition for eventually applying the perturbation tools in Dontchev and Rockafellar (2009) for understanding the stability of GEI equilibrium. Results already obtained about the stability of equilibrium in one-stage models of exchange, without future states, suggest a high potential for new insights from such a coming project, cf. Dontchev and Rockafellar (2012), Jofre et al. (2013).

Theorem 6 (variational representation of enhanced equilibrium). *Elements p, q, w_i, c_i and λ_i furnish an enhanced equilibrium if and only if they solve the composite variational inequality with the following components for the agents i and contracts k:*

$$-\Big(\lambda_i(0)p(0) - \sum_{s>0}\lambda_i(s)p(s)A_i(s), \lambda_i(0)p(0); \dots; \lambda_i(s)p(s), \lambda_i(s)p(s); \dots\Big)$$
$$\in \partial[-u_i]\Big(w_i(0), c_i(0); \dots; w_i(s), c_i(s); \dots\Big), \tag{A}$$

$$\begin{cases} \sum_{s>0}\lambda_i(s)p(s)D_k(s, p(s)) - \lambda_i(0)q_k \in N_+(z_{ik}^+), \\ \lambda_i(0)[q_k - p(0)D_k(0, p(0))] \\ \quad - \sum_{s>0}\lambda_i(s)p(s)D_k(s, p(s)) \in N_+(z_{ik}^-) \text{ when } k \notin K_i, \end{cases} \tag{B}$$

$$\begin{cases} p(0)\, d_i(0, p(0)) + q[z_i^+ - z_i^-] \in N_+(\lambda_i(0)), \\ p(s)\, d_i(s, p(s)) \in N_+(\lambda_i(s)) \quad \text{for } s > 0, \end{cases} \tag{C}$$

$$\sum_i d_{il}(s, p(s)) \in N_+(p_l(s)) \quad \text{for the goods } l = 1, \dots, L \text{ in all states } s, \tag{D}$$

$$\sum_i z_{ik}^+ - \sum_i z_{ik}^- \in N_+(q_k). \tag{E}$$

Condition (A) here simply re-expresses (A) of Theorem 2 by subgradients of the convex functions $-u_i$. Conditions (B), (D), and (E) come unchanged from (B) of Theorem 2 and the market-clearing conditions (13) and (14). In (C), we have the complementary slackness form of the budget conditions in (C) of Theorem 2, and this is equivalent because the multipliers $\lambda_i(s)$ must in fact be positive in optimality. The only feature to raise a question is the absence in (D) of the equilibrium equations for good 0 in (13). However, those equations follow from the other conditions. Specifically, because the budget constraints must hold with equality, we have for $s = 0$ that

$$0 = \sum_i \Big[p(0)\, d_i(0, p(0)) + q\left[z_i^+ - z_i^-\right]\Big]$$
$$= \sum_i d_{i0}(0, p(0)) + \sum_{l>0} p_l(0) \sum_i d_{il}(0, p(0)) + q\Big[\sum_i z_i^+ - \sum_i z_i^-\Big]$$

where all terms but $\sum_i d_{i0}(0, p(0))$ are known to vanish by complementary slackness. Then $\sum_i d_{i0}(0, p(0)) = 0$ as well. The same argument works for $s > 0$, without the q part.

It has to be underscored here that explicit inclusion of the money rates $\lambda_i(s)$ was a key to achieving this representation (hence the notion of an *enhanced* equilibrium). Furthermore, this relied on concavity rather than just quasi-concavity of the utility functions u_i. Note that the portfolio variables z_{ik}^- for $k \notin K_i$ have effectively been suppressed in light of the obligation constraints (5) on the agents.

Representing an equilibrium by a variational inequality is one thing, and proving its existence as a solution to that variational inequality is another. The basic tool is the following "structured" version of a fixed-point theorem.

Existence criterion. A functional variational inequality $-F(x) \in \partial f(x)$ for a proper, lower semicontinuous, convex function f has a solution, in particular, when

(a) dom f is bounded,
(b) F is continuous relative to the closure of dom f.

In the special case of a geometric variational inequality $-F(x) \in N_C(x)$ for a nonempty, closed, convex set C, both dom f and its closure are replaced by C.

We established this criterion in Jofre et al. (2007) with an argument which invokes a basic fixed-point theorem in the context of special properties enjoyed by subgradient mappings ∂f,[43] and wish to put it to work here. Condition (b) of the criterion poses no difficulties in our equilibrium context, but an immediate impediment is a lack of the boundedness demanded by (a). Indeed, *none* of the components in Theorem 6 has bounded domain. But this is a familiar circumstance of in equilibrium theory, even if previously approached by economists from other directions. Carefully articulated *truncations* must be introduced, and that is how the proof of Theorem 1 is achieved.

Appendix: truncations and the existence proof

Let $\mathcal{V}_0$ denote the variational inequality of Theorem 6 for which we are seeking a solution. Step by step, we will replace $\mathcal{V}_0$ by other variational inequalities with smaller domains until we arrive at one with bounded domain, which therefore has a solution. We will execute this in such a manner that the solution we get must also be a solution to $\mathcal{V}_0$.

To get started down this track, we consider what happens when a complementary slackness condition (11), corresponding to $N_+ = N_{[0,\infty)}$, is replaced by $N_+^\eta = N_{[0,\eta]}$ for some $\eta \in (0, \infty)$:

$$\begin{aligned} &\beta \in N_+^\eta(\alpha) \iff \beta \le 0 \quad \text{for}\, \alpha = 0, \quad \beta = 0 \quad \text{for}\, 0 < \alpha < \eta, \\ &\beta \ge 0 \quad \text{for}\, \alpha = \eta. \end{aligned} \tag{33}$$

[43] The derivation is simple because, in terms of the "resolvent" $P_f = (I + \partial f)^{-1}$, the condition $-F(x) \in \partial f(x)$ is equivalent to having $M(x) = x$ for $M(x) = P_f(-F(x))$. The resolvent P_f maps the whole space single-valued into dom f and is Lipschitz continuous with constant 1. If F is continuous, M therefore maps the closure of the convex set dom f continuously into itself and has to have a fixed-point when dom f is bounded.

It's important to observe that

$$\beta \in N_+^\eta(\alpha) \implies \alpha\beta = \eta \max\{0, \beta\}. \tag{34}$$

For any $\eta \in (0, \infty)$, let $\mathcal{V}_1(\eta)$ denote the variational inequality obtained from $\mathcal{V}_0$ through replacement of (D) and (E) by

$$\sum_i d_{il}(s, p(s)) \in N_+^\eta(p_l(s)) \quad \text{for the goods } l = 1, \ldots, L, \tag{D$_\eta$}$$

$$\sum_i z_{ik}^+ - \sum_i z_{ik}^- \in N_+^\eta(q_k), \tag{E$_\eta$}$$

which entail by (34) that

$$\begin{aligned} p_l(s) \sum_i d_{il}(s, p(s)) &= \eta \max\Big\{0, \sum_i d_{il}(s, p(s))\Big\}, \\ q_k\Big[\sum_i z_{ik}^+ - \sum_i z_{ik}^-\Big] &= \eta \max\Big\{0, \sum_i z_{ik}^+ - \sum_i z_{ik}^-\Big\}. \end{aligned} \tag{35}$$

Obviously, since this modification has no effect on (A), (B) and (C) of Theorem 6, which are equivalent to the saddle point expression of optimality in Theorem 2. When we pass from $\mathcal{V}_0$ to $\mathcal{V}_1(\eta)$, we are thus dealing with a modified formulation of economic equilibrium in which the agents are confronted with the same utility maximization problems $\mathcal{P}_i(p, q)$, but the market-clearing requirements have undergone a sort of "η-relaxation."

In what follows, however, we also wish to contemplate truncations with respect to goods and portfolios in the agents' problems. Assistance will come from the notation that

$$G_\mu = \big\{\text{the vectors in } I\!R^{1+L} \text{ having all components } \leq \mu\big\} \tag{36}$$

We fix $\bar{\eta} \in (0, \infty)$ and deal with elements $\hat{w}_i$ and $\hat{c}_i$ such as appear in the assumption of ample survivability. As observed ahead of Theorem 1, there is no loss of generality in supposing for these elements that actually

$$\hat{d}_{i0}(s) < 0 \quad \text{for } s = 1, \ldots, S, \quad \text{as well as for } s = 0. \tag{37}$$

Choose $\bar{\mu}$ high enough that

$$\hat{w}_i(s) \in G_{\bar{\mu}} \quad \text{and} \quad \hat{c}_i(s) \in G_{\bar{\mu}} \quad \text{for all } s. \tag{38}$$

For $\mu \in [\bar{\mu}, \infty)$, we define potential substitutes u_i^μ for the utility functions u_i by

$$u_i^\mu(w_i, c_i) = \begin{cases} u_i(w_i, c_i) \text{ if } w_i(s) \in G_\mu \text{ and } c_i(s) \in G_\mu \text{ for all } s \\ \quad \text{along with } u_i(w_i, c_i) \geq u_i(\hat{w}_i, \hat{c}_i) - 1, \\ -\infty \quad \text{otherwise.} \end{cases} \tag{39}$$

Then u_i^μ, like u_i, is concave and upper semicontinuous, and its associated domain U_i^μ (i.e., the set where u_i^μ is finite) is nonempty, convex and bounded. The subgradient condition

$$\begin{aligned} &- \Big(\lambda_i(0)p(0) - \sum_{s>0} \lambda_i(s)p(s)A_i(s), \lambda_i(0)p(0); \ldots; \lambda_i(s)p(s), \lambda_i(s)p(s); \ldots\Big) \\ &\quad \in \partial\left[-u_i^\mu\right]\Big(w_i(0), c_i(0); \ldots; w_i(s), c_i(s); \ldots\Big), \end{aligned} \tag{A_μ}$$

can potentially serve therefore as a substitute for (A) which fits with our modular variational inequality scheme.

We denote by $\mathcal{V}_2(\eta, \mu)$ the variation inequality obtained from $\mathcal{V}_1(\eta)$ by substituting (A_μ) for (A) and at the same time replacing (B) by

$$\begin{cases} \sum_{s>0} \lambda_i(s)p(s)D_k(s, p(s)) - \lambda_i(0)q_k \in N_+^\mu\left(z_{ik}^+\right), \\ \lambda_i(0)[q_k - p(0)D_k(0, p(0))] \\ \quad - \sum_{s>0} \lambda_i(s)p(s)D_k(s, p(s)) \in N_+^\mu\left(z_{ik}^-\right), \quad \text{when } k \notin K_i \end{cases} \tag{B_μ}$$

Step 1 For the problems $\mathcal{P}_i^\mu(p, q)$ obtained by substituting u_i^μ for u_i, conditions (A_μ), (B_μ) and (C) characterize optimality in terms of a saddle point of the corresponding Lagrangian L_i^μ just as (A), (B) and (C) do in Theorem 2 for the problems $\mathcal{P}_i(p, q)$.

This is elementary but underscores the fact that $\mathcal{V}_2(\eta, \mu)$ stands for a version of "η-relaxed" equilibrium in which the agents' problems have undergone truncation.

Step 2 There exists $\bar{\eta} \in (0, \infty)$ such that, for all $\eta \in [\bar{\eta}, \infty)$ and $\mu \in [\bar{\mu}, \infty)$, the solutions to the variational inequality $\mathcal{V}_1(\eta)$ (if any) are the same as those of the variational inequality $\mathcal{V}_2(\eta, \mu)$.

A solution to $\mathcal{V}_2(\eta, \mu)$ will also solve $\mathcal{V}_1(\eta)$ if the additional bounds in the truncated problems $\mathcal{P}_i^\mu(p, q)$ are not active. This will certainly be true for the utility bound entering the definition of u_i^μ in (38): namely since $(\hat{w}_i, \hat{c}_i)$ satisfies (38) and thus, together with the (0, 0) portfolio, furnishes a feasible solution to $\mathcal{P}_i^\mu(p, q)$, any optimal solution (w_i, c_i, z_i^+, z_i^-) to $\mathcal{P}_i^\nu(p, q)$ must have $u_i(w_i, c_i) \geq u_i(\hat{w}_i, \hat{c}_i)$, not merely $u_i(w_i, c_i) \geq u_i(\hat{w}_i, \hat{c}_i) - 1$.

The issue in Step 2 can be settled, therefore, by demonstrating that the conditions (D_η) and (E_η) that are common to $\mathcal{V}_1(\eta)$ and $\mathcal{V}_2(\eta, \mu)$ already produce, by themselves, bounds on goods and portfolios which make the further bounds introduced with μ be inactive when μ is high enough. For this we first note that, by adding over all agents i the budget equations that are guaranteed by (C), we must have

$$\begin{aligned} 0 &= p(0)\sum_i d_i(0, p(0)) + q\Big[\sum_i z_i^+ - \sum_i z_i^-\Big] \\ &= \sum_i d_{i0}(0, p(0)) + \sum_{l>0} p_l(0)\sum_i d_{il}(0, p(0)) \\ &\quad + \sum_k q_k\Big[\sum_i z_{ik}^+ - \sum_i z_{ik}^-\Big], \\ 0 &= \sum_i d_{i0}(s, p(s)) + \sum_{l>0} p_l(s)\sum_i d_{il}(s, p(s)) \quad \text{for } s > 0, \end{aligned} \tag{40}$$

where, in the notation of (7),

$$\begin{aligned} d_i(0, p(0)) &= w_i(0) + c_i(0)) + D(0, p(0))z_i^- - e_i(0), \\ d_i(s, p(s)) &= w_i(s) + c_i(s) - D(s, p(s))\left[z_i^+ - z_i^-\right] - e_i(s) - A_i(s)w_i(0) \quad \text{for } s > 0, \\ &\quad \text{with goods components } \ d_{il}(s, p(s)), \ \ d_{il}(0, p(0)), \quad \text{for } l = 0, 1, \ldots, L. \end{aligned}$$

The relations in (35) coming from (D_η) and (E_η) translate (40) into

$$\begin{aligned} -\sum_i d_{i0}(0, p(0)) \;&=\; \eta\sum_{l>0}\max\Big\{0, \sum_i d_{il}(0, p(0))\Big\} \\ &\qquad + \eta\sum_k q_k \max\Big\{0, \sum_i z_{ik}^+ - \sum_i z_{ik}^-\Big\}, \\ -\sum_i d_{i0}(s, p(s)) \;&=\; \eta\sum_{l>0}\max\Big\{0, \sum_i d_{il}(s, p(s))\Big\}. \end{aligned} \tag{41}$$

In the first equation of (41), we have $-\sum_i d_{i0}(0, p(0)) \le \sum_i e_{i0}(0)$. Recalling our assumption in the specification of $D(0, p(0))$ that there exists, independently of $p(0)$, of a lower bound $D(0, p(0)) \ge D^*(0) \ge 0$ in which the matrix $D^*(0)$ has at least one positive entry in each column, we see that the first equation in (41), after being turned into an inequality by lowering η to $\bar\eta$, places upper bounds on the nonnegative vectors $w_i(0)$, $c_i(0)$ and z_i^- which are independent of the particular $\eta \ge \bar\eta$. The $w_i(0)$ bounds then induce an upper bound on the left side of the second equation in (41), and with η again lowered to $\bar\eta$, that yields upper bounds independent of the particular $\eta \ge \bar\eta$ for the vectors $w_i(s)$ and $c_i(s)$ as well as, through our assumptions on the matrices $D(s, p(s))$, an estimate for the size $\sum_i z_{ik}^+ - \sum_i z_{ik}^-$. That estimate, with the bounds already obtained for the vectors z_{ik}^- places bounds on the vectors z_{ik}^-. We now merely have to take μ high enough that none of these bounds can be active.

Step 3 For $\bar\mu$ as in Step 2, there further exists $\bar\zeta \in (0, \infty)$ large enough that, for any $\eta \in [\bar\eta, \infty)$ and $\mu \in [\bar\mu, \infty)$, solutions to $\mathcal{V}_2(\eta, \mu)$ are sure to have

$$u_i(w_i, c_i) \le \bar\zeta \ \ \text{and} \ \ \lambda_i(s) < \bar\zeta \quad \text{for } s = 0, 1, \ldots, S. \tag{42}$$

The first upper bound in (42) results from the bounds in Step 2 for $\bar{\mu}$ and the upper semicontinuity of u_i. In terms of $\bar{\zeta}_i$ being the max of u_i over a closed set associated with those bounds, we can take $\bar{\zeta} > \max_i \bar{\zeta}_i$. For the bounds on $\lambda_i(s)$, we appeal to the saddle point condition for optimality in $\mathcal{P}_i^{\mu}(p, q)$ mentioned in Step 1. That condition says, in part, that

$$L_i^{\mu}(w_i, c_i, z_{i+}, z_i^-; \lambda_i) \;\geq\; L_i^{\mu}(\hat{w}_i, \hat{c}_i, 0, 0; \lambda_i).$$

Because the budget constraints in $\mathcal{P}_i^{\mu}(p, q)$ must hold as equations in optimality by (C), we have

$$L_i^{\mu}(w_i, c_i, z_{i+}, z_i^-; \lambda_i) \;=\; u_i(w_i, c_i) \leq \bar{\zeta}_i.$$

via (38). Consequently, though the formula for $L_i^{\mu}(\hat{w}_i, \hat{c}_i, 0, 0; \lambda_i)$ corresponding to the one in (15) for L_i in terms of excess demands, we have

$$\begin{aligned} \bar{\zeta}_i \;\geq\; u_i(\hat{w}_i, \hat{c}_i) - \sum_s \lambda_i(s) p(s) \hat{d}_i(s) \;&=\; u_i(\hat{w}_i, \hat{c}_i) \\ - \sum_s \lambda_i(s) \Big[\hat{d}_{i0}(s) + \sum_{l>0} p_l(s) \hat{d}_{il}(s) \Big]&. \end{aligned} \tag{43}$$

Condition (a) of ample survivability allows the sums over $l > 0$ to be dropped without upsetting the inequality, and as enhanced in (37), provides us then with the upper bounds $\lambda_i(s) \leq \bar{\zeta}_i / |\hat{d}_{i0}(s)|$. Taking $\bar{\zeta}$ greater than these bounds produces the desired result.

The bounds achieved in Step 3 furnish the platform for truncating the one condition in $\mathcal{V}_0$ that has not been modified until now, namely (C), to

$$p(0)\, d_i(0, p(0)) + q[z_i^+ - z_i^-] \in N_+^{\zeta}(\lambda_i(0)), \quad p(s)\, d_i(s, p(s)) \in N_+^{\zeta}(\lambda_i(s)) \quad \text{for } s > 0, \tag{C_ζ}$$

Let $\mathcal{V}_3(\eta, \mu, \zeta)$ be the variational inequality obtained from $\mathcal{V}_2(\eta, \mu)$ with (C_ζ) replacing (C).

Step 4 For $\bar{\mu}$ and $\bar{\zeta}$ as in Steps 2 and 3 and the variational inequality $\mathcal{V}_3(\eta, \mu, \zeta)$ with respect to any choice of $\eta \in [\bar{\eta}, \infty)$, $\mu \in [\bar{\mu}, \infty)$ and $\zeta \in [\bar{\zeta}, \infty)$,

(a) solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ are the same as the solutions to $\mathcal{V}_1(\eta)$,
(b) a solution to $\mathcal{V}_3(\eta, \mu, \zeta)$ exists.

Here (a) summarizes what we already know from Step 3, whereas (b) holds by the existence criterion above, inasmuch as truncations have made the domain in $\mathcal{V}_3(\eta, \mu, \zeta)$ be bounded. Only one thing still remains: demonstrating that by taking η large enough we can ensure that the price bounds from (D_η) and (E_η) will be inactive, so that the solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ must actually be solutions to original variational inequality $\mathcal{V}_0$. A lower bound on the multipliers, complementary to the upper bound in Step 3, will help us toward this goal.

Step 5 There exists $\varepsilon > 0$ such that, as long as $\mu \in [\bar{\mu} + 1, \infty)$ and $\zeta \in [\bar{\zeta}, \infty)$ as well as $\eta \in [\bar{\eta}, \infty)$, solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ will have

$$\lambda_i(s) \geq \varepsilon \quad \text{for } s = 0, 1, \ldots, S. \tag{44}$$

To see this, fix an s, initially >0 because that case is easier, and let w_i^+ denote for any w_i the modification in which the component $w_{i0}(s)$ is replaced by $w_{i0}(s) + 1$ but all other components are kept the same. Our focus is on condition (A_μ), which implies for the elements (w_i, c_i) in solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ that

$$u_i^\mu(w_i^+, c_i) \leq u_i^\mu(w_i, c_i) + \lambda_i(s). \tag{45}$$

We know from Step 2 that in such a solution the vector components of w_i and c_i in the various states must lie in $G_{\bar{\mu}}$, in the notation (36), and the corresponding vector components of w_i^+ will then lie in G_μ, inasmuch as $\mu \geq \bar{\mu}+1$. In that case we have from the definition of u_i^μ in (39) that $u_i^\mu(w_i, c_i) = u_i(w_i, c_i)$ and $u_i^\mu(w_i^+, c_i) = u_i(w_i^+, c_i)$, along with $u_i(\hat{w}_i, \hat{c}_i) - 1 \leq u_i(w_i^+, c_i)$, so that (45) yields

$$u_i(\hat{w}_i, \hat{c}_i) - 1 \leq u_i(w_i^+, c_i) \leq u_i(w_i, c_i) + \lambda_i(s). \tag{46}$$

We claim that for w_i and c_i having vector components in $G_{\bar{\mu}}$, whether or not they are part of a solution to $\mathcal{V}_2(\eta, \mu, \zeta)$, there is a positive lower bound to the values of $\lambda_i(s)$ occurring in (46).

Indeed, if a lower bound were not available, there would be a sequence of elements (w_i^n, c_i^n) with vector components in $G_{\bar{\mu}}$ such that

$$u_i(\hat{w}_i, \hat{c}_i) - 1 \ \leq \ u_i([w_i^n]^+, c_i^n) \leq u_i(w_i^n, c_i^n) + \lambda_i^n(s) \ \text{ for } n = 1, 2, \ldots, \ \text{ with } \lambda_i^n(s) \to 0. \tag{47}$$

The boundedness of the goods vectors allows us to suppose, without loss of generality that (w_i^n, c_i^n) converges as $n \to \infty$ to some (w_i^∞, c_i^∞), in which case $([w_i^n]^+, c_i^n)$ converges to $([w_i^\infty]^+, c_i^\infty)$. Under our assumptions, u_i is continuous relative to the set $\{(w_i, c_i) \,|\, u_i(w_i, c_i) \geq u_i(\hat{w}_i, \hat{c}_i) - 1\}$, which is closed, so we get in (47) as $n \to \infty$ that $u_i([w_i^\infty]^+, c_i^\infty) \leq u_i(w_i^\infty, c_i^\infty)$. This contradicts the insatiability of u_i with respect to good 0.

The argument for the case of $s = 0$ is essentially the same, but with $\lambda_i(0)$ initially replaced by $\lambda_i(0) - \theta_i$, where θ_i is the component for good 0 in the vector $\sum_{s>0} \lambda_i(s) p(s) A_i(s)$ appearing in (A), or for that matter, (A_μ). Since $\theta_i \geq 0$, it can be removed and we can proceed with $\lambda_i(0)$ by itself just as in the argument already given.

Step 6 There is a bound ψ such that, in any solution to the variational inequality $\mathcal{V}_3(\eta, \mu, \zeta)$ with $\eta \in [\bar{\eta}, \infty)$, $\mu \in [\bar{\mu} + 1, \infty)$ and $\zeta \in [\bar{\zeta}, \infty)$, the prices satisfy

$$p_l(s) < \psi \text{ for all } l > 0 \text{ and states } s = 0, 1, \ldots, S, \text{ and } q_k < \psi \text{ for all } k.$$

In order to confirm this, we return to the inequalities in (43), where we have through (a) of ample survivability that $\hat{d}_{i0}(s) < 0$ and $\hat{d}_{il}(s) \leq 0$ for $l > 0$ and therefore

$$\bar{\zeta}_i \ \geq \ u_i(\hat{w}_i, \hat{c}_i) - \varepsilon \sum_s \Big[\hat{d}_{i0}(s) + \sum_{l>0} p_l(s)\hat{d}_{il}(s)\Big]$$

when $\lambda_i(s)$ is replaced by the lower bound in Step 5. This implies that

$$\sum_{l>0} p_l(s)[-\hat{d}_{il}(s)] \ \leq \ [\bar{\zeta}_i - u_i(\hat{w}_i, \hat{c}_i)]/\varepsilon \text{ for } s = 0, 1, \ldots, S.$$

Adding now over i and invoking from part (b) in the assumption of ample survivability the property that $\sum_i[-\hat{d}_{il}(s)] > 0$, we obtain upper bounds on the prices $p_l(s)$.

Condition (B_μ) in $\mathcal{V}_3(\eta, \mu, \zeta)$ now has a role for the prices q_k. We already know that it reduces to (B) of $\mathcal{V}_0$ as a property of solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$, because the μ upper bounds on the portfolio variables are inactive in a solution to $\mathcal{V}_3(\eta, \mu, \zeta)$ for the choices stipulated for μ. Condition (B) entails

$$\lambda_i(0)[q_k - p(0)D_k(0, p(0))] - \sum_{s>0} \lambda_i(s)p(s)D_k(s, p(s)) \leq 0 \quad \text{when } k \notin K_i.$$

There is at least one i with $k \notin K_i$ by (6), and for that i then we have

$$q_k \leq \frac{1}{\lambda_i(0)}\Big[p(0)D_k(0, p(0)) + \sum_{s>0} \lambda_i(s)p(s)D_k(s, p(s))\Big]$$

Utilizing the lower bound ε on $\lambda_i(0)$ in Step 5 together with the upper bound in Step 3 on $\lambda_i(s)$ for $s > 0$ and the upper bound on the p prices that we have just produced, and recalling the continuous dependence of the D_k vectors on those prices, we arrive at an upper bound on q_k.

Concluding argument. We already knew from Step 4 that, by taking μ and ζ large enough, we could get the solutions to the fully truncated variational inequality $\mathcal{V}_3(\eta, \mu, \zeta)$ to come out the same as the solutions to $\mathcal{V}_1(\eta)$ for all $\eta \in [\bar{\eta}, \infty)$. Now, though, we know further that by taking η larger than the bound ψ in Step 6, we can make the truncations in (D_η) and (E_η) be inactive in solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ and hence also in $\mathcal{V}_1(\eta)$. In this case, the solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ can be identified with the solutions to $\mathcal{V}_0$. Since the existence of a solution to $\mathcal{V}_3(\eta, \mu, \zeta)$ has been established, this verifies the existence of a solution to $\mathcal{V}_0$, which we set out to prove.

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