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RESEARCH ARTICLE

# The value of public information in common-value Tullock contests

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**Abstract** Consider a symmetric common-value Tullock contest with incomplete information in which the players' cost of effort is the product of a random variable and a deterministic real function of effort, $d$. We show that the *Arrow–Pratt curvature* of $d$, $R_d$, determines the effect on equilibrium efforts and payoffs of the increased flexibility/reduced commitment that more information introduces into the contest: If $R_d$ is increasing, then effort decreases (increases) with the level of information when the cost of effort (value) is independent of the state of nature. Moreover, if $R_d$ is increasing (decreasing), then the value of public information is nonnegative (nonpositive).

**Keywords** Tullock contests · Common values · Value of public information

**JEL Classfication** C72 · D44 · D82

## 1 Introduction

We study how changes in the information available to the players of a symmetric common-value Tullock contest with incomplete information affect their equilibrium payoffs and their incentives to exert effort. In a *Tullock contest* a player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players—see Tullock (1980). In a symmetric common-value contest with incomplete

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information, players have a common state-dependent value for the prize and a common state-dependent cost of effort, and all players have the same information.

There are a variety of economic settings (rent-seeking, innovation tournaments, patent races) in which agents face a game strategically equivalent to a Tullock contest—see Baye and Hoppe (2003). Tullock contests may also arise by design, e.g., in sport competition or internal labor markets—see Konrad (2008) for a general survey. Skaperdas and Gan (1995) and Clark and Riis (1998) provide alternative axiomatizations of Tullock contests.

In our setting, players' uncertainty about their common value and common cost is described by a probability space, and players' information is described by a subfield of the field on which players' common prior is defined. Representing players' information as a $\sigma$-subfield (rather than as a partition) allows us to capture situations in which, for example, players' value and/or cost are continuous random variables and players' information comes from observing a continuous signal. (In the setting considered by Wasser (2013), for which we derive results in Proposition 5.2, players are uncertain about their constant marginal cost of effort, which is the realization of a continuous random variable. If players observe a noisy public signal of their marginal cost, then we may not be able to represent their information as a partition.) In this model, changes in the level of information are conveniently represented as changes in the subfield describing the players' information. (When players' uncertainty can be represented as a partition, our model is equivalent to Harsanyi's model—see Jackson (1993) and Vohra (1999).)

We begin by showing that every contest in which players' cost of effort is a twice differentiable, strictly increasing, and convex function in every state has a unique equilibrium in pure strategies, which is symmetric and interior. We establish this result by first showing along lines of the proof of Szidarovszky and Okuguchi (1997)'s Theorem 1 that the complete information game defined by the realized state of nature has a unique equilibrium, which is symmetric and interior. (Establishing this result when the cost function is convex, rather than strictly convex, allows as to deal with the linear case.) Then, we construct an equilibrium of the Bayesian game of incomplete information associated with the contest appealing to the argument of Theorem 3.1 in Einy et al. (2003)—EMS (2003) henceforth. Our existence result implies those obtained by Warneryd (2003) and Wasser (2013), which deal with the two polar cases in which players have either full information or just the prior information. Einy et al. (2015) have recently established a general existence theorem for Tullock contests with incomplete information when the private information of each player is described by a countable partition of the space of states of nature, and have provided conditions for uniqueness of equilibrium. These results do not apply to our setting, in which the players' information may not be generated by a countable partition of the space of states of nature.

There is a well-known formal equivalence between Tullock contests and the Cournot model. This equivalence allows us to use some auxiliary results obtained in EMS (2003), which studies the value of public information in a Cournot duopoly. Unlike EMS (2003), however, we do not assume that the cost function is linear, but allow instead for any convex function. Also, our results apply to generalized Tullock contest, for which a player's probability of wining the prize is the ratio between her *score* and the sum of the scores of all the players, provided the score is a twice differentiable, increasing, and concave function of effort. In contrast, EMS (2003) assumes that the demand function, whose role in the Cournot model is akin to that of the contest success function in a Tullock contest, is log concave. Also, unlike EMS (2003), we allow for any finite number of players instead of just two, and we derive results about the impact of information on players' equilibrium expected efforts as well as on their payoffs—EMS (2003) is concerned exclusively with the value of public information.

For the class of contests in which equilibrium is unique and symmetric, the question "how changes in the level of information available to the players affects their equilibrium expected payoffs and efforts" is well posed. We are able to provide an answer to this question when the players' cost of effort is a multiplicative function, that is, when it is the product of a random variable and a deterministic real-valued function $d$ of the player's effort. Following EMS (2003), given a function $d$ and a pair of random variables $(v, w)$ describing, respectively, the players' common value and common cost, which are the uncertain elements of the contest, we define a binary relation that ranks information subfields according to the level of information they contain: A subfield $\mathcal{H}$ is more informative than some other subfield $\mathcal{G}$ if the predictions of the value and cost are the same whether players' information is given by $\mathcal{H}$ or by the aggregate information in $\mathcal{H}$ and $\mathcal{G}$.

More information allows the contest's participants more flexibility when choosing how much effort they want to exert, but reduces their ability to commit to exert a low effort when, e.g., the value of the prize is high. We define two auxiliary real-valued functions, $S$ and $U$, which provide the equilibrium expected effort and payoff, respectively, in a contest in which $v$ and $w$ are positive constant random variables and players have full information (i.e., their information field is the field on which the common prior is defined). It turns out that the curvature of these functions determines the effect on equilibrium expected effort and payoff, respectively, of the increased flexibility/reduced commitment that more information introduces into the contest: If $S$ is convex (concave), then the players' expected effort increases (decreases) with the level of information. Likewise, if $U$ is convex (concave), then the players' expected payoff increases (decreases) with the level of information, i.e., the value of public information is nonnegative (nonpositive). Moreover, the conditions leading to either of these functions been either concave or convex are related to the *Arrow–Pratt curvature* of the function $d$, the deterministic component of the cost function. (In expected utility theory, the Arrow–Pratt curvature of an individual's utility function is a measure of his relative risk aversion.)

Using our results relating the curvature of the auxiliary functions $S$ and $U$ to the effect of changes in information on equilibrium expected efforts and payoffs, we show that if the Arrow–Pratt curvature of $d$ is increasing, then the equilibrium expected effort decreases with the level of information in contests in which the cost of effort is independent of the state of nature, and increases with the level of information in contests in which the value is independent of the state of nature. Moreover, if the Arrow–Pratt curvature of $d$ is increasing (decreasing), then the value of public information is nonnegative (nonpositive) in every symmetric common-value Tullock contest with incomplete information in the class defined by the function $d$.

An interesting implication of our results is that if players' efforts are monetary, i.e., if the function $d$ is the identity, then the Arrow–Pratt curvature of $d$ is constant, and therefore the value of information is zero (i.e., payoffs are invariant to changes in information). If the cost of effort is independent of the state of nature, then the equilibrium expected effort is also invariant to changes in information, whereas if the common value is independent of the state of nature, then the equilibrium expected effort increases with the level of information—see Example 5.4.

In contrast, if $d$ is a convex quadratic cost function, for example, then the Arrow–Pratt curvature of $d$ is increasing, and therefore the value of information is nonnegative. If in addition the cost of effort is independent of the state of nature, then players exert less effort the better informed they are. It is not difficult, however, to find examples in which the cost of effort is state-dependent, and players exert more effort the better informed they are—see Example 5.4.

The impact of public information on the equilibrium expected payoffs and efforts in Tullock contests has been seldom studied in the literature. For two-player *generalized* Tullock contests, in which the prize is allocated using some score function $g$, and efforts are monetary (i.e., the cost of effort is independent of the state of nature and $d$ is the identity), Warneryd (2003) studies the equilibrium expected efforts for two polar information structures: When players' information about the value is just their common prior, and when they observe the value. Warneryd (2003) finds that whether the equilibrium expected effort is greater or less for one or the other information structure depends on whether the ratio $g/g'$ is a concave or convex function. This result is easily derived in our setting and extended to contests with any number of players and arbitrary information structures. Moreover, we are able to evaluate as well the impact of changes in the level of information on the equilibrium payoffs of generalized Tullock contests—see Proposition 5.3.

Wasser (2013) studies Tullock contests in which the players' constant marginal cost of effort is uncertain—see also Myerson and Warneryd (2006). For symmetric contests, Wasser (2013)'s Proposition 3 shows that when players' information about the common marginal cost is just their prior information, they exert less effort than when they observe the marginal cost. We show that this conclusion, which is an implication of our results, extends to any two comparable information structures. Moreover, we show that in these contests, the value of public information is zero, i.e., equilibrium payoffs are invariant to changes in the level of information—see Proposition 5.2.

Other related work includes Morath and Münster (2013) and Kovenock et al. (2013), who study the incentives for information acquisition and information sharing, respectively, in all-pay auction contests, and Denter et al. (2011), who identify conditions under which a mandated transparency policy on lobbying leads to an increase in efforts. Of course, there is a large literature studying the value of information and the incentives for information acquisition in auctions.

## 2 Symmetric common-value Tullock contests

In a Tullock contest, a group of players $N = \{1, \ldots, n\}$, with $n \geq 2$, compete for a prize by choosing a level of *effort* in $\mathbb{R}_+$. Given a profile of players' efforts

$x \in \mathbb{R}^n_+ \backslash \{0\}$, the prize is allocated to player $i \in N$ with probability $\bar{\rho}_i(x) = x_i/\bar{x}$, where $\bar{x} \equiv \sum_{j=1}^N x_j$, whereas if $x = 0$, i.e., if players exert no effort, then the prize is allocated using some predetermined probability vector $\bar{\rho}(0) \in \Delta^n$. We assume that players are uncertain about their common value for the prize and their common cost function. This uncertainty is described by a probability space $(\Omega, \mathcal{F}, p)$, where $\Omega$ is the set of states of nature, $\mathcal{F}$ is a $\sigma$-field of subsets of $\Omega$, and $p$ is $\sigma$-additive probability measure on $\mathcal{F}$. We interpret $p$ as the players' common prior belief about the realized state of nature. The players' value of the prize is described by an integrable function $v : \Omega \to \mathbb{R}_{++}$. The player's cost is described by a function $c : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for every integrable function $s : \Omega \to \mathbb{R}_+$, $c(\cdot, s(\cdot))$ is integrable. The players' information about the state of nature is described by a $\sigma$-subfield of $\mathcal{F}$, $\mathcal{G}$, specifying the event observed by players following each realization of the state of nature. We therefore identify a *symmetric common-value Tullock contest with incomplete information* with a collection $T = (N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G})$. (The description of a Tullock contests omits any reference to the probability distribution $\bar{\rho}(0)$ used to allocate the prize when players exert no effort since, as we show in the proof of Lemma 6.1 in the Appendix, under our assumptions the unique equilibrium of $T$ is independent of $\bar{\rho}(0)$.)

A symmetric common-value Tullock contest with incomplete information $T = (N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G})$ defines a Bayesian game $G(T)$ in which the set of actions of each player is $\mathbb{R}_+$, and the payoff function of each player $i \in N$ is $u_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}$ given for every $\omega \in \Omega$ and $x \in \mathbb{R}^n_+$ by

$$u_i(\omega, x) = \bar{\rho}_i(x)v(\omega) - c(\omega, x_i).$$

In this game, a pure strategy of player $i \in N$ is an integrable $\mathcal{G}$-measurable function $s_i : \Omega \to \mathbb{R}_+$ specifying player $i$'s effort in each state of nature. (Requiring that a strategy be $\mathcal{G}$-measurable restricts the events on which a player may condition her actions to those that she observes.) Given a strategy profile $s = (s_1, \dots, s_n)$, we denote by $s_{-i}$ the profile obtained from $s$ by suppressing the strategy of player $i \in N$. A (pure strategy) *Bayesian Nash equilibrium* of a symmetric common-value Tullock contest $T$ is a (pure strategy) Bayesian Nash equilibrium of $G(T)$. Throughout the paper, we restrict attention to pure strategy equilibria. An explicit definition of equilibrium follows.

Let $T = (N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G})$ be a symmetric common-value Tullock with incomplete information. If $X$ is an integrable random variable on $(\Omega, \mathcal{F}, p)$, and $\mathcal{H}$ is a $\sigma$-subfield of $\mathcal{F}$, we write $\mathbb{E}[X \mid \mathcal{H}]$ for the conditional expectation of $X$ with respect to $\mathcal{H}$. A profile of strategies $s^* = (s_1^*, \dots, s_n^*)$ is *Bayesian Nash equilibrium* of $T$ if for every player $i \in N$, every pure strategy $s_i$ of player $i$, and almost all $\omega \in \Omega$,

$$\mathbb{E}[u_i(\cdot, s^*(\cdot)) \mid \mathcal{G}](\omega) \geq \mathbb{E}[u_i(\cdot, s_{-i}^*(\cdot), s_i(\cdot)) \mid \mathcal{G}](\omega).$$

Our first result establishes conditions implying the existence and uniqueness of a pure strategy equilibrium in symmetric common-value Tullock contests with incomplete information.

**Theorem 2.1** *A symmetric common-value Tullock contest with incomplete information in which the players' cost function $c(\omega, \cdot)$ is twice differentiable, strictly increasing, convex, and satisfies $c(\omega, 0) = 0$ for all $\omega \in \Omega$ has a unique (pure strategy) Bayesian Nash equilibrium, $s^*$. Moreover, $s^*$ is symmetric and interior, i.e., $s_1^*(\omega) = s_2^*(\omega) = \ldots = s_n^*(\omega) > 0$ for all $\omega \in \Omega$.*

*Proof* For every $\omega \in \Omega$, define the $n$-person complete information game $G(\omega, T)$ in which the set of pure strategies of every player is $\mathbb{R}_+$ and the payoff function of each player $i \in N$, $h_i(\omega, \cdot) : \mathbb{R}_+^n \to \mathbb{R}_+$, is given for $x \in \mathbb{R}_+^n$ by

$$h_i(\omega, x) = \mathbb{E}[u_i(\cdot, x) \mid \mathcal{G}](\omega).$$

The game $G(\omega, T)$ has a unique Nash equilibrium $t^*(\omega) = (t_1^*(\omega), \ldots, t_n^*(\omega))$, which is symmetric and interior, i.e., $t_1^*(\omega) = t_2^*(\omega) = \ldots = t_n^*(\omega) > 0$. (We establish this result in the Appendix, Lemma 6.1, along the lines of Szidarovszky and Okuguchi (1997)'s Theorem 1.) Using an argument analogous to that of the proof in Theorem 3.1 in Einy et al. (2003), one can show that the strategy profile $s^* \in S^n$ given for $\omega \in \Omega$ by $s^*(\omega) = t^*(\omega)$ is a Bayesian Nash equilibrium of the Bayesian game $G(T)$. Uniqueness, symmetry, and interiority follow from the fact that for all $\omega \in \Omega$, the profile $t^*(\omega) \in \mathbb{R}_+^n$ is the unique Nash equilibrium of $G(\omega, T)$, and $t_1^*(\omega) = t_2^*(\omega) = \ldots = t_n^*(\omega) > 0$. □

Theorem 2.1 holds on a broader class of *generalized* symmetric common-value Tullock contests with incomplete information in which the prize is allocated according to a *contests success function* $\rho : \Omega \times \mathbb{R}_+^n \to \Delta^n$ given for $(\omega, x) \in \Omega \times \mathbb{R}_+^n \backslash \{0\}$ and $i \in N$ by

$$\rho_i(\omega, x) = \frac{g(\omega, x_i)}{\sum_{j=1}^n g(\omega, x_j)},$$

where $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *score* function such that for all $\omega \in \Omega$, $g(\omega, \cdot)$ is twice differentiable, strictly increasing and concave, and satisfies $g(\omega, 0) = 0$. In the Bayesian game defined by a generalized Tullock contest, $(T, g)$, where $T = (N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G})$, the payoff function of each player $i \in N$ is given for all $(\omega, x) \in \Omega \times \mathbb{R}_+^n$ by

$$u_i(\omega, x) = \rho_i(\omega, x)v(\omega) - c(\omega, x_i).$$

Hence, there is a bijection between the equilibrium sets of this contest $(T, g)$ and the Tullock contest $\hat{T} = (N, (\Omega, \mathcal{F}, p), v, \hat{c}, \mathcal{G})$, in which the cost function is $\hat{c}(\omega, \cdot) = g^{-1}(\omega, \cdot) \circ c(\omega, \cdot)$ for all $\omega \in \Omega$ . The next remark, which makes precise this relation, will be useful to derive the implications for generalized Tullock contests of the results obtained in Sects. 3 and 4 for Tullock contests.

*Remark 2.2* A symmetric common-value generalized Tullock contest with incomplete information $((N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G}), g)$ in which the players' cost function satisfies the assumptions of Theorem 2.1 and the score function $g(\omega, \cdot)$ is twice differentiable, strictly increasing and concave, and satisfies $g(\omega, 0) = 0$ for all $\omega \in \Omega$ has a unique (pure strategy) Bayesian Nash equilibrium $\hat{s}^*$ . Moreover, $\hat{s}^*$ is symmetric and interior, and is given for all $\omega \in \Omega$ by $\hat{s}^*(\omega) = g^{-1}(\omega, s^*(\omega))$, where $s^*$ is the unique Bayesian Nash equilibrium of the Tullock contest $(N, (\Omega, F, p), v, \hat{c}, \mathcal{G})$, with $\hat{c}(\omega, \cdot) = g^{-1}(\omega, \cdot) \circ c(\omega, \cdot)$ for all $\omega \in \Omega$.

In order to study the effect of information on equilibrium efforts and payoffs, we restrict attention to the class of symmetric common-value Tullock contests with incomplete information in which for all $(\omega, x) \in \Omega \times \mathbb{R}_+$ , the players' cost is

$$c(\omega, x) = w(\omega)d(x),$$

where $w$ is a nonnegative integrable random variable, i.e., $w \in L^1_+(\Omega, \mathcal{F}, p)$, and $d$ is a deterministic real-valued function.

Let $d$ be a twice differentiable, strictly increasing, and convex function such that $d(0) = 0$. We denote by $\mathcal{T}(d)$ the family of all symmetric common-value Tullock contests with incomplete information, $(N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G})$, defined by a pair of nonnegative integrable random variables $(v, w) \in L^1_+(\Omega, \mathcal{F}, p) \times L^1_+(\Omega, \mathcal{F}, p)$, and a $\sigma$-subfield of $\mathcal{F}$, $\mathcal{G}$.

Let $(v, w) \in L^1_+(\Omega, \mathcal{F}, p) \times L^1_+(\Omega, \mathcal{F}, p)$, and let $\mathcal{G}$ and $\mathcal{H}$ be any two $\sigma$-subfields of $\mathcal{F}$. We say that $\mathcal{H}$ is *weakly more informative* than $\mathcal{G}$, and we write $\mathcal{H} \succsim \mathcal{G}$, if

$$\mathbb{E}(v \mid \mathcal{H}) = \mathbb{E}(v \mid \mathcal{G} \vee \mathcal{H}) \text{ and } \mathbb{E}(w \mid \mathcal{H}) = \mathbb{E}(w \mid \mathcal{G} \vee \mathcal{H}),$$

where $\mathcal{G} \vee \mathcal{H}$ is the smallest $\sigma$-subfield of $\mathcal{F}$ that contains both $\mathcal{G}$ and $\mathcal{H}$. That is, $\mathcal{H}$ is weakly more informative than $\mathcal{G}$ if the predictions of the value and the cost (the uncertain elements of the contest) are the same whether players' information is given by $\mathcal{H}$ or it is given by the aggregate information in $\mathcal{G}$ and $\mathcal{H}$. Note that $\mathcal{H} \succsim \mathcal{G}$ whenever $\mathcal{H}$ is finer than $\mathcal{G}$.

For any $\sigma$-subfield of $\mathcal{F}$, $\mathcal{G}$, we denote by $s^*_{\mathcal{G}}$ and $u^*_{\mathcal{G}}$ the equilibrium strategy and payoff of every player in the Bayesian Nash equilibrium of the contest $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$. (These mappings are well defined because by Theorem 2.1 each contest has a unique and symmetric equilibrium.) Let $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$. We say that the *value of public information* in $T$ *is nonnegative (nonpositive)* if for every contest $(N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{H}) \in \mathcal{T}(d)$,

$$\mathcal{H} \succsim \mathcal{G} \Rightarrow \mathbb{E}(u^*_{\mathcal{H}}) \geq \mathbb{E}(u^*_{\mathcal{G}}) \; (\mathbb{E}(u^*_{\mathcal{H}}) \leq \mathbb{E}(u^*_{\mathcal{G}})).$$

Also, we say that the *equilibrium expected effort is decreasing (increasing) with the level of information* in $T$ if for every contest $(N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{H}) \in \mathcal{T}(d)$,

$$\mathcal{H} \succsim \mathcal{G} \Rightarrow \mathbb{E}(s^*_{\mathcal{G}}) \geq \mathbb{E}(s^*_{\mathcal{G}}) \; (\mathbb{E}(s^*_{\mathcal{H}}) \leq \mathbb{E}(s^*_{\mathcal{G}})).$$

## 3 Information and effort

We study the effect of changes in the level of information on the equilibrium expected effort. For each $(a, b) \in \mathbb{R}^2_{++}$, we denote by $s(a, b)$ the strategy of each player in the unique Bayesian Nash equilibrium of the contest $(N, (\Omega, \mathcal{F}, p), a1_\Omega, (b1_\Omega)\, d, \mathcal{F})$ and write

$$S(a, b) := \mathbb{E}(s(a, b)) \tag{1}$$

for the equilibrium expected effort. Proposition 3.1 establishes an auxiliary result relating the effect of changes in the level of information on the equilibrium expected effort to the curvature of the function $S$ (specifically, whether it is convex or concave). We omit the proof of this proposition since it is identical to that of Proposition 3.3 in EMS (2003), which establishes this result by a simple argument involving the Law of Iterated Expectations and Jensen's Inequality.

**Proposition 3.1** *Assume that $d$ is twice differentiable, strictly increasing, and convex, and such that $d(0) = 0$. If the function $S$ is convex (concave) on $\mathbb{R}^2_{++}$, then the equilibrium expected effort increases (decreases) with the level of information in every symmetric common-value Tullock contest with incomplete information $T \in \mathcal{T}(d)$.*

Let $d$ be a twice differentiable, strictly increasing, and convex function satisfying $d(0) = 0$. For all $(a, b) \in \mathbb{R}^2_{++}$ the contest $(N, (\Omega, \mathcal{F}, p), a1_\Omega, (b1_\Omega)\, d, \mathcal{F})$ has a unique symmetric Bayesian Nash equilibrium by Theorem 2.1. In this equilibrium, the strategy of every player satisfies $s(a, b) > 0$. Therefore,

$$\bar{\rho}_i(s(a, b)(\omega), \ldots, s(a, b)(\omega)) = \frac{1}{n} \tag{2}$$

for all $i \in N$ and $\omega \in \Omega$, i.e., in equilibrium all players win the prize with the same probability. Moreover, since $s(a, b)$ maximizes

$$\mathbb{E}\left[u_i(\cdot, s(a, b), \ldots, s(a, b), x_i) \mid \mathcal{G}\right](\omega) = \mathbb{E}\left[\frac{x_i}{(n-1)s(a, b)+x_i} a - bd(x_i) \mid \mathcal{G}\right](\omega)$$

for all $\omega \in \Omega$, the first-order condition

$$\mathbb{E}\left[\frac{a\,(n-1)}{n^2 s(a, b)} \mid \mathcal{G}\right](\omega) = b\mathbb{E}\left[d'(s(a, b)) \mid \mathcal{G}\right](\omega)$$

holds for all $\omega \in \Omega$. Since $s(a, b)$ is $\mathcal{G}$ -measurable, then

$$s(a, b)d'(s(a, b)) = \frac{n-1}{n^2 b} a. \tag{3}$$

Proposition 3.2 provides conditions under which the curvature of the deterministic component of the players' cost (specifically the sign of the second derivative of the product $xd'(x)$) determines the effects of changes in the level of information on the equilibrium expected effort. For all $a, b \in \mathbb{R}_{++}$, let

$$\bar{S}(a) := S(a, 1), \text{ and } \hat{S}(b) := S(1, b).$$

The functions $\bar{S}$ and $\hat{S}$ identify the equilibrium expected effort of every player in the unique Bayesian Nash equilibrium of the contests $(N, (\Omega, \mathcal{F}, p), a1_\Omega, (1_\Omega)\, d, \mathcal{F})$ and $(N, (\Omega, \mathcal{F}, p), 1_\Omega, (b1_\Omega)\, d, \mathcal{F})$, respectively.

**Proposition 3.2** *Assume that d is thrice differentiable, strictly increasing and convex, and satisfies $d(0) = 0$, and let $T \in \mathcal{T}(d)$ be a symmetric common-value Tullock contest with incomplete information.*

(3.2.1) *If $w$ is constant on $\Omega$ and $\left(xd'(x)\right)''$ is nonpositive (nonnegative) on $\mathbb{R}_+$, then the equilibrium expected effort increases (decreases) with the level of information in $T$.*

(3.2.2) *If $v$ is constant on $\Omega$ and $\left(xd'(x)\right)''$ is nonpositive on $\mathbb{R}_+$, then the equilibrium expected effort increases with the level of information in $T$.*

*Proof* We prove Proposition 3.2.1. Differentiating Eq. (3) with respect to $a$ we get

$$\left(sd'(s)\right)'(a,b)s_a(a,b) = \frac{n-1}{n^2 b}.$$

Hence,

$$s_a(a,b) = \frac{n-1}{n^2 b\,(sd'(s))'(a,b)} > 0, \tag{4}$$

i.e., the equilibrium effort increases with the players' common value of the prize, $a$. Differentiating this expression we get

$$s_{aa}(a,b) = -\frac{n-1}{n^2 b}\frac{\left(sd'(s)\right)''(a,b)s_a(a,b)}{\left((sd'(s))'(a,b)\right)^2} = -\frac{\left(sd'(s)\right)''(a,b)}{(sd'(s))'(a,b)}\,(s_a(a,b))^2. \tag{5}$$

W.l.o.g. assume that $w(\cdot) = 1$ on $\Omega$. Since $s_a(a,1) > 0$ by (4), then Eq. (5) implies

$$s_{aa}(a,1) \lesseqqgtr 0 \Leftrightarrow \left(sd'(s)\right)''(a,1) \gtreqqless 0.$$

Hence,

$$\left(sd'(s)\right)''(a,1) \geq 0 \Rightarrow \bar{S}''(a) = \mathbb{E}(s_{aa}(a,1)) \leq 0,$$

and

$$\left(sd'(s)\right)''(a,1) \leq 0 \Rightarrow \bar{S}''(a) = \mathbb{E}(s_{aa}(a,1)) \geq 0.$$

Therefore, the conclusion of Proposition 3.2.1 follows from Proposition 3.1.

We prove Proposition 3.2.2. Differentiating (3) with respect to $b$ we get

$$s_b(a,b) = -\frac{(n-1)a}{n^2}\frac{1}{(sd'(s))'(a,b)}\frac{1}{b^2} < 0, \tag{6}$$

i.e., the equilibrium effort decreases with $b$ (hence with the cost of effort). Differentiating this expression with respect to $b$ again yields

$$s_{bb}(a,b) = \frac{(n-1)a}{n^2}\frac{1}{b^2}\frac{1}{(sd'(s))'(a,b)}\left(\frac{2}{b} + \frac{\left(sd'(s)\right)''(a,b)s_b(a,b)}{(sd'(s))'(a,b)}\right). \tag{7}$$

W.l.o.g. assume that $v(\cdot) = 1$ on $\Omega$. Since $s_b(1, b) < 0$ as shown in (6), then

$$\left(sd'(s)\right)''(1, b) \leq 0 \Rightarrow s_{bb}(1, b) > 0 \Rightarrow \hat{S}''(b) = \mathbb{E}(s_{bb}(1, b)) > 0,$$

where the first implication follows from Eq. (7). Thus, the equilibrium expected effort increases with the level of information in $T$ by Proposition 3.1. □

For any twice differentiable strictly increasing function $d : \mathbb{R}_+ \to \mathbb{R}_+$, the *Arrow–Pratt curvature* of $d$ is given for $x \in \mathbb{R}_+$ by

$$R_d(x) = \frac{xd''(x)}{d'(x)}.$$

In expected utility theory, $-R_d$ is interpreted as a measure of relative risk aversion for an individual with preferences represented by a concave von Neumann–Morgenstern utility function $d$. In our setting, however, $d$ is the deterministic component of the players' cost and is assumed to be convex (rather than concave) to assure that an equilibrium exists (by Theorem 2.1). Also, the players' utility function is state-dependent. Thus, interpreting $R_d$ as a measure of relative risk aversion would be a stretch. Nevertheless, as Proposition 3.3 below shows, the derivative of $R_d$ identifies conditions on the curvature of $d$ which allow us to determine the impact of changes in the level of information on the equilibrium expected efforts (as well as on payoffs, as we shall see in the next section).

**Proposition 3.3** *Let $d$ be a thrice differentiable, strictly increasing, and convex function satisfying $d(0) = 0$. If $R_d$ is increasing, then in every symmetric common-value Tullock contest with incomplete information $T \in \mathcal{T}(d)$ in which $v$ (respectively, $w$) is constant on $\Omega$, the equilibrium expected effort increases (decreases) with the level of information.*

*Proof* For all $x \in \mathbb{R}_+$,

$$\begin{aligned}(1 + R_d(x))' &= \frac{\left(2d''(x) + xd'''(x)\right)d'(x) - (d'(x) + xd''(x))d''(x)}{(d'(x))^2} \\ &= \frac{(xd'(x))'' - d'(x)d''(x) - x\left(d''(x)\right)^2}{(d'(x))^2}.\end{aligned}$$

If $R_d$ is increasing, then

$$R'_d(x) = (1 + R_d(x))' \geq 0 \Rightarrow (xd'(x))'' \geq d'(x)d''(x) + x\left(d''(x)\right)^2 \geq 0,$$

and therefore by Proposition 3.2.1, the equilibrium expected effort decreases with the level of information in $T$ whenever $w$ is constant on $\Omega$.

Assume that $v$ is constant on $\Omega$, and w.l.o.g. set $v(\cdot) = 1$. Taking log in Eq. (3) yields

$$\ln s(a, b) + \ln d'(s(a, b)) = \ln \frac{n-1}{n^2} + \ln a - \ln b.$$

for all $(a, b) \in \mathbb{R}^2_{++}$. Setting $a = 1$ and differentiating with respect to $b$ yields

$$-\frac{1}{b} = \frac{s_b(1,b)}{s(1,b)} + \frac{s_b(1,b)d''(s(1,b))}{d'(s(1,b))} = \frac{s_b(1,b)}{s(1,b)}\,(1 + R_d(s(1,b))\,.$$

Hence, $s_b(1, b) < 0$. Differentiating with respect to $b$ again yields

$$\frac{1}{b^2} = \frac{s_{bb}(1,b)s(1,b) - s_b(1,b)^2}{s(1,b)^2}\,(1 + R_d(s(1,b)) + \frac{s_b(1,b)}{s(1,b)}\,(1 + R_d(s(1,b))'\,.$$

If $R_d$ is increasing, then the second term in the right-hand side is nonpositive, and therefore the first term is positive, i.e.,

$$s_{bb}(1,b) > \frac{s_b(1,b)^2}{s(1,b)} > 0.$$

Hence,

$$\hat{S}''(b) = \mathbb{E}(s_{bb}(1,b)) > 0,$$

and thus the equilibrium expected effort increases with the level of information in $T$ by Proposition 3.1. □

## 4 The value of public information

In this section we study the value of public information. Let $d$ be a twice differentiable, strictly increasing, and convex function satisfying $d(0) = 0$. For each $(a, b) \in \mathbb{R}^2_{++}$, we write $U(a, b)$ for the expected equilibrium payoff in the unique Bayesian Nash equilibrium of the contest $(N, (\Omega, \mathcal{F}, p), a1_\Omega, (b1_\Omega)\, d, \mathcal{F})$. Proposition 4.1 establishes an auxiliary result relating the curvature of the function $U$ to the sign of the value of information. This result on the value of public information is the counterpart of Proposition 3.1. Its proof is also omitted.

**Proposition 4.1** *Assume that $d$ is twice differentiable, strictly increasing and convex, and satisfies $d(0) = 0$, and let $T \in \mathcal{T}(d)$ be a symmetric common-value Tullock contest with incomplete information. If the function $U$ is convex (concave) on $\mathbb{R}^2_{++}$, then the value of public information in $T$ is nonnegative (nonpositive).*

Our main result in this section establishes that the Arrow–Pratt curvature of the deterministic component of the cost function determines the value of information in a symmetric common-value Tullock contest with incomplete information. In establishing this result, the homogeneity of degree one of $U$ plays a key role.

**Theorem 4.2** *If $d$ is thrice differentiable, strictly increasing and convex, and such that $d(0) = 0$ and $R_d$ is increasing (decreasing), then the value of public information in every symmetric common-value Tullock contest with incomplete information $T \in \mathcal{T}(d)$ is nonnegative (nonpositive).*

*Proof* For $(a, b) \in \mathbb{R}^2_{++}$ the unique equilibrium of the contest $(N, (\Omega, \mathcal{F}, p), a1_\Omega, (b1_\Omega) d, \mathcal{F})$ is symmetric and interior by Theorem 2.1. Hence, in equilibrium all players win the price with the same probability (see Eq. (2)), and therefore

$$U(a, b) = \frac{a}{n} - b\mathbb{E}(k(a, b)), \tag{8}$$

where

$$k(a, b) := d(s(a, b)).$$

Let $\lambda \in \mathbb{R}_{++}$. Since the payoff function of a player in the Bayesian game associated with the contest $(N, (\Omega, \mathcal{F}, p), \lambda a1_\Omega, (\lambda b1_\Omega) d, \mathcal{F})$ is

$$u_i(\omega, x) = \bar{\rho}_i(x)\lambda a - \lambda bd(x_i) = \lambda[\bar{\rho}_i(x)a - bd(x_i)],$$

then $s(\lambda a, \lambda b) = s(a, b)$ and $U(\lambda a, \lambda b) = \lambda U(a, b)$, i.e., $s$ is homogeneous of degree zero and $U$ is homogeneous of degree one on $\mathbb{R}^2_{++}$. Hence, $U$ is convex (concave) if and only if $U_{aa}(a, b) \geq 0$ ($U_{aa}(a, b) \leq 0$)—see Lemma 6.2 in the Appendix.

Differentiating (8) we get

$$U_{aa}(a, b) = -b\mathbb{E}(k_{aa}(a, b)).$$

We show below that

$$k_{aa}(a, b) \lesseqqgtr 0 \Leftrightarrow R'_d \gtreqqless 0.$$

Hence,

$$R'_d \geq 0 \Rightarrow \mathbb{E}(k_{aa}(a, b)) \leq 0 \Rightarrow U_{aa}(a, b) \geq 0,$$

and

$$R'_d \leq 0 \Rightarrow \mathbb{E}(k_{aa}(a, b)) \geq 0 \Rightarrow U_{aa}(a, b) \leq 0,$$

which completes the proof by Theorem 4.2.

Differentiating $k$, we get

$$k_a(a, b) = d'(s(a, b))s_a(a, b).$$

Differentiating again and using Eq. (5), we get

$$\begin{aligned} k_{aa}(a, b) &= d''(s(a, b))\,(s_a(a, b))^2 + d'(s(a, b))s_{aa}(a, b) \\ &= \left(d''(s(a, b)) - d'(s(a, b))\frac{\left(sd'(s)\right)''(a, b)}{(sd'(s))'(a, b)}\right)(s_a(a, b))^2. \end{aligned}$$

Hence,

$$\begin{aligned} k_{aa}(a, b) \lesseqqgtr 0 &\Leftrightarrow \frac{d''(s(a, b))}{d'(s(a, b))} - \frac{\left(sd'(x)\right)''(a, b)}{(sd'(x))'(a, b)} \lesseqqgtr 0 \\ &\Leftrightarrow \left(\ln d'(s(a, b))\right)' - \left(\ln\left(sd'(s)\right)'(a, b)\right)' \lesseqqgtr 0 \end{aligned}$$

$$\Leftrightarrow \left(\ln \frac{\left(sd'(s)\right)'(a,b)}{d'(s(a,b))}\right)' \gtreqless 0$$

$$\Leftrightarrow \left(\frac{\left(sd'(s)\right)'(a,b)}{d'(s(a,b))}\right)' \gtreqless 0$$

$$\Leftrightarrow \left(1 + \frac{s(a,b)d''(s(a,b))}{d'(s(a,b))}\right)' \gtreqless 0$$

$$\Leftrightarrow R'_d(s(a,b)) \gtreqless 0.$$

□

## 5 Applications and examples

Our next proposition derives the implications of our results for *classic* Tullock contests, in which the players' marginal cost is equal to one independently of the state, i.e., for contests in the class $T = N, (\Omega, \mathcal{F}, p), v, (1_\Omega)\bar{d}, \mathcal{G}) \in \mathcal{T}(\bar{d})$, where $\bar{d}$ is the identity function. Since $R'_{\bar{d}}(x) = \left(x\bar{d}'(x)\right)'' = 0$ for all $x \in \mathbb{R}_+$, these results follow immediately from Proposition 3.2.1 and Theorem 4.2.

**Proposition 5.1** *In every symmetric common-value classic Tullock contest with incomplete information the value of public information is zero, and the equilibrium expected effort is invariant to changes in the players' information.*

Wasser (2013) studies symmetric common-value Tullock contest with incomplete information in which the players' value is $v(\cdot) = 1$ on $\Omega$, and their constant marginal cost of effort is a random variable, i.e., contests in the class $T = (N, (\Omega, \mathcal{F}, p), 1_\Omega, w\bar{d}, \mathcal{G}) \in \mathcal{T}(\bar{d})$. Wasser (2013)'s Proposition 3 establishes that players exert less effort when their information about their constant marginal cost of effort is just their prior than when they observe it. Proposition 5.2, which follows immediately from Proposition 3.2.2 and Theorem 4.2, extends this result to general information structures, e.g., to the case in which players observe a noisy public signal of their common constant marginal cost of effort, and also establishes results about the value of public information in these contests.

**Proposition 5.2** *In every symmetric common-value Tullock contest $T \in \mathcal{T}(\bar{d})$ in which the value $v$ is constant on $\Omega$ the equilibrium expected effort increases with the level of information, and the value of public information is zero.*

Warneryd (2003) studies two-player generalized Tullock contests in which the players' cost of effort is $c(\omega, x) = x$ for all $\omega \in \Omega$, and shows that if the score function is a state-independent, thrice differentiable, increasing, and concave function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(0) = 0$, then players' exert less (more) effort when their information about the value is just their prior than when they observe the value whenever the function $g/g'$ is convex (concave). Proposition 5.3 derives this result, extends it to contests with more than two players and general information structures, and establishes results about the value of public information.

**Proposition 5.3** *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a thrice differentiable, strictly increasing, and concave function satisfying $g(0) = 0$, and let $((N, (\Omega, \mathcal{F}, p), v, (1_\Omega)\bar{d}, \mathcal{G}), g)$ be a symmetric common-value generalized Tullock contest with incomplete information. If the function $g/g'$ is concave (convex), then the equilibrium expected effort increases (decreases) with the level of information, and the value of public information is nonnegative (nonpositive).*

*Proof* Let $a \in \mathbb{R}_+$. By Remark 2.2, the contest $T = ((N, (\Omega, \mathcal{F}, p), a1_\Omega, (1_\Omega)\bar{d}, \mathcal{G}), g)$ has a unique equilibrium. Denote by $\tilde{s}(a)$ the strategy of every player in this equilibrium. Since $\tilde{s}(a)$ maximizes

$$\mathbb{E}\left[\frac{g(x_i)}{(n-1)g(\tilde{s}(a)) + g(x_i)}a - x_i \mid \mathcal{G}\right](\omega),$$

for all $\omega \in \Omega$, the first-order condition

$$\mathbb{E}\left[\frac{n-1}{n^2}\frac{g'(\tilde{s}(a))}{g(\tilde{s}(a))}a \mid \mathcal{G}\right](\omega) = 1$$

holds for all $\omega \in \Omega$. Since $\tilde{s}(a)$ is $\mathcal{G}$ -measurable, then

$$\frac{g(\tilde{s}(a))}{g'(\tilde{s}(a))} = \frac{n-1}{n^2}a.$$

Differentiating this equation we get

$$\left(\frac{g(\tilde{s}(a))}{g'(\tilde{s}(a))}\right)'\tilde{s}'(a) = \frac{n-1}{n^2}.$$

Since $g$ is concave, then $\tilde{s}'(a) > 0$, i.e., the equilibrium effort increases with the players' common value of the prize, $a$. Differentiating again yields

$$\left(\frac{g(\tilde{s}(a))}{g'(\tilde{s}(a))}\right)'\tilde{s}''(a) + \left(\frac{g(\tilde{s}(a))}{g'(\tilde{s}(a))}\right)''\tilde{s}'(a) = 0.$$

Hence,

$$\left(\frac{g(\tilde{s}(a))}{g'(\tilde{s}(a))}\right)'' \gtreqless 0 \Leftrightarrow \tilde{s}''(a) \lesseqgtr 0.$$

Thus, $\tilde{S}(a) := \mathbb{E}[\tilde{s}(a)]$ is convex (concave) whenever $g/g'$ is concave (convex), and therefore the equilibrium expected effort increases (decreases) with the level of information by Proposition 3.1.

Moreover, $\tilde{s}(a) > 0$ by Remark 2.2, and therefore since $\bar{d}$ is the identity, the equilibrium expected payoff is

$$\tilde{U}(a) := \frac{a}{n} - \mathbb{E}(\tilde{s}(a)).$$

Hence, $\tilde{U}(a)$ is convex (concave) whenever $g/g'$ is convex (concave), and thus the equilibrium payoff increases (decreases) with the level of information by Proposition 4.1, i.e., the value of information is nonnegative (nonpositive). □

We conclude discussing examples that illustrate other interesting features.

*Example 5.4* Consider two-player Tullock contests in which $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = \{\varnothing, \Omega, \{\omega_1\}, \{\omega_2\}\}$, $p(\omega_1) = p(\omega_2) = 1/2$, and $d(x) = x^2/2 + \beta x$, where $\beta \geq 0$. Hence, $R_d'(x) > 0$ if $\beta > 0$, and $R_d'(x) = 0$ if $\beta = 0$. We calculate the equilibria in contests in which players' information is $\mathcal{F}$ and $\mathcal{G} = \{\varnothing, \Omega\}$, and the value and the random component of the cost are $(\hat{v}, \hat{w})$ and $(\tilde{v}, \tilde{w})$ described in the following table

| | $\omega_1$ | $\omega_2$ | $\mathbb{E}(\cdot)$ |
|---|---|---|---|
| $\hat{v}, \hat{w}$ | $1, 1/4$ | $3, 1/4$ | $2, 1/4$ |
| $\tilde{v}, \tilde{w}$ | $2, 1/8$ | $2, 3/8$ | $2, 1/4$ |

The table below describes the equilibrium efforts and payoffs for $\beta = 2$.

| $\beta = 2$ | $\omega_1$ | $\omega_2$ | $\mathbb{E}(\cdot)$ |
|---|---|---|---|
| $\hat{s}^*_{\mathcal{G}}$ | $\sqrt{3} - 1$ | $\sqrt{3} - 1$ | $\sqrt{3} - 1$ |
| $\hat{s}^*_{\mathcal{F}}$ | $\sqrt{2} - 1$ | $1$ | $1/\sqrt{2}$ |
| $\hat{u}^*_{\mathcal{G}}$ | $1/2 - \sqrt{3}/4$ | $3/2 - \sqrt{3}/4$ | $1 - \sqrt{3}/4$ |
| $\hat{u}^*_{\mathcal{F}}$ | $5/8 - \sqrt{2}/4$ | $7/8$ | $3/4 - \sqrt{2}/8$ |
| $\tilde{s}^*_{\mathcal{G}}$ | $\sqrt{3} - 1$ | $\sqrt{3} - 1$ | $\sqrt{3} - 1$ |
| $\tilde{s}^*_{\mathcal{F}}$ | $\sqrt{5} - 1$ | $\sqrt{7/3} - 1$ | $\sqrt{7/12} + \sqrt{5/4} - 1$ |
| $\tilde{u}^*_{\mathcal{G}}$ | $1 - \sqrt{3}/8$ | $1 - 3\sqrt{3}/8$ | $1 - \sqrt{3}/4$ |
| $\tilde{u}^*_{\mathcal{F}}$ | $7/8 - \sqrt{5}/8$ | $9/8 - \sqrt{21}/8$ | $1 - (\sqrt{5} + \sqrt{21})/16$ |

Thus, $\mathbb{E}(\hat{s}^*_{\mathcal{F}}) < \mathbb{E}(\hat{s}^*_{\mathcal{G}})$, and $\mathbb{E}(\tilde{s}^*_{\mathcal{F}}) > \mathbb{E}(\tilde{s}^*_{\mathcal{G}})$, i.e., effort decreases with the level of information in $(N, (\Omega, \mathcal{F}, p), \hat{v}, \hat{w}d, \mathcal{G})$, and increases in $(N, (\Omega, \mathcal{F}, p), \tilde{v}, \tilde{w}d, \mathcal{G})$. In both contests, consistently with Theorem 4.1, the value of information is positive.

The table below describes the equilibrium efforts and payoffs for $\beta = 0$.

| $\beta = 0$ | $\omega_1$ | $\omega_2$ | $\mathbb{E}(\cdot)$ |
|---|---|---|---|
| $\hat{s}^*_{\mathcal{G}}$ | $\sqrt{2}$ | $\sqrt{2}$ | $\sqrt{2}$ |
| $\hat{s}^*_{\mathcal{F}}$ | $1$ | $\sqrt{3}$ | $1/2 + \sqrt{3}/2$ |
| $\hat{u}^*_{\mathcal{G}}$ | $1/4$ | $5/4$ | $3/4$ |
| $\hat{u}^*_{\mathcal{F}}$ | $3/8$ | $9/8$ | $3/4$ |
| $\tilde{s}^*_{\mathcal{G}}$ | $\sqrt{2}$ | $\sqrt{2}$ | $\sqrt{2}$ |
| $\tilde{s}^*_{\mathcal{F}}$ | $2$ | $2/\sqrt{3}$ | $1 + 1/\sqrt{3}$ |
| $\tilde{u}^*_{\mathcal{G}}$ | $7/8$ | $5/8$ | $3/4$ |
| $\tilde{u}^*_{\mathcal{F}}$ | $3/4$ | $3/4$ | $3/4$ |

Thus, $\mathbb{E}(\hat{s}^*_{\mathcal{F}}) < \mathbb{E}(\hat{s}^*_{\mathcal{G}})$, and $\mathbb{E}(\tilde{s}^*_{\mathcal{F}}) > \mathbb{E}(\tilde{s}^*_{\mathcal{G}})$, i.e., effort decreases with the level of information in $(N, (\Omega, \mathcal{F}, p), \hat{v}, \hat{w}d, \mathcal{G})$ and decreases in $(N, (\Omega, \mathcal{F}, p), \tilde{v}, \tilde{w}d, \mathcal{G})$. In both contests, again consistently with Theorem 4.1, the value of information is zero.

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## 6 Appendix

**Lemma 6.1** *A symmetric common-value Tullock contest with complete information in which the players' cost of effort is a twice differentiable strictly increasing and convex function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $c(0) = 0$ has a unique pure strategy Nash equilibrium. Moreover, this equilibrium is symmetric and interior.*

*Proof* In the game associated with a symmetric common-value Tullock contest with complete information the set of pure strategies of every player is $\mathbb{R}_+$, and the payoff function of each player $i \in N$ is $h_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ given for $x \in \mathbb{R}^n_+ \backslash \{0\}$ by

$$h_i(x) = \frac{x_i}{\bar{x}} v - c(x_i),$$

where $v > 0$ is the players' common value and $\bar{x} = \sum_{j=1}^n x_j$, and

$$h_i(0) = \rho_i v - c(0) = \rho_i v,$$

where $\rho \in \Delta^n$ is predetermined. Thus, $h_i(\cdot, x_{-i})$ is twice differentiable and concave on $\mathbb{R}_{++}$, and

$$\frac{\partial h_i(x)}{\partial x_i} = \frac{\bar{x} - x_i}{\bar{x}^2} v - c'(x_i).$$

Let $x^* \in \mathbb{R}^n_+$ be a pure strategy Nash equilibrium. Then, for all $i \in N$, player $i$'s effort $x^*_i$ solves the problem

$$\max_{x_i \in \mathbb{R}_+} h_i\left(x_i, x^*_{-i}\right),$$

i.e., $(\partial h_i(x^*)/\partial x_i)\, x^*_i = 0$. Clearly, $x^* \neq 0$; since $n \geq 2$, then $\rho_i < 1$ for some $i \in N$, and therefore

$$h_i(0, 0) = \rho_i v < v - c(\varepsilon) = h_i(\varepsilon, 0)$$

for $\varepsilon > 0$ sufficiently small. Let $k \in N$ be such that $x^*_k > 0$. Then,

$$\frac{\partial h_i(0, x^*_{-i})}{\partial x_i} = \frac{v}{\bar{x}^*} - c'(0) > \frac{\bar{x}^* - x^*_k}{(\bar{x}^*)^2} v - c'(x^*_k) = 0$$

for all $i \in N\backslash\{k\}$. Thus, $x_i^* > 0$, and therefore $\partial h_i(x^*)/\partial x_i = 0$, i.e.,

$$\frac{\bar{x}^* - x_i^*}{(\bar{x}^*)^2} v = c'(x_i^*)$$

for all $i \in N$. Hence, $x_i^* = t^* > 0$ for all $i \in N$, where $t^*$ is the unique solution (recall that $c'' \geq 0$) to the equation

$$\frac{(n-1)}{n^2 t} v = c'(t). \tag{9}$$

Obviously, the profile $(t^*, \ldots, t^*)$, where $t^*$ is the solution to Eq. (9), is a pure strategy equilibrium. Hence, the contest has a unique pure strategy Nash equilibrium, which is symmetric and interior. □

**Lemma 6.2** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice differentiable and homogeneous of degree one. Then, $f$ is convex (concave) on $\mathbb{R}^2$ if and only if $f_{xx}(x, y) \geq 0$ ($f_{xx}(x, y) \leq 0$) for all $(x, y) \in \mathbb{R}^2$.*

*Proof* By Euler's Theorem,

$$f(x, y) = x f_x(x, y) + y f_y(x, y).$$

Differentiating with respect to $x$ on both sides on this equation and simplifying yields

$$x f_{xx}(x, y) + y f_{yx}(x, y) = 0. \tag{10}$$

Likewise,

$$x f_{xy}(x, y) + y f_{yy}(x, y) = 0. \tag{11}$$

Hence,

$$x^2 f_{xx}(x, y) = y^2 f_{yy}(x, y),$$

and therefore

$$f_{xx}(x, y) \lesseqqgtr 0 \Leftrightarrow f_{yy}(x, y) \lesseqqgtr 0.$$

Further, (10) and (11) imply

$$f_{xx}(x, y) f_{yy}(x, y) - f_{xy}(x, y) f_{yx}(x, y) = 0.$$

Thus, the eigenvalues of the Hessian matrix of $f$ are nonnegative (nonpositive) when $f_{xx}$ is a nonnegative (nonpositive) function on $\mathbb{R}^2$. □

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