Econ Theory (2016) 62:567–586
DOI 10.1007/s00199-015-0915-6

RESEARCH ARTICLE

Second-price auctions with sequential and costly participation

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Received: 25 July 2014 / Accepted: 5 September 2015 / Published online: 15 September 2015

Abstract This paper studies second-price auctions in which bidders make participation decisions sequentially in an exogenous order and participating bidders incur bidding costs. When bidders decide whether to participate or not, they know their own valuations as well as earlier bidders' participation decisions. To analyze bidders' participation and bidding decisions, we study equilibria in cutoff strategies with which a bidder participates and bids his valuation if his valuation exceeds a cutoff given his observation on earlier bidders' participation. Focusing on the case of two bidders, we present two main results on comparative statics and revenue comparison. In the comparative statics analysis, we study the effects of a change in bidders' characteristics on equilibrium cutoffs. In revenue comparison, we show that the considered sequential entry format yields lower revenue than the simultaneous entry counterpart. Finally,

We are grateful to Daniel Ackerberg, Yeon-Koo Che, Myeonghwan Cho, Kyungmin (Teddy) Kim, an associate editor, an anonymous referee, and participants at the 15th Korean Economic Association International Conference, the 8th Biennial Conference on Economic Design, and the 2015 Winter Meeting of the Korean Econometric Society for helpful comments. Lee's research was supported by the year of 2012 Research Fund of the UNIST (Ulsan National Institute of Science and Technology). Park's research was supported by the Yonsei University Research Grant of 2012.

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we discuss the difficulties in generalizing these two results for the case of more than two bidders.

Keywords Second-price auctions · Sequential participation · Participation costs · Cutoff equilibria

JEL Classification D44 · D82

1 Introduction

Participating in an auction and making bids is a costly activity. It costs bidders' resources such as their time, effort and money to travel to the auction site and to express their bids in the required format. Participation costs do not disappear completely in online auctions as bidders need to register at the online auction website, learn the auction rules, and spend time to choose and make their bids. Facing these participation costs, bidders compare the benefit and cost of participating in the auction and decide to participate only when they find participation profitable. In other words, in the presence of participation costs, the set of participating bidders is determined endogenously given the auction format. In order to determine the set of participating bidders, most of the literature on so-called auctions with entry assumes that potential bidders make participation decisions *simultaneously*. However, in auctions for government procurements or corporate takeovers, it is possible that potential bidders enter *sequentially* observing previous entrants, and this is our point of departure in this paper.

We study a single object independent private value auction model where bidders' valuation distributions and participation costs are common knowledge. We consider the following procedure to determine the set of participating bidders. The seller contacts potential bidders sequentially in an exogenous order.[1] When contacted by the seller, each bidder learns the participation decisions of earlier bidders and then decides whether to participate or not knowing his valuation. After contacting all the bidders, the seller holds a second-price auction among participating bidders to determine the winner of the object and his payment. Bidders participating in the auction incur participation costs, which can be interpreted as "bidding costs."[2]

The considered auction procedure can be divided into two stages: the entry stage in which bidders decide whether to participate or not and the bidding stage in which participating bidders choose their bids. Since we consider second-price auctions with independent private values, analyzing the bidding stage is straightforward. It is optimal for each participating bidder to bid his valuation regardless of others' bids. Given

[1] Crémer et al. (2007, 2009) study the case where the seller can choose the order in which she contacts potential bidders, while considering different kinds of participation costs than the one considered in this paper. See Sect. 4.3 for a discussion.

[2] There is an alternative scenario in which bidders need to incur costs to learn their own valuations after deciding to participate and before choosing bids. In this case, participation costs can be interpreted as "information acquisition costs." See, for example, McAfee and McMillan (1987), Persico (2000) and Crémer et al. (2009).

this bidding behavior, a bidder's expected utility in the auction is increasing in his valuation when he has a chance to win. Hence, to analyze the entry stage, we look for a cutoff equilibrium in which each bidder decides to participate if and only if his valuation exceeds a cutoff given his information set. The cutoff of a bidder at a certain information set is determined endogenously as a best response to the other bidders' cutoffs, and an equilibrium occurs when all the cutoffs best respond to each other. Using this relationship, we obtain the conditions that characterize equilibrium cutoffs.

Focusing on the two-bidder case, we obtain two main results on comparative statics and revenue comparison. In the comparative statics analysis, we study the effects of the two bidders' valuation distributions and participation costs on their equilibrium participation cutoffs. An interesting finding is that, as a bidder's valuation becomes stochastically higher (to be made precise in our formal statement), his cutoff changes in opposite directions depending on whether he is the first mover (bidder 1) or the second (bidder 2). If the bidder is bidder 1, his cutoff decreases, whereas if the bidder is bidder 2, his cutoff increases. When bidder 1's valuation becomes stochastically higher conditionally on his participation, bidder 2 expects a lower profit from participation when observing bidder 1 participate, which increases bidder 2's cutoff. Then bidder 1 faces weaker competition with bidder 2, and thus, bidder 1 becomes more aggressive lowering his cutoff. On the other hand, when bidder 2's valuation becomes stochastically higher, bidder 1 expects a lower profit from participation, which increases bidder 1's cutoff. Then, when observing bidder 1 participate, bidder 2 expects a lower profit from participation, which induces him to increase his cutoff as well.

In revenue comparison, we compare the seller's expected revenue under sequential and simultaneous participation and show that sequential participation yields lower revenue than simultaneous participation when there are two homogeneous bidders. The main reason for this result can be explained as follows. Under sequential participation, a bidder can adjust his cutoff to earlier bidders' participation decisions. Hence, in the two-bidder case, bidder 2 is less likely to participate when bidder 1 participates than when he does not. As a result, compared with simultaneous participation, sequential participation tends to reduce the likelihood that both bidders participate. Since the seller obtains positive revenue only when both bidders participate under second-price auctions, the sequential entry format is revenue-dominated by the simultaneous entry counterpart. Meanwhile, sequential participation tends to increase the likelihood that only one bidder participates. If the seller uses a reserve price or an entry fee, she can receive positive revenue even when there is only one participant, and sequential participation can yield higher revenue than simultaneous participation.

We also consider the three-bidder case and discuss the difficulties in generalizing our two main results for this case. First, in the comparative statics analysis, it is not possible to determine the direction of change in some cutoff due to counteracting effects generated by a chain of responses in cutoffs. For example, suppose that bidder 1's valuation becomes stochastically higher conditionally on his participation. Then bidders 2 and 3 expect a lower profit from participation when observing bidder 1 participate, and they increase their cutoffs following bidder 1's participation. However, there is a secondary effect due to the increase in bidder 3's cutoff. In response to the increase in bidder 3's cutoff, bidder 2 now has an incentive to lower his cutoff. Hence, bidder 2 faces two counteracting effects on his cutoff, and we cannot determine the direction of the net effect from the equilibrium conditions. Second, in revenue comparison, sequential entry has counteracting effects on revenue, leaving its net effect ambiguous. When participation costs are not so large, sequential participation tends to increase the likelihood that exactly two bidders participate, while it tends to reduce the likelihood that all three bidders participate, compared with simultaneous participation. The first change has a positive effect on revenue, while the second has a negative one. Hence, whether sequential participation yields higher revenue than simultaneous participation depends on which effect is dominant.

1.1 Related literature

In the literature, auctions where bidders incur bidding costs have been studied in different auction environments. Samuelson (1985) studies first-price auctions in the context of competitive procurement and focuses on symmetric equilibria. Cao and Tian (2010) also consider first-price auctions, studying not only symmetric equilibria but also asymmetric ones. Campbell (1998) studies second-price auctions with two potential bidders, providing a sufficient condition for the existence of asymmetric equilibria and discussing coordination between the two bidders. Tan and Yilankaya (2006) consider second-price auctions with a general number of bidders and examine both symmetric and asymmetric equilibria mainly assuming that valuation distributions and participation costs are the same across bidders. Cao and Tian (2013) allow different participation costs across bidders and analyze the same setup as in Tan and Yilankaya (2006). Unlike most studies that assume privately known valuations and commonly known participation costs, Green and Laffont (1984) study second-price auctions when both valuations and participation costs are private information and uniformly distributed, while Kaplan and Sela (2006) examine second-price auctions when valuations are common knowledge and participation costs are private information. Hausch and Li (1993) consider common value auctions where bidders incur not only bidding costs but also information acquisition costs. Miralles (2010) analyzes collusion in weakly efficient auctions with costly entry. Also, there are studies that consider the problem of finding optimal or efficient auctions (see, for example, Stegeman 1996; Menezes and Monteiro 2000; Lu 2009; and Celik and Yilankaya 2009).

All the aforementioned works assume that bidders make participation decisions simultaneously, while there are only a relatively few works that study "auctions with sequential entry." Fishman (1988) analyzes a takeover bidding process in which the target firm solicits bids from two bidders sequentially and shows that the first bidder may make a preemptive initial bid. Ehrman and Peters (1994) and Crémer et al. (2007, 2009) consider selling mechanisms in which the seller contacts potential buyers sequentially and establish optimal mechanisms. Segev and Sela (2014) examine all-pay contests in which contestants determine their efforts sequentially observing earlier contestants' efforts. A closely related work to ours is Tian and Xiao (2011). Our work extends and complements their work in the following two senses. First, they assume identical participation costs, whereas we allow different participation costs. As argued in Cao and Tian (2013), the assumption of identical participation costs is restrictive and unrealistic as bidders from different locations incur different travel costs to the auction site and time opportunity costs may vary across bidders. There is also a theoretical merit. In their comparative statics results, Tian and Xiao (2011) analyze the aggregate effects of the common participation cost on bidders' equilibrium cutoffs. By allowing different participation costs, we can disentangle these effects and study the effects of each individual bidder's participation cost separately. Second, Tian and Xiao (2011) compare bidders' payoffs while we examine the seller's revenue with two homogeneous bidders. If the seller wants to favor one bidder over the other, their analysis will be relevant. However, the seller's revenue is a natural and widely studied measure of an auction's performance, and thus, it is important to study revenue with sequential participation.

The remainder of this paper is organized as follows. Section 2 describes the basic setup, and Sect. 3 provides the characterizing conditions of cutoff equilibria. Section 4 investigates the case of two bidders studying comparative statics and revenue comparison. Section 5 considers the case of three bidders and discusses how the previous results change in this case. Section 6 concludes the paper. All the proofs are relegated to the "Appendix."

2 The setup

There are $n \geq 2$ potential bidders (or simply bidders) and a seller. The seller has an indivisible object which has no value to her. Each bidder is risk-neutral and has a valuation on the object. The seller desires to sell the object via a second-price sealed-bid auction. Participating in the auction is costly to the bidders, and the set of participating bidders is determined by the following procedure. The seller contacts the bidders sequentially, one bidder at a time, in an exogenous order and asks them whether they are willing to participate in the auction. When contacting each bidder, the seller reveals the participation decisions of the earlier bidders. Each bidder, knowing his own valuation as well as the participation decisions of the earlier bidders, decides whether to participate or not. A bidder who agrees to participate commits to participation. After contacting all the bidders, the seller holds a second-price auction among participating bidders to determine the winner of the object and his payment. That is, the bidder who submits the highest bid wins the object and pays the second highest bid. In the case that there are multiple bidders submitting the highest bid, the seller selects one of them as the winner with equal probability.

We refer to the ith bidder to be contacted by the seller as bidder i, for $i = 1, \ldots, n$. The valuations of the bidders are determined independently, and the valuation of bidder i, denoted by v_i, is distributed according to a continuously differentiable cumulative distribution function (cdf) $F_i(\cdot)$ with density $f_i(\cdot)$ that has full support on [0, 1]. The participation cost of bidder i is denoted by $c_i \in (0, 1)$.[3] We refer to the pair (F_i, c_i) as the characteristic of bidder i. The characteristics of the bidders are common knowledge among the bidders and the seller, while valuations are private information. We write the set of actions for each bidder as $A = \{out\} \cup [0, \infty)$; *out* represents the action of

[3] If $c_i \geq 1$, bidder i will never participate in the auction, and thus we can proceed with the analysis without him.

not participating, while a nonnegative real number represents the action of participating and bidding that number in the auction. Let $\{out, in\}$ be the information partition of A where in corresponds to $[0, \infty)$.[4] Then the collection of all possible information sets for bidder $i \geq 2$ is given by $H_i = \{out, in\}^{i-1}$, while we write that for bidder 1 as $H_1 = \{\varnothing\}$. A (pure) strategy for bidder i is a mapping $b_i : [0, 1] \times H_i \to A$. That is, $b_i(v_i, h_i)$ represents the action of bidder i who has valuation v_i and is at information set h_i.

3 Cutoff equilibria

Participating bidders face a second-price auction, and thus, bidding his own valuation is weakly better for a participating bidder than bidding other values, regardless of the bids of other participating bidders. Moreover, when each participating bidder bids his valuation, a bidder's expected profit from participation is increasing in his valuation when he has a chance to win. Hence, it is natural to restrict attention to equilibria in which bidders use cutoff strategies.[5] A cutoff strategy for bidder i is described by cutoffs $\{v_i^*(h_i)\}_{h_i \in H_i}$ where $v_i^*(h_i) \in [0, 1]$ for all $h_i \in H_i$, and it can be written as

$$b_i(v_i, h_i) = \begin{cases} out & \text{if } v_i \leq v_i^*(h_i), \\ v_i & \text{if } v_i > v_i^*(h_i). \end{cases}$$

That is, with a cutoff strategy, a bidder participates and bids his valuation if it exceeds the cutoff given his information set, and he does not participate otherwise. $v_i^*(h_i) = 1$ means that bidder i does not participate at information set h_i regardless of his valuation. To take into account the sequential nature of bidders' decision making, we use the notion of perfect Bayesian equilibrium and look for an equilibrium in cutoff strategies or a *cutoff equilibrium*.[6]

A bidder compares the expected benefit of participation and the participation cost, and he enters the auction only when the benefit is large enough to cover the cost. Below we derive the expression for the benefit that a bidder can expect from participating in the auction, which we refer to as the bidder's expected utility in the auction. Suppose that bidder i participates, and consider the bidding stage. Let $N = \{1, \ldots, n\}$ be the set of all bidders, and let $P \subset N \setminus \{i\}$ be the set of other participating bidders than bidder i. Let $U_i(v_i; \{v_j^*\}_{j \in P})$ be bidder i's expected utility in the auction (without taking into account his participation cost) when his valuation is v_i and the cutoffs

[4] To be precise, the information partition should be written as $\{\{out\}, [0, \infty)\}$, but for simplicity we use out instead of $\{out\}$ with an abuse of notation.

[5] There is also a preemptive equilibrium at which the first bidder among those whose valuations exceed their participation costs participates and bids 1, while the rest of the bidders do not participate. We can exclude such an equilibrium by requiring equilibrium strategies to be undominated strategies.

[6] In our formulation, a zero-probability event occurs when a bidder who is not supposed to participate (i.e., a bidder with cutoff 1) decides to participate. Since perfect Bayesian equilibrium allows players to hold an arbitrary belief at a zero-probability information set, we prescribe that bidder i believes $v_j = 1$ with probability 1 when he observes bidder j with cutoff 1 participating, which can be justified by a continuity argument. Given this belief, it is optimal for bidder i not to participate regardless of his valuation. Thus, at equilibrium we have $v_i^*(h_i) = 1$ for every zero-probability information set h_i.

of other participating bidders are given by $\{v_j^*\}_{j\in P}$. If there is no other participant (i.e., $P = \emptyset$), bidder i wins the object at zero price, and thus $U_i(v_i; \{v_j^*\}_{j\in P}) = v_i$. Suppose that there are other participating bidders than bidder i (i.e., $P \neq \emptyset$). Let $v_P^* = \max_{j\in P}\{v_j^*\}$ be the highest cutoff among the cutoffs adopted by other participating bidders. Since the bidder with the highest valuation wins the object in the second-price auction, the probability that bidder i obtains the object is 0 if $v_i \leq v_P^*$ and $\prod_{j\in P}[F_j(v_i) - F_j(v_j^*)]/[1 - F_j(v_j^*)]$ if $v_i > v_P^*$. Then by applying the envelope theorem, we obtain

$$U_i(v_i; \{v_j^*\}_{j\in P}) = \begin{cases} 0 & \text{if } v_i \leq v_P^*, \\ \int_{v_P^*}^{v_i} \left(\prod_{j\in P} \frac{F_j(v)-F_j(v_j^*)}{1-F_j(v_j^*)}\right) \mathrm{d}v & \text{if } v_i > v_P^*. \end{cases}$$

It can be checked that, when $U_i(v_i; \{v_j^*\}_{j\in P})$ is positive, it is increasing in v_i and decreasing in v_j^* for any $j \in P$. That is, the expected utility of a bidder increases as his valuation becomes higher, and it reduces as the cutoff of any other participating bidder becomes higher, as long as the bidder has a valuation higher than the cutoff of any other participating bidder. Since the expression of U_i does not involve the characteristic of bidder i, we can drop the subscript i in U_i and just write it as U.

In the following lemma, we present a basic property of equilibrium cutoffs, which plays a critical role in deriving the equilibrium conditions.

Lemma 1 (Increasing Property of Equilibrium Cutoffs) *Let $\{v_i^*(h_i)\}_{h_i\in H_i, i\in N}$ be a cutoff equilibrium. Suppose that $i > j \geq 1$ and h_i contains (h_j, in) as its first j components. Then $v_i^*(h_i) \geq v_j^*(h_j)$ with strict inequality if and only if $v_j^*(h_j) < 1$.*

The idea behind Lemma 1 is straightforward. If a bidder's valuation is smaller than the cutoff of an earlier participating bidder, there is no chance for the bidder to win the object, and thus, it is not profitable for him to participate in the auction. As a result, the cutoff of the current bidder must exceed the cutoff of any earlier participating bidder.

Now we derive the conditions that characterize equilibrium cutoffs. Let $\{v_i^*(h_i)\}_{h_i\in H_i, i\in N}$ be a cutoff equilibrium. Consider bidder i at information set h_i. Let $P(h_i)$ be the set of participating bidders in information set h_i. Also, given h_i, let h_i^j be the information set consisting of the first $j - 1$ components of h_i, for $j = 1, \ldots, i - 1$. Suppose that $v_i^*(h_i) < 1$. Then by Lemma 1, bidder i with valuation $v_i^*(h_i)$ at information set h_i obtains a positive expected utility in the auction only when all the remaining bidders do not participate. Hence, the indifference condition for bidder i with valuation $v_i^*(h_i) < 1$ at information set h_i is given by

$$U(v_i^*(h_i); \{v_j^*(h_i^j)\}_{j\in P(h_i)}) F_{i+1}(v_{i+1}^*(h_i, in)) F_{i+2}(v_{i+2}^*(h_i, in, out)) \\ \cdots F_n(v_n^*(h_i, in, out, \ldots, out)) = c_i.$$

Suppose that $v_i^*(h_i) = 1$. If a bidder with valuation 1 participates, all the remaining bidders do not participate, believing that the bidder has valuation 1. Hence, the no profitable participation condition for bidder i with valuation $v_i^*(h_i) = 1$ at information set h_i is given by

$$U(1; \{v_j^*(h_i^j)\}_{j\in P(h_i)}) \leq c_i. \tag{1}$$

Combining the two, we can express the indifference or no profitable participation condition for bidder i at h_i as

$$U(v_i^*(h_i); \{v_j^*(h_i^j)\}_{j \in P(h_i)}) F_{i+1}(v_{i+1}^*(h_i, in)) F_{i+2}(v_{i+2}^*(h_i, in, out)) \cdots F_n(v_n^*(h_i, in, out, \ldots, out)) \leq c_i,$$

with equality if $v_i^*(h_i) < 1$. Considering all i and h_i , we obtain $2^n - 1$ conditions for $2^n - 1$ cutoffs, and a profile of cutoffs that satisfy all the $2^n - 1$ conditions as well as the increasing property in Lemma 1 constitutes an equilibrium.

The increasing property in Lemma 1 can be strengthened as follows. Consider bidder i at information set h_i, and let $v_{P(h_i)}^*$ be the highest cutoff among the cutoffs adopted by the participating bidders in h_i. In case there is no participating bidder in h_i, we set $v_{P(h_i)}^* = 0$. If bidder i participates and wins the object, his payment is at least $v_{P(h_i)}^*$. Hence, in order for bidder i to have an incentive to participate, his valuation should be at least $v_{P(h_i)}^* + c_i$, which leads to $v_i^*(h_i) \geq \min\{v_{P(h_i)}^* + c_i, 1\}$. Suppose that every bidder has participation cost c (or, more generally, every bidder's participation cost is bounded below by c). If there are m participating bidders in bidder i's information set h_i, we have $v_i^*(h_i) \geq \min\{(m+1)c, 1\}$. Let $\overline{m}$ be the smallest integer m such that $m \geq 1/c - 1$. Then, at equilibrium, we cannot have more than $\overline{m}$ participating bidders. That is, after observing sufficiently many bidders participating, no bidder finds it profitable to participate in the auction. As a result, if the seller finds sufficiently many participants, she can stop contacting the remaining bidders and move on to the bidding stage. For example, if $c = 1/2$, we have $\overline{m} = 1$ and there cannot be more than one participating bidder at equilibrium. Notice, however, that $\overline{m}$ is an upper bound that holds for any profile of valuation distributions of the bidders, and with a particular distribution profile, the maximum number of participating bidders can be smaller than $\overline{m}$. For example, when the valuation of each bidder is uniformly distributed and each bidder has participation cost $1/3$, there can be at most one participating bidder at equilibrium, while we have $\overline{m} = 2$. Hence, when to stop contacting the bidders should be determined depending on their valuation distributions as well as their participation costs.

Another observation we can make is that, assuming identical participation costs, if a bidder finds participation not profitable even with the highest possible valuation 1, then so do the later bidders. This can be seen directly from the no profitable participation condition (1). This result may not hold with different participation costs since a bidder with a low participation cost may find participation profitable even when an earlier bidder with a higher participation cost has no profitable participation.

Remark 1 In the terminology of Tian and Xiao (2011), bidder i is *driven out* at information set h_i if $v_i^*(h_i) = 1$. Assuming homogeneous bidders, they show that the smallest value of the common participation cost that drives out at least one bidder decreases as the number of bidders increases from two to three. Our lower bound of equilibrium cutoffs implies that if $c \geq 1/n$, the last bidder (bidder n) never participates when all the other bidders participate, regardless of bidders' valuation distributions. Thus, our result provides a range of the common participation costs that drive out at least one bidder for all valuation distributions.

4 Two bidders

In this section, we study the case of two bidders and present our main results.

4.1 Comparative statics of equilibrium cutoffs

When there are two bidders, we have three cutoffs: $v_1^*(\varnothing)$, $v_2^*(out)$ and $v_2^*(in)$. For notational convenience, we will use v_1^* instead of $v_1^*(\varnothing)$. Noting that neither v_1^* nor $v_2^*(out)$ can be 1, we can write the three indifference or no profitable participation conditions to determine the cutoffs as

$$v_1^* F_2(v_2^*(in)) = c_1, \tag{2}$$

$$v_2^*(out) = c_2, \tag{3}$$

$$\int_{v_1^*}^{v_2^*(in)} \frac{F_1(v) - F_1(v_1^*)}{1 - F_1(v_1^*)} \, \mathrm{d}v \leq c_2 \text{ (with equality if } v_2^*(in) < 1). \tag{4}$$

Since $v_1^* < 1$, Lemma 1 implies $v_1^* < v_2^*(in)$. We can show that a cutoff equilibrium exists and is unique as follows. Since (3) determines $v_2^*(out)$ uniquely, it remains to show that (2) and (4) together determine v_1^* and $v_2^*(in)$ uniquely. Using (2), we can express v_1^* as a decreasing function of $v_2^*(in)$. Substituting this relationship into v_1^*, we can express the left-hand side of (4) as an increasing function of $v_2^*(in)$. From this we obtain a unique value of $v_2^*(in)$, which in turn yields the unique value of v_1^*.

Next we present our first main result, which studies how equilibrium cutoffs change with valuation distributions and participation costs.

Proposition 1 (Comparative Statics) *Suppose that there are two bidders, and let* $\{v_1^*, v_2^*(out), v_2^*(in)\}$ *be the cutoff equilibrium. Assume that* $v_2^*(in) < 1$.

(i) Let $\tilde{F}_1(\cdot)$ be a cdf obtained by taking a convex transformation of $F_1(\cdot)$. That is, $\tilde{F}_1 = g \circ F_1$ for some differentiable convex function g defined on [0,1] *such that $g(0) = 0$, $g(1) = 1$ and $g'(\cdot) > 0$. As $F_1(\cdot)$ changes to $\tilde{F}_1(\cdot)$, v_1^* decreases, $v_2^*(out)$ remains the same, and $v_2^*(in)$ increases. As $F_2(\cdot)$ changes to $\tilde{F}_2(\cdot)$ where $\tilde{F}_2(v) < F_2(v)$ for all $v \in (0, 1)$, v_1^* and $v_2^*(in)$ increase, while $v_2^*(out)$ remains the same. (In both scenarios, v_1^* and $v_2^*(in)$ remain the same if $v_2^*(in) = 1$.)*

(ii) As c_1 increases, v_1^* and $v_2^*(in)$ increase, while $v_2^*(out)$ remains the same. ($v_2^*(in)$ remains the same if $v_2^*(in) = 1$.) As c_2 increases, v_1^* decreases, while $v_2^*(out)$ and $v_2^*(in)$ increase. (v_1^* and $v_2^*(in)$ remain the same if $v_2^*(in) = 1$.)

Proposition 1(i) shows among others that, as a bidder's valuation becomes stochastically higher, his cutoff changes in opposite directions depending on whether he is the first mover or the second. If the bidder is the first mover, his cutoff decreases, whereas if the bidder is the second mover, his cutoff increases after bidder 1's participation. Note that we use a stronger notion of stochastic dominance for bidder 1's valuation distribution than for bidder 2's. The reason for using the different notions can be explained as follows. When bidder 2 computes his expected utility in the auction, he uses the distribution of v_1 conditional on bidder 1's participation (see (4)), and thus in order to obtain the desired results, we need first-order stochastic dominance conditional on bidder 1's participation. However, assuming that $\tilde{F}_1(\cdot)$ first order stochastically dominates $F_1(\cdot)$ unconditionally does not guarantee that the same relation holds conditionally on bidder 1's participation.[7] One way to guarantee that conditional stochastic dominance holds for any value of bidder 1's cutoff is to assume that $\tilde{F}_1(\cdot)$ is a convex transformation of $F_1(\cdot)$, as imposed in Proposition 1(i). On the other hand, when bidder 1 computes his expected utility in the auction, he uses the unconditional distribution of v_2 (see (2)) since he makes a participation decision before bidder 2. Thus, it suffices to use the standard notion of first-order stochastic dominance for bidder 2's valuation distribution.

The main channel that generates the comparative statics results is the following. As can be seen from the equilibrium conditions, a bidder's expected utility in the auction increases as a later bidder increases his cutoff and as an earlier participating bidder lowers his cutoff. Hence, other things being equal, a bidder reacts to an increase in a later bidder's cutoff by lowering his cutoff, while he reacts to an increase in an earlier participating bidder's cutoff by raising his cutoff as well. A change in characteristics brings about an adjustment by the affected bidder, and the other bidder responds to this adjustment as described above. As bidder 2's valuation becomes stochastically higher in the sense of first-order stochastic dominance and as bidder 1's participation cost increases, participating in the auction becomes less profitable for bidder 1, as can be seen from (2). This induces bidder 1 to raise his cutoff v_1^*, which in turn makes bidder 2 behave more conservatively and raise his cutoff $v_2^*(in)$ as well by (4). Similarly, as bidder 1's valuation becomes stochastically higher conditionally on his participation and as bidder 2's participation cost increases, participating in the auction becomes less profitable for bidder 2, as can be seen from (4). Consequently, bidder 2 raises his cutoff $v_2^*(in)$, which induces bidder 1 to behave more aggressively and lower his cutoff v_1^* by (2).

Remark 2 Our result on the existence and uniqueness of cutoff equilibria can be considered as an extension of Propositions 1 and 3 of Tian and Xiao (2011) in that they assume identical participation costs while we allow different participation costs. When the two bidders have the same participation cost, Proposition 1(ii) implies that, as we increase the common participation cost, both $v_2^*(out)$ and $v_2^*(in)$ increase while the change in v_1^* is ambiguous. This is consistent with Proposition 2 of Tian and Xiao (2011), which shows that, with two homogeneous bidders with characteristic (F, c), $v_2^*(out)$ and $v_2^*(in)$ increase in the common participation cost c, while v_1^* increases in c as well when $F(\cdot)$ is concave. By focusing on identical participation costs, Tian and Xiao (2011) analyze the aggregate effect of changing the participation costs of both bidders simultaneously. In contrast, by allowing different participation costs, we can examine the effect of changing one bidder's participation cost separately, which provides us with sharper predictions, as shown in Proposition 1(ii).

[7] For example, consider $F_1(v) = v$ for $v \in [0, 1]$ and $\tilde{F}_1(v) = \frac{7}{9}v^2 + \frac{1}{3}v$ for $v \in [0, \frac{3}{4})$ and $-v^2+3v-1$ for $v \in [\frac{3}{4}, 1]$. $\tilde{F}_1(v)$ first order stochastically dominates $F_1(v)$, but $F_1(v)$ first order stochastically dominates $\tilde{F}_1(v)$ conditionally on $v \geq v_1^*$ for any $v_1^* \in [\frac{3}{4}, 1)$.

4.2 Revenue comparison with simultaneous participation

A natural benchmark scenario is the one where bidders make participation decisions simultaneously. The main goal of this subsection is to compare revenue generated by sequential and simultaneous participation. For analytic simplicity, we focus on the case where two bidders have the same characteristic (F, c).[8]

In order to compare revenue, we need to compare equilibrium cutoffs first. Under simultaneous participation, bidders have no information about the participation decisions of other bidders when they decide whether to participate or not. Thus, a bidder uses a single cutoff in a cutoff strategy. Tan and Yilankaya (2006) show that, when bidders have the same characteristic, there exists a unique symmetric equilibrium in which bidders use the same cutoff. With two homogeneous bidders, the cutoff v^s in the symmetric equilibrium is the solution to the equation $v^s F(v^s) = c$. They further show that (i) if $F(\cdot)$ is concave, there is no other equilibrium and that (ii) if $F(\cdot)$ is strictly convex, there are also asymmetric equilibria in which bidders use different cutoffs. With two homogeneous bidders, the conditions for the cutoffs in an asymmetric equilibrium $\{v_1^a, v_2^a\}$ are given by

$$v_1^a F(v_2^a) = c,$$
$$v_1^a F(v_1^a) + \int_{v_1^a}^{v_2^a} F(v)\,\mathrm{d}v \le c \text{ (with equality if } v_2^a < 1), \tag{5}$$

with $v_1^a < v_2^a$. Following the argument of Cao and Tian (2013), we can show that, if $F(\cdot)$ is strictly convex with the nonincreasing reverse hazard rate $f(\cdot)/F(\cdot)$, there is only one asymmetric equilibrium. The following lemma compares equilibrium cutoffs in the two scenarios of simultaneous and sequential participation.

Lemma 2 *Suppose that there are two bidders with the same characteristic* (F, c). *Let* $\{v_1^*, v_2^*(out), v_2^*(in)\}$ *be the cutoff equilibrium when participation is sequential. Let* v^s *and* $\{v_1^a, v_2^a\}$ *be the symmetric equilibrium and an asymmetric equilibrium, respectively, when participation is simultaneous. Then* $v_1^* \le v_1^a < v^s < v_2^a \le v_2^*(in)$ *with equalities if and only if* $v_2^a = 1$.

Note that all the three pairs of cutoffs, (v^s, v^s), (v_1^a, v_2^a) and $(v_1^*, v_2^*(in))$, satisfy the equation $v_1 F(v_2) = c$. Lemma 2 shows that the gap between the cutoffs used by the two bidders is larger in $(v_1^*, v_2^*(in))$ than in (v_1^a, v_2^a). Bidder 2 behaves more conservatively when he observes bidder 1 participating than when he obtains no information about bidder 1's participation. This in turn induces bidder 1 to behave more aggressively when his participation decision is revealed to bidder 2.

Now we examine revenue that the seller expects to collect from the auction. With two bidders, the seller receives payment only when both bidders participate, and thus, the cutoff $v_2^*(out)$ does not affect the seller's revenue in the case of sequential participation.

[8] Dealing with heterogeneous bidders increases multiplicity of equilibria, which obscures comparison. Thus, we focus on homogeneous bidders not only for analytic tractability but also for clarity of comparison.

Define

$$R(v_1^*, v_2^*) = \int_{v_2^*}^{1} \int_{v_2^*}^{v_1} v_2 \, \mathrm{d}F(v_2)\mathrm{d}F(v_1) + \int_{v_2^*}^{1} \int_{v_1^*}^{v_2} v_1 \, \mathrm{d}F(v_1)\mathrm{d}F(v_2)$$

for (v_1^*, v_2^*) such that $0 \leq v_1^* \leq v_2^* \leq 1$. Then $R(v_1^*, v_2^*)$ gives the expected revenue when bidder 1 uses cutoff v_1^* and bidder 2 uses cutoff v_2^* (at information set (in) in the case of sequential participation). In the following proposition, we present our second main result that compares revenue under sequential and simultaneous participation.

Proposition 2 (Revenue Comparison) *Suppose that there are two bidders with the same characteristic* (F, c). *Let* $\{v_1^*, v_2^*(out), v_2^*(in)\}$ *be the cutoff equilibrium when participation is sequential. Let* v^s *and* $\{v_1^a, v_2^a\}$ *be the symmetric equilibrium and an asymmetric equilibrium, respectively, when participation is simultaneous.*

(i) Suppose that $F(\cdot)$ *is concave. Then* $R(v^s, v^s) > R(v_1^*, v_2^*(in))$.
(ii) Suppose that $F(\cdot)$ *is strictly convex. Then* $R(v^s, v^s) > R(v_1^a, v_2^a) \geq R(v_1^*, v_2^*(in))$ *with equality if and only if* $v_2^a = 1$. *Moreover, if* $\{v_1^a, v_2^a\}$ *and* $\{\tilde{v}_1^a, \tilde{v}_2^a\}$ *are two asymmetric equilibria with* $v_2^a < \tilde{v}_2^a$, *then* $R(v_1^a, v_2^a) > R(\tilde{v}_1^a, \tilde{v}_2^a)$.

When $F(\cdot)$ is concave, there is no asymmetric equilibrium. Thus, we can only compare revenue under sequential participation and revenue at the symmetric equilibrium under simultaneous participation, and Proposition 2(i) shows that simultaneous participation yields higher revenue than sequential participation. When $F(\cdot)$ is strictly convex, we also have asymmetric equilibria. Proposition 2(ii) shows that revenue decreases as the cutoffs of the two bidders become more asymmetric. As the two bidders use more asymmetric cutoffs, the likelihood of both bidders participating gets lower, which reduces revenue. In the case of sequential participation, bidder 2 behaves more conservatively when he observes bidder 1 participating and more aggressively when he observes bidder 1 not participating, compared to the case of simultaneous participation. As a result, sequential participation tends to reduce the probability that no bidder or both bidders participate and to increase the probability that only one bidder participates. Since revenue is positive only when both bidders participate, sequential participation yields lower revenue than simultaneous participation.

4.3 Extension with reserve price and entry subsidy

With two homogeneous bidders, sequential participation cannot improve the seller's revenue over simultaneous participation because it tends to reduce the likelihood of both bidders participating. At the same time, it tends to increase the probability that exactly one bidder participates. If the seller sets a reserve price or charges an entry fee, she can receive positive revenue even when there is only one participating bidder, which creates the possibility that the seller obtains higher revenue under sequential participation than under simultaneous participation. Indeed, we can show that, when the seller uses a reserve price and an entry fee (or an entry subsidy), sequential participation can yield higher revenue than simultaneous participation.[9]

When the seller can use entry subsidies, she can reimburse bidders for their participation costs, and thus, it becomes irrelevant who incurs participation costs between the seller and a bidder. Hence, we can also cover the situation where the seller needs to incur costs to induce the participation of bidders, as considered by Crémer et al. (2007). In a related work, Crémer et al. (2009) consider an alternative scenario where bidders need to incur costs to discover their valuations. The models of Crémer et al. (2007, 2009) are more general than ours in that they allow the seller to choose the sequence of bidders to be contacted and to use an arbitrary mechanism. They show that, under certain conditions, optimal mechanisms can be implemented by a sequence of buy now offers or second-price auctions with reserve prices. However, there are subtle differences in the nature of participation costs between our model and their models, which prevents us from applying their results directly in order to find optimal mechanisms in our setting. First, as mentioned in the Introduction, we consider bidding costs while Crémer et al. (2009) consider information acquisition costs. Consequently, we use interim participation constraints, while Crémer et al. (2009) use ex ante participation constraints. Second, in our model participation costs are incurred only when bidders choose to participate, whereas in Crémer et al. (2007) participation costs are incurred when the seller contacts a bidder regardless of whether the bidder eventually participates or not. Thus, participation costs in Crémer et al. (2007) can be interpreted as "bidder invitation costs." In Crémer et al. (2007), it is without loss of generality to assume that every invited bidder participates, and the seller's expected revenue does not depend on disclosure policies because incurred participation costs do not depend on disclosed information. In contrast, in our model, bidders' participation decisions and incurred participation costs depend on disclosed information as well as bidders' valuations. This will complicate the analysis, and we leave it for future research to investigate optimal mechanisms with bidding costs.

5 Three bidders

In this section, we consider the case where there are three bidders and discuss the difficulties in generalizing the two main results for this case. With three bidders, there are seven cutoffs to be determined. Suppose that bidder 1 decides not to participate. Then the remaining two bidders are in the same situation as the case where there are two bidders to begin with. Hence, we can determine the cutoffs $v_2^*(out)$, $v_3^*(out, out)$ and $v_3^*(out, in)$ uniquely by considering the two-bidder case with characteristics (F_2, c_2) and (F_3, c_3). Moreover, the effects of the characteristics of bidders 2 and 3 on these three cutoffs can be analyzed as in Proposition 1, while the characteristic of bidder 1 has no effect on them. The remaining four cutoffs are determined by the following conditions:

[9] We can also show that a cutoff equilibrium exists uniquely and that the same comparative statics results as in Proposition 1 hold as well. A formal treatment and proofs of these results can be provided upon request.

$$v_1^* F_2(v_2^*(in)) F_3(v_3^*(in, out)) = c_1, \tag{6}$$

$$F_3(v_3^*(in, in)) \int_{v_1^*}^{v_2^*(in)} \frac{F_1(v) - F_1(v_1^*)}{1 - F_1(v_1^*)} \, \mathrm{d}v \le c_2$$
$$\text{(with equality if } v_2^*(in) < 1), \tag{7}$$

$$\int_{v_1^*}^{v_3^*(in,out)} \frac{F_1(v) - F_1(v_1^*)}{1 - F_1(v_1^*)} \, \mathrm{d}v \le c_3$$
$$\text{(with equality if } v_3^*(in, out) < 1), \tag{8}$$

$$\int_{v_2^*(in)}^{v_3^*(in,in)} \frac{F_1(v) - F_1(v_1^*)}{1 - F_1(v_1^*)} \frac{F_2(v) - F_2(v_2^*(in))}{1 - F_2(v_2^*(in))} \, \mathrm{d}v \le c_3$$
$$\text{(with equality if } v_3^*(in, in) < 1), \tag{9}$$

together with the increasing property.

There are at least two difficulties that arise when we conduct a comparative statics analysis with three bidders. First, we cannot guarantee that a cutoff equilibrium is unique, unlike in the two-bidder case, and the direction of changes in cutoffs can vary across different equilibria. Second, even when a cutoff equilibrium is unique, a change in characteristics can create counteracting effects on a bidder's cutoff, making the direction of the net effect indeterminate. Below we elaborate on these issues.

We can determine the four cutoffs as follows. Note that the conditions (6)–(9) provide four equations in four unknowns. We choose one cutoff such that given any value of the cutoff we can determine the values of the other three cutoffs uniquely using three equations. Then we can express the three cutoffs as functions of the chosen cutoff. We plug these expressions into the remaining equation to obtain one equation in one unknown. If this final equation is monotonic in its unknown, we have a unique solution, from which we can obtain a unique equilibrium. However, in general, we cannot guarantee that the final equation is monotonic, and thus, we may have multiple equilibria. Let us illustrate this point with v_1^* as the chosen cutoff. From (8), we can express $v_3^*(in, out)$ as an increasing function of v_1^*. Using this relationship in (6), we can express $v_2^*(in)$ as a decreasing function of v_1^*, and then, we can use (7) to express $v_3^*(in, in)$ as an increasing function of v_1^*. Finally, using the dependence of $v_2^*(in)$ and $v_3^*(in, in)$ on v_1^*, we can express (9) in terms of v_1^* only. As v_1^* increases, $[F_1(v) - F_1(v_1^*)]/[1 - F_1(v_1^*)]$ becomes smaller, which will reduce the left-hand side of (9). At the same time, increasing v_1^* decreases $v_2^*(in)$ and increases $v_3^*(in, in)$, which will increase the left-hand side of (9). Due to these counteracting effects, we cannot guarantee that the left-hand side of (9) expressed in terms of v_1^* only is monotonic. If it is non-monotonic, we have multiple values of v_1^* satisfying the final equation (9) for some range of c_3.

To illustrate that the direction of changes in cutoffs may differ across different equilibria, let us consider an increase in c_1. This change does not affect $v_3^*(in, out)$ as a function of v_1^* obtained from (8), while it induces $v_2^*(in)$ to increase given v_1^* from (6), which in turn makes $v_3^*(in, in)$ decrease given v_1^* from (7). As a result, as c_1 increases, the left-hand side of (9) expressed in terms of v_1^* only will become smaller. Then, v_1^* will increase if the left-hand side of (9) is increasing at the considered solution, while it will decrease otherwise. This observation suggests that we cannot obtain clean comparative statics results with three bidders due to the possibility of multiple equilibria. Even when there always exists a unique equilibrium, a problem still remains. We can apply the same idea behind Proposition 1 to the three-bidder case, but with more than two bidders there can be a chain of responses that create counteracting effects on a bidder's cutoff, leading to ambiguity in the direction of change. To illustrate this point, let us consider an increase in c_1 again. Also, to eliminate the complication due to multiple equilibria, suppose that the left-hand side of (9) expressed in terms of v_1^* only is increasing in v_1^*. Then an increase in c_1 will make v_1^* higher. In response to a higher cutoff adopted by an earlier bidder, bidders 2 and 3 will increase their cutoffs $v_2^*(in)$, $v_3^*(in, out)$ and $v_3^*(in, in)$ as well. However, there is a further effect due to the increase in $v_3^*(in, in)$. As a later bidder becomes more conservative, bidder 2 wants to be more aggressive lowering his cutoff $v_2^*(in)$. Thus, there are two counteracting effects on $v_2^*(in)$, and it is not possible to determine the direction of the net effect on $v_2^*(in)$ from the equilibrium conditions. Similar problems arise when we change the other characteristics.

Now we discuss revenue comparison between sequential and simultaneous participation. Under sequential participation, bidders can adjust their participation decisions to the decisions of earlier bidders, and roughly speaking, a bidder will behave conservatively (resp. aggressively) when he observes many bidders (resp. few bidders) participating before him. As a result, in general, sequential participation tends to reduce the likelihood that too few or too many bidders participate and to increase the likelihood that an intermediate number of bidders participate, compared with simultaneous participation. We can apply this tendency to the three-bidder case. Recall that, with the considered auction format, the seller receives positive revenue only when there are at least two participants. As long as the participation costs are not too large, sequential participation will tend to increase the probability that exactly two bidders participate while reducing the probability that all three bidders participate. Having a higher probability that exactly two bidders participate will produce a positive effect on revenue, while having a lower probability that all three bidders participate will exert a negative one. Whether sequential or simultaneous participation yields higher revenue depends on which effect is dominant. Hence, unlike in the two-bidder case, we cannot exclude the possibility that sequential participation yields higher revenue than simultaneous participation when there are three or more bidders.[10]

6 Conclusion

In this paper, we have studied a second-price auction where bidders bear participation costs. In our model, bidders make participation decisions sequentially in an exogenous

[10] At this point, we are unable to find an example with three bidders in which sequential participation yields higher revenue than simultaneous participation. With more than two bidders, the expression for revenue under sequential participation involves a lot of terms taking into account all possible combinations of who wins the object at whose bid (compared to just two terms in the two-bidder case). Thus we leave it as an open question for future research to prove whether or not Proposition 2 extends to the case of more than two bidders.

order observing the participation decisions of earlier bidders. The distinctive feature of our model is that bidders can adjust their participation decisions to those made by earlier bidders. This feature leads to our two main results with two bidders. First, when a bidder's valuation becomes stochastically higher, the direction of change in his equilibrium cutoff depends on whether he is the first mover (bidder 1) or the second (bidder 2). As bidder 1's valuation becomes stochastically higher conditionally on his participation, bidder 2's expected profit from participating in the auction decreases. This makes bidder 2 more conservative when seeing bidder 1 participate, which in turn induces bidder 1 to become more aggressive and lower his cutoff. On the other hand, as bidder 2's valuation becomes stochastically higher, bidder 1's expected profit from participating in the auction decreases. Then bidder 1 becomes more conservative, which makes bidder 2 become more conservative as well and raise his cutoff when seeing bidder 1 participate. Second, with two homogeneous bidders, the sequential entry format is revenue-dominated by the simultaneous entry counterpart. In a second-price auction with zero reserve price, the seller receives positive revenue only when there are at least two participants. Since bidder 1 participating induces bidder 2 to become conservative, sequential entry reduces the likelihood that both bidders participate compared to simultaneous entry, which results in lower revenue.

There are difficulties in generalizing these two results for the case of more than two bidders. With more than two bidders, a change in characteristics can create counteracting effects on some bidder's cutoff leaving the direction of the net effect ambiguous. Also, with more than two bidders, sequential entry can increase the probability that two or more bidders participate although it will reduce the probability that most of bidders participate. Again, these aspects produce two counteracting effects on revenue, making the direction of the net effect indeterminate. A direction for future research is to study endogenous entry of bidders where bidders decide optimally when to enter the auction with their entry decisions revealed to other bidders. In such a scenario, bidders will face a trade-off; they may want to enter early in order to preempt other bidders' participation, while they may want to wait to acquire more information about others' valuations.

7 Appendix

Proof of Lemma 1: If $v_j^*(h_j) = 1$, then h_i is a zero-probability information set, and thus, we have $v_i^*(h_i) = 1$ by our belief specification. Suppose that $v_j^*(h_j) < 1$ but $v_i^*(h_i) \leq v_j^*(h_j)$. At information set h_i, bidder i believes that $v_j > v_j^*(h_j)$ since bidder j participates. Hence, at h_i, bidder i with valuation $v_i^*(h_i)$ has no chance to win the object when bidding his valuation, and thus, his profit from participation is $-c_i < 0$. However, since $v_i^*(h_i) \leq v_j^*(h_j) < 1$, bidder i with valuation $v_i^*(h_i)$ should be indifferent between participating and not participating at h_i, which is a contradiction. Therefore, $v_i^*(h_i) > v_j^*(h_j)$. □

Proof of Proposition 1: We first describe how to obtain v_1^* and $v_2^*(in)$ from conditions (2) and (4). Let $\tilde{v}$ be the unique value satisfying $\tilde{v}F_2(\tilde{v}) = c_1$. Note that $\tilde{v} \in (0, 1)$.

Define functions ϕ and ψ on $[\tilde{v}, 1]$ by $\phi(y) = c_1/F_2(y)$ and

$$\psi(y) = \int_{\phi(y)}^{y} \frac{F_1(v) - F_1(\phi(y))}{1 - F_1(\phi(y))}\, \mathrm{d}v.$$

Note that $v_1^* = \phi(v_2^*(in))$ by (2) and $\psi(v_2^*(in))$ represents the left-hand side of (4) taking into account the relationship $v_1^* = \phi(v_2^*(in))$. It is easy to check that $\phi(\cdot)$ is continuously differentiable and decreasing, $\phi(\tilde{v}) = \tilde{v}$, $\psi(\cdot)$ is continuously differentiable, and $\psi(\tilde{v}) = 0$. Also,

$$\psi'(y) = \frac{F_1(y) - F_1(\phi(y))}{1 - F_1(\phi(y))} - \frac{f_1(\phi(y))\phi'(y)}{[1 - F_1(\phi(y))]^2} \int_{\phi(y)}^{y} [1 - F_1(v)]\, \mathrm{d}v > 0$$

for all $y \in (\tilde{v}, 1)$. Thus, $\psi(\cdot)$ is increasing on its domain. Suppose that $\psi(1) \leq c_2$. Then we have $v_2^*(in) = 1$ and $v_1^* = \phi(1) = c_1$. Now suppose that $\psi(1) > c_2$. Since $\psi(\tilde{v}) = 0 < c_2$, there exists a unique value $y^* \in (\tilde{v}, 1)$ such that $\psi(y^*) = c_2$. Then we have $v_2^*(in) = y^*$ and $v_1^* = \phi(y^*) \in (c_1, \tilde{v})$.

(i) Since $v_2^*(out) = c_2$, neither $F_1(\cdot)$ nor $F_2(\cdot)$ affects $v_2^*(out)$. Suppose that $F_1(\cdot)$ changes to $\tilde{F}_1(\cdot)$ described in the proposition. Fix $v_0 \in (0, 1)$, and define

$$K(v) = \frac{F_1(v) - F_1(v_0)}{1 - F_1(v_0)} - \frac{\tilde{F}_1(v) - \tilde{F}_1(v_0)}{1 - \tilde{F}_1(v_0)}$$

for $v \in [v_0, 1]$. Then we have

$$K'(v) = \frac{f_1(v)[1 - g(F_1(v_0)) - (1 - F_1(v_0))g'(F_1(v))]}{(1 - F_1(v_0))(1 - g(F_1(v_0)))}.$$

Since $g'(F_1(v))$ is increasing in v, $1 - g(F_1(v_0)) - (1 - F_1(v_0))g'(F_1(v))$ is decreasing in v. Since $K(v_0) = K(1) = 0$, $1 - g(F_1(v_0)) - (1 - F_1(v_0))g'(F_1(v))$ should change its sign once on $[v_0, 1]$. This implies $K(v) > 0$ for all $v \in (v_0, 1)$. Then $\psi(y)$ decreases for any $y \in (\tilde{v}, 1]$, while $\phi(y)$ remains the same. If $v_2^*(in) = 1$ with $F_1(\cdot)$, then v_1^* and $v_2^*(in)$ remain the same. If $v_2^*(in) < 1$ with $F_1(\cdot)$, then $v_2^*(in)$ increases which reduces v_1^* by (2). Suppose that $F_2(\cdot)$ changes to $\tilde{F}_2(\cdot)$ where $\tilde{F}_2(v) < F_2(v)$ for all $v \in (0, 1)$. Now $\phi(y) = c_1/\tilde{F}_2(y)$ and it is defined for $y \in [\tilde{v}', 1]$, where $\tilde{v}'$ is the unique value satisfying $\tilde{v}'\tilde{F}_2(\tilde{v}') = c_1$. Note that $\tilde{v}' > \tilde{v}$ and the value of $\phi(y)$ for any $y \in [\tilde{v}', 1)$ increases as a result of the change in $F_2(\cdot)$. Then $\psi(y)$ decreases for any $y \in [\tilde{v}', 1)$. If $v_2^*(in) = 1$ with $F_2(\cdot)$, then v_1^* and $v_2^*(in)$ remain the same. If $v_2^*(in) < 1$ with $F_2(\cdot)$, then $v_2^*(in)$ increases which increases v_1^* by (4).

(ii) Since $v_2^*(out) = c_2$, $v_2^*(out)$ is increasing in c_2 and independent of c_1. Suppose that c_1 increases to $c_1' > c_1$. Now $\phi(y) = c_1'/F_2(y)$ and it is defined for $y \in [\tilde{v}'', 1]$, where $\tilde{v}''$ is the unique value satisfying $\tilde{v}''F_2(\tilde{v}'') = c_1'$. Note that $\tilde{v}'' > \tilde{v}$, and the value of $\phi(y)$ for any $y \in [\tilde{v}'', 1]$ increases as a result of the increase in c_1. Then $\psi(y)$ decreases for any $y \in [\tilde{v}'', 1]$. Suppose that $v_2^*(in) = 1$ with c_1 . Then we still have $v_2^*(in) = 1$, which gives $v_1^* = c_1'$. In this case, $v_2^*(in)$ remains the same, while v_1^*

increases from c_1 to c_1'. Now suppose that $v_2^*(in) < 1$ with c_1. Then $v_2^*(in)$ increases. In order to satisfy (4), v_1^* should increase as well. Suppose that c_2 increases to $c_2' > c_2$. Then $\phi(y)$ and $\psi(y)$ remain the same for all $y \in [\tilde{v}, 1]$. Suppose that $v_2^*(in) = 1$ with c_2. Then $\psi(1) \leq c_2 < c_2'$ and we still have $v_2^*(in) = 1$, which gives $v_1^* = c_1$. In this case, both $v_2^*(in)$ and v_1^* remain the same. Now suppose that $v_2^*(in) < 1$ with c_2. Then $v_2^*(in)$ increases. In order to satisfy (2), v_1^* should decrease. □

Proof of Lemma 2: Since $vF(v)$ is continuous and increasing from 0 to 1 as v increases from 0 to 1, there exists unique $v^s \in (0, 1)$ satisfying $v^s F(v^s) = c$. We define a function ϕ on $[v^s, 1]$ by $\phi(y) = c/F(y)$. Note that we have $v^s = \phi(v^s)$, $v_1^a = \phi(v_2^a)$ and $v_1^* = \phi(v_2^*(in))$. Since $\phi(\cdot)$ is decreasing and $v_1^a < v_2^a$, it must be that $v_1^a < v^s < v_2^a$, and $v_1^* < v_1^a$ and $v_1^* = v_1^a$ follow if we show $v_2^a < v_2^*(in)$ and $v_2^a = v_2^*(in)$, respectively. Define

$$\psi(y) = \int_{\phi(y)}^{y} \frac{F(v) - F(\phi(y))}{1 - F(\phi(y))} \, \mathrm{d}v$$

and

$$\lambda(y) = \phi(y)F(\phi(y)) + \int_{\phi(y)}^{y} F(v) \, \mathrm{d}v$$

for $y \in [v^s, 1]$.

Suppose that $v_2^a < 1$. Then $\lambda(v_2^a) = c$. If $v_2^*(in) = 1$, then we have $v_2^a < v_2^*(in)$ and we are done. Thus suppose that $v_2^*(in) < 1$. Then $\psi(v_2^*(in)) = c$. Note that

$$\psi(y) = \frac{\phi(y)F(\phi(y)) + \int_{\phi(y)}^{y} F(v) \, \mathrm{d}v - yF(\phi(y))}{1 - F(\phi(y))} = \frac{\lambda(y) - yF(\phi(y))}{1 - F(\phi(y))}.$$

Thus, $\psi(v_2^a) = [c - v_2^a F(\phi(v_2^a))]/[1 - F(\phi(v_2^a))] < c$ since $v_2^a > v_1^a \geq c$. Since $\psi(\cdot)$ is increasing, we have $v_2^a < v_2^*(in)$. Now suppose that $v_2^a = 1$. Then $\lambda(1) \leq c$, and thus $\psi(1) \leq [c - F(\phi(1))]/[1 - F(\phi(1))] < c$. This proves that $v_2^*(in) = 1$. □

Proof of Proposition 2: Define $\tilde{R}(v_2) = R(c/F(v_2), v_2)$ for $v_2 \in [v^s, 1]$.

(i) Suppose that $F(\cdot)$ is concave. We can show that

$$\tilde{R}'(v_2) = -f(v_2)\left[v_2 - v_1 F(v_1) - \int_{v_1}^{v_2} F(v)\mathrm{d}v - (v_1)^2 f(v_1)\frac{1}{F(v_2)} + (v_1)^2 f(v_1)\right],$$

where $v_1 = c/F(v_2)$. Since $\int_{v_1}^{v_2} F(v)\mathrm{d}v \leq F(v_2)(v_2 - v_1)$, we have

$$\tilde{R}'(v_2) \leq -v_2 f(v_2)(1 - F(v_2))\left[1 - \frac{(v_1)^2 f(v_1)}{v_2 F(v_2)}\right] - v_1 f(v_2)(F(v_2) - F(v_1)).$$

Since $F(\cdot)$ is concave, we have $F(v) \geq vf(v)$ for all $v \in [0, 1]$. Thus, we have $(v_1)^2 f(v_1) \leq v_1 F(v_1)$. For $v_2 > v^s$, we have $v_1 F(v_1) < v_2 F(v_2)$. This implies $\tilde{R}'(v_2) < 0$ for all $v_2 \in (v^s, 1)$. Since $v^s < v_2^*(in) \leq 1$, we have $\tilde{R}(v^s) > \tilde{R}(v_2^*(in))$.

(ii) Suppose that $F(\cdot)$ is strictly convex. For any v_2 and v_2' such that $v^s \le v_2 < v_2' \le 1$, we have

$$\tilde{R}(v_2) - \tilde{R}(v_2') = (1 - F(v_2'))\left[\int_{v_2}^{v_2'} v\mathrm{d}F(v) - \int_{v_1'}^{v_1} v\mathrm{d}F(v)\right] + \int_{v_2}^{v_2'}\int_{v_2}^{u} v\mathrm{d}F(v)\mathrm{d}F(u) + \int_{v_2}^{v_2'}\int_{v_1}^{u} v\mathrm{d}F(v)\mathrm{d}F(u),$$

where $v_1 = c/F(v_2)$ and $v_1' = c/F(v_2')$. Noting that the last two terms are positive and using integration by parts, we have

$$\begin{aligned}\tilde{R}(v_2) - \tilde{R}(v_2') &> (1 - F(v_2'))\left[\int_{v_2}^{v_2'} v\mathrm{d}F(v) - \int_{v_1'}^{v_1} v\mathrm{d}F(v)\right] \\ &= (1 - F(v_2'))[v_2'F(v_2') - v_2F(v_2) - G(v_2') + G(v_2) \\ &\quad - v_1F(v_1) + v_1'F(v_1') + G(v_1) - G(v_1')],\end{aligned}$$

where $G(v) = \int_0^v F(u)du$ for $v \in [0, 1]$.

Let $v_2' = 1$. Then $\tilde{R}(v_2) - \tilde{R}(1) > 0$. If $v_2^a = 1$, then $v_2^*(in) = 1$ and we have $\tilde{R}(v^s) > \tilde{R}(v_2^a) = \tilde{R}(v_2^*(in))$. If $v_2^a < 1$ and $v_2^*(in) = 1$, we have $\tilde{R}(v_2^a) > \tilde{R}(v_2^*(in))$. If $v_2^a < 1$ and $\tilde{v}_2^a = 1$, we have $\tilde{R}(v_2^a) > \tilde{R}(\tilde{v}_2^a)$.

Let $v_2 = v^s$ and $v_2' = v_2^a < 1$. Note that $G(v_2^a) - G(v_1^a) = c - v_1^aF(v_1^a)$ by (5), and using this, we obtain

$$\tilde{R}(v^s) - \tilde{R}(v_2^a) > (1 - F(v_2^a))\left\{[v_2^aF(v_2^a) - 2G(v_2^a)] - [v^sF(v^s) - 2G(v^s)]\right\}.$$

The derivative of $vF(v) - 2G(v)$ is $vf(v) - F(v)$. Since $F(\cdot)$ is strictly convex, the derivative is positive for $v > 0$ and thus $vF(v) - 2G(v)$ is increasing in v. Hence, $\tilde{R}(v^s) > \tilde{R}(v_2^a)$.

Let $v_2 = v_2^a$ and $v_2' = v_2^*(in) < 1$. Note that $G(v_2^*(in)) - G(v_1^*) = F(v_1^*)(v_2^*(in) - v_1^*) + c(1 - F(v_1^*))$ by (4), and using this, we obtain

$$\begin{aligned}\tilde{R}(v_2^a) - \tilde{R}(v_2^*(in)) > (1 - F(v_2^*(in)))\{&[v_2^*(in)F(v_2^*(in)) - 2G(v_2^*(in))] \\ &- [v_2^aF(v_2^a) - 2G(v_2^a)] + (v_2^*(in) - c)F(v_1^*)\} > 0.\end{aligned}$$

Thus, $\tilde{R}(v_2^a) > \tilde{R}(v_2^*(in))$.

Let $v_2 = v_2^a$ and $v_2' = \tilde{v}_2^a$ where $v_2^a < \tilde{v}_2^a < 1$. Then we have

$$\tilde{R}(v_2^a) - \tilde{R}(\tilde{v}_2^a) > (1 - F(\tilde{v}_2^a))\left\{[\tilde{v}_2^aF(\tilde{v}_2^a) - 2G(\tilde{v}_2^a)] - [v_2^aF(v_2^a) - 2G(v_2^a)]\right\} > 0.$$

Thus, $\tilde{R}(v_2^a) > \tilde{R}(\tilde{v}_2^a)$. □

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