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SYMPOSIUM

Existence of Nash equilibrium in ordinal games with discontinuous preferences

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Abstract We provide conditions guaranteeing the existence of Nash equilibrium in games in which players' preferences can be arbitrary binary relations. Our main result generalizes Reny's (Economic Theory, forthcoming) existence result for games with ordered preferences and He and Yannelis' (Economic Theory, forthcoming) existence result for abstract economies with non-ordered preferences.

Keywords Nash equilibrium · Discontinuous games · Ordinal games

JEL Classification C72

1 Introduction

Since the pioneering works of Dasgupta and Maskin (1986) and Reny (1999), there has been much progress regarding the problem of existence of equilibrium in games with discontinuous preferences. For instance, McLennan et al. (2011) and Barelli and Meneghel (2013) have considerably extended Reny's (1999) result. Barelli and

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Meneghel's (2013) result has in turn been extended by Carmona and Podczeck (2014) to games with a measure space of players, by Reny (2013) to ordinal games with ordered preferences (i.e., preferences described by complete pre-orders), and by He and Yannelis (2014) and Scalzo (2015) to abstract economies (i.e., generalized games) with non-ordered preferences.[1]

In this paper, we establish the existence of Nash equilibrium in ordinal games where players' preferences can be arbitrary binary relations. For instance, a class of preferences which is not covered by the known results from the literature on existence of Nash equilibrium, but is covered by our results, is that of discontinuous preferences that are reflexive and complete but need not be transitive. Reflexive and complete but non-transitive preferences can emerge by aggregation when a player is not one individual but represents a group of individuals.[2]

The key condition of our existence result is called point target security, or, on some more general level, correspondence target security. It can be roughly described as replacing the value function which appears in several results for games with discontinuous payoff functions by a map which sets targets in ordinal terms.

Our existence result has two particular implications. First, both the existence results in the tradition of Reny (1999) for discontinuous games and those in the tradition of Shafer and Sonnenschein (1975) for abstract economies are covered.[3] In particular, our existence result generalizes that obtained in Reny (2013) for games with ordered preferences and those obtained in Scalzo (2015) and He and Yannelis (2014) for games with non-ordered preferences.

Second, our existence result applies whenever a game is such that for each player, besides of the strategy set being compact and convex, the best-reply correspondence has a well-behaved sub-correspondence, where "well-behaved" means "closed with non-empty convex values." Of course, existence of Nash equilibrium in this case is part of the standard theory. Nevertheless, there is a point. In fact, the formulation and the proof of Reny's (2013) existence result requires a division into two cases: that in which best-reply correspondences are well-behaved, and that in which this is not true. So one may ask whether there is a deeper level of abstraction which integrates the two cases. Our result gives an affirmative answer.

The paper is organized as follows. Section 2 introduces general notation and terminology. Our notions of point target security and correspondence target security are presented in Sect. 3. Our existence result is in Sect. 4. In Sect. 5 we relate our notions of point and correspondence target security to other notions from the literature to deal with discontinuous preferences in the context of ordinal games. Some concluding remarks are in Sect. 6. The proofs of our results may be found in Sect. 7.

1 There has been many other recent developments; e.g., Allison and Lepore (2014), Bagh and Jofre (2006), Balder (2011), Bich (2009), Bich and Laraki (2012), Carmona (2009, 2011), de Castro (2011), Nessah (2011), Prokopovych (2011, 2013, 2015) and Reny (2009, 2011).

2 For an example in the context our paper, see Sect. 6.

3 Other papers in these traditions, which have not been mentioned before, are Borglin and Keiding (1976), Yannelis and Prabhakar (1983), and Simon (1987).

2 Notation and definitions

An (*ordinal*) *game* $G = (X_i, R_i)_{i\in I}$ is given by a finite set $I = \{1, \ldots, n\}$ of players, and a pure strategy space X_i and a binary relation R_i on X for each $i \in I$, where $X = \prod_{i\in I} X_i$. It is assumed that for all $i \in I$, X_i is a non-empty subset of a Hausdorff locally convex space. As usual, given a player $i \in I$, the symbol "$-i$" means "all players but i"; in particular, $X_{-i} = \prod_{j\neq i} X_j$.

Given a game $G = (X_i, R_i)_{i\in I}$, it is assumed that the set I of players is partitioned into two sets I^w and I^s where for each $i \in I^w$, $x R_i y$ means x is at least as good as y (i.e., R_i is a weak preference relation), while for each $i \in I^s$, $x R_i y$ means x is strictly preferred to y (i.e., R_i is a strict preference relation). Now a *Nash equilibrium* of G is an $x^* \in X$ such that (a) for each $i \in I^w$, $x^* R_i (x_i, x^*_{-i})$ for all $x_i \in X_i$, and (b), for each $i \in I^s$, there is no $x_i \in X_i$ such that $(x_i, x^*_{-i}) R_i x^*$. We let $E(G)$ denote the set of Nash equilibria of a game $G = (X_i, R_i)_{i\in I}$.

We say that a game $G = (X_i, R_i)_{i\in I}$ is a *game with ordered preferences* if R_i is a complete pre-order for each $i \in I$, in which case, of course, we take $I^w = I$, and that the game $G = (X_i, R_i)_{i\in I}$ is a *game with payoff functions* if for each $i \in I$, R_i has a utility representation $u_i \colon X \to \mathbb{R}$. If the latter case occurs, and it is appropriate to signify this, we will write $G = (X_i, u_i)_{i\in I}$ instead of $G = (X_i, R_i)_{i\in I}$.

Finally, we say that the game $G = (X_i, R_i)_{i\in I}$ is *compact* if X_i is compact for all $i \in I$, and that the game is *convex* if X_i is convex for all $i \in I$. The game G is said to have *convex preferences* if for all $i \in I$, X_i is convex and $\{x'_i \in X_i \colon (x'_i, x_{-i}) R_i y\}$ is a convex set for any $x, y \in X$.

3 Correspondence target security

In this section we introduce our notion of correspondence target security. For sake of illustration, we start in Sect. 3.1 by looking at the stronger but easier notion of point target security.

3.1 Definition of point target security

The definition of point target security is as follows.

Definition 1 A game $G = (X_i, R_i)_{i\in I}$ is *point target secure* if for each compact set $K \subseteq E(G)^c$ there is a function $\pi = (\pi^1, \ldots, \pi^n) \colon X \to X^I$ such that for all $x \in K$ there is an open neighborhood O of x and, for each $i \in I$, an $\tilde{x}_i \in X_i$ such that

(a) (i) $(\tilde{x}_i, x'_{-i}) R_i \pi^i(x')$ for all $x' \in O \cap K$ and all $i \in I$, and (ii) there exists an $i \in I$ such that $x_i \notin \mathrm{co}\big(\{w_i \in X_i \colon (w_i, x_{-i}) R_i \pi^i(x)\}\big)$;

or

(b) there exists an $i \in I$ such that (i) $(\tilde{x}_i, x'_{-i}) R_i \pi^i(x')$ for all $x' \in O \cap K$, and (ii) $x'_i \notin \mathrm{co}\big(\{w_i \in X_i \colon (w_i, x'_{-i}) R_i \pi^i(x')\}\big)$ for all $x' \in O \cap K$.

Some comments are in order. A central notion in conditions for existence of Nash equilibrium in games with discontinuous payoff functions is that of the value function of a player, i.e., the function which assigns to each strategy profile the supremum of payoffs the player can obtain by unilaterally changing his strategy. This notion is, in general, not available in the context of ordinal games. The functions π^i may be viewed as a surrogate, specifying targets in ordinal terms. In this sense, the above definition says that whenever $x \in X$ is a non-equilibrium point, then every player i can reach his target on some neighborhood of x by means of the same strategy $\tilde{x}_i$, and there is a player i for whom the target is non-trivial at x, i.e., who does not reach his target at x (this is (a)), or there is at least one player who can reach his target on some neighborhood of x by means of the same $\tilde{x}_i$, but for whom the target is non-trivial in the stronger sense that it cannot be reached at any point in some neighborhood of x (this is (b)).

Note that there are elements of reciprocity and security in the definition of point target security. Reciprocity is present in (ii) of both (a) and (b) because different players may be involved at different non-equilibrium strategy profiles. Security is present in (i) of (a) and (b) because with strategy $\tilde{x}_i$ player i obtains an outcome better than $\pi^i(x')$ for any strategy profile x' in some neighborhood of x (whether such outcome is weakly or strictly better than $\pi^i(x')$ depends on whether R_i is a weak or a strict preference relation).

It might appear more natural to formulate the notion of point target security so as to involve just one target function that is defined on all of $E(G)^c$. However, having a family of target functions, each being defined on some compact subset of $E(G)^c$, adds some extra generality, and in particular gives a notion that is weaker than Reny's (2013) point security (see Sect. 5.1).

3.2 Definition of correspondence target security

Here is the definition of correspondence target security.

Definition 2 A game $G = (X_i, R_i)_{i \in I}$ is *correspondence target secure* if for each compact set $K \subseteq E(G)^c$ there is a correspondence $\pi = (\pi^1, \ldots, \pi^n): X \twoheadrightarrow X^I$ such that for all $x \in K$ there is an open neighborhood O of x, and, for each $i \in I$, a closed correspondence $\psi_i: O \twoheadrightarrow X_i$, with non-empty and convex values, such that

(a) (i) $\psi_i(x') \subseteq \mathrm{co}\big(\bigcup_{v \in \pi^i(x')} \{w_i \in X_i: (w_i, x'_{-i}) R_i v\}\big)$ for all $x' \in O \cap K$ and all $i \in I$, and (ii) there exists an $i \in I$ such that $x_i \notin \mathrm{co}\big(\bigcup_{v \in \pi^i(x)} \{w_i \in X_i: (w_i, x_{-i}) R_i v\}\big)$,

or

(b) there exists an $i \in I$ such that (i) $\psi_i(x') \subseteq \mathrm{co}\big(\bigcup_{v \in \pi^i(x')} \{w_i \in X_i: (w_i, x'_{-i}) R_i v\}\big)$ for all $x' \in O \cap K$, and (ii) $x'_i \notin \mathrm{co}\big(\bigcup_{v \in \pi^i(x')} \{w_i \in X_i: (w_i, x'_{-i}) R_i v\}\big)$ for all $x' \in O \cap K$.

The difference between the conditions of point target security and correspondence target security is just that in the latter there is a target correspondence instead of a target function and that the securing strategy is allowed to vary along some correspondence, rather than having to be fixed; modulo this, there is the same interpretation.

Remark 1 A sufficient condition for a game $G = (X_i, R_i)_{i \in I}$ to be correspondence target secure is: There is a function $\pi = (\pi^1, \dots, \pi^n): X \to X^I$ such that (1) for all $x \in X$ and all $i \in I$, $x_i \notin \text{co}\big(\{w_i \in X_i : (w_i, x_{-i}) R_i \pi^i(x)\}\big)$, and (2) for each $x \in E(G)^c$ there is a player $i \in I$, an open neighborhood O of x, and a non-empty-valued correspondence $\psi_i : O \twoheadrightarrow X_i$ such that co ψ_i is closed and for all $x' \in O \cap E(G)^c$, $\psi_i(x') \subseteq \{w_i \in X_i : (w_i, x'_{-i}) R_i \pi^i(x')\}$.

The condition in Remark 1 is tailored for the case where the R_i's are strict preference relations, and will be used in the proof of Theorem 4 below.

If the actions sets X_i are metrizable, we will also consider the following weakening of the notion of correspondence target security.

Definition 3 A game $G = (X_i, R_i)_{i \in I}$ is *weakly correspondence target secure* if for each compact $K \subseteq E(G)^c$ there is a correspondence $\pi = (\pi^1, \dots, \pi^n): X \twoheadrightarrow X^I$ such that for each $x \in K$ there is a player $i \in I$, an open neighborhood O of x, and a closed correspondence $\psi_i : O \twoheadrightarrow X_i$, with non-empty and convex values, such that:

(i) $\psi_i(x') \subseteq \text{co}\big(\bigcup_{v \in \pi^i(x')} \{w_i \in X_i : (w_i, x'_{-i}) R_i v\}\big)$ for all $x' \in O \cap K$;
(ii) $x_i \notin \text{co}\big(\bigcup_{v \in \pi^i(x)} \{w_i \in X_i : (w_i, x_{-i}) R_i v\}\big)$.

4 Existence of equilibrium

In this section we state our main result on the existence of Nash equilibrium.

Theorem 1 *Let $G = (X_i, R_i)_{i \in I}$ be a compact and convex game. Suppose that one of the following conditions is true:* (i) *G is correspondence target secure;* (ii) *G is weakly correspondence target secure and X is metrizable. Then $E(G) \neq \emptyset$.*

In view of Theorem 3 in Sect. 5.2, Theorem 1 generalizes the main existence result of Reny (2013) by allowing for games with non-ordered preferences.[4] It follows from Remark 4 in that section that Theorem 1, in fact, strictly generalizes Reny's (2013) existence result, even when preferences are given by payoff functions. By what will be pointed out in Sect. 5.3, Theorem 1 also generalizes the existence result in He and Yannelis (2014, Theorem 1).

Correspondence target security is a weak condition, but is not necessary for the existence of Nash equilibrium. This is illustrated by the following example.

Example 1 Let $G = (X_i, u_i)_{i=1,2}$ be a two-player game with $X_1 = X_2 = [0, 1]$ and with payoff functions defined by $u_1 \equiv 0$ and

$$u_2(x) = \begin{cases} 1 & \text{if } x_1 < 1 \text{ and } x_2 = x_1, \\ 1 & \text{if } x_1 = 1 \text{ and } x_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

[4] Actually, Theorem 1 includes Theorem 4.2 in Reny (2013) when correspondence security is assumed to hold with respect to the entire set of players. It is straightforward to define the notion of correspondence target security with respect to a subset of players and to obtain a corresponding extension of Theorem 1, so that Theorem 4.2 in Reny (2013) is covered without the above additional assumption; see Sect. 7.9.

Clearly, $E(G) = \{x \in X : u_2(x) = 1\}$. However, G is not weakly correspondence target secure, therefore not correspondence target secure, regardless of the way players' preferences are represented (see Sect. 7.3 for a proof).

5 Implications

In this section we relate our target security notions and our main existence result to corresponding notions and result from the literature.

5.1 Point security

Reny (2013) introduced the notion of point security. A game $G = (X_i, R_i)_{i \in I}$ is *point secure* if whenever $x \in E(G)^c$ there is an $\hat{x} \in X$ and an open neighborhood U of x such that for each $y \in U$ there is an $i \in I$ such that $y_i \notin \text{co}\big(\{w_i \in X_i : (w_i, y_{-i}) R_i (\hat{x}_i, x'_{-i})\}\big)$ for all $x' \in U$.

Theorem 2 shows that point secure games with ordered preferences are point target secure.[5]

Theorem 2 *Let $G = (X_i, R_i)_{i \in I}$ be a point secure game with ordered preferences. Then given any compact subset K of $E(G)^c$ there is a function $\pi = (\pi^1, \ldots, \pi^n) : X \to X^I$ such that for all $x \in K$ there is an open neighborhood O of x and an $\tilde{x} \in X$ such that*

(i) $(\tilde{x}_i, x'_{-i}) R_i \pi^i(x')$ *for all $x' \in O \cap K$ and all $i \in I$;*
(ii) *for each $y \in O$ there is an $i \in I$ such that $y_i \notin co\big(\{w_i \in X_i : (w_i, y_{-i}) R_i \pi^i(x)\}\big)$.*

In particular, there is a player $i \in I$ for whom (ii) *holds at x. Consequently, G is point target secure.*

Together with this theorem, the following example shows that point target security is indeed strictly weaker than point security.

Example 2 Let $G = (X_i, u_i)_{i=1,2}$ be the two-player game with $X_1 = X_2 = [0, 1]$ and payoff functions $u_1 = 1_{\{(1,1/2)\}}$ and $u_2 = 1_D$, where $D = \{(x_1, x_2) \in X : x_2 = 1/2\}$. Note that for each $i = 1, 2$, $u_i(\cdot, x_{-i})$ is quasiconcave for any $x_{-i} \in X_{-i}$. Evidently $E(G) = \{(1, 1/2)\}$. To see that the game is point target secure, consider the functions π^1, π^2 defined by setting $\pi^1(x) = (1, x_2)$ and $\pi^2(x) = (x_1, 1/2)$ for each $x \in X$, and given $x \in E(G)^c$, let $O = X$, $\tilde{x}_1 = 1$, and $\tilde{x}_2 = 1/2$. For this specifications, it is easily seen that (a) in Definition 1 holds at each $x \in K \subseteq E(G)^c$ (for (a)(ii), consider $i = 1$ if $x_2 = 1/2$, and $i = 2$ if $x_2 \neq 1/2$). However, G is not point secure. To see this, note first that $y_i \notin \text{co}\big(\{w_i \in X_i : (w_i, y_{-i}) R_i (\hat{x}_i, x'_{-i})\}\big)$ is equivalent to $u_i(\hat{x}_i, x'_{-i}) > u_i(y)$ as u_i is own-strategy quasiconcave, $i = 1, 2$. Now let $x = (1/2, 1/2) \in E(G)^c$. Then for $y = x$, $u_2(y) \geq u_2(\hat{x}_2, x'_{-2})$ for all $\hat{x}_2 \in X_2$ and all $x' \in X$. Thus, as $u_1(\hat{x}_1, x'_{-1}) = 0$ for all $\hat{x}_1 \in X_1$ and all $x' \in X$ such that $x'_2 \neq 1/2$, G is not point secure.

[5] The proof of this result is analogous to that of Lemma 2 in Carmona (2014), which in turn builds on the proof of Reny's (2013) existence result.

Remark 2 For fairness of comparison, we have shown that the game in Example 2 actually satisfies a version of point target security that is stronger than that in Definition 1 and which parallels Reny's (2013) notion of point security: there is a function $\pi: X \to X^n$ such that for all $x \in E(G)^c$ there is an open neighborhood O of x and, for each $i \in I$, an $\tilde{x}_i \in X$ such that (i) $(\tilde{x}_i, x'_{-i}) R_i \pi^i(x')$ for all $x' \in O$ and all $i \in I$, and (ii) there is an $i \in I$ such that $x_i \notin \text{co}\big(\{w_i \in X_i : (w_i, x_{-i}) R_i \pi^i(x)\}\big)$.

Another way of making the formulations of point target security and point security be parallel is to weaken Reny's (2013) notion of point security as follows: Whenever $K \subseteq E(G)^c$ is compact and $x \in K$, there is an $\hat{x} \in X$ and an open neighborhood U of x such that for each $y \in U \cap K$ there is an $i \in I$ such that $y_i \notin \text{co}\big(\{w_i \in X_i : (w_i, y_{-i}) R_i (\hat{x}_i, x'_{-i})\}\big)$ for all $x' \in U \cap K$. With this weakening of point security, Theorem 2 still holds, as may be seen from the proof, and looking in Example 2 at $x = (1/2, 1/2) \in K = \{z \in X : z_1 = 1/2\} \subseteq E(G)^c$ shows that point target security is still strictly weaker.

Remark 3 Our notion of point target security is more universal than that of point security in the following sense. Consider a game $G = (X_i, R_i)_{i \in I}$ with ordered preferences. For each $i \in I$, let R'_i be the asymmetric part of R_i. Then $G' = (X_i, R'_i)_{i \in I}$ is again a game according to our definition. Moreover, as each R_i is a complete preorder, G and G' are equivalent in the sense that an $x^* \in X$ is an equilibrium of G if and only if it is an equilibrium of G' (recall the equilibrium definition stated in Sect. 2). Now point security and point target security as defined above may apply to both G and G'. However, it might happen that G' is point secure but that G', and hence also G, has no equilibrium. For example, suppose G has the form $G = (X_i, u_i)_{i=1,2}$, with $X_1 = X_2 = [0, 1]$, $u_1 \equiv 0$, and u_2 being such that for each $x_1 \in X_1$ the problem $\max_{x_2 \in X_2} u_2(x_1, x_2)$ has no solution (e.g., for each $x_1 \in X_1$, take $u_2(x_1, 1) = 0$ and $u_2(x_1, x_2) = x_2$ whenever $0 \leq x_2 < 1$). Then G has no Nash equilibrium by the choice of u_2, but G' is point secure as R'_1 is empty by the choice of u_1.

The example shows that for a game G with ordered preferences the game G' associated with G according to the previous paragraph may be point secure, but G may fail to have an equilibrium. This, however, cannot happen regarding point target security. In fact, Theorem 1 above implies that for a game G with ordered preferences to have a Nash equilibrium, it suffices that one of G or G' is point target secure, and it even suffices that any game G'' is point target secure if G'' is constructed by replacing the preference relation by its asymmetric part for each member of an arbitrary subset of the players in G.

5.2 Correspondence security

In this section we show that correspondence target security covers Reny's (2013) correspondence security. A game $G = (X_i, R_i)_{i \in I}$ is said to be *correspondence secure* if whenever $x \in E(G)^c$ there is an open neighborhood U of x and a non-empty-valued correspondence $\varphi: U \twoheadrightarrow X$ such that $\text{co}\,\varphi$ is closed and, for each $y \in U$, there is an $i \in I$ such that $y_i \notin \text{co}\big(\{w_i \in X_i : (w_i, y_{-i}) R_i (z_i, x'_{-i})\}\big)$ for all $x' \in U$ and $z_i \in \varphi_i(x')$, where φ_i is the composition of φ with the projection of X onto X_i.

Theorem 3 *Every correspondence secure game with ordered preferences is correspondence target secure.*

Remark 4 The argument used to show that the game in Example 2 is not point secure also shows that it is not correspondence secure.

5.3 The continuous inclusion property

In this section, we show that the scope of our notion of correspondence target security encompasses abstract economies where players' preferences have the continuous inclusion property defined by He and Yannelis (2014). Following the tradition of Shafer and Sonnenschein (1975), He and Yannelis (2014) specify an *abstract economy* as a list $\mathscr{E} = (X_i, A_i, P_i)_{i \in I}$ where, for each $i \in I$, X_i is an action set, assumed to be a non-empty subset of a Hausdorff locally convex space, $A_i : X \twoheadrightarrow X_i$ is a constraint correspondence, assumed to be non-empty-valued, and P_i is a preference correspondence $P_i : X \twoheadrightarrow X_i$ with the interpretation that $x'_i \in P_i(x)$ if and only if player i strictly prefers (x'_i, x_{-i}) to x. An *equilibrium* of $\mathscr{E}$ is $x^* \in X$ such that, for each $i \in I$, $x^*_i \in \bar{A}_i(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$. Here and below, $\bar{A}_i$ denotes the correspondence sending each $x \in X$ to the closure of $A_i(x)$. Let $E(\mathscr{E})$ denote the set of equilibria of $\mathscr{E}$.

He and Yannelis (2014, Theorem 1) established the existence of an equilibrium for abstract economies satisfying the following conditions:

(i) For each $i \in I$, X_i is non-empty, compact, convex and metrizable;
(ii) For each $i \in I$, A_i is non-empty and convex-valued;
(iii) For each $i \in I$, $\bar{A}_i$ is upper hemicontinuous;
(iv) For each $i \in I$ and each $x \in X$ with $P_i(x) \cap A_i(x) \neq \emptyset$, there exists an open neighborhood O of x and a non-empty-valued correspondence $F_i : O \twoheadrightarrow X_i$ such that co F_i is closed and $F_i(x') \subseteq P_i(x') \cap A_i(x')$ for any $x' \in O$;[6]
(v) For each $i \in I$, $x_i \notin \mathrm{co}(P_i(x) \cap A_i(x))$ for all $x \in X$.

We weaken conditions (iv) as follows.

Definition 4 Let $\mathscr{E} = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy, and for each $x \in X$, let $I(x) = \{i \in I : P_i(x) \cap A_i(x) \neq \emptyset\}$. We say that $\mathscr{E}$ has the *weak continuous inclusion property* if for each $x \in X$ such that $I(x) \neq \emptyset$ and $x_i \in \bar{A}_i(x)$ for all $i \in I$, there is an $i \in I$, an open neighborhood O of x, and a non-empty-valued correspondence $F_i : O \twoheadrightarrow X_i$ such that co F_i is closed and $F_i(x') \subseteq P_i(x') \cap A_i(x')$ for each $x' \in O \cap E(\mathscr{E})^c$.

To see that this definition indeed weakens conditions (iv) in He and Yannelis (2014, Theorem 1), note that it makes a requirement only at non-equilibrium points x where $x_i \in \bar{A}_i(x)$ for all $i \in I$, and only for some $i \in I(x)$,[7] rather than for all of them. That

[6] In the terminology of He and Yannelis (2014), this means that $P_i \cap A_i$ has the continuous inclusion property at x.

[7] While our definition of the weak continuous inclusion property does not explicitly require such i to belong to $I(x)$, this is of course a consequence of the definition.

this weakening actually has substance is pointed out in Remark 7 below. A further weakening is obtained by requiring the condition "$F_i(x') \subseteq P_i(x') \cap A_i(x')$" to hold only at non-equilibrium points which, in particular, implies that the set of equilibria need not be closed.

To an abstract economy $\mathscr{E}$ we associate a game $G_{\mathscr{E}} = (X_i, R_i)_{i \in I}$ as follows. First, we set $I = I^s$ (recall from Sect. 2 that this means that each R_i is interpreted as a strict preference relation). Second, for each $i \in I$ and $x \in X$, we set

$$P_i'(x) = \begin{cases} \bar{A}_i(x) & \text{if } x_i \notin \bar{A}_i(x), \\ P_i(x) \cap A_i(x) & \text{otherwise,} \end{cases}$$

and then R_i is defined by setting $x' R_i x$ if and only if $x'_{-i} = x_{-i}$ and $x'_i \in P_i'(x)$; in particular, for any $i \in I$ and any $x \in X$, the upper section of R_i at x has the form $P_i'(x) \times \{x_{-i}\}$. It is easy to see that x^* is an equilibrium of $\mathscr{E}$ if and only if x^* is a Nash equilibrium of $G_{\mathscr{E}}$.

Our next result shows that the game $G_{\mathscr{E}}$ associated with an abstract economy $\mathscr{E}$ is correspondence target secure if $\mathscr{E}$ has the weak continuous inclusion property and if, as in He and Yannelis (2014, Theorem 1), the constraint correspondences in $\mathscr{E}$ satisfy (ii) and (iii) above and the irreflexivity assumption (v) above holds.

Theorem 4 *Let $\mathscr{E} = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$, A_i takes non-empty and convex values, $\bar{A}_i$ is closed, and $x_i \notin co(P_i(x) \cap A_i(x))$ for all $x \in X$. If $\mathscr{E}$ has the weak continuous inclusion property, then $G_{\mathscr{E}}$ is correspondence target secure.*

Remark 5 The equilibrium existence result by He and Yannelis (2014, Theorem 1) for abstract economies requires the assumption that players' action sets be metrizable. In contrast, combining Theorem 4 and Theorem 1 gives an equilibrium existence result for abstract economies without that assumption.

Remark 6 If the action sets X_i are metrizable, it is sufficient for equilibrium existence if, instead of "$x_i \notin \mathrm{co}(P_i(x) \cap A_i(x))$ for all $x \in X$ and all $i \in I$," it is just assumed that given any non-equilibrium point $x \in X$, $x_i \notin \mathrm{co}(P_i(x) \cap A_i(x))$ for some $i \in I$ for whom the weak continuous inclusion property holds at this x. Under this condition it follows by arguments similar to those of the proof of Theorem 4 that the game is weakly correspondence target secure, so that Theorem 1 applies with condition (ii) to give an equilibrium.

Remark 7 One may apply an equilibrium existence result for abstract economies to get a result on existence of Walrasian equilibrium for exchange economies. In such applications, the constraint correspondences A_i of the players are their budget correspondences, where prices are set by an additional auctioneer-player, who ranks price vectors according to the value they give to the excess demand resulting from the choices of the consumers. However, if the endowments of the consumers may be on the boundary of their consumption sets, this approach has problems if it is done using the continuous inclusion property and, as in He and Yannelis (2014, Theorems 1), the continuous inclusion property is required to hold in the form of condition (iv) above.

In fact, as may easily be seen by simple examples, if the endowment of a consumer is on the boundary of his consumption set and the price vector is such that there are no cheaper points in his budget set, then, in general, condition (iv) fails at the endowment point of this consumer. This is so even when preferences and the aggregate endowment are such that existence of a Walrasian equilibrium follows from the standard theory. In contrast, it is not hard to see that if the continuous inclusion property is required only in the form of Definition 4, then the standard results on existence of Walrasian equilibrium for exchange economies are covered, including the case in which individual endowments may be on the boundary of the consumption sets.

5.4 Games with well-behaved best-reply correspondences

In this section, we show that correspondence target security covers games with well-behaved best-reply correspondences.

For a game $G = (X_i, R_i)_{i \in I}$, with $I = I^w$ we write $B_{G,i} \colon X_{-i} \twoheadrightarrow X_i$ for the best-reply correspondence of i, i.e., $B_{G,i}(x_{-i}) = \{y_i \in X_i \colon (y_i, x_{-i}) R_i (x_i', x_{-i})$ for all $x_i' \in X_i\}$. We say that G is *well-behaved* if, for each $i \in I$, there exists a non-empty and convex-valued closed correspondence $b_i \colon X_{-i} \twoheadrightarrow X_i$ such that $b_i(x_{-i}) \subseteq B_{G,i}(x_{-i})$ for all $x_{-i} \in X_{-i}$.

Theorem 5 *Let $G = (X_i, R_i)_{i \in I}$ be a game with ordered preferences.*

1. *If G is well-behaved and $B_{G,i}$ is convex-valued for each $i \in I$, then G is correspondence target secure.*
2. *Assume that X_i is metrizable, compact and convex for each $i \in I$. Then G is well-behaved if and only if there exists a game $G' = (X_i, u_i)_{i \in I}$ such that for each $i \in I$, $B_{G',i}(x_{-i}) \subseteq B_{G,i}(x_{-i})$ for all $x_{-i} \in X_{-i}$, $u_i(X) \subseteq \{0, 1\}$, u_i is upper semicontinuous, and $u_i(\cdot, x_{-i})$ is quasiconcave for all $x_{-i} \in X_{-i}$.*

Games satisfying the properties noted for G' in part 2 of Theorem 5 are considered in McLennan et al. (2011, Section 6) and Barelli and Meneghel (2013, Proposition 4.1). As shown in Example 2, such games need not be correspondence secure. However, by Theorem 5, they are correspondence target secure.

6 Conclusion

We have obtained a general existence result that applies to games in which players' preferences need not be complete pre-orders. The novelty of our approach consists in the introduction of the notions of point target security and correspondence target security. These conditions require the existence of a function or a correspondence π from X to X^I such that certain local properties hold (these local properties are analogous to those in standard notions such as McLennan et al.'s (2011) C-security or Barelli and Meneghel's (2013) continuous security).

The generality of our approach arises because we do not fix such a π a priori. Specific choices of π yield specific existence results. These include: (1) Reny's (2013) existence results for games with ordered preferences; for this, π^i is constructed based on properties of point security or correspondence security; (2) He and Yannelis's (2014) existence result for abstract economies; for this case, π^i is taken to be the identity function; (3) the classical existence result for well-behaved games; for this case π is defined from players' best-reply correspondences.

We conclude with an example of a game that is covered by our existence result, but not by any of those mentioned in the previous paragraph.

The set of players is $I = \{1, 2\}$, and the action sets are $X_1 = X_2 = [0, 1]$. The preferences R_2 of player 2 are given in terms of the utility function $u_2 = 1_D$, where $D = \{x = (x_1, x_2) \in X: x_2 = 1/2\}$.

Player 1 represents a group of two individuals $h = k, l$. Individual k has the utility function $u_k = 1_{\{(1,1/2)\}} + \frac{1}{2}1_{[1/2,1)\times\{1/2\}} + \rho$, and individual l the utility function $u_l = 1_{[1/2,1)\times\{1/2\}} + \frac{1}{2}1_{X\setminus([1/2,1]\times\{1/2\})} + \rho$, where $\rho: X \to [0, 1/2)$ is given by setting $\rho(x) = x_1$ if $x = (x_1, x_2) \in [0, 1/2) \times \{0\}$, and $\rho(x) = 0$ otherwise. The preferences R_1 of player 1 are given by aggregation according to the rule

$$xR_1y \iff u_h(x) \geq u_h(y) \text{ for both } h = l \text{ and } h = k\,, \text{ or } u_h(x) > u_h(y) \text{ for some } h.$$

The asymmetric part R_1' of R_1 is then given by

$$xR_1'y \iff u_h(x) \geq u_h(y) \text{ for both } h = l \text{ and } h = k\,, \\ \text{and } u_h(x) > u_h(y) \text{ for some } h.$$

Clearly R_1 is reflexive and complete, and R_1' is transitive. But R_1 is not transitive. To see this, take $x \in X\setminus([1/2, 1]\times\{1/2\})$, let $y = (1, 1/2)$, and take $z \in [1/2, 1)\times\{1/2\}$. Then xR_1y, yR_1z, but xR_1z fails.

Let the functions π^1, π^2 from X to X^I be defined by setting $\pi^1(x) = (3/4, x_2)$ and $\pi^2(x) = (x_1, 1/2)$ for each $x \in X$. Let $\tilde{x}_1 = 3/4$, and $\tilde{x}_2 = 1/2$. Then for each $x' \in X$, $(\tilde{x}_i, x'_{-i})R_i\pi^i(x')$ for both $i = 1$ and $i = 2$. Observe that the equilibrium set $E(G)$ of the game is $[1/2, 1] \times \{1/2\}$. If $x \in E(G)^c$ with $x_2 \neq 1/2$, then by quasiconcavity of u_2, $x_2 \notin \text{co}(\{w_2 \in X_2: (w_2, x_{-2})R_2\pi^2(x)\})$. If $x \in E(G)^c$ with $x_2 = 1/2$, then $x_1 \in [0, 1/2)$ and $\pi^1(x) = (3/4, 1/2)$; since $\{w_1 \in X_1: (w_1, 1/2)R_1(3/4, 1/2)\} = [1/2, 1]$ it follows that $x_1 \notin \text{co}(\{w_1 \in X_1: (w_1, x_{-1})R_1\pi^1(x)\})$. Thus the game is point target secure.

Reny's (2013) existence result does not apply to this game, as preferences are not transitive. At $x_2 = 0$, the best-reply set of player 1 is empty, so the game is not well-behaved in the sense of Sect. 5.4, and thus the standard theory does not apply. In regard to the result of He and Yannelis (2014), define preference correspondences $P_i: X \twoheadrightarrow X_i$, $i = 1, 2$, by setting $P_i(x) = \{x_i' \in X_i : (x_i', x_{-i})R_i'x\}$ for $x \in X$ (where, as before, R_i' denotes the asymmetric part of R_i). Note that for $x = (1/4, 1/2)$, $P_2(x) = \emptyset$ and $P_1(x) \neq \emptyset$, but any neighborhood of x contains points x' with $x_2' \neq 1/2$ and $x_2' \neq 0$, and thus points x' such that $P_1(x') = \emptyset$. Thus the continuous inclusion property fails at $x = (1/4, 1/2)$ for both P_1 and P_2, so the result of He and Yannelis (2014) has nothing to say to the situation. In fact, the same holds regarding the weak continuous inclusion property defined above and our Theorem 4.

7 Proofs

In this section, we present the proofs of our results. In Sect. 7.9, we present the notion of correspondence target security with respect to a subset of players and an existence result for games satisfying this condition.

7.1 Lemmas

Lemma 1 *Let Z be a topological space, Y a compact convex subset of a Hausdorff topological vector space, and for each h in a non-empty finite set H, let $\psi_h\colon Z \twoheadrightarrow Y$ be a closed correspondence with non-empty and convex values. Then the correspondence $\psi\colon Z \twoheadrightarrow Y$, defined by setting $\psi(x) = co\big(\bigcup_{h\in H}\psi_h(x)\big)$ for all $x \in Z$, is closed.*

Proof Write Δ for the unit simplex in $\mathbb{R}^H$ and define $f\colon \Delta \times Y^H \to Y$ by setting $f(\alpha, y) = \sum_{h\in H}\alpha_h y_h$ for $(\alpha, y) \in \Delta \times Y^H$. Define $\psi'\colon Z \twoheadrightarrow \Delta \times Y^H$ by setting $\psi'(x) = \Delta \times \prod_{h\in H}\psi_h(x)$, $x \in X$. As each ψ_h takes convex values, we have $\psi = f \circ \psi'$. Thus since Y is compact and f is continuous, the fact that the ψ_h's are closed implies that ψ is closed. □

Lemma 2 *Let X be a topological space, H a non-empty finite set, Y_h a compact convex subset of a Hausdorff topological vector space for each $h \in H$, and $\psi\colon X \twoheadrightarrow \prod_{h\in H} Y_h$ a correspondence with non-empty values such that co ψ is closed. Then the correspondence co $\mathrm{proj}_{Y_h} \circ \psi\colon X \twoheadrightarrow Y_h$ is closed for each $h \in H$.*

Proof Note that $\mathrm{co}\,\mathrm{proj}_{Y_h} \circ \psi = \mathrm{proj}_{Y_h} \circ \mathrm{co}\,\psi$. Using this fact, the lemma easily follows because the closed correspondence co ψ takes values in the compact set $\prod_{h\in H} Y_h$ and because proj_{Y_h} is continuous for each $h \in H$. □

7.2 Proof of Theorem 1

Let $G = (X_i, R_i)_{i\in I}$ be a compact and convex game. Suppose (i) of the theorem holds. Arguing by way of contradiction, suppose that $E(G) = \emptyset$. Choose π, O^x, and $(\psi_i^x)_{i\in I}$, $x \in X$, according to the definition of correspondence target security (apply this definition with $K = X$). For each $x \in X$ let I^x be the (non-empty) set consisting of those $i \in I$ for which $\psi_i^x(x') \subseteq \mathrm{co}\big(\bigcup_{v\in\pi^i(x')}\{w_i \in X_i\colon (w_i, x'_{-i}) R_i v\}\big)$ for all $x' \in O^x$. Let $\langle(V^x, C^x)\rangle_{x\in X}$ be a family of subsets of X such that for each $x \in X$, $x \in V^x \subseteq C^x \subseteq O^x$, V^x is open, and C^x is closed. As X is compact, the family $\langle(I^x, V^x, C^x, O^x, \langle\psi_i^x\rangle_{i\in I})\rangle_{x\in X}$ has a finite subfamily $\langle(I^{x_j}, V^{x_j}, C^{x_j}, O^{x_j}, \langle\psi_i^{x_j}\rangle_{i\in I})\rangle_{j=1}^m$ such that $\langle V^{x_j}\rangle_{j=1}^m$ is an open cover of X. For each $x \in X$ and each $i \in I$, let $J(x) = \{j \in \{1, \dots, m\}\colon x \in V^{x_j}\}$ and $J_i(x) = \{j \in J(x)\colon i \in I^{x_j}\}$. Similarly, let $\bar{J}(x) = \{j \in \{1, \dots, m\}\colon x \in C^{x_j}\}$ and $\bar{J}_i(x) = \{j \in \bar{J}(x)\colon i \in I^{x_j}\}$. Clearly, for each $x \in X$ and $i \in I$, $J(x) \subseteq \bar{J}(x)$ and $J_i(x) \subseteq \bar{J}_i(x)$.

For each $i \in I$, define $\varphi_i\colon X \twoheadrightarrow X_i$ by setting, for each $x \in X$,

$$\varphi_i(x) = \begin{cases} \mathrm{co}\Big(\bigcup_{j\in \bar{J}_i(x)} \psi_i^{x_j}(x)\Big) & \text{if } J_i(x) \neq \emptyset, \\ X_i & \text{otherwise.} \end{cases}$$

Clearly φ_i has non-empty and convex values. Moreover, φ_i is closed. To see this, write $A = \{x \in X\colon J_i(x) \neq \emptyset\}$ and note first that A is open, because if $x' \in A$ and $j \in J_i(x')$, then $x' \in V^{x_j}$ and $i \in I^{x_j}$, and because $i \in I^{x_j}$ implies that V^{x_j} is included in A. Fix an $x_0 \in A$. Observe that $\psi_i^{x_j}$ takes non-empty values on some neighborhood of x_0 for every $j \in \bar{J}_i(x_0)$. Using this fact and Lemma 1, we see that the correspondence $x \mapsto \mathrm{co}\big(\bigcup_{j\in\bar{J}_i(x_0)} \psi_i^{x_j}(x)\big)$ is closed at x_0. Now we have $\bar{J}_i(x) \subseteq \bar{J}_i(x_0)$ for each x in the neighborhood $U = (\bigcap_{j\in\bar{J}(x_0)} O^{x_j}) \cap (\bigcap_{j\in\{1,\ldots,m\}\setminus\bar{J}(x_0)} (C^{x_j})^c)$ of x_0, and it follows that the correspondence $x \mapsto \mathrm{co}\big(\bigcup_{j\in\bar{J}_i(x)} \psi_i^{x_j}(x)\big)$ is closed at x_0. As $x_0 \in A$ was arbitrary, $x \mapsto \mathrm{co}\big(\bigcup_{j\in\bar{J}_i(x)} \psi_i^{x_j}(x)\big)$ is closed on A. As A is open, φ_i must be closed.

It follows that the correspondence $\varphi = \prod_{i\in I}\varphi_i$ has a fixed point x^* say. Now if $J_i(x^*) \neq \emptyset$, then $x_i^* \in \mathrm{co}\big(\bigcup_{j\in\bar{J}_i(x^*)} \psi_i^{x_j}(x^*)\big)$, by the definition of φ_i. Also, if $j \in \bar{J}_i(x^*)$, then $x^* \in O^{x_j}$ and $i \in I^{x_j}$, so that $\psi_i^{x_j}(x^*) \subseteq \mathrm{co}\big(\bigcup_{v\in\pi^i(x^*)}\{w_i \in X_i\colon (w_i, x_{-i}^*) R_i v\}\big)$. Consequently, for each $i \in I$ such that $J_i(x^*) \neq \emptyset$,

$$x_i^* \in \mathrm{co}\left(\bigcup_{v\in\pi^i(x^*)} \{w_i \in X_i\colon (w_i, x_{-i}^*) R_i v\}\right). \tag{1}$$

Suppose there is a $j^* \in J(x^*)$ such that (b) in Definition 2 holds at x_{j^*}. Let $i^* \in I$ be chosen according to this condition. Then $i^* \in I^{x_{j^*}}$, so $j^* \in J_{i^*}(x^*)$. Consequently (1) holds for i^*. But this is impossible by (b)(ii) in Definition 2, because $j^* \in J(x^*)$ implies $x^* \in V^{x_{j^*}} \subseteq O^{x_{j^*}}$.

Thus (a) in Definition 2 must hold at x_j for each $j \in J(x^*)$. Thus, for each $i \in I$, $i \in I^{x_j}$ for all $j \in J(x^*)$ so $J_i(x^*) \neq \emptyset$. Thus (1) holds for all $i \in I$. But this contradicts (a)(ii) in Definition 2, and we conclude that $E(G) \neq \emptyset$ if (i) of the theorem holds.

Suppose (ii) of the theorem holds. Again arguing by way of contradiction, suppose $E(G) = \emptyset$. Choose π, i^x, O^x, and $\psi_{i^x}^x$, $x \in X$, according to the definition of weak correspondence target security. For each $i \in I$, let K_i be the set of those $x \in X$ at which $i = i^x$, and let $O^i = \bigcup_{x\in K_i} O^x$.

Fix any $i \in I$. As X is metrizable, O^i is paracompact. Let $\langle\beta_s^i\rangle_{s\in S^i}$ be a locally finite partition of unity subordinated to the open cover $\langle O^x\rangle_{x\in K_i}$ of O^i (see Engelking 1989, Theorem 5.1.9). For each $s \in S^i$ choose $x_s^i \in K_i$ so that $(\beta_s^i)^{-1}((0,1]) \subseteq O^{x_s^i}$. Define $\varphi_i\colon X \twoheadrightarrow X_i$ by setting $\varphi_i(x') = \sum_{s\in S^i} \beta_s^i(x')\psi_i^{x_s^i}(x')$ if $x' \in O^i$, and $\varphi_i(x') = X_i$ otherwise. By choice of the correspondences $\psi_{i^x}^x$, φ_i is closed with non-empty convex values (use the fact that O^i is open to see closedness).

Do this construction for each $i \in I$ and let $\varphi = \prod_{i\in I}\varphi_i$. Then φ has a fixed point, again say x^*. Now for player $i = i^{x^*}$, we have $x^* \in K_i$ and thus $x^* \in O^i$.

By construction of the correspondence φ, and by choice of the correspondences $\psi^x_{i^x}$, it follows that $x^*_i \in \varphi_i(x^*) \subseteq \text{co}\big(\bigcup_{v\in\pi^i(x^*)}\{w_i \in X_i: (w_i, x^*_{-i})R_i v\}\big)$ for $i = i^{x^*}$. But this gives a contradiction to (ii) in the definition of weak correspondence target security. Thus $E(G) \neq \emptyset$ and the proof of Theorem 1 is complete.

7.3 Proof that the game in Example 1 is not weakly correspondence target secure

Let G be the game in Example 1. For each $i = 1, 2$, let R_i be either player i's ordered preference relation defined by u_i, or its asymmetric part. Suppose that G is weakly correspondence target secure. Let K be a compact subset of $E(G)^c$ such that its projection onto $X_1 = [0, 1]$ includes an open neighborhood of 1; e.g., let $K = [1-\varepsilon, 1]\times\{1/2\}$ for some $0 < \varepsilon < 1/2$. Choose π and a family $\langle i^x, O^x, \psi^x\rangle_{x\in K}$ according to definition of weak correspondence target security. Fix $x \in K$. As $u_1 \equiv 0$, we must have $i^x = 2$ (use (ii) in Definition 3 if $1 \in I^w$, and (i) if $1 \in I^s$). If $2 \in I^s$, (i) in Definition 3 implies that $u_2(v) = 0$ for some $v \in \pi^2(x)$, whereas if $2 \in I^w$, (ii) in Definition 3 implies that $u_2(v) = 1$ for all $v \in \pi^2(x)$. In both cases, it follows that $\text{co}\left(\bigcup_{v\in\pi^2(x)}\{w_2 \in X_2: (x_1, w_2)R_2 v\}\right) = \{w_2 \in X_2: u_2(x_1, w_2) = 1\}$. Hence, if $x \in K$ is such that $x_1 = 1$ and $x' \in O^x \cap K$, (i) in Definition 3 implies that $\psi^x_2(x') = \{x'_1\}$ if $x'_1 < 1$, and $\psi^x_2(x') = \{0\}$ if $x'_1 = 1$. Consequently, if $x \in K$ satisfies $x_1 = 1$, then, given our choice of K, ψ^x is not closed at x, contradicting the choice of ψ^x. This contradiction establishes that G is not weakly correspondence target secure.

7.4 Proof of Theorem 2

Let $G = (X_i, R_i)$ be a point secure game with ordered preferences, and K a compact subset of $E(G)^c$. For each $x \in K$, choose $(U^x, \hat{x}^x)$ according to the definition of point security. Since K is compact, there exists a finite sub-collection $\langle(U^k, \hat{x}^k)\rangle^m_{k=1}$ such that $K \subseteq \bigcup^m_{k=1} U^k$. Of course, we may assume that U^k is non-empty for all $k = 1, \ldots, m$. Note that by the definition of point security,

$$\begin{aligned}&\text{for all } k = 1, \ldots, m \text{ and } y \in U^k, \text{ there exists } i \in I \text{ such that}\\ &\quad y_i \notin \text{co}\big(\{w_i \in X_i: (w_i, y_{-i})R_i(\hat{x}^k_i, x'_{-i})\}\big) \quad \text{for all } x' \in U^k. \end{aligned} \tag{2}$$

Let $\langle C^k\rangle^m_{k=1}$ be a family of closed subsets of X such that $C^k \subseteq U^k$ for each $k = 1, \ldots, m$ and $K \subseteq \bigcup^m_{k=1} C^k$ (cf. the proof of Theorem 1). For each $x \in K$, let $K(x) = \{k \in \{1, \ldots, m\}: x \in C^k\}$ and $O^x = (\bigcap_{k\in K(x)} U^k) \cap (\bigcap_{k\in K(x)^c}(C^k)^c)$.

For each $i \in I$, define a binary relation $\succeq_i$ on $\{1, \ldots, m\}$ by setting $k' \succeq_i k$ if for all $x' \in U^{k'}$ there exists $x \in U^k$ such that $(\hat{x}^{k'}_i, x'_{-i})R_i(\hat{x}^k_i, x_{-i})$. Then $\succeq_i$ is complete, reflexive and transitive. For each $x \in K$ and $i \in I$, let k^x_i be a greatest element of $\succeq_i$ in $K(x)$, and set $\tilde{x}^x = (\hat{x}^{k^x_i}_i)_{i\in I}$.

For each $i \in I$, $x \in K$, and $k \in K(x)$, let

$$K_i(k, x) = \{k' \in \{1, \ldots, m\}: k' \succeq_i k \text{ and } x \in U^{k'}\},$$

and for $k' \in K_i(k, x)$ choose $y(i, k', k, x) \in U^k$ so that $(\hat{x}_i^{k'}, x_{-i}) R_i (\hat{x}_i^k, y_{-i}(i, k', k, x))$.

Now to define the function π, for each $i \in I$ let $\pi^i(x) = x$ if $x \in K^c$. If $x \in K$, let $(\hat{x}_i^k, y_{-i}(i, k, x))$ be a least element for R_i in the set $\{(\hat{x}_i^k, y_{-i}(i, k', k, x)): k' \in K_i(k, x)\}$, $k \in K(x)$, and let $\pi^i(x)$ be a greatest element for R_i in $\{(\hat{x}_i^k, y_{-i}(i, k, x)): k \in K(x)\}$.

Fix any $x \in K$. To see that (i) in the statement of the theorem hold, let $x' \in O^x \cap K$ and $i \in I$. Then $K(x') \subseteq K(x)$ and $x' \in U^{k_i^x}$. Thus, for each $k \in K(x')$, $k_i^x \in K_i(k, x')$. This implies that

$$(\tilde{x}_i^x, x'_{-i}) = (x_i^{k_i^x}, x'_{-i}) R_i (\hat{x}_i^k, y_{-i}(i, k_i^x, k, x')) R_i (\hat{x}_i^k, y_{-i}(i, k, x'))$$

for each $k \in K(x')$, and hence that $(\tilde{x}_i^x, x'_{-i}) R_i (\hat{x}_i^k, y_{-i}(i, k, x'))$ for each $k \in K(x')$. As $\pi^i(x') \in \{(\hat{x}_i^k, y_{-i}(i, k, x')): k \in K(x')\}$, it follows that $(\tilde{x}_i^x, x'_{-i}) R_i \pi^i(x')$. Thus (i) holds.

As for (ii), let $y \in O^x$ and $k \in K(x)$ be given and note that $y \in U^k$. By (2) there is an $i \in I$ such that $y_i \notin \text{co}\big(\{w_i \in X_i: (w_i, y_{-i}) R_i (\hat{x}_i^k, x'_{-i})\}\big)$ for all $x' \in U^k$. In particular, $y_i \notin \text{co}\{w_i \in X_i: (w_i, y_{-i}) R_i (\hat{x}_i^k, y_{-i}(i, k, x))\}$ because $y(i, k, x) \in U^k$. Since $\pi^i(x) R_i (\hat{x}_i^k, y_{-i}(i, k, x))$, it follows that $y_i \notin \text{co}\big(\{w_i \in X_i: (w_i, y_{-i}) R_i \pi^i(x)\}\big)$. Thus (ii) holds. This completes the proof.

7.5 Proof of Theorem 3

Let $G = (X_i, R_i)$ be a correspondence secure game with ordered preferences, and K a compact subset of $E(G)^c$. We will show that there exists a non-empty-valued correspondence $\pi = (\pi^1, \ldots, \pi^n): X \twoheadrightarrow X^I$ such that for every $x \in K$ there is an open neighborhood O of x and, for each $i \in I$, a non-empty-valued correspondence $\psi_i: O \twoheadrightarrow X_i$ such that $\text{co}\, \psi_i$ is closed and

(i) for all $i \in I$, $x' \in O \cap K$, and $z_i \in \psi_i(x')$, there is a $v \in \pi^i(x')$ such that $(z_i, x'_{-i}) R_i v$;
(ii) for all $y \in O$ there is an $i \in I$ such that $y_i \notin \text{co}\big(\bigcup_{v \in \pi^i(x)} \{w_i \in X_i: (w_i, y_{-i}) R_i v\}\big)$.

Clearly, Theorem 3 then follows.

For each $x \in K$, choose (U^x, φ^x) according to the definition of correspondence security. Recall that φ_i^x denotes the composition of φ^x with the projection of X onto X_i. By Lemma 2, the properties of φ^x imply that φ_i^x is non-empty-valued and $\text{co}\, \varphi_i^x$ is closed for each $x \in X$ and each $i \in I$.

Now since K is compact, there is a finite sub-collection $\langle (U^k, \varphi^k) \rangle_{k=1}^m$ such that $K \subseteq \bigcup_{k=1}^m U^k$. We may assume that U^k is non-empty for all $k = 1, \ldots, m$. Note that by the definition of correspondence security, for all $k = 1, \dots, m$ and $y \in U^k$, there exists $i \in I$ such that

$$y_i \notin \mathrm{co}\big(\{w_i \in X_i : (w_i, y_{-i}) R_i (z_i, x'_{-i})\}\big) \quad \text{for all } x' \in U^k \text{ and } z_i \in \varphi_i^k(x'). \quad (3)$$

As in the proof of Theorem 2, let $\langle C^k \rangle_{k=1}^m$ be a family of closed subsets of X such that $C^k \subseteq U^k$ for each $k = 1, \dots, m$ and such that $K \subseteq \bigcup_{k=1}^m C^k$. For each $x \in K$, let $K(x) = \{k \in \{1, \dots, m\} : x \in C^k\}$ and $O^x = (\bigcap_{k \in K(x)} U^k) \cap (\bigcap_{k \in K(x)^c} (C^k)^c)$.

For each $i \in I$, define a binary relation $\succeq_i$ on $\{1, \dots, m\}$ by setting $k' \succeq_i k$ if for all $(x', z'_i) \in \mathrm{graph}(\varphi_i^{k'})$ there exists $(x, z_i) \in \mathrm{graph}(\varphi_i^k)$ such that $(z'_i, x'_{-i}) R_i (z_i, x_{-i})$. Then $\succeq_i$ is complete, reflexive and transitive. For each $x \in K$ and $i \in I$, let k_i^x be a greatest element of $\succeq_i$ in $K(x)$ and define $\psi_i^x : O^x \twoheadrightarrow X_i$ by setting $\psi_i^x(x') = \varphi_i^{k_i^x}(x')$ for all $x' \in O^x$. Note that the correspondence ψ_i^x is non-empty-valued and such that $\mathrm{co}\, \psi_i^x$ is closed, because $\varphi_i^{k_i^x}$ has these properties by what has been noted above.

For each $i \in I$, $x \in K$, $k \in K(x)$, and $z_i \in X_i$, let

$$K_i(k, x, z_i) = \{k' \in \{1, \dots, m\} : k' \succeq_i k, x \in U^{k'} \text{ and } z_i \in \varphi_i^{k'}(x)\},$$

and choose points $y(i, k', k, x, z_i) \in U^k$ and $w_i(i, k', k, x, z_i) \in \varphi_i^k(y(i, k', k, x, z_i))$ for each $k' \in K_i(k, x, z_i)$, so that

$$(z_i, x_{-i}) R_i (w_i(i, k', k, x, z_i), y_{-i}(i, k', k, x, z_i)).$$

Note that, for each $i \in I$,

$$k_i^x \in K_i(k, x', z_i) \text{ for each } x \in K, x' \in O^x \cap K, k \in K(x') \text{ and } z_i \in \psi_i^x(x'). \quad (4)$$

To see this, fix such x, x', k, and z_i. By definition of O^x, we have $K(x') \subseteq K(x)$, and therefore $k_i^x \succeq_i k$ by definition of k_i^x. Moreover, as $k_i^x \in K(x)$, we have $x' \in U^{k_i^x}$, again by definition of O^x. Finally, we have $z_i \in \varphi_i^{k_i^x}(x')$ by definition of ψ_i^x.

For each $x \in K$, let $Z_i(x) = \{z_i \in X_i : K_i(k, x, z_i) \neq \emptyset \text{ for each } k \in K(x)\}$. Note that $Z_i(x) \neq \emptyset$ for each $x \in K$, because $x \in O^x$ and hence by (4), $\psi_i^x(x) \subseteq Z_i(x)$, and because the correspondence ψ_i^x takes non-empty values.

Now to define the correspondence π, for each $i \in I$ let $\pi^i(x) = \{x\}$ if $x \in K^c$. If $x \in K$, define π^i as follows. First, for each $z_i \in Z_i(x)$ and each $k \in K(x)$, let $(w_i(i, k, x, z_i), y_{-i}(i, k, x, z_i))$ be a least element for R_i in the set

$$\{(w_i(i, k', k, x, z_i), y_{-i}(i, k', k, x, z_i)) : k' \in K_i(k, x, z_i)\};$$

then, for each $z_i \in Z_i(x)$, let $\pi^i(x, z_i)$ be a greatest element for R_i in

$$\{(w_i(i, k, x, z_i), y_{-i}(i, k, x, z_i)) : k \in K(x)\}, \quad z_i \in Z_i(x);$$

finally, let

$$\pi^i(x) = \big\{\pi^i(x, z_i) : z_i \in Z_i(x)\big\}.$$

Note that

$$y(i,k,x,z_i) \in U^k \quad \text{and} \quad w_i(i,k,x,z_i) \in \varphi_i^k(y(i,k,x,z_i)) \tag{5}$$

for each $k \in K(x)$ since $y(i,k',k,x,z_i) \in U^k$ and $w_i(i,k',k,x,z_i) \in \varphi_i^k(y(i,k',k,x,z_i))$ for all $k' \in K_i(k,x,z_i)$.

Fix any $x \in K$. To see that statement (i) above holds, let $x' \in O^x \cap K$, $i \in I$, and $z_i \in \psi_i^x(x') = \varphi_i^{k_i^x}(x')$. By (4), $k_i^x \in K_i(k,x',z_i)$ for each $k \in K(x')$, which implies that $z_i \in Z_i(x')$ and hence that $\pi^i(x',z_i) \in \pi^i(x')$. Moreover, for each $k \in K(x')$,

$$\begin{aligned}(z_i, x'_{-i})\ R_i\ &\left(w_i(i,k_i^x,k,x',z_i), y_{-i}(i,k_i^x,k,x',z_i)\right)\\ R_i\ &\left(w_i(i,k,x',z_i), y_{-i}(i,k,x',z_i)\right).\end{aligned}$$

Thus $(z_i, x'_{-i}) R_i (w_i(i,k,x',z_i), y_{-i}(i,k,x',z_i))$ holds for all $k \in K(x')$, and it follows that $(z_i, x'_{-i}) R_i \pi^i(x',z_i)$. Hence, (i) above holds, as $\pi^i(x',z_i) \in \pi^i(x')$.

We next establish (ii) above. Let $y \in O^x$ and $k \in K(x)$ be given, and note that $y \in U^k$. It follows from (3) that there exists $i \in I$ such that

$$y_i \notin \text{co}\{w_i \in X_i \colon (w_i, y_{-i}) R_i (z_i, x'_{-i})\} \quad \text{for all } x' \in U^k \text{ and } z_i \in \varphi_i^k(x'). \tag{6}$$

Arguing by contradiction, suppose $y_i \in \text{co}\big(\bigcup_{v\in\pi^i(x)}\{w_i \in X_i \colon (w_i, y_{-i}) R_i v\}\big)$. Then there exists $J \in \mathbb{N}$, $\{v^j\}_{j=1}^J \subseteq \pi^i(x)$ and $\{y_i^j\}_{j=1}^J \subseteq X_i$ such that $(y_i^j, y_{-i}) R_i v^j$ and $y_i \in \text{co}\{y_i^j\}_{j=1}^J$. Let $v \in \{v^j\}_{j=1}^J$ be such that $v^j R_i v$ for all $j = 1, \ldots, J$. Since $v \in \pi^i(x)$, then $v = \pi^i(x,z_i)$ for some $z_i \in Z_i(x)$. Thus, for each $j = 1, \ldots, J$,

$$(y_i^j, y_{-i}) R_i \pi^i(x,z_i) R_i (w_i(i,k,x,z_i), y_{-i}(i,k,x,z_i)),$$

and therefore $y_i \in \text{co}\{w_i \in X_i \colon (w_i, y_{-i}) R_i (w_i(i,k,x,z_i), y_{-i}(i,k,x,z_i))\}$. But this contradicts (6), because by (5), $y(i,k,x,z_i) \in U^k$ and $w_i(i,k,x,z_i) \in \varphi_i^k(y(i,k,x,z_i))$. This contradiction establishes that $y_i \notin \text{co}\big(\bigcup_{v\in\pi^i(x)}\{w_i \in X_i \colon (w_i, y_{-i}) R_i v\}\big)$. This completes the proof.

7.6 Proof of Theorem 4

Let $\mathscr{E} = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy such that the assumptions of Theorem 4 are satisfied, and let $G_{\mathscr{E}} = (X_i, R_i)_{i\in I}$ be the associated game. Recall that $E(G_{\mathscr{E}}) = E(\mathscr{E})$. Define π by identifying π^i with the identity on X for all $i \in I$; thus, for each $x \in X$ and $i \in I$, $\{w_i \in X_i \colon (w_i, x_{-i}) R_i \pi^i(x)\} = \{w_i \in X_i \colon (w_i, x_{-i}) R_i x\} = P'_i(x)$, i.e., $(z_i, x_{-i}) R_i \pi^i(x)$ is equivalent to $z_i \in P'_i(x)$ for each $z_i \in X_i$. (The correspondences P'_i were defined prior to the statement of Theorem 4.)

By hypothesis, $x_i \notin \text{co}(P_i(x) \cap A_i(x))$ for all $x \in X$ and all $i \in I$. Also by hypothesis, for all $x \in X$ and all $i \in I$, $A_i(x)$ is convex, and hence so is $\bar{A}_i(x)$, being the closure of $A_i(x)$. Thus for all $x \in X$ and all $i \in I$, $x_i \notin \text{co}\big(P'_i(x)\big)$, by the definition of P'_i.

Consider any $x \in E(G_{\mathscr{E}})^c$. By Remark 1 and what was pointed out so far in this proof, we need to show that there is an $i \in I$, a neighborhood O of x, and a correspondence $\psi_i : O \twoheadrightarrow X_i$ such that (1) co ψ_i is closed and has non-empty values, and (2) $\psi_i(x') \subseteq P_i'(x')$ for all $x' \in O \cap E(G_{\mathscr{E}})^c$.

Now $x \in E(G_{\mathscr{E}})^c$ means that x is not an equilibrium of the abstract economy $\mathscr{E}$. Thus we have two cases. The first is given when $x_i \notin \bar{A}_i(x)$ for some $i \in I$. For such an i, since $\bar{A}_i$ is closed, there is an open neighborhood O of x such that $x_i' \notin \bar{A}_i(x')$ for all $x' \in O$. By the definition of P_i', this means $P_i'(x') = \bar{A}_i(x')$ for all $x' \in O$. Thus, letting $\psi_i : O \twoheadrightarrow X_i$ be the restriction of $\bar{A}_i$ to O, (2) holds, and so does (1) because, as noted above, $\bar{A}_i$ takes convex values.

The second case is given when $x_i \in \bar{A}_i(x)$ for all $i \in I$. In this case, the fact that x is not an equilibrium of $\mathscr{E}$ implies that the set $I(x)$ in Definition 4 is non-empty; let i, O and F_i be chosen according to Definition 4, and set $\psi_i = F_i$. In particular, (1) holds for ψ_i. Moreover, for any $x' \in O \cap E(G_{\mathscr{E}})^c$, $\psi_i(x') \subseteq P_i(x') \cap A_i(x') \subseteq \bar{A}_i(x')$ and thus (2) follows, because $P_i'(x') = P_i(x') \cap A_i(x')$ or $P_i'(x') = \bar{A}_i(x')$, by the definition of P_i'.

7.7 Proof of Part 1 of Theorem 5

Let $G = (X_i, R_i)_{i \in I}$ be as in the statement of the theorem, and for each $i \in I$, let $b_i : X_{-i} \twoheadrightarrow X_i$ be as in the definition of a well-behaved game. Define $\pi^i(x) = \{(y_i, x_{-i}) \in X : y_i \in b_i(x_{-i})\}$ for all $i \in I$ and $x \in X$. Furthermore, for each $x \in E(G)^c$, let $O = X$ and $\psi_i(x') = b_i(x'_{-i})$ for each $x' \in O$ and $i \in I$. Then, for each $i \in I$, $x' \in O$, and $z_i \in \psi_i(x') = b_i(x'_{-i})$, we have $(z_i, x'_{-i}) \in \pi^i(x')$ and $(z_i, x'_{-i}) R_i (z_i, x'_{-i})$. Thus (a)(i) in the definition of correspondence target security holds at each $x \in E(G)^c$.

Let $x \in E(G)^c$. Then $x_i \notin B_{G,i}(x_{-i})$ for some $i \in I$. As R_i is transitive, we have $B_{G,i}(x_{-i}) = \bigcup_{v \in \pi^i(x)} \{w_i \in X_i : (w_i, x_{-i}) R_i v\}$. As $B_{G,i}$ is convex-valued, it follows that $\bigcup_{v \in \pi^i(x)} \{w_i \in X_i : (w_i, x_{-i}) R_i v\} = \mathrm{co}\big(\bigcup_{v \in \pi^i(x)} \{w_i \in X_i : (w_i, x_{-i}) R_i v\}\big)$. Thus also (a)(ii) in the definition of target correspondence security holds at each $x \in E(G)^c$.

7.8 Proof of Part 2 of Theorem 5

Suppose that G is well-behaved and, for each $i \in I$, let $b_i : X_{-i} \twoheadrightarrow X_i$ be as in the definition of a well-behaved game. Fix $i \in I$ and define, for each $x \in X$,

$$u_i(x) = \begin{cases} 1 & \text{if } x_i \in b_i(x_{-i}), \\ 0 & \text{otherwise.} \end{cases}$$

Then $G' = (X_i, u_i)_{i \in I}$ is such that for each $i \in I$, $B_{G',i} = b_i$, u_i is upper semicontinuous, $u_i(X) \subseteq \{0, 1\}$, and $u_i(\cdot, x_{-i})$ is quasiconcave for each $x_{-i} \in X_{-i}$. Note that $B_{G',i}(x_{-i}) \subseteq B_{G,i}(x_{-i})$ for each $x_{-i} \in X_{-i}$ because $B_{G',i} = b_i$.

Conversely, suppose that G satisfies the condition in 2. We will show that for each $i \in I$ there exists a non-empty and convex-valued closed correspondence $b_i: X_{-i} \twoheadrightarrow X_i$ such that $b_i(x_{-i}) \subseteq B_{G',i}(x_{-i})$ for all $x_{-i} \in X_{-i}$. Since $B_{G',i}(x_{-i}) \subseteq B_{G,i}(x_{-i})$ for all $x_{-i} \in X_{-i}$, this establishes that G is well-behaved.

Fix $i \in I$ and let $\varphi_i: X_{-i} \twoheadrightarrow X_i$ be any upper hemicontinuous correspondence with non-empty and closed values that satisfies $\varphi_i(x_{-i}) = \{x_i \in X_i: u_i(x_i, x_{-i}) = 1\}$ for all $x_{-i} \in F_i$, where $F_i = \{x_{-i} \in X_{-i}: \max_{x_i \in X_i} u_i(x_i, x_{-i}) = 1\}$. That such a correspondence exists can be seen as follows. If $F_i = \emptyset$, set $\varphi_i(x_{-i}) = X_i$ for each $x_{-i} \in X_{-i}$. Otherwise, let $A = \{x \in X: u_i(x) = 1\}$. As u_i is upper semicontinuous, A is closed. Let d be a metric for the topology of X, and define $f: X \to \mathbb{R}$ by setting $f(z) = -\inf\{d(z, z'): z' \in A\}, z \in X$. Then f is continuous, and since $X = X_{-i} \times X_i$, the existence of a φ_i with the desired properties follows by Berge's maximum theorem.

Note that $B_{G',i}(x_{-i})$ equals X_i if $x_{-i} \in X_{-i} \setminus F_i$, and it equals $\varphi_i(x_{-i})$ if $x_{-i} \in F_i$. Thus $\varphi_i(x_{-i}) \subseteq B_{G',i}(x_{-i})$ for all $x_{-i} \in X_{-i}$. Since $u_i(\cdot, x_{-i})$ is upper semicontinuous and quasiconcave for each $x_{-i} \in X_{-i}$, $B_{G',i}$ has closed and convex values, and it follows that $\overline{\text{co}}\,\varphi_i(x_{-i}) \subseteq B_{G',i}(x_{-i})$ for all $x_{-i} \in X_{-i}$. By Theorem 17.35 in Aliprantis and Border (2006), $\overline{\text{co}}\,\varphi_i$ is closed. Set $b_i = \overline{\text{co}}\,\varphi_i$.

7.9 Correspondence target security with respect to a subset of players

In this section, we present Reny's (2013) notion of correspondence security with respect to a subset of players and introduce the notion of correspondence target security with respect to a subset of players. We then show that every game with ordered preferences satisfying correspondence security with respect to a subset of players is correspondence target secure with respect to the same subset of players, provided that the best-reply correspondences of the remaining players are well-behaved.

Let $G = (X_i, R_i)_{i \in I}$ be a game. For $J \subseteq I$, let B_J denote the set of strategy profiles at which every player $i \in J^c$ plays a best reply, i.e., $B_J = \{x \in X: x R_i (x_i', x_{-i})$ for all $i \in J^c$ and $x_i' \in X_i\}$.

The following is Reny's (2013) notion of correspondence security with respect to a subset of players. A game G is *correspondence secure with respect to* J if for all $x \in E(G)^c \cap B_J$ there exists an open neighborhood U of x and a non-empty-valued correspondence $\varphi: U \twoheadrightarrow X$ such that $\text{co}\,\varphi$ is closed and for all $y \in U \cap B_J$ there is a player $i \in J$ for whom $y_i \notin \text{co}\big(\{w_i \in X_i: (w_i, y_{-i}) R_i (z_i, x'_{-i})\}\big)$ for all $x' \in U \cap B_J$ and $z_i \in \varphi_i(x')$.

Definition 5 A game $G = (X_i, R_i)_{i \in I}$ is *correspondence target secure with respect to* J if for each compact $K \subseteq E(G)^c$ there is a correspondence $\pi = (\pi^1, \ldots, \pi^n): X \twoheadrightarrow X^I$ such that, for all $x \in K \cap B_J$, there is an open neighborhood O of x and, for each $i \in I$, a closed correspondence $\psi_i: O \twoheadrightarrow X_i$, with non-empty and convex values, such that

(a) (i) $\psi_i(x') \subseteq \text{co} \bigcup_{v \in \pi^i(x')} \{w_i \in X_i: (w_i, x'_{-i}) R_i v\}$ for all $x' \in O \cap K \cap B_J$ and $i \in I$, and (ii) there exists an $i \in I$ such that $x_i \notin \text{co}\big(\bigcup_{v \in \pi^i(x)} \{w_i \in X_i: (w_i, x_{-i}) R_i v\}\big)$,

or

(b) there exists an $i \in I$ such that (i) $\psi_i(x') \subseteq \text{co}\big(\bigcup_{v\in\pi^i(x')}\{w_i \in X_i: (w_i, x'_{-i})R_i v\}\big)$ for all $x' \in O \cap K \cap B_J$ and (ii) $x'_i \notin \text{co}\big(\bigcup_{v\in\pi^i(x')}\{w_i \in X_i: (w_i, x'_{-i})R_i v\}\big)$ for all $x' \in O \cap K \cap B_J$.[8]

We obtain the following result.

Theorem 6 *Let $G = (X_i, R_i)_{i\in I}$ be a game with ordered preferences and $J \subseteq I$. If G is correspondence secure with respect to J and, for each $i \in J^c$, the best-reply correspondence B_i is closed with non-empty and convex values, then G is target correspondence secure with respect to J.*

Proof Let $G = (X_i, R_i)_{i\in I}$ be as in the statement of the theorem. By Theorems 3 and 5, we may assume that $J \neq \emptyset$ and $J \neq I$.

Let K be a compact subset of $E(G)^c$. Since for each $i \in J^c$ the best-reply correspondence B_i is closed, B_J is compact. Hence $K \cap B_J$ is compact, and Theorem 3, together with Lemma 2, implies that there exists a correspondence $\pi = (\pi^1, \ldots, \pi^n): X \twoheadrightarrow X^I$ such that for all $x \in K \cap B_J$ there exists an open neighborhood O of x, and for each $i \in J$, a non-empty-valued correspondence $\psi_i: O \twoheadrightarrow X_i$ such that co ψ_i is closed and

(i′) for all $i \in J$, $x' \in O \cap K \cap B_J$ and $z_i \in \psi_i(x')$, there exists $v \in \pi^i(x')$ such that $(z_i, x'_{-i})R_i v$, and

(ii′) for all $y \in O$, there is $i \in J$ such that $y_i \notin \text{co}\big(\bigcup_{v\in\pi^i(x)}\{w_i \in X_i: (w_i, x_{-i})R_i v\}\big)$.

Because $J \neq \emptyset$, (i′) and (ii′) imply that condition (a) in the definition of correspondence target security holds. As K was an arbitrary compact subset of $E(G)^c$, G is correspondence target secure with respect to J. □

The notion of weak correspondence target secure with respect to a subset of players is as follows.

Definition 6 A game $G = (X_i, R_i)_{i\in I}$ is *weakly correspondence target secure with respect to J* if for each compact $K \subseteq E(G)^c$ there is a correspondence $\pi = (\pi^1, \ldots, \pi^n): X \twoheadrightarrow X^I$ such that, for all $x \in K \cap B_J$, there is a player $i \in I$, an open neighborhood O of x and a closed correspondence $\psi_i: O \twoheadrightarrow X_i$, with non-empty and convex values, such that

(i) for all $x' \in O \cap K \cap B_J$, $\psi_i(x') \subseteq \text{co}\big(\bigcup_{v\in\pi^i(x')}\{w_i \in X_i: (w_i, x'_{-i})R_i v\}\big)$;

(ii) $x_i \notin \text{co}\big(\bigcup_{v\in\pi^i(x)}\{w_i \in X_i: (w_i, x_{-i})R_i v\}\big)$.

The following results extends Theorem 1.

Theorem 7 *Let $G = (X_i, R_i)_{i\in I}$ be a compact and convex game and $J \subseteq I$ be such that, for each $i \in J^c$, the best-reply correspondence B_i is closed with non-empty and convex values. Suppose that one of the following conditions is true:* (i) *G is correspondence target secure with respect to J;* (ii) *G is weakly correspondence target secure with respect to J and X is metrizable. Then $E(G) \neq \emptyset$.*

[8] Note that the definition of correspondence target security with respect to J does not explicitly require the player for whom conditions (a) and (b) in Definition 5 hold to belong to J. However, this may be shown to be a consequence of the definition.

Proof The proof is analogous to that of Theorem 1, and therefore we simply indicate how one needs to change it to prove the above statement. For both part (i) and part (ii), and for each $i \in J^c$, change the definition of φ_i as follows: $\varphi_i(x) = B_i(x)$ for all $x \in X$.

The arguments in the proof of Theorem 1 still apply to show that φ has a fixed point x^*. By the definition of B_J, we have $x^* \in B_J$. Thus the proof can be completed in the same way as the proof of Theorem 1. □

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