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RESEARCH ARTICLE

Rational housing bubble

Bo Zhao[1]

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Abstract This paper studies an economy inhabited by overlapping generations of households and investors, with the only difference between the two being that households derive utility from housing services, whereas investors do not. Tight collateral constraint limits the borrowing capacity of households and drives the equilibrium interest rate level down to the housing price growth rate, which makes housing attractive as a store of value for investors. A housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and without the expectation of receiving a dividend either in terms of utility or in terms of rent. Pension reform that reduces the contribution rate may increase the supply of credit and create the housing bubble. Empirical findings from China are consistent with theoretical predictions.

Keywords Housing bubble · Collateral constraint · Pension reform · Chinese economy

JEL Classification G12 · E20 · R21

1 Introduction

Housing assets play a dual role. These assets are not only an investment good but also a consumption good. With the first property alone, housing assets, such as fiat money, can have a rational bubble in the overlapping generation model developed by Samuelson (1958). People are willing to hold housing assets as a store of value,

✉ Bo Zhao
bo.robert.zhao@gmail.com

1 China Center for Economic Research, National School of Development, Peking University, Beijing, China

although their intrinsic value is zero. However, with the second property alone, housing assets, such as a Lucas tree, cannot have a rational bubble in Samuelson's model.[1]

My research question is the following: Can housing assets have a rational bubble with both properties described above? This paper departs from the two-period consumption-loan model developed by Samuelson (1958) with only one twist: The economy consists of two types of agents, households and investors, with the only difference between the two being that households derive utility from housing services, whereas investors do not. With two types of agents coexisting in the model, the equilibrium can have two possible outcomes, which depend on the degree of collateral constraint.

If the collateral constraint is loose, the model economy ultimately arrives at a bubbleless equilibrium, in which investors lend to workers at an interest rate that is higher than the growth rate. Because the equilibrium interest rate is higher than the return rate to housing assets (which is equal to the growth rate), investors have no incentives to hold the housing assets.

Tight collateral constraint limits the borrowing capacity of households and drives the equilibrium interest rate level down to the housing price growth rate, which makes housing attractive as a store of value for investors. There is an excess supply of funds from the investors and asset shortage because households are borrowing-constrained at the equilibrium interest rate. In the equilibrium, investors use the excess funds to purchase houses that are useless to them and expect that the young investors will purchase the housing assets from them in the future.

As long as the rental housing market friction is high enough, the rental market cannot absorb all of the housing assets bought by investors, and the investors will hold some empty houses in the equilibrium. This behavior occurs because high rental market friction implies a higher rental price-to-housing-price ratio, which has households substitute rental housing for owner-occupied housing. However, investors are always indifferent between leaving houses empty or renting them out in the bubbly equilibrium. This suggests that the supply of rental houses is infinitely elastic, and the amount of rental houses in the equilibrium is determined by the demand of households. Therefore, vacant houses can appear when the demand is less than the supply of rental housing. According to the definition by Arce and Lopez-Salido (2011), a housing bubble arises in the equilibrium where there are empty houses because investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or in terms of rent.[2]

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[1] With a positive growth rate, the model economy has two stationary equilibria with an interest rate that is either above or below the growth rate. (If the growth rate is zero, there is only one equilibrium with a positive interest rate.) In the bubbly equilibrium, the growth rate of the bubble is equal to the interest rate, and the size of the bubble cannot grow more rapidly than the economy does. Therefore, only the lower interest rate is possible in the bubbly equilibrium. Moreover, positive dividends (either in terms of rent or in terms of utility) rule out negative equilibrium interest rate. Hence, the growth rate of the bubble must be positive and lower than the growth rate, which implies that the size of the bubble as a proportion of the economy approaches zero in the stationary equilibrium.

[2] The definition of housing bubble used in this paper is different from the traditional definition of bubbles. Asset bubbles are usually defined as the difference between the fundamental and market values of assets, and it is often assumed that bubbles are intrinsically valueless, e.g., Tirole (1985). The two definitions are

The main contribution of the paper is the extension of Samuelson (1958) to include two types of agents with preference heterogeneity and to show that a housing bubble is possible even if only part of the population derives dividends from housing assets. Bubbly equilibrium in which constrained households coexist with empty housing will arise if the collateral constraint is tight and rental market friction is high. The market frictions are necessary because they create asset shortage and bring down the equilibrium interest rate to the growth rate of the economy. Therefore, any shocks that tighten the collateral constraint or create the asset shortage, e.g., the removal of pension system, will trigger the housing bubble in the model. The idea is similar to mechanism through which bubbles occur in the rational bubble literature, in which market frictions guarantee that the equilibrium interest rate (available to outside investors) will be smaller or equal to the growth rate of the economy (Samuelson 1958; Tirole 1985; Farhi and Tirole 2012).[3] Instead of focusing on the general types of bubble, this paper pays special attention to the dual role of housing assets and to its interaction with both financial market friction and rental market friction in determining housing bubble.

The second contribution of the paper is the demonstration that a housing bubble can exist in a production economy à la Diamond (1965). In the similar framework, Tirole (1985) studies the existence of a bubble in the presence of a Lucas tree and shows that a bubble absorbs the excess savings and removes dynamic inefficiency. This paper extends Tirole (1985) to the study of housing assets, the rent value of which is endogenous, and grows as rapidly as the economy does. It shows that a housing bubble absorbs the excess savings from investors and achieves constrained dynamic efficiency, although the sources of inefficiency are different.[4]

This paper adopted the two-period overlapping generation with collateral constraint for several reasons. First, the two-period model with two types of agents is the simplest modeling device that allows us to characterize the basic needs for the store of value from

Footnote 2 continued
not always equivalent, and bubbles can be attached to assets that are intrinsically valuable. See Arce and Lopez-Salido (2011) for the case that housing price is lower than the fundamental value, i.e., the discounted value of utility flows generated by the housing assets, in the bubbly equilibrium. Miao and Wang (2011) study bubbles in stocks' prices whose payoffs are endogenously determined by investment and affected by bubbles. Our paper shares some similarities with Miao and Wang (2011). Both studies investigate bubbles on assets that are intrinsically valuable. In our paper, housing assets provide utility flows to a subgroup of population. In Miao and Wang (2011), bubbles are attached to productive assets, which can be used as a collateral. Both papers consider the impact of credit constraint on asset bubbles. However, our model abstracts from credit constraint to investment, and housing asset is not an input to production. Therefore, two models have different implications of asset bubbles on investment.

[3] For example, in Samuelson (1958) and Tirole (1985), the market friction comes from the market incompleteness of OLG structure. In Farhi and Tirole (2012), the market friction comes from the borrowing constraint. See Brunnermeier and Oehmke (2013) for other frictions, such as informational frictions and heterogeneous beliefs.

[4] The constrained dynamic efficiency is a weaker notion of efficiency than the dynamic efficiency. An allocation is constrained dynamically efficient if there is no other resource feasible allocation that increases the lifetime utility of some agents without reducing that of another, which satisfies the collateral constraint. For its definitions in other model environment, see Farhi and Tirole (2012) and Kunieda (2008). I should thank one anonymous referee for pointing out this. In Tirole (1985), the source of dynamic inefficiency is inherited from standard overlapping generation model with production à la Diamond (1965). In this paper, the constrained dynamic inefficiency is caused by the presence of collateral constraint.

investors in closed form. Second, the overlapping generation framework highlights the intergenerational link and becomes the ideal model to study the effect of pension reform on the possibility of housing bubbles, a channel which is shown to be relevant for China's Housing Boom in the empirical part. The theoretical model of Arce and Lopez-Salido (2011) is most similar to that presented in my paper. Arce and Lopez-Salido (2011) introduces housing assets in a three-period overlapping generation model, in which multiple stationary equilibria exist depending on the financial constraint. This paper constructs a two-period overlapping generation model with two types of agents and a production sector. It shows that multiple equilibria do not necessarily appear in the overlapping generation model. Arce and Lopez-Salido (2011) does not consider the production sector and therefore is silent about investment and capital accumulation.

There is extensive literature on rational asset bubbles.[5] It is well known that rational asset bubbles cannot arise in the simple infinite-horizon model with finite number of households because the transversality condition rules out exploding asset prices path (Scheinkman 1977; Brock 1978, 1982; Tirole 1982). Therefore, many papers follow Samuelson (1958) to study bubbles (or fiat money) in the overlapping generation economy (Wallace 1978; Bewley 1979; Tirole 1985; Weil 1987; Grossman and Yanagawa 1993). Due to the incompleteness of market structure, asset bubbles serve as a store of value and solve the problem of dynamic inefficiency.

However, bubble can exist in the infinite-horizon economy under certain conditions pointed out by Kocherlakota (1992) and extended by Santos and Woodford (1997); Hellwig and Lorenzoni (2009). This is because there are some similarities between the standard overlapping generation economy and the infinite-horizon economy with borrowing constraint or other financial frictions (Aiyagari and McGrattan 1998; Cozzi 2001; Farhi and Tirole 2012).[6] Recent studies start to introduce borrowing constraint or other financial frictions into the infinite-horizon model, e.g., Kocherlakota (2009), Hirano and Yanagawa (2010), Miao and Wang (2011), Wang and Wen (2012), Farhi and Tirole (2012), Kiyotaki and Moore (2012), and Martin and Ventura (2012) introduce credit constraint and investor heterogeneity into a production economy. Bubbles serve as a collateral asset that helps alleviate the financial constraint of productive firms. Caballero and Krishnamurthy (2006) and Caballero et al. (2008) argue that speculative bubbles alleviate the asset scarcity problem in an emerging market and explain global imbalance. Instead of focusing on the role of bubbles in alleviating the borrowing constraint of investors, this paper focuses on the roles of bubbles as a store of value for household consumption. It introduces collateral constraint and heterogenous agents in an otherwise standard two-period overlapping generation model with housing assets. Tight collateral constraint creates the possibility of constrained dynamic inefficiency and housing assets as a store of value for investors remove the constrained dynamic inefficiency, a result consistent with the standard overlapping generation model of rational bubbles.

[5] See Brunnermeier and Oehmke (2013) for other forms of bubbles.

[6] The borrowing constraint in the infinite-horizon models essentially shortens the life span of households and makes an infinitely lived household's behavior similar to that of a sequence of finitely lived households without altruism.

In terms of the consequence of bubble, classic theory of rational bubble usually predicts that bubbles increase interest rate and crowd out unproductive investment (Tirole 1985). Other studies challenge this idea by proposing the crowd-in effect of bubbles (Olivier 2000; Ventura 2002). Recent studies find that bubbles can also crowd in the investment by financially constrained entrepreneurs, e.g., Kocherlakota (2009), Hirano and Yanagawa (2010), Miao and Wang (2011), Wang and Wen (2012), Farhi and Tirole (2012), and Martin and Ventura (2012). This paper abstracts from the financial friction for investment and features the crowd-out effect of bubble only. However, the model economy can generate positive co-movement between investment and housing prices during the transition from the bubbleless stationary equilibrium to the bubbly stationary equilibrium after an exogenous liquidity supply shock. The liquidity shock, which takes the form of the removal of pension system in the policy experiment, creates a downward pressure on interest rate and raises the possibility of housing bubbles. In the empirical section, we apply the model to China, where the housing boom in China can be partly attributed to the rapid decline in the pension system.[7] Moreover, the empirical study finds the negative correlation between investment-to-GDP ratio and housing price, an evidence that the crowd-out effect of housing bubble dominates the crowd-out effect of housing bubble in China.

In terms of welfare implications, the conventional wisdom is that bubble is Pareto improving and efficient. However, there are other potential costs even before the bubble burst. Grossman and Yanagawa (1993) and Hirano and Yanagawa (2010) emphasize the externality of investment that the social return on investment exceeds the private return. Olivier (2000) shows that the welfare implications of bubbles depend on the type of asset that is being speculated on. The speculation on the unproductive assets would be welfare-reducing, whereas the speculation on the productive assets would be welfare-improving. Miao et al. (2014) compare the welfare gains of housing bubble from the relaxation of credit constraint with the welfare losses of housing bubble from inefficient overinvestment and argue that net the effect of housing bubble is to reduce welfare. Miao and Wang (2014) analyzes the welfare loss due to the capital misallocation caused by sectoral bubbles. This paper argues that a bubble is good for investors because it is a good substitute for consumption loans. However, the housing bubble reduces the welfare of households. It raises the borrowing cost of households and reduces the amount of housing services they consumed.

The structure of this paper is organized as follows. Section 1 constructs an overlapping generation model with exogenous endowment growth to illustrate the existence of housing bubble. It also considers a policy experiment of pension reform that may cause the emergence of housing bubble. Section 2 discusses the model extension which includes the rental housing market and production sector. Section 3 presents empirical evidence from China to test the implications of theoretical model. Concluding remarks are provided in Sect. 4.

[7] The reasons we consider the pension reform in the policy experiment are twofolded. The first reason is its empirical relevance to China. The second reason is that the pension wealth (including PAYG pension wealth as well as fully funded pension wealth) and housing wealth are the two largest components in households' balance sheet. Therefore, they are the most important stores of value for average households.

2 Benchmark model

The benchmark model is a two-period overlapping generation model based on the consumption-loan model by Samuelson (1958).

2.1 Preference and endowment

The economy is inhabited by two types of households, investors and households, denoted by subscript i and h, respectively. Both types live for two periods.

Investors have the utility function

$$u_i\left(c_{i,t}^t, c_{i,t+1}^t\right) = \ln c_{i,t}^t + \beta \ln c_{i,t+1}^t \tag{1}$$

where $\beta > 0$. Let $c_{i,t}^t$ and $c_{i,t+1}^t$ denote the non-durable consumption of investors born at t at time t and $t+1$, respectively.

The households derive utility not only from non-durable consumption $c_{h,t}^t$, $c_{h,t+1}^t$, but also from housing services $h_{h,t+1}^t$ when they are young.[8] Households' utility has the following form

$$u_h\left(c_{h,t}^t, c_{h,t+1}^t, h_{h,t+1}^t\right) = \ln c_{h,t}^t + \beta\left(1-\zeta\right)\ln c_{h,t+1}^t + \beta\zeta \ln h_{h,t+1}^t \tag{2}$$

where $0 < \zeta < 1$.

Both investors and households receive same positive income y_t^t when they are young and 0 when they are old.[9] Denote the growth rate of output per capita by g. Hence,

$$\frac{y_{t+1}^{t+1}}{y_t^t} = 1 + g \tag{3}$$

In each period, there are $N_t\omega$ young households and $N_t\left(1-\omega\right)$ young investors, where ω is an exogenous parameter, $0 < \omega < 1$. The population growth rate is

$$\frac{N_{t+1}}{N_t} = 1 + n \tag{4}$$

2.2 Pension system

The government is running a pay-as-you-go (PAYG) social security plan. It collects τy_t^t from each young individual at period t and pays $\tau\left(1+n\right) y_t^t$ to each old generation,

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8 The discount factor for households should be $\beta\left(1-\zeta\right)$, not β. The reason that I did not assume a different utility function for investors, e.g., $u_h\left(c_{h,t}^t, c_{h,t+1}^t, h_{h,t+1}^t\right) = \ln c_{h,t}^t + \beta \ln c_{h,t+1}^t + \gamma \ln h_{h,t+1}^t$, is that the current utility form in Eq. 2 can greatly simplify the analytical expressions in equilibrium. It would not affect the qualitative results.

9 Section 2 includes the production sector and endogenous wage rate. Since I introduce pay-as-you-go social security in the model, the old will receive positive pension benefit. Hence, I can normalize the labor income of the elderly to zero without loss of generality.

where $\tau \geq 0$. Hence, the gross return on PAYG system is given by $(1+g)(1+n)$. The government has no consumption and keeps budget constraint balanced each period.

In the benchmark model, there are three ways to transfer resources intertemporally, i.e., the private IOUs, the housing assets, and the pension system. The PAYG pension plan serves as the forced saving by the government, which reduces private savings. In the policy experiment part, the reduction in pension system serves as an exogenous liquidity shock to the economy.

2.3 Asset market

The price of owner-occupied houses in terms of non-durable consumption goods is given by p_t. Housing assets are completely divisible. For simplicity, I assume away rental market in the benchmark model. It can be considered as the extreme case where rental market friction is infinitely large. See the extension of the model in Sect. 2 for the active rental market.

Both households and investors are subject to the same borrowing constraint

$$a^t_{s,t+1} \geq -(1-\theta)\, p_t h^t_{s,t+1} \tag{5}$$

where $s = i, h$. Housing is the only collateral in this economy. The downpayment ratio θ satisfies $0 < \theta < 1$.[10]

The total stock of housing stock at period t is H_t. The model abstracts from housing construction decision, and H_t is exogenously given. It simply assumes that the government build new houses each period, sell them to the young households or investors, and earn a zero profit. Incorporating the housing construction explicitly, e.g., assuming that H_t is a continuous and differentiable function of p_t, will not affect the qualitative conclusion of the paper.

2.4 Investors

The problem of investors who are born after time $t \geq 1$ can be written as

$$\max_{c^t_{i,t},\, c^t_{i,t+1},\, h^t_{i,t+1},\, a^t_{i,t+1}} \ln c^t_{i,t} + \beta \ln c^t_{i,t+1} \tag{6}$$

subject to the following constraints

$$c^t_{i,t} + a^t_{i,t+1} + p_t h^t_{i,t+1} = (1-\tau)\, y^t_t \tag{7}$$

$$c^t_{i,t+1} = \tau(1+n)\, y^{t+1}_{t+1} + R_{t+1} a^t_{i,t+1} + p_{t+1} h^t_{i,t+1} \tag{8}$$

$$a^t_{i,t+1} \geq -(1-\theta)\, p_t h^t_{i,t+1} \tag{9}$$

$$c^t_{i,t},\, c^t_{i,t+1},\, h^t_{i,t+1} \geq 0 \tag{10}$$

[10] If $\theta = 0$, borrowing-constrained households can buy infinite amount of housing without violating the borrowing constraint. If $\theta = 1$, this endowment economy becomes autarky, and the equilibrium interest rate is not well defined.

2.5 Households

The problem of households who are born after time $t \geq 1$ can be formulated similarly

$$\max_{c^t_{h,t}, c^t_{h,t+1}, h^t_{h,t+1}, a^t_{h,t+1}} \ln c^t_{h,t} + \beta (1-\zeta) \ln c^t_{h,t+1} + \beta\zeta \ln h^t_{h,t+1} \tag{11}$$

subject to the following constraints

$$c^t_{h,t} + a^t_{h,t+1} = (1-\tau)\, y^t_t - p_t h^t_{h,t+1} \tag{12}$$

$$c^t_{h,t+1} = \tau (1+n)\, y^{t+1}_{t+1} + R_{t+1} a^t_{h,t+1} + p_{t+1} h^t_{h,t+1} \tag{13}$$

$$a^t_{h,t+1} \geq -(1-\theta)\, p_t h^t_{h,t+1} \tag{14}$$

$$c^t_{h,t}, c^t_{h,t+1}, h^t_{h,t+1} \geq 0 \tag{15}$$

2.6 Competitive equilibrium

Definition 1 Given the financial asset $a^1_{s,1}$ and housing stocks $h^0_{s,1}$ for the initial old, the distribution of generation t, $\{\mu_{s,t}\}^{\infty}_{t=1}$ with total mass equal to N_t, the initial interest rate R_1, pension system τ, housing stock $\{H_t\}^{\infty}_{t=1}$, the competitive equilibrium consists of the endowment sequences $\{y^t_t\}^{\infty}_{t=1}$, prices $\{p_t, R_{t+1}\}^{\infty}_{t=1}$, allocations $\left\{c^t_{s,t}, c^t_{s,t+1}, h^t_{s,t+1}\right\}^{\infty}_{t=1}$, and the initial consumption $c^0_{s,1}$, $s = i, h$, such that

1. The allocations solve the problem of investors (6) and households (11)
2. The housing market, credit market, and non-durable goods market clear

$$\int h^t_{s,t+1} \mathrm{d}\mu_{s,t} = H_{t+1} \tag{16}$$

$$\int a^t_{s,t+1} \mathrm{d}\mu_{s,t} = 0 \tag{17}$$

$$\int y^t_t \mathrm{d}\mu_{s,t} = \int c^t_{s,t} \mathrm{d}\mu_{s,t} + \int c^{t-1}_{s,t} \mathrm{d}\mu_{s,t-1} + p_t (H_{t+1} - H_t) \tag{18}$$

Proposition 1 gives investors' optimal decisions. The solutions are derived in the "Appendix".

Proposition 1 *Given $\tau, g, n, \{R_t, p_t, y^t_t\}^{\infty}_{t=1}$, the optimal decisions of investors are the followings:*

1. If $R_{t+1} = \frac{p_{t+1}}{p_t}$, then

$$c^t_{i,t} = \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_{t+1}}\right] y^t_t$$

$$c^t_{i,t+1} = \frac{\beta R_{t+1}}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_{t+1}}\right] y^t_t$$

$$a_{i,t+1}^{t} + p_t h_{i,t+1}^{t} = (1-\tau)\, y_t^t - c_{i,t}^t$$
$$a_{i,t+1}^{t} > -(1-\theta)\, p_t h_{i,t+1}^{t}$$
$$h_{i,t+1}^{t} \geq 0$$

2. *If* $R_{t+1} > \frac{p_{t+1}}{p_t}$, *then*

$$c_{i,t}^t = \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t$$
$$c_{i,t+1}^t = \frac{\beta R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t$$
$$a_{i,t+1}^t = (1-\tau)\, y_t^t - c_{i,t}^t \geq 0$$
$$h_{i,t+1}^t = 0$$

3. *If* $R_{t+1} < \frac{p_{t+1}}{p_t}$, *then*

$$c_{i,t}^t = \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{i,t+1}}\right] y_t^t$$
$$c_{i,t+1}^t = \frac{\beta\gamma_{i,t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{i,t+1}}\right] y_t^t$$
$$a_{i,t+1}^t = -(1-\theta)\, p_t h_{i,t+1}^t$$
$$p_t h_{i,t+1}^t = \frac{\beta\gamma_{i,t+1}\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta\gamma_{i,t+1}\,(1+\beta)} y_t^t$$
$$h_{i,t+1}^t > 0$$

where $\gamma_{i,t+1} \equiv \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t}$

The solutions to households' problem are given in the "Appendix". Proposition 2 summarizes the results.

Proposition 2 *Given* $\tau, g, n, \left\{R_t, p_t, y_t^t\right\}_{t=1}^{\infty}$, *the optimal decisions of households are the followings:*

1. *If households are not borrowing-constrained, the optimal allocations are*

$$c_{h,t}^t = \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t$$
$$c_{h,t+1}^t = \frac{\beta\,(1-\zeta)\,R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t$$
$$p_t h_{h,t+1}^t = \frac{1}{1-\frac{p_{t+1}}{p_t R_{t+1}}}\frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t$$
$$a_{h,t+1}^t = (1-\tau)\, y_t^t - p_t h_{h,t+1}^t - c_{h,t}^t$$

2. *If households are borrowing-constrained, the optimal allocations are*

$$
\begin{aligned}
c^t_{h,t} &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{h,t+1}}\right]y^t_t \\
c^t_{h,t+1} &= \frac{\beta(1-\zeta)\gamma_{h,t+1}}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{h,t+1}}\right]y^t_t \\
p_t h^t_{h,t+1} &= \frac{\Psi_t+\Phi_t}{2\theta\varphi(1+\beta)} \\
a^t_{h,t+1} &= -(1-\theta)\,p_t h^t_{h,t+1}
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_{h,t+1} &\equiv \frac{b+\frac{\Psi_t+\Phi_t}{2\theta(1+\beta)}}{\beta(1-\zeta)\left(a-\frac{\Psi_t+\Phi_t}{2\varphi(1+\beta)}\right)} \\
\Psi_t &\equiv a\varphi\beta - b\theta(1+\beta\zeta) \\
\Phi_t &\equiv \sqrt{\Psi_t^2+4ab\theta\beta\zeta\varphi(1+\beta)} \\
\varphi &\equiv \frac{p_{t+1}}{p_t}-(1-\theta)R_{t+1} \\
a &\equiv (1-\tau)\,y^t_t \\
b &\equiv \tau(1+n)(1+g)\,y^t_t
\end{aligned}
$$

In order to characterize the existence and uniqueness of the stationary equilibrium, we first study the properties of optimal decision rules. Lemma 1 describes the shapes of the supply curve and demand curve in the credit market. The proof can be found in the "Appendix".

Lemma 1 *The credit demand (credit supply) of households (investors) is always a strictly decreasing (increasing) function of interest rate.*

In the rest part of this section, we focus on the properties of stationary equilibrium. To simplify the problem, we can detrend the allocations and prices using their growth rate along the balanced growth path. The rest of the paper assumes $g = n = 0$ but keep in mind that all the variables are detrended.[11]

Lemma 2 states one important feature of this overlapping generation economy. The real interest rate cannot be negative in the stationary equilibrium. The proof can be found in the "Appendix". The intuition is the following. If the interest rate is negative, then both investors and households want to borrow against housing assets. The credit market cannot clear in this case. However, it cannot rule out zero net interest rate because investors are indifferent between investing in housing asset and lending to

[11] We can define $\tilde{y}^t_t \equiv \frac{y^t_t}{(1+g)^t}$, $\tilde{c}^t_{s,t} \equiv \frac{c^t_{s,t}}{(1+g)^t}$, $\tilde{c}^{t-1}_{s,t} \equiv \frac{c^{t-1}_{s,t}}{(1+n)(1+g)^t}$, $\tilde{a}^t_{s,t+1} \equiv \frac{a^t_{s,t+1}}{(1+g)^t}$, $\tilde{p}_t \equiv \frac{p_t}{(1+n)^t(1+g)^t}$, $\tilde{R}_{t+1} \equiv \frac{R_{t+1}}{(1+n)(1+g)}$, $\tilde{h}^t_{s,t+1} \equiv h^t_{s,t+1}(1+n)^t$, $\tilde{H}_{t+1} \equiv H_{t+1}$, $\tilde{\varphi} \equiv \frac{\varphi}{(1+n)(1+g)}$, $s = i, h$.

households in the equilibrium. Recall that the normalized interest rate is given by $\tilde{R}_{t+1} \equiv \frac{R_{t+1}}{(1+n)(1+g)}$. Therefore, Lemma 2 actually states that the equilibrium interest rate R is greater than or equal to the growth rate $n+g$ in the stationary equilibrium.

Lemma 2 *If* $0 < \omega, \theta < 1$*, there is no stationary equilibrium with gross interest rate* $R < 1$.

Proposition 3 characterizes the uniqueness of stationary equilibrium. It states that the allocation of housing assets in this economy depends on the tightness of collateral constraint. There are two threshold levels for collateral constraint, denoted by θ_L and θ_H, and three different cases.

Proposition 3 *There exists a unique stationary equilibrium.*

1. *If* $\theta \leq \theta_L$*, there are unconstrained households and unconstrained investors holding zero housing assets.*
2. *If* $\theta_L < \theta \leq \theta_H$*, there are borrowing-constrained households and unconstrained investors holding zero housing assets.*
3. *If* $\theta > \theta_H$*, there are constrained households and unconstrained investors holding housing assets.*
   where

$$\theta_L = \omega$$

and θ_H *is determined by*

$$(1-\omega)\left(1-\tau-\frac{1}{1+\beta}\right)y-\omega\left(\frac{1-\theta_H}{\theta_H}\right)\frac{\Psi\left(\theta_H\right)+\Phi\left(\theta_H\right)}{2\theta_H\left(\beta+1\right)}=0$$

where Ψ *and* Φ *are defined in Proposition 2.*

We provide some intuition here (see the "Appendix" for the proof). It is clear that the optimal demand and supply of credit are continuous from Propositions 1 and 2. Lemma 1 proves that the demand of credit from households is monotonically decreasing in the interest rate, and the supply of credit from investors is a monotonically increasing function of interest rate. From Lemma 2, there exists a unique stationary equilibrium with $R^* \geq 1$. Households always borrow from investors in the equilibrium because they also consume housing services. Investors will not be borrowing-constrained when $R^* \geq 1$. They supply credit in the market. θ will only affect the optimal decision of households, who are the demand side of credit market. As the borrowing constraint becomes tighter (higher θ), households are going to be borrowing-constrained first. High θ reduces the borrowing limit of constrained households. If θ is high enough, the total borrowings from households become less than the total credit supply from investors. Interest rate has to be lower in order to clear the credit market. Therefore, tighter borrowing constraint reduces the credit demand from households and drives the equilibrium interest rate down. When the gross interest rate drops to one, housing assets become attractive as an alternative saving mean to the investors. The credit market clearing condition requires that the extra supply of credit coming from investors to be invested in the housing assets, which are the only alternative assets in this economy.

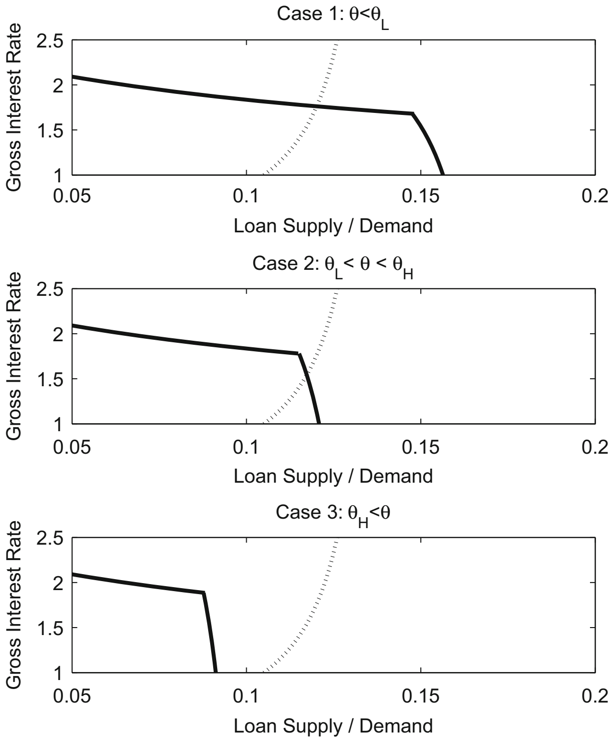

Fig. 1 Three cases of stationary equilibrium. The fraction of households $\omega = 0.65$, payroll tax $\tau = 0.2$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$

Figure 1 exhibits three cases in Proposition 3. The dotted line is the credit supply of investors. The minimum equilibrium gross interest rate is 1. The solid line is the credit demand from households. As proved by Lemma 1, it is a decreasing function of interest rate. It is kinked because it consists of two parts. The flatter part is the credit demand of unconstrained households. The steeper part is the credit demand of borrowing-constrained households. The intersection point pins down the equilibrium interest rate.

Although our model shares some similarities with Arce and Lopez-Salido (2011), the two models have different implications on the multiplicity of stationary equilibria. Depending on the parameters, Arce and Lopez-Salido (2011) can have at most two bubbleless stationary equilibria and one bubbly equilibrium. The key difference lies in the shape of credit supply curve. In Arce and Lopez-Salido (2011), the credit supply curve is kinked because the assumption of a three-period OLG economy, in which households consume housing in the first period and sell their housing in second period to finance their old-age consumption. When the interest rate is high, the credit supply is an increasing function of interest rate, as we have in our two-period model. However, the supply of credit will start to increase as the interest rate becomes lower enough because the borrowing-constrained young households can afford supplying more credit when they are middle-aged if the interest rate on their mortgage is low. Therefore, the credit supply curve can cross the credit demand curve twice under positive interest rate, which in turn implies two bubbleless stationary equilibrium.

Corollary 1 *The third case of stationary equilibrium, i.e., constrained households and unconstrained investors with empty housing, is a bubbly equilibrium for investors, but not for households.*

This paper defines housing bubble according to Arce and Lopez-Salido (2011), i.e., a housing bubble arises in an equilibrium in which the agents hold houses for resale purposes only and without the expectation of receiving a dividend either in terms of utility or in terms of rent. Therefore, the third case of stationary equilibrium is bubbly in the sense that unconstrained investors with empty housing coexist with constrained households in the economy.[12] Corollary 1 describes the special feature of the equilibrium with bubble, i.e., it is a bubble for the investors only (see "Appendix" for the proof). It may seem strange. However, in order to understand the intuition, let me quote a paragraph from Tirole (1985). He described two views of money: the fundamentalist view and the bubbly view of money. The fundamentalist view argues that "money is held to finance transactions (or to pay taxes or to satisfy a reserve requirement). To this purpose, money must be a store of value. However, it is not held for speculative purposes as there is no bubble on money." The bubbly view argues that "money is a pure store value à la Samuelson (1958). It does not serve any transaction purpose at least in the long run. This view implies that price of money (bubble) grows at the real rate of interest, and that money is held entirely for speculation." "The two representations are in the long run inconsistent."

This paper combines the two views together in one model by assuming different preferences on housing assets. Households derive utility from housing assets. This is similar to the fundamentalist view. Investors treat housing assets as investment tools and a store of value. This is same as the bubbly view. Therefore, this paper shows that the two views on money can be reconciled in one model.

2.7 Social welfare

We want to understand the welfare implications of housing bubble for the whole economy before housing bubble bursts.[13] Grossman and Yanagawa (1993) and Hirano and

[12] This paper does not model the collapse of the bubble. In the benchmark model, the current young investors are willing to hold empty housing assets because they expect that the future young generations will purchase housing assets from them. We can introduce the crash of bubble following Caballero and Krishnamurthy (2006). Suppose we introduce another useless asset, say fiat money, as an alternative store of value for investors in this economy. The fiat money may also be valued by investors and becomes a pure bubble using the definition of Arce and Lopez-Salido (2011). After the introduction of fiat money, there could be multiple stationary equilibria. The conjecture is that investors only purchase fiat money as a store of value in one equilibrium and only purchase housing as a store of value in the other equilibrium. Hence, housing price crash is possible when there is a coordination failure among investors. Under this assumption, the expected housing price growth rate will be higher than the interest rate before the bubble crashes. This is because the risk-averse investors need to be compensated for the housing price risks.

[13] Obviously, the collapse of housing bubble implies welfare losses for those who hold the assets and welfare gains for the future young generation who will purchase the assets.

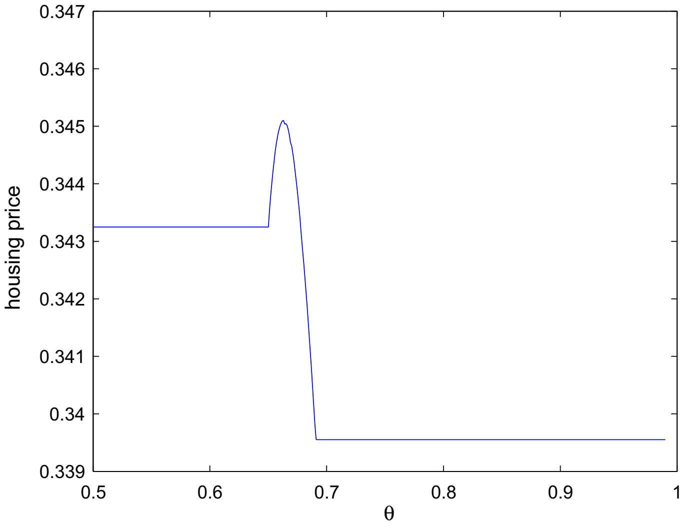

Fig. 2 Equilibrium housing prices under different θ. The fraction of households $\omega = 0.65$, payroll tax $\tau = 0.2$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$

Yanagawa (2010) emphasize the externality of investment that the social return on investment exceeds the private return. Olivier (2000) shows that the welfare implications of bubbles depend on the type of asset that is being speculated on. The speculation on the unproductive assets would be welfare-reducing, whereas the speculation on the productive assets would be welfare-improving. Miao and Wang (2014) analyzes the welfare loss due to the capital misallocation caused by sectoral bubbles.

In this paper, the stationary equilibrium is unique and depends on the collateral constraint. Therefore, the way we evaluate the welfare implications of housing bubble is to compare welfare for households and investors under different degrees of collateral constraint. This paper argues that a bubble is good for investors because it is a good substitute for consumption loans. However, the housing bubble reduces the welfare of households. It raises the borrowing cost of households and reduces the amount of housing services they consumed.

To understand this, we consider three cases under Proposition 3. What is the welfare implications of tightening collateral constraint? Figs. 2, 3, and 4 illustrate the housing prices, gross interest rate, and social welfare under different values of θ, the downpayment ratio. It is clear that under the first case, in which both households and investors are unconstrained, the tightening of collateral constraint does not affect the social welfare, which can be defined as follows:

$$\begin{aligned}&(1-\omega)\, u_i + \omega u_h \\ &= (1-\omega)\left[\ln c_{i,t}^t + \beta \ln c_{i,t+1}^t\right] + \omega\left[\ln c_{h,t}^t + \beta\,(1-\zeta)\ln c_{h,t+1}^t + \beta\zeta \ln h_{h,t+1}^t\right]\end{aligned}$$

Under the second case, in which households are borrowing-constrained and investors do not purchase housing assets, the tightening of borrowing constraint implies

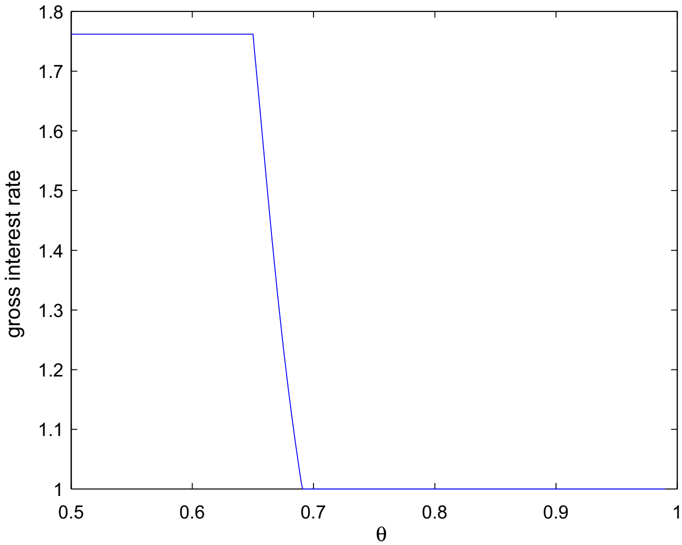

Fig. 3 Equilibrium interest rate under different θ. The fraction of households $\omega = 0.65$, payroll tax $\tau = 0.2$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$

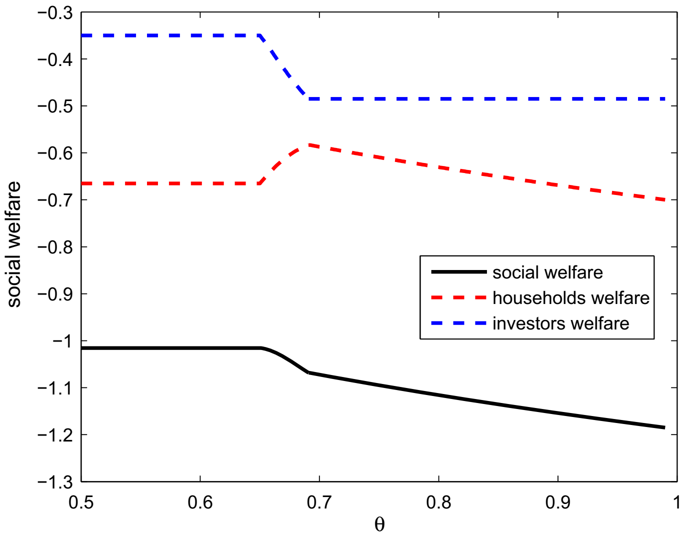

Fig. 4 Social welfare under different θ. The fraction of households $\omega = 0.65$, payroll tax $\tau = 0.2$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$

social welfare losses due to the fact that value function of constrained maximization is smaller than the unconstrained maximization. However, the tightening of borrowing constraint may have different welfare implications for investors and households. First, investors will lose. This is because investors supply credit in the market, and the return on that loan decreases as the collateral constraint becomes tighter. Second, households' welfare may increase or decrease. As shown in the proof of Proposition 3, interest rate will decrease and housing prices may increase or decrease as the collateral constraint becomes tighter.[14] Hence, households may benefit from the tightening of borrowing constraint by enjoying both cheaper housing assets and lower interest rate on their mortgages. However, households may also lose if the housing price becomes higher enough, which dominates the gains from lower interest rate.

Under the third case, in which households are borrowing-constrained and investors purchase housing assets, the tightening of borrowing constraint also implies social welfare losses. This is because investors are not affected by the tightening of borrowing constraint any more as the return on their asset is constant. Households will lose as their housing consumption becomes smaller due to the fact that investors demand more empty housing assets.

2.8 Pension reform

How can the housing bubble arise in the economy where there was no bubble in the first place? The previous discussion emphasizes the role of excess supply of credit. This section studies one way of generating excess supply of credit, the pension reform, in which the government removes the PAYG system completely by setting the pension tax and pension benefit to zero.

Figure 5 illustrates the pension reform in the benchmark economy. The dotted line denotes the demand and supply of credit before the pension reform. The solid line denotes the credit demand and supply after the pension reform. The removal of PAYG increases the supply of credit and reduces the borrowing of unconstrained households. However, for the constrained households, the reform increases their credit demand because they can use extra resources from tax reduction to purchase housing assets. If the interest rate after the pension reform becomes zero, investors want to purchase housing assets as their return is as high as the return on loans to the households. Whether the interest rate after the pension reform will be pushed down toward zero depends on the tightness of collateral constraint. If the borrowing constraint is tight enough, the increase in the credit supply will surpass the increase in credit demand from the borrowing-constrained households.

Proposition 4 *Suppose the government removes the PAYG system in the endowment economy. Bubble will arise if and only if* $\theta > \omega$. *If* $\tau > \frac{\theta-\omega}{1-\omega}$, *the housing wealth-to-GDP ratio is higher than the ratio before the reform.*

Proposition 4 gives a sufficient condition for the housing bubble to arise after the pension reform (See "Appendix" for the proof). It says that investors will purchase housing assets after the reform if the population share of young households is less than the downpayment ratio. In other words, bubbles are more likely to arise after pension reform in the economy where the proportion of young investors is large or

[14] On one hand, tighter collateral constraint reduces total borrowing from households, which pushes the housing prices down. On the other hand, interest rate becomes smaller due to the general equilibrium effect which tends to raise housing prices by allowing households to pay less interest on their mortgages.

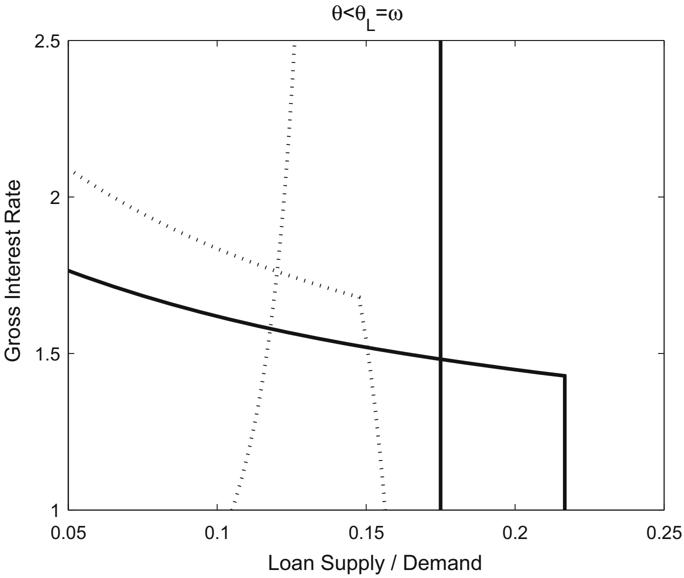

Fig. 5 An illustration of pension reform. The fraction of households $\omega = 0.65$, payroll tax $\tau = 0.2$, downpayment ratio $\theta = 0.60$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$. The *dotted line* denotes the credit demand and supply before the pension reform. The *solid line* denotes the credit demand and supply after the pension reform

the financial friction is large. The condition is very intuitive. Either larger share of investors or more strict collateral constraint increases the demand for housing assets as a store of value. It is worth mentioning that housing price does not necessarily increase after pension reform. The housing wealth-to-GDP ratio will be higher than the ratio before the reform if the reduction in the pension is large enough. The intuition is that the condition is more likely to be satisfied when higher pension tax reduces the housing assets bought by the constrained households.

Figure 6 exhibits the policy experiments in all three cases, i.e., $\theta < \theta_L$, $\theta_L < \theta < \theta_H$, and $\theta > \theta_H$, where θ_L and θ_H are defined in Proposition 3. According to Proposition 4, only pension reform in case 2 and case 3 can trigger housing bubble.

3 Model extension

This section extends the benchmark model to include the rental market and production sector. It shows that the qualitative results in the previous section still hold.

3.1 Rental market

This section presents a two-period endowment economy with rental market. The investors' problem can be written as

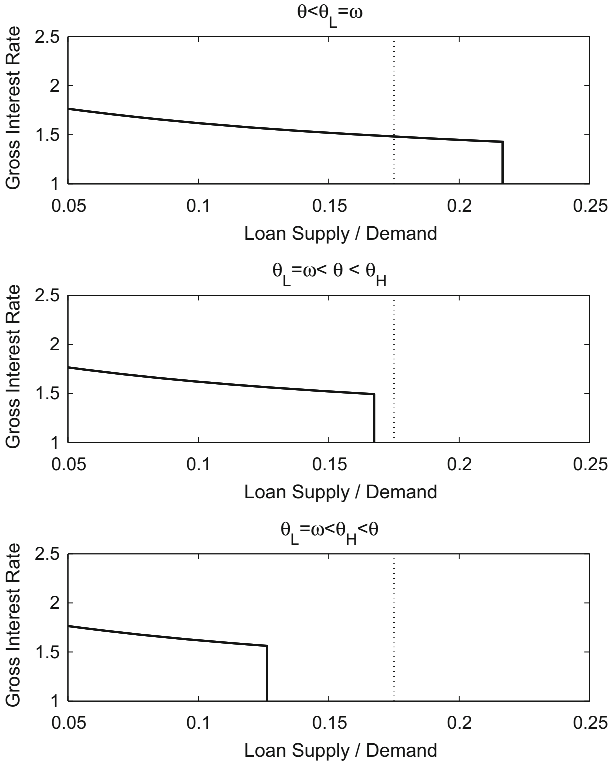

Fig. 6 Stationary equilibrium after the pension reform in three cases. The fraction of households $\omega = 0.65$, payroll tax $\tau = 0$, downpayment ratio $\theta = 0.60, 0.66, 0.72$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$

$$\max_{c^t_{i,t}, c^t_{i,t+1}, h^t_{i,t+1}, h^R_{i,t+1}, a^t_{i,t+1}} \ln c^t_{i,t} + \beta \ln c^t_{i,t+1} \tag{19}$$

subject to the following constraints

$$c^t_{i,t} + a^t_{i,t+1} + p_t h^t_{i,t+1} = (1-\tau)\, y^t_t + p^r_t h^R_{i,t+1} \tag{20}$$

$$c^t_{i,t+1} + \delta_r p_{t+1} h^R_{i,t+1} = \tau\,(1+n)\, y^{t+1}_{t+1} + R_{t+1} a^t_{i,t+1} + p_{t+1} h^t_{i,t+1} \tag{21}$$

$$h^t_{i,t+1} \geq h^R_{i,t+1} \tag{22}$$

$$a^t_{i,t+1} \geq -(1-\theta)\, p_t h^t_{i,t+1} \tag{23}$$

$$c^t_{i,t}, c^t_{i,t+1}, h^t_{i,t+1}, h^R_{i,t+1} \geq 0 \tag{24}$$

where p^r_t is the housing rental price and $h^R_{i,t+1}$ denotes the amount of rental housing supplied by investors. $\delta_r > 0$ denotes the depreciation rate of rental housing. There exists rental market frictions, in the sense that the owner-occupied housing has a smaller depreciation rate than rental housing. This can be interpreted as the moral hazard problem of tenants. Without loss of generality, the depreciation rate of owner-occupied housing is normalized to zero.

Because we assume that investors cannot derive utility flow directly from rental housing, the investors will not rent houses in the model. Since all the households are homogenous, they will not provide positive rental housing in the equilibrium. Hence, the households are the demand side of rental market. The households' optimization problem becomes

$$\max_{c_{h,t}^t, c_{h,t+1}^t, h_{h,t+1}^r, h_{h,t+1}^t, a_{h,t+1}^t} \ln c_{h,t}^t + \beta(1-\zeta)\ln c_{h,t+1}^t + \beta\zeta \ln\left(h_{h,t+1}^r + h_{h,t+1}^t\right) \tag{25}$$

subject to the following constraints

$$c_{h,t}^t + a_{h,t+1}^t = (1-\tau)\, y_t^t - p_t h_{h,t+1}^t - p_t^r h_{h,t+1}^r \tag{26}$$

$$c_{h,t+1}^t = \tau(1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{h,t+1}^t + p_{t+1} h_{h,t+1}^t \tag{27}$$

$$a_{h,t+1}^t \geq -(1-\theta)\, p_t h_{h,t+1}^t \tag{28}$$

$$c_{h,t}^t,\, c_{h,t+1}^t,\, h_{h,t+1}^t,\, h_{h,t+1}^r \geq 0 \tag{29}$$

where h_{t+1}^r is the amount of rental housing demanded by households. We can similarly define the competitive equilibrium.

Definition 2 Given the financial asset $a_{s,1}^1$ and housing stocks $h_{s,1}^0$ and $h_{s,1}^r$ for the initial old, the distribution of generation t, $\left\{\mu_{s,t}\right\}_{t=1}^{\infty}$ with total mass equal to N_t, the initial interest rate R_1, pension system τ, housing stocks $\{H_t\}_{t=1}^{\infty}$, the competitive equilibrium is the sequence of endowment $\left\{y_{s,t}^t\right\}_{t=1}^{\infty}$, prices $\left\{p_t, R_{t+1}, p_t^r\right\}_{t=1}^{\infty}$, allocations $\left\{c_{s,t}^t, c_{s,t+1}^t, h_{s,t+1}^t, h_{i,t+1}^R, h_{h,t+1}^{r,i}\right\}_{t=1}^{\infty}$, and the initial consumption $c_{s,1}^0$, $s = i, h$, such that

1. The allocations solve the problem of investors (19) and households (25)
2. The housing market, financial market, rental market, and non-durable goods market clear

$$H_{t+1} = \int h_{s,t+1}^t \mathrm{d}\mu_{s,t} \tag{30}$$

$$0 = \int a_{s,t+1}^t \mathrm{d}\mu_{s,t} \tag{31}$$

$$\int h_{i,t+1}^R \mathrm{d}\mu_{i,t} = \int h_{h,t+1}^r \mathrm{d}\mu_{h,t} \tag{32}$$

$$\int y_t^t \mathrm{d}\mu_{s,t} = \int c_{s,t}^t \mathrm{d}\mu_{s,t} + \int c_{s,t}^{t-1} \mathrm{d}\mu_{s,t-1} + p_t\left(H_{t+1} - H_t\right) \tag{33}$$

The policy functions for the problem of investors (19) and households (25) are solved in the "Appendix". The following Lemma 3 can simplify our analysis (see "Appendix" for the proof). The unconstrained households will not rent houses in the stationary equilibrium. The intuition is that the rental market frictions require a larger rental price, and only those constrained households are willing to pay this extra cost.

Lemma 3 *The unconstrained households will not rent houses in the stationary equilibrium.*

We are interested in whether the rental market can remove the equilibrium with housing bubble. To simplify the analysis, we focus on the economy after the pension reform. Proposition 5 states that a housing bubble exists in the equilibrium after the pension reform if the collateral constraint θ and the rental market friction δ_r are large enough (see "Appendix" for the proof).

Proposition 5 *If the collateral constraint $\theta > \omega$ and the rental market friction δ_r are large enough, there exists a bubble equilibrium after the pension reform.*

1. *If $\delta_r \geq \theta\zeta$, households will not rent houses and investors will hold empty houses. There exists a housing bubble for investors.*
2. *If $\theta\zeta > \delta_r \geq \omega\zeta$, households will rent some houses and investors will still hold some empty houses. There exists a housing bubble for investors.*
3. *If $\delta_r < \omega\zeta$, investors will rent all the houses to households, and there is no housing bubble.*

3.2 Production sector

The benchmark model can be extended to include the production sector à la Diamond (1965). Suppose there exists a production sector with production function written as:

$$Y_t = F(K_t, A_t L_t) \tag{34}$$

where the growth rate of labor-augmented technology is given by $A_{t+1}/A_t = 1 + g$. Suppose $F(K_t, A_t L_t) = K_t^\alpha (A_t L_t)^{1-\alpha}$, the profit maximization of the firm implies that

$$R_t = 1 + \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} - \delta$$
$$w_t = (1 - \alpha) A_t K_t^\alpha (A_t L_t)^{-\alpha}$$

where δ is the depreciation rate for capital.

Now, the investors' problem becomes

$$\max \ln c_{i,t}^t + \beta \ln c_{i,t+1}^t \tag{35}$$

subject to the following constraints

$$c_{i,t}^t + a_{t+1}^t + p_t h_{i,t+1}^t = (1 - \tau) w_t \tag{36}$$
$$c_{i,t+1}^t = \tau (1 + n) w_{t+1} + R_{t+1} a_{i,t+1}^t + p_{t+1} h_{i,t+1}^t \tag{37}$$
$$c_{i,t}^t, c_{i,t+1}^t, h_{i,t+1}^t \geq 0 \tag{38}$$

The households' problem becomes

$$\max \ln c_{h,t} + \beta (1-\zeta) \ln c^t_{h,t+1} + \beta\zeta \ln h^t_{h,t+1} \tag{39}$$

subject to the following constraints

$$c^t_{h,t} + a^t_{t+1} = (1-\tau) w_t - p_t h^t_{h,t+1} \tag{40}$$
$$c^t_{h,t+1} = \tau (1+n) w_{t+1} + R_{t+1} a^t_{h,t+1} + p_{t+1} h^t_{h,t+1} \tag{41}$$
$$a^t_{h,t+1} \geq -(1-\theta) p_t h^t_{h,t+1} \tag{42}$$
$$c^t_{h,t}, c^t_{h,t+1}, h^t_{h,t+1} \geq 0 \tag{43}$$

We can similarly define the competitive equilibrium.

Definition 3 Given the financial asset $a^1_{s,1}$ and housing stocks $h^0_{s,1}$ for the initial old, the distribution of generation t, $\left\{\mu_{s,t}\right\}^{\infty}_{t=1}$ with total mass equal to N_t, the initial interest rate R_1, pension system τ, housing stocks $\{H_t\}^{\infty}_{t=1}$, the competitive equilibrium consists of prices $\{p_t, R_{t+1}\}^{\infty}_{t=1}$, allocations $\left\{c^t_{s,t}, c^t_{s,t+1}, h^t_{s,t+1}, K_{t+1}\right\}^{\infty}_{t=1}$, and the initial consumption $c^0_{s,1}$, $s = i, h$, such that

1. The allocations solve the problem of investors (35) and households (39)
2. Firm rents capital and hires labor from households to maximize profit.
3. The housing market, financial market, labor market, and non-durable goods market clear

$$H_{t+1} = \int h^t_{s,t+1} \mathrm{d}\mu_{s,t} \tag{44}$$
$$K_{t+1} = \int a^t_{s,t+1} \mathrm{d}\mu_{s,t} \tag{45}$$
$$N_t = L_t \tag{46}$$
$$Y_t = \int c^t_{s,t} \mathrm{d}\mu_{s,t} + \int c^{t-1}_{s,t} \mathrm{d}\mu_{s,t-1} + p_t (H_{t+1} - H_t) + I_t \tag{47}$$

where $I_t = K_{t+1} - (1-\delta) K_t$

We are interested in whether the production sector can remove the equilibrium with housing bubble. To simplify the analysis, we focus on the economy after the pension reform. Proposition 6 proves that a housing bubble can exist even under the presence of a production sector as long as the collateral constraint is binding enough (see "Appendix" for the proof). Under such condition, the model economy will become constrained dynamic inefficient after the pension reform. Housing bubble solves the constrained dynamic inefficiency problem by absorbing the excess supply of credit in the market.

Proposition 6 *If the following condition holds, there exists a bubble equilibrium after the pension reform.*

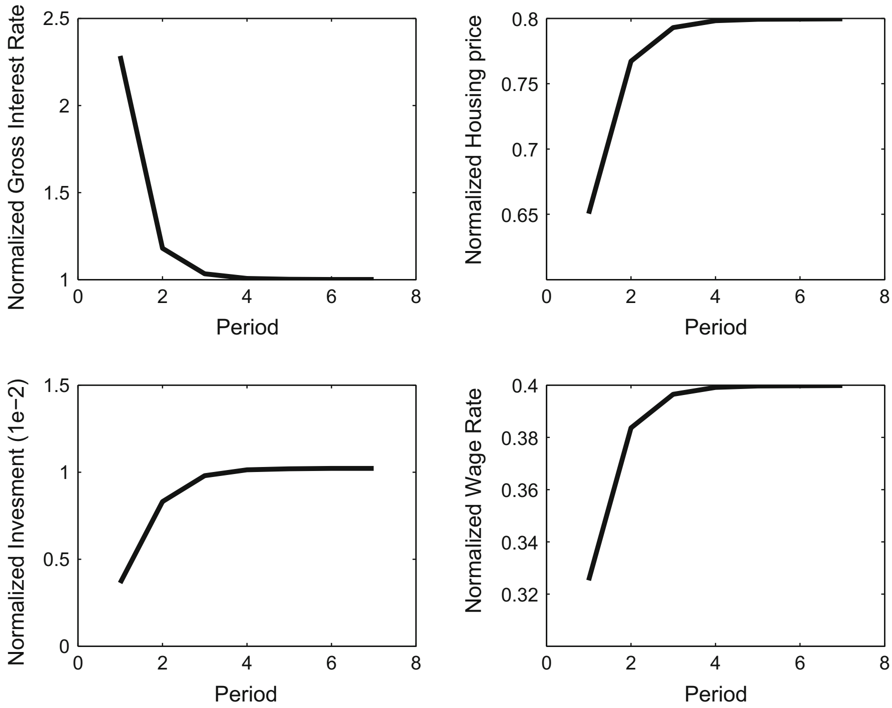

Fig. 7 Transitional dynamics after the pension reform. Model period equals 30 years. The fraction of households $\omega = 0.33$, payroll tax decreases to zero from $\tau = 0.40$ after the reform, the downpayment ratio $\theta = 0.70$, discount factor $\beta = 1$, and $\zeta = 0.5$, the annual population growth rate is 2 % and the productivity growth rate is 5 %

$$\theta > \omega \frac{1}{1 - \alpha \frac{1+\beta}{\beta} \frac{n+g+1}{n+g+\delta}}$$

It would be interesting to study the transition dynamics after the pension reform in this production economy. However, the analytical solution to characterize such transition is not possible. Proposition 7 describes the transitional dynamics after the pension reform, and Fig. 7 shows a numerical example of the transition path after the pension reform (see "Appendix" for the proof). The normalized interest rate is defined as the gross interest rate divided by the gross population growth rate plus the productivity growth rate.[15] The normalized housing price growth rate is the housing price sequence divided by the current population and productivity level. The investment is normalized in the similar way. The normalized wage rate is defined as the wage rate divided by the current productivity. The proof of the proposition shows that the housing price growth

[15] We can normalize all economic variables by their growth rate along the balanced growth path. Denote $\tilde{y}_t^t \equiv \frac{y_t^t}{(1+g)^t}$, $s,\tilde{} c_t^t \equiv \frac{c_{s,t}^t}{(1+g)^t}$, $\tilde{c}_{s,t}^{t-1} \equiv \frac{c_{s,t}^{t-1}}{(1+n)(1+g)^t}$, $\tilde{a}_{s,t+1}^t \equiv \frac{a_{s,t+1}^t}{(1+g)^t}$, $\tilde{k}_{t+1} \equiv \frac{k_{t+1}}{(1+g)^t(1+n)^t}$, $\tilde{p}_t \equiv \frac{p_t}{(1+n)^t(1+g)^t}$, $\tilde{R}_{t+1} \equiv \frac{R_{t+1}}{(1+n)(1+g)}$, $\tilde{h}_{s,t+1}^t \equiv h_{s,t+1}^t (1+n)^t$, $\tilde{H}_{t+1} \equiv H_{t+1}$, $\tilde{\varphi} \equiv \frac{\varphi}{(1+n)(1+g)}$, $s = i, h$.

rate is equal to the gross interest rate during the transition. Therefore, investors will hold housing assets right after the pension reform.

Proposition 7 *In the production economy, suppose the government removes the PAYG system, and there exists a housing bubble in the new stationary equilibrium. Both housing price and interest rate converge monotonically to the unique new steady state.*

Caballero and Krishnamurthy (2006) mentions that the investment-related demand for a store of value can generate positive co-movement between investment and asset prices. The consumption-related demand for a store of value usually crowds out savings and reduces investment. However, as shown in Fig. 7, the extended model with production sector is able to generate the right co-movement between investment and housing prices after an exogenous shock to the liquidity supply, i.e., the pension reform. The main driven force is the increased supply of credit coming from the declined pension system. It is worth mentioning that after the transition, the economy with housing bubble has the usual crowding-out effect on investment as described in the bubble literature.

4 Empirical evidence

4.1 Chinese housing market

This section provides some empirical evidence for housing assets being used as a store of value for the financially less developed countries such as China. Although the USA has already experienced a burst in housing bubble in 2008, housing prices in China have been increasing strongly over the past decade. Figure 8 shows that the average land selling price in China increases at an annual rate 15.7 % from 2000 to 2009. It also draws the average commodity building selling price for 35 major Chinese cities, which exhibits a slower annual growth rate, 7 % , from year 2000 to 2009.[16] Wu et al. (2012) constructs a constant-quality price index for newly built private housing in 35 major Chinese cities. According to their estimates, the annual price growth is nearly 10 % from year 2000 to 2009.

There are rapid growth in the real estate investment and homeownership rate. Figure 9 shows that the share of real estate investment in total fixed investment increases from 13 % at year 1999 to 20 % at year 2010. In the year 2010, The homeownership rate for the urban households is nearly 90 %, which is among the highest in the world.[17] These two facts imply that many Chinese households own more than one apartment.

[16] The average selling price does not take into account the quality changes in the housing market. Unfortunately, there is no official constant-quality housing price index for China.

[17] The urban home ownership rate increases from less than 30 to 70 % during 1994–1999, a period when the housing reform takes place. Before the housing reform, it is the state-owned enterprises (SOE) that are responsible for providing employee housing to workers, with a little or no charge for rents. The government liberalizes the housing market in 1994 by selling the public housing to the current employee in state-owned enterprises at heavily subsidized price. Newly employed workers in SOE and workers in the private sectors have to purchase houses that are provided by private real estate developers. The transition into the new housing system ends around 1999, after which no SOE are allowed to provide employee housing to their

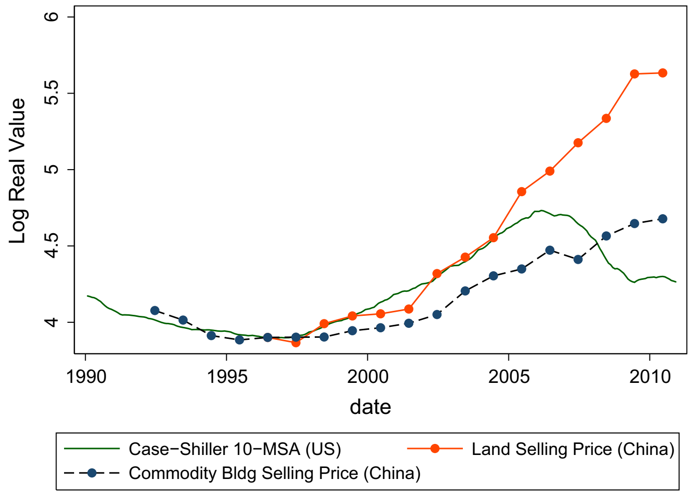

Fig. 8 Housing Price and Land Price: China and the USA. The US Housing price index is from S&P/Case-Shiller 10-MSA Index. The land selling price is computed by author using data from China Satistics Year Book. The land price is defined as the total value of land purchased divided by total land space purchased. The commodity building selling prices are based on the 35-city average selling price series from National Bureau of Statistics. All series are in log real value deflated by CPI (Urban CPI for Chinese data) and normalized to the same level at year 1996

Fig. 9 Urban residential investment and homeownership rate. The share of urban residential investment is defined as the real estate development (including land purchase) divided by the total investment in fixed assets in the whole country. Homeownership rate is from China urban households survey

Footnote 17 continued
workers. At the end of the year 2010, the home ownership rate of urban households in China is 89.3 %. 40.1 % of them own privatized houses which previously are owned by the government or state-owned enterprises. 38 % of households have bought houses that are provided at a market price.

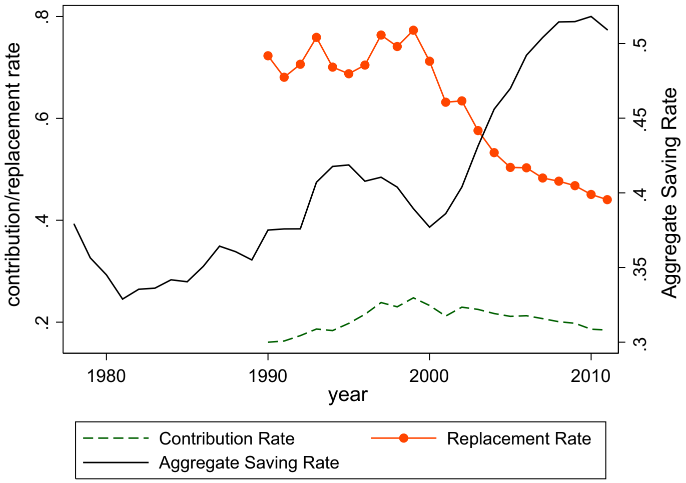

Fig. 10 Social security replacement rate and contribution rate. Data are from China Statistics Year Books 1990–2011. Replacement rate is defined as the total pension benefit payment per urban retiree covered in the pension system divided by the average urban wage rate. The contribution rate is the total contribution per urban worker covered in the pension system divided by the average urban wage rate

Popular wisdoms say that there is a housing bubble in China. Although it is hard to detect the bubble before it bursts, we can look at the vacancy rate in the housing market according to our definition of the housing bubble.[18] In the USA, the vacancy rate rises from 12.7 to 14.5 during 2005–2010. In China, according to the China Family Panel Studies 2011, the vacancy rate in year 2010 is 11 % according to the author's estimate.[19]

One of the reasons that households hold empty housing is the conflict between the shortage of assets and the need for a store of value, which is strengthened by the pension reform in China. Therefore, Chinese households purchase housing assets as a store of value to finance their old-age consumption. Figure 10 plots the pension replacement rate and contribution rate for urban households in China. The pension reform starts from 1999, which shifts the traditional pay-as-you-go (PAYG) system to a mixture of PAYG system and fully funded system. From then on, the replacement rate decreases from around 75 % to only 45 % in 2009.[20] During the same period, the national saving rate in China increases by 15 %, which suggests that Chinese households increase savings partly to compensate the decline in the pension benefit.

Although the capital return in China is high, financial frictions prevent the productive private firms from borrowing from financial intermediaries and create a huge gap between capital return and interest rate (Bai et al. 2006; Song et al. 2011). Therefore, asset shortage arises and reduces investment opportunities for Chinese households. Many households can only invest in bank deposit and government bond which deliver

[18] A vacant house or apartment is a unit that has been sold but is not occupied by anybody. The vacancy rate is defined as the proportion of vacant units in total housing units.

[19] 22 % of urban households own more than one apartment. Among them, only 25 % households rent their apartments out.

[20] See Song et al. (2012) for the detailed descriptions of China's pension system.

almost zero return, which reflects the huge demand for assets or investment tools in China. What if Chinese households turn their savings into stocks instead of housing assets? Because the poor development in the financial market, the average return of stocks over the past twenty years is very low.[21] If the capital account were fully open, Chinese households would have purchased huge amount of assets abroad directly. Hence, according to our theoretical model, the constrained dynamic inefficiency due to financial frictions and excess supply of liquidity will create rooms for speculative bubbles. Recent studies by Chen and Wen (2014) provide the crowding-out evidence of housing bubble on real investment in China. Also a working paper version of Miao and Wang (2014) shows that housing bubbles reduce R&D in China.

4.2 Regression results

In the section, we use Chinese city-level and provincial-level data to study the impact of pension reform on Chinese housing market. Unfortunately, housing vacancy rates at the provincial or city levels are not available. Instead, we directly estimate the effect of pension reform on housing prices. The Chinese pension system mainly operates at either city or provincial levels, i.e., each city or province has its own pension fund account and balance sheet. Therefore, we can exploit the regional variations in pension system to identify its effect on the savings and housing prices. The housing price data we used are the average selling price in 35 major cities from 2000 to 2013 published by the NBS of China. We linked the all the 35 cities to 31 provinces and compute the average contribution rate using provincial data. Housing prices, provincial GDP, and provincial average wage payment for urban workers are deflated by provincial CPI index.

Before we go to the regression results, we use simple scatter plot to verify several correlations which predicted the theoretical model. Figure 11 plots 10-year changes in the housing prices, average provincial contribution rate, and investment-to-GDP ratio during 2001–2011.[22] There are two main findings. First, there are larger housing price appreciation in the province where the pension contribution rate declines more, where the contribution rate is defined as the pension contribution per urban worker covered in the pension system in each province divided by the average wage rate for urban workers in the same province.[23] Second, there are larger housing price appreciation in the province where the investment-to-GDP ratio decreases more, a fact that is consistent with the crowding-out effect of bubble on real investment.[24]

[21] The average real return on Shanghai stock market index is only 2 % from 2000 to 2009.

[22] The 35 cities are Beijing (BJ), Tianjing (TJ), Shijiazhuang (SJZ), Taiyuan (TY), Huhehaote (HHHT), Shenyang (SY), Changchun (CC), Haerbing (HEB), Shanghai (SH), Nanjing (NJ), Hangzhou (HZ), Hefei (HF), Fuzhou (FZ), Nanchang (NC), Jinan (JN), Zhengzhou (ZZ), Wuhan (WH), Changsha (CS), Nanning(NN), Haikou (HK), Chongqing (CQ), Chengdu (CD), Guiyang (GY), Kunming(KM), Xian(XA), Lanzhou (LZ), Xining (XN), Yinchuan (YC), Wulumuqi (WLMQ), Dalian (DL), Qingdao (QD), Ningbo (NB), Xiamen (XM), Shengzhen (SZ), and Guangzhou (GZ).

[23] The simple OLS univariate regression has a coefficient −7.06, which is significant at 1 % confidence level. The adjusted R-squared is 0.33.

[24] The simple OLS univariate regression has a coefficient −2.64, which is also significant at 1 % confidence level. The adjusted R-square is 0.31.

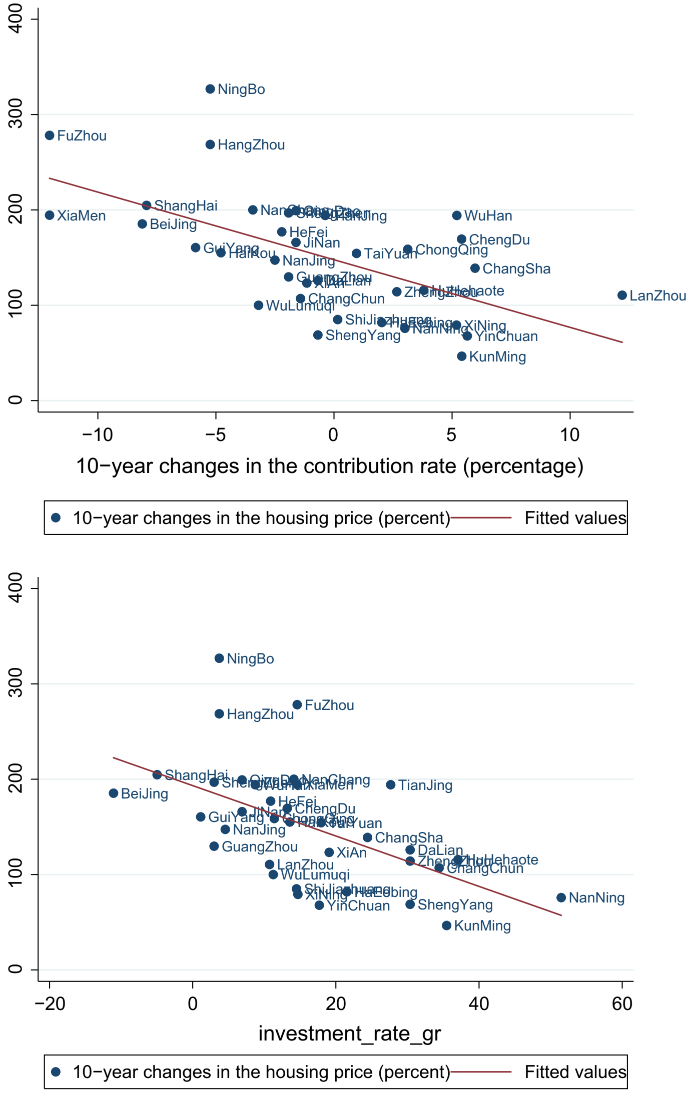

Fig. 11 Ten-year changes in the housing prices, contribution rate, and investment-to-GDP ratio during 2001–2011

The next step is to run the following full-fledged fixed-effect panel regression to estimate the effect of contribution rate and investment-to-GDP ratio on housing price. The benchmark regression is specified as follows:

$$\ln\left(P_t^i\right) = \alpha^i + \beta_t + f(y_t^i) + \theta\tau_t^i + \eta I_t^i + \epsilon_t^i$$

where α^i is the city fixed-effect and β_t is the year fixed effect. $f(y_t^i)$ is third-order polynomial of real city GDP. τ_t^i is the average contribution rate to pension, and I_t^i

Table 1 Fixed-effect regression result

Depend Var. ln (housing price)	(1)	(2)	(3)	(4)
Contribution rate	−1.17***	−1.02**	−0.86**	−0.86**
	(−2.84)	(−2.54)	(−2.30)	(−2.16)
investment/GDP	–	−0.434***	−0.47***	−0.23**
		(3.20)	(3.67)	(1.74)
Pension balance/GDP	–	–	−2.34*	−1.19
			(−1.92)	(−0.90)
ln (population/construction land)	–	–	–	0.057**
				(2.1)
Year effect	Yes	Yes	Yes	Yes
$f(y_t^i)$	Yes	Yes	Yes	Yes
City dummies	Yes	Yes	Yes	Yes
R-Squared	0.69	0.73	0.70	0.40
No. of obs.	490	490	490	385

is the investment-to-GDP ratio in each province. The regression results are given by Table 1. All standard errors are clustered at city level. The t statistics are reported in the parentheses.

The coefficient before the contribution rate is smaller than the slope in Fig 11. The benchmark specification (specification 2) shows that a 10 percentage point decline in the contribution rate is correlated with a 10.2 % increase in real housing price level. It also shows that a 10 percentage point increase in the investment-to-GDP ratio is correlated with a 4.3 % decline in the housing price level.[25]. In specification 3, we add the pension balance-to-GDP ratio, which captures the total amount of forced pension saving by the government. Our last specification controls the population density of the city, which is defined as the number of city population divided by the total amount of construction land in the city. It reduces the coefficient before investment-to-GDP ratio but does not change the impact of contribution rate on housing prices.

5 Conclusion

This paper studies an economy inhabited by overlapping generations of households and investors, with the only difference between the two being that households derive utility from housing services, whereas investors do not. Tight collateral constraint limits the borrowing capacity of households and drives the equilibrium interest rate level down to the housing price growth rate, which makes housing attractive as a store of value for investors. As long as the rental market friction is high enough, the investors

[25] One standard deviation of contribution rate in the sample is 6.4 percentage points. One standard deviation of investment-to-GDP ratio is 14 percentage points. Strictly speaking, we only identify the correlation rather the causality in the regression. Future works require finding valid instruments to control the endogeneity issues of explanatory variables, such as investment-to-GDP ratio and the contribution rate.

will hold a positive number of vacant houses in equilibrium. A housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and without the expectation of receiving a dividend either in terms of utility or in terms of rent. The paper also shows that the theory predictions are consistent with empirical evidence from China.

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Mathematical appendix

Proof of Proposition 1

The Lagrangian function is

$$\begin{aligned} L &= \ln c_{i,t}^{t} + \beta \ln c_{i,t+1}^{t} \\ &\quad + \lambda_1 \left[(1-\tau)\, y_t^t - c_{i,t}^t - a_{i,t+1}^t - p_t h_{i,t+1}^t \right] \\ &\quad + \lambda_2 \left[\tau\,(1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{i,t+1}^t + p_{t+1} h_{i,t+1}^t - c_{i,t+1}^t \right] \\ &\quad + \mu_1 \left[a_{i,t+1}^t + (1-\theta)\, p_t h_{i,t+1}^t \right] \\ &\quad + \nu_1 h_{i,t+1}^t \end{aligned}$$

The FOCs become

$$\begin{aligned} c_{i,t}^t &: \frac{1}{c_{i,t}^t} - \lambda_1 = 0 \\ c_{i,t+1}^t &: \frac{\beta}{c_{i,t+1}^t} - \lambda_2 = 0 \\ a_{i,t+1}^t &: -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\ h_{i,t+1}^t &: -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1\,(1-\theta)\, p_t + \nu_1 = 0 \end{aligned}$$

where

$$\begin{aligned} &\mu_1 \geq 0, \quad \text{if } a_{i,t+1}^t + (1-\theta)\, p_t h_{i,t+1}^t > 0, \quad \text{then } \mu_1 = 0 \\ &\nu_1 \geq 0, \quad \text{if } h_{i,t+1}^t > 0, \quad \text{then } \nu_1 = 0 \end{aligned}$$

The lifetime budget constraint for the investors is

$$c_{i,t}^t + \frac{c_{i,t+1}^t}{R_{t+1}} = (1-\tau)\, y_t^t + \frac{\tau\,(1+n)\, y_{t+1}^{t+1}}{R_{t+1}} + \left(\frac{p_{t+1}}{R_{t+1}} - p_t \right) h_{i,t+1}^t$$

1. $a^t_{i,t+1} + (1-\theta)\, p_t h^t_{i,t+1} > 0$, i.e., the borrowing constraint of the investors is not binding; $h^t_{i,t+1} > 0$, i.e., the unconstrained investors hold positive amount of housing. Therefore, $\mu_1 = \nu_1 = 0$. Plug them into the FOCs

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$-\lambda_1 p_t + \lambda_2 p_{t+1} = 0$$

The following equality holds $R_{t+1} = \frac{p_{t+1}}{p_t}$, and the optimal consumption rules are

$$c^t_{i,t} = \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y^t_t$$
$$c^t_{i,t+1} = \frac{\beta R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y^t_t$$

The allocation between the loans and housing assets is indeterminate. The total saving is determined by

$$a^t_{i,t+1} + p_t h^t_{i,t+1} = (1-\tau_t)\, y^t_t - c^t_{i,t}$$

2. $a^t_{i,t+1} + (1-\theta)\, p_t h^t_{i,t+1} > 0$, i.e., the borrowing constraint of investor is not binding; $h^t_{i,t+1} = 0$, i.e., the investor holds zero amount of housing. Therefore, $\mu_1 = 0$, $\nu_1 \geq 0$. Plug them into the FOCs,

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$-\lambda_1 p_t + \lambda_2 p_{t+1} + \nu_1 = 0$$

Hence, $R_{t+1} \geq \frac{p_{t+1}}{p_t}$
(a) If $\nu_1 = 0$, then we go back to case 1
(b) If $\nu_1 > 0$, then $R_{t+1} > \frac{p_{t+1}}{p_t}$. The purchase of housing is less attractive than lending to the others.

$$a^t_{i,t+1} = (1-\tau)\, y^t_t - c^t_{i,t}$$
$$h^t_{i,t+1} = 0$$

3. $a^t_{i,t+1} + (1-\theta)\, p_t h^t_{i,t+1} = 0$, i.e., the borrowing constraint of the investors is binding; $h^t_{i,t+1} > 0$, i.e., the constrained investors hold positive amount of housing. Therefore, $\mu_1 \geq 0$, $\nu_1 = 0$.
(a) If $\mu_1 = \upsilon_1 = 0$, we go back to case 1. If $\mu_1 > 0$, $\nu_1 = 0$, then

$$\frac{\lambda_1}{\lambda_2} > R_{t+1}$$
$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1}}{p_t}$$

$$\frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t}$$

Suppose $\frac{p_{t+1}}{p_t} < R_{t+1} < \frac{\lambda_1}{\lambda_2}$, then $R_{t+1} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t} < \frac{p_{t+1}-(1-\theta)p_{t+1}}{\theta p_t} = \frac{p_{t+1}}{p_t}$, a contradiction! Therefore,

$$R_{t+1} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t}$$

Let $\gamma_{i,t} \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t}$. Rewrite the budget constraints as

$$\begin{aligned} c_{i,t}^t &= (1-\tau)\, y_t^t - \theta p_t h_{i,t+1}^t \\ c_{i,t+1}^t &= \tau\,(1+n)\,(1+g)\, y_t^t + \theta \gamma_{i,t} p_t h_{i,t+1}^t \end{aligned}$$

Solve for $p_t h_{i,t+1}^t$

$$p_t h_{i,t+1}^t = \frac{\beta\gamma_{i,t}\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta\gamma_{i,t}\,(1+\beta)} y_t^t$$

Therefore,

$$\begin{aligned} c_{i,t}^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{i,t}}\right] y_t^t \\ c_{i,t+1}^t &= \frac{\beta\gamma_{i,t}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{i,t}}\right] y_t^t \\ a_{i,t+1}^t &= -\,(1-\theta)\, p_t h_{i,t+1}^t \\ p_t h_{i,t+1}^t &= \frac{\beta\gamma_{i,t}\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta\gamma_{i,t}\,(1+\beta)} y_t^t \end{aligned}$$

4. $a_{i,t+1}^t + (1-\theta)\, p_t h_{i,t+1}^t = 0$, i.e., the borrowing constraint of the investors is binding; $h_{i,t+1}^t = 0$, i.e., the investors hold zero amount of housing

$$\begin{aligned} c_{i,t}^t &= (1-\tau)\, y_t^t \\ c_{i,t+1}^t &= \tau\,(1+n)\,(1+g)\, y_t^t \end{aligned}$$

Then, $\mu_1, v_1 \geq 0$.

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1\,(1-\theta)\, p_t + v_1 &= 0 \end{aligned}$$

(a) If $\mu_1, \nu_1 > 0$, either investors have too little endowment when they are young and do not want to save

$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1} - R_{t+1}(1-\theta)p_t}{\theta p_t} > \frac{p_{t+1}}{p_t} > R_{t+1}$$

or investors' borrowing cost is too large

$$\frac{\lambda_1}{\lambda_2} > R_{t+1} > \frac{p_{t+1}}{p_t} > \frac{p_{t+1} - R_{t+1}(1-\theta)p_t}{\theta p_t}$$

In this article, I assume the young has enough endowment and wants to save. Therefore, I rule out the case $\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1}-R_{t+1}(1-\theta)p_t}{\theta p_t} > \frac{p_{t+1}}{p_t} > R_{t+1}$.

(b) If $\mu_1 > 0$, $\upsilon_1 = 0$, we go back to case 3

(c) If $\mu_1 = 0$, $\nu_1 > 0$, we go back to case 2

(d) If $\mu_1 = 0$, $\upsilon_1 = 0$, we go back to case 1

Proof of Proposition 2

The Lagrangian function is

$$\begin{aligned}
L &= \ln c^t_{h,t} + \beta\zeta \ln\left(h^t_{h,t+1}\right) + \beta(1-\zeta)\ln c^t_{h,t+1} \\
&\quad +\lambda_1\left[(1-\tau)y^t_t - p_t h^t_{h,t+1} - c^t_{h,t} - a^t_{h,t+1}\right] \\
&\quad +\lambda_2\left[\tau(1+n)y^{t+1}_{t+1} + R_{t+1}a^t_{h,t+1} + p_{t+1}h^t_{h,t+1} - c^t_{t+1}\right] \\
&\quad +\mu_1\left[a^t_{h,t+1} + (1-\theta)p_t h^t_{h,t+1}\right]
\end{aligned}$$

The FOCs become

$$\begin{aligned}
c^t_{h,t} &: \frac{1}{c^t_{h,t}} - \lambda_1 = 0 \\
c^t_{h,t+1} &: \frac{\beta(1-\zeta)}{c^t_{h,t+1}} - \lambda_2 = 0 \\
a^t_{h,t+1} &: -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\
h^t_{h,t+1} &: \frac{\beta\zeta}{h^t_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1(1-\theta)p_t = 0
\end{aligned}$$

where

$$\mu_1 \geq 0, \quad \text{if } a^t_{t+1} + (1-\theta)p_t h^t_{h,t+1} > 0, \quad \text{then } \mu_1 = 0$$

and the lifetime budget constraint is given by

$$c^t_{h,t} + \frac{c^t_{h,t+1}}{R_{t+1}} + \left(p_t - \frac{p_{t+1}}{R_{t+1}}\right)h^t_{h,t+1} = (1-\tau)y^t_t + \frac{\tau(1+n)y^{t+1}_{t+1}}{R_{t+1}}$$

1. $a^t_{h,t+1} + (1-\theta) p_t h^t_{h,t+1} > 0$, i.e., the borrowing constraint of the households is not binding. Therefore, $\mu_1 = 0$. Hence,

$$\frac{\lambda_1}{\lambda_2} = R_{t+1} = \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c^t_{h,t+1}}{h^t_{h,t+1}}}{p_t}$$

The optimal decision rules are

$$c^t_{h,t} = \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right] y^t_t$$
$$c^t_{h,t+1} = \frac{\beta(1-\zeta)R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right] y^t_t$$
$$p_t h^t_{h,t+1} = \frac{1}{1-\frac{p_{t+1}}{p_t R_{t+1}}}\frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right] y^t_t$$
$$a^t_{h,t+1} = (1-\tau) y^t_t - p_t h^t_{h,t+1} - c^t_{h,t}$$

2. $a^t_{h,t+1} + (1-\theta) p_t h^t_{h,t+1} = 0$, i.e., the borrowing constraint of the households is binding. Therefore, $\mu_1 \geq 0$
   (a) If $\mu_1 = 0$, then we go back to case 1.
   (b) If $\mu_1 > 0$

$$-\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0$$
$$\frac{\beta\zeta}{h^t_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1(1-\theta) p_t = 0$$

Hence, the condition for R_{t+1} is given by

$$R_{t+1} < \frac{\lambda_1}{\lambda_2}$$

Let $\frac{\lambda_1}{\lambda_2} \equiv \gamma_{h,t}$, then from the budget constraint

$$c^t_{h,t} = (1-\tau) y^t_t - \theta p_t h^t_{h,t+1}$$

and

$$c^t_{h,t+1} = \tau(1+n)(1+g) y^t_t + (p_{t+1} - R_{t+1}(1-\theta) p_t) h^t_{h,t+1}$$

From the FOC w.r.t. $h^t_{h,t+1}$, we have

$$\frac{\beta\zeta}{h^t_{h,t+1}} - \lambda_1 \theta p_t + \lambda_2 (p_{t+1} - R_{t+1}(1-\theta) p_t) = 0$$

Use the expression for λ_1, λ_2, we have

$$\begin{aligned} 1 &= \lambda_1 (1-\tau) y_t^t - \lambda_1 \theta p_t h_{h,t+1}^t \\ \beta(1-\zeta) &= \lambda_2 \tau (1+n)(1+g) y_t^t + \lambda_2 (p_{t+1} - R_{t+1}(1-\theta) p_t) h_{h,t+1}^t \\ \beta\zeta &= \lambda_1 \theta p_t h_{h,t+1}^t - \lambda_2 (p_{t+1} - R_{t+1}(1-\theta) p_t) h_{h,t+1}^t \end{aligned}$$

Therefore,

$$1+\beta = \lambda_1 (1-\tau) y_t^t + \lambda_2 \tau (1+n)(1+g) y_t^t$$

Note that

$$\begin{aligned} 1+\beta &= \frac{(1-\tau) y_t^t}{(1-\tau) y_t^t - \theta p_t h_{t+1}^t} \\ &\quad + \beta(1-\zeta) \frac{\tau(1+n)(1+g) y_t^t}{\tau(1+n)(1+g) y_t^t + (p_{t+1} - R_{t+1}(1-\theta) p_t) h_{h,t+1}^t} \end{aligned}$$

This is a quadratic equation for $p_t h_{h,t+1}^t$. Let

$$\begin{aligned} x &= p_t h_{h,t+1}^t \\ \varphi &= \frac{p_{t+1}}{p_t} - (1-\theta) R_{t+1} \\ a &= (1-\tau) y_t^t \\ b &= \tau(1+n)(1+g) y_t^t \end{aligned}$$

Then,

$$1+\beta = \frac{a}{a-\theta x} + \frac{\beta(1-\zeta) b}{b+\varphi x}$$

Positive consumption in both periods requires that

$$\begin{aligned} a - \theta x &> 0 \\ b + \varphi x &> 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} x &< \frac{a}{\theta} \quad \text{if } \varphi > 0 \\ x &< \min\left(\frac{a}{\theta}, -\frac{b}{\varphi}\right) \quad \text{if } \varphi < 0 \end{aligned}$$

The above equation can be written as:

$$(1+\beta)(a-\theta x)(b+\varphi x) - a(b+\varphi x) - \beta(1-\zeta) b(a-\theta x) = 0$$

Let

$$\Pi(x) = (1+\beta)(a-\theta x)(b+\varphi x) - a(b+\varphi x) - \beta(1-\zeta)b(a-\theta x)$$
$$= -\theta\varphi(1+\beta)x^2 + (a\varphi\beta - b\theta(1+\beta\zeta))x + \beta\zeta ab$$

i. If $\varphi > 0$, because $\Pi(0) = \beta\zeta ab > 0$, it has one positive solution. The positive solution must satisfy $x < \frac{a}{\theta}$ because

$$\Pi\left(\frac{a}{\theta}\right) = -\theta\varphi(1+\beta)\frac{a^2}{\theta^2} + (a\varphi\beta - b\theta(1+\beta\zeta))\frac{a}{\theta} + \beta\zeta ab$$
$$= -\frac{\varphi a^2}{\theta} - ab < 0$$

The positive solution is the relative larger solution, which is given by

$$p_t h^t_{h,t+1} = x = \frac{-\Psi_t - \Phi_t}{-2\theta\varphi(1+\beta)} = \frac{\Psi_t + \Phi_t}{2\theta\varphi(1+\beta)}$$

where $\Psi_t = a\varphi\beta - b\theta(1+\beta\zeta)$, $\Phi_t = \sqrt{\Psi_t^2 + 4ab\theta\beta\zeta\varphi(\beta+1)}$.

ii. If $\varphi < 0$, there are two positive solutions or two negative solutions because the product of two solutions is equal to

$$\frac{\beta\zeta ab}{-\theta\varphi(1+\beta)x^2} > 0$$

If there are two positive solutions, the sum of two solutions has to satisfy

$$\frac{(a\varphi\beta - b\theta(1+\beta\zeta))}{\theta\varphi(1+\beta)} > 0$$

because $\varphi < 0$, it implies that $a\varphi\beta - b\theta(1+\beta\zeta) < 0$.

A. If $\frac{-\varphi}{b} > \frac{\theta}{a}$, then $x < \min\left(\frac{a}{\theta}, -\frac{b}{\varphi}\right) = -\frac{b}{\varphi}$

$$\Pi\left(-\frac{b}{\varphi}\right) = -\theta\varphi(1+\beta)b^2\frac{1}{\varphi^2} - (a\varphi\beta - b\theta - \beta\zeta b\theta)\frac{b}{\varphi} + \beta\zeta ab$$
$$= b\theta(1-\zeta)\beta\left(\frac{b}{-\varphi} - \frac{a}{\theta}\right) < 0$$

Therefore, only the smaller solution satisfies $x < -\frac{b}{\varphi}$. The unique solution is given by

$$x = \frac{-\Psi_t - \Phi_t}{-2\theta\varphi(1+\beta)} = \frac{\Psi_t + \Phi_t}{2\theta\varphi(1+\beta)}$$

B. If $\frac{-\varphi}{b} < \frac{\theta}{a}$, then $x < \min\left(\frac{a}{\theta}, -\frac{b}{\varphi}\right) = \frac{a}{\theta}$

$$\Pi\left(\frac{a}{\theta}\right) = \frac{a^2 b}{\theta}\left(\frac{-\varphi}{b} - \frac{\theta}{a}\right) < 0$$

Therefore, only the smaller solution satisfies $x < \frac{a}{\theta}$. The unique solution is given by

$$x = \frac{-\Psi_t - \Phi_t}{-2\theta\varphi(1+\beta)} = \frac{\Psi_t + \Phi_t}{2\theta\varphi(1+\beta)}$$

In the end, we can define $\gamma_{h,t}$

$$\gamma_{h,t} \equiv \frac{\lambda_1}{\lambda_2} = \frac{c^t_{h,t+1}}{\beta(1-\zeta)c^t_{h,t}} = \frac{b+\varphi x}{\beta(1-\zeta)(a-\theta x)}$$

and

$$c^t_{h,t} = \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{h,t}}\right]y^t_t$$
$$c^t_{h,t+1} = \frac{\beta(1-\zeta)\gamma_{h,t}}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{h,t}}\right]y^t_t$$
$$p_t h^t_{h,t+1} = \frac{\Psi_t + \Phi_t}{2\theta\varphi(1+\beta)}.$$

Proof of Lemma 1

We start first by looking the saving function of the unconstrained investor. Investors are not constrained if and only if $R_{t+1} > \frac{p_{t+1}}{p_t}$. From Proposition 1, we can write down the saving function of an unconstrained investor as

$$a^t_{i,t+1} = (1-\tau)y^t_t - \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y^t_t$$

It is obvious to see that the saving function of the unconstrained investor is a decreasing function of interest rate. Investor is borrowing-constrained if and only if $R_{t+1} < \frac{p_{t+1}}{p_t}$. From Proposition 1, we can write down the saving function of an constrained investor as

$$a^t_{i,t+1} = -(1-\theta)\frac{\beta\gamma_{i,t+1}(1-\tau) - \tau(1+n)(1+g)}{\theta\gamma_{i,t+1}(1+\beta)}y^t_t$$

where $\gamma_{i,t+1} \equiv \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t} > 0$ Lower interest rate increases $\gamma_{i,t}$ and implies more borrowing, or equivalently, less saving. Hence, the credit supply of investors is always a decreasing function of interest rate.

For the unconstrained households, its credit demand is given by $-a^t_{h,t+1} = c^t_{h,t} + p_t h^t_{h,t+1} - (1-\tau)\, y^t_t$. From Proposition 2, we know that both $c^t_{h,t}$ and $p_t h^t_{h,t+1}$ are decreasing function of interest rate. When the household is borrowing-constrained, the credit demand function becomes complicated.

$$
\begin{aligned}
p_t h^t_{h,t+1} &= \frac{\Psi_t + \sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\,(\beta+1)}}{2\theta\varphi\,(\beta+1)} \\
&= 2ab\beta\zeta \frac{1}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\,(\beta+1)} - \Psi_t}
\end{aligned}
$$

Differentiate $p_t h^t_{h,t+1}$ directly w.r.t. φ Then,

$$
\begin{aligned}
\frac{\partial p_t h^t_{h,t+1}}{\partial \varphi} &= -2ab\beta\zeta \left(\frac{1}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\,(\beta+1)} - \Psi_t} \right)^2 \\
&\quad \times \left(\frac{d}{d\varphi}\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\,(\beta+1)} - \frac{d}{d\varphi}\Psi_t \right)
\end{aligned}
$$

Note that $\Psi_t = a\varphi\beta - b\theta\,(1+\beta\zeta)$ and $\frac{d}{d\varphi}\Psi_t = a\beta$

$$
\begin{aligned}
&\frac{d}{d\varphi}\sqrt{\Psi_t^2 + 4ab\theta\beta\zeta\varphi\,(\beta+1)} \\
&\quad = a\beta \frac{(a\varphi\beta - b\theta\,(1+\beta\zeta)) + 2b\zeta\theta\,(\beta+1)}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\,(\beta+1)}} < a\beta
\end{aligned}
$$

because of

$$
\begin{aligned}
&((a\varphi\beta - b\theta\,(1+\beta\zeta)) + 2b\zeta\theta\,(\beta+1))^2 - \left(\Psi_t^2 + 4ab\beta\zeta\theta\varphi\,(\beta+1)\right) \\
&\quad = -4b^2\zeta\theta^2\,(\beta+1)\,(1-\zeta) < 0
\end{aligned}
$$

We have $\frac{\partial p_t h^t_{h,t+1}}{\partial \varphi} > 0$, $\frac{\partial p_t h^t_{h,t+1}}{\partial R} < 0$. The credit demand of constrained household is an decreasing function of interest rate.

Proof of Lemma 2

The stationary equilibrium is defined as the competitive general equilibrium in which all individual allocations and prices are time-invariant. We need to further assume that $H_t = \bar{H}$ in the stationary equilibrium to get constant housing price. Denote the constant housing price by p^*. Obviously, we have $p^* > 0$. Otherwise, workers would purchase infinite amount of houses. Suppose the equilibrium gross interest $R^* < 1$. The gross return of housing for the investors is 1, which is higher than the gross return

R^* on consumption loans. From the previous decision rules, the borrowing constraint for both types of households would be binding. The total borrowing of workers is positive, and the total borrowing of investors is nonnegative. Therefore, the market for credit cannot clear at $R^* < 1$. Equilibrium interest rate has to be higher, and $R^* < 1$ cannot be a equilibrium interest rate. Note that if $\theta = 1$, both investors and households cannot borrow in the equilibrium. Therefore, the economy becomes autarky, and the interest rate is not well defined. We rule out this case by requiring that $\theta < 1$.

Proof of Proposition 3

Proposition 3 characterizes the uniqueness of stationary equilibrium. It states that the allocation of housing assets in this economy depends on the tightness of collateral constraint. We provide some intuition here (please see the appendix for the proof). We have shown that the optimal demand and supply of credit are continuous. Lemma 1 proves that the demand of credit from households is monotonically decreasing in the interest rate, and the supply of credit from investors is a monotonically increasing function of interest rate. From Lemma 2, there exists a unique stationary equilibrium with $R^* \geq 1$. Households always borrow from investors in the model because they consume housing services. Investors will not be borrowing-constrained when $R^* \geq 1$. They supply credit in the market. θ will only affect the optimal decision of households, who are the demand side of credit market. As the borrowing constraint becomes tighter (higher θ), households are going to be borrowing-constrained first. High θ reduces the borrowing limit of constrained households. If θ is high enough, the total borrowing from households become less than the total credit supply from investors. Interest rate has to be lower in order to clear the consumption loan market. Therefore, tighter borrowing constraint reduces the credit demand from households and drives the equilibrium interest rate down. When the gross interest rate drops to one, housing assets become attractive as an alternative saving mean to the investors. The credit market clearing condition requires that the extra supply of credit coming from investors to be invested in the housing assets, which are the only alternative assets in this economy. Therefore, there are two threshold levels for collateral constraint, denoted by θ_L and θ_H and three different cases which we analyze one by one.

1. Unconstrained households and unconstrained investors without housing. In the stationary equilibrium, $y_t^t = y$, $H_t = H$. The equilibrium prices $\left(p_1^*, R_1^*\right)$ are determined by

$$H = \omega \frac{1}{p_1} \frac{R_1}{R_1 - 1} \frac{\beta\zeta}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1}\right) y$$

$$0 = 1 - \tau - \frac{1}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1}\right) \left(1 + \omega \frac{\beta\zeta R_1}{R_1 - 1}\right)$$

The second equation determines a unique $R_1^* > 1$.[26] Hence, housing price can be determined by

$$p_1^* = \omega \frac{y}{H} \frac{R_1^*}{R_1^* - 1} \frac{\beta\zeta}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1^*}\right)$$

Note that θ cannot affect either p_1^* or R_1^*. Now, we can solve for the first threshold θ_L when households is borrowing-constrained

$$(1-\tau) - \frac{1}{1+\beta}\left(1 - \tau + \frac{\tau}{R_1^*}\right) = \theta_L \frac{R_1^*}{R_1^* - 1} \frac{\beta\zeta}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1^*}\right)$$

Using the credit market clearing condition, we have $\theta_L = \omega$. Therefore, $\frac{\partial \theta_L}{\partial \omega} = 1$. The intuition is that more households will increase the equilibrium interest rate. When the interest rate becomes higher, households will reduce the consumption and housing expenditure. They will be borrowing-constrained under a stricter borrowing constraint.

2. Constrained households and unconstrained investors without housing. The equilibrium prices $\left(p_2^*, R_2^*\right)$ are determined by

$$\omega \frac{1}{p_2} \frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} = H$$
$$(1-\omega)\left[1 - \tau - \frac{1}{1+\beta}\left(1 - \tau + \frac{\tau}{R_2}\right)\right] y - \omega(1-\theta)\frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} = 0$$

The two equations imply two implicit functions $p_2^*\left(R_2^*, \theta\right)$ and $R_2^*(\theta)$. The effect of θ on equilibrium housing price is given by

$$\frac{dp_2^*\left(R_2^*, \theta\right)}{d\theta} = \frac{\partial p_2^*\left(R_2^*, \theta\right)}{\partial R_2^*} \frac{dR_2^*}{d\theta} + \frac{\partial p_2^*\left(R_2^*, \theta\right)}{\partial \theta}$$

On one hand, tighter credit constraint reduces the housing demand, which tends to reduce the price. However, tighter credit constraint also reduces interest rate, which in turn encourages housing consumption. Hence, the total effect is indeterminate.

3. Constrained households and unconstrained investors with empty housing. When $R_3^* = \frac{p_{t+1}}{p_t} = 1$, the market clearing conditions become

$$\omega \frac{1}{p_3} \frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} + (1-\omega)\frac{I}{p_3} = H$$
$$(1-\omega)\left[(1-\tau)y - \frac{1}{1+\beta}y - I\right] - \omega(1-\theta)\frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} = 0$$

[26] The other solution $R < 1$ cannot be an equilibrium interest rate.

where I denotes the investor's purchase of housing assets. Combine the two conditions and note that $\varphi = \theta$ when $R = 1$.

$$(1-\omega)\left(1-\tau-\frac{1}{1+\beta}\right)y+\omega\frac{\Psi+\Phi}{2\theta\left(\beta+1\right)}=p_3H$$

which suggests that p_3^* is independent of θ since $\left(\Psi+\Phi\right)/\theta$ does not depend on θ. The total amount of savings is invested in housing assets. The threshold θ_H for investors to hold housing assets is determined by

$$(1-\omega)\left(1-\tau-\frac{1}{1+\beta}\right)y-\omega\left(\frac{1-\theta_H}{\theta_H}\right)\frac{\Psi+\Phi}{2\theta_H\left(\beta+1\right)}=0$$

It is also true that $\frac{\partial\theta_H}{\partial\omega} > 0$. This is because high ω implies fewer credit supply from investors. The collateral constraint has to be higher to clear the credit market.

Proof of Proposition 1

Suppose there is a useless asset called paper. In case 3, it has positive value in the equilibrium. This is because investor has excess supply of credit in the market, which can be invested in the paper. Since the equilibrium interest rate is 1, the price of paper remains constant in the equilibrium. The size of the paper bubble is given by

$$B=(1-\omega)\left(1-\tau-\frac{1}{1+\beta}\right)y-\omega\left(\frac{1-\theta}{\theta}\right)\frac{\Psi+\Phi}{2\theta\left(\beta+1\right)}>0 \text{ for } \theta>\theta_H$$

This is called pure bubble. However, the bubble can also take the form of housing assets. If the investors purchase the housing assets I instead, then

$$B=(1-\omega)\,I$$

which means bubble can shift from paper market to the housing market. If we define the bubble as the case in which investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or in terms of rent, then the case 3 satisfies this definition because we rule out the rental market. The next question is whether there is bubble for households? The answer is no. First of all, we define the fundamental value of housing assets to households, and then, we show that under properly adjusted interest rate, the housing price is equal to its fundamental value for households in all three cases.

1. Unconstrained households and unconstrained investors without housing. The fundamental value of housing is defined as

$$p_t^F=\frac{p_{t+1}+\frac{\zeta}{1-\zeta}\frac{c_{h,t+1}^t}{h_{h,t+1}^t}}{R_{t+1}}$$

$$= \sum_{\tau=0}^{\infty} \frac{1}{R_{t+1} \ldots R_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{h,t+\tau+1}^{t+\tau}}{h_{h,t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{R_{t+1} \ldots R_{t+T-1}}$$

Using the first-order condition of households

$$p_t^F = \textstyle\sum_{\tau=0}^{\infty} \frac{1}{R_{t+1}\ldots R_{t+\tau}} \left(p_{t+\tau} R_{t+\tau} - p_{t+\tau+1}\right) + \displaystyle\lim_{T\to\infty} p_{t+T} \frac{1}{R_{t+1} \ldots R_{t+T-1}}$$

In the stationary equilibrium, $R_1^* > 1$, $\lim_{T\to\infty} p_1^* \frac{1}{(R_1^*)^T} = 0$

$$p^F = \sum_{\tau=0}^{\infty} \frac{1}{\left(R_1^*\right)^{\tau+1}} \left(p_1^* R_1^* - p_1^*\right) = p_1^* \sum_{\tau=0}^{\infty} \frac{R_1^* - 1}{\left(R_1^*\right)^{\tau+1}} = p_1^*$$

2. Constrained households and unconstrained investors without housing. The fundamental value of housing can be defined as

$$p_t^F = \frac{p_{t+1} + \frac{\zeta}{1-\zeta} \frac{c_{h,t+1}^t}{h_{h,t+1}^t}}{\hat{R}_t}$$
$$= \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{h,t+\tau+1}^{t+\tau}}{h_{h,t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+T-1}}$$

where $\hat{R}_t = \theta \frac{\lambda_1}{\lambda_2} + (1-\theta) R_{t+1}$. This measures the effective interest rate that households face. It takes into account the shadow value of borrowing constraint. If the borrowing constraint is not binding, $\lambda_1/\lambda_2 = R_{t+1} = \hat{R}_t$. If the borrowing constraint is binding, the effect interest rate is a weighted average of λ_1/λ_2 and R_{t+1}. Therefore, $R_{t+1} < \hat{R}_t < \lambda_1/\lambda_2$. Using the first-order condition of constrained households

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+\tau}} \frac{\lambda_1 p_t - \lambda_2 p_{t+1} - \mu_1 (1-\theta)\, p_t}{\lambda_2} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+T-1}}$$

In the stationary equilibrium, $\hat{R}_2^* = \theta \frac{\lambda_1}{\lambda_2} + (1-\theta)\, R_2^* > 1$, $\lim_{T\to\infty} p_2^* \frac{1}{\left(\hat{R}_2^*\right)^T} = 0$

$$p^F = \sum_{\tau=0}^{\infty} \frac{1}{\left(\hat{R}_2^*\right)^{\tau+1}} \frac{\lambda_1 p_2^* - \lambda_2 p_2^* - \left(\lambda_1 - \lambda_2 R_2^*\right)(1-\theta)\, p_2^*}{\lambda_2}$$
$$= p_2^* \sum_{\tau=0}^{\infty} \frac{1}{\left(\hat{R}_2^*\right)^{\tau+1}} \left(\frac{\lambda_1}{\lambda_2}\theta + R_2^* (1-\theta) - 1\right)$$

$$= p_2^* \sum_{\tau=0}^{\infty} \frac{\hat{R}_2^* - 1}{\left(\hat{R}_2^*\right)^{\tau+1}} = p_2^*$$

3. Constrained households and unconstrained investors with empty housing. The fundamental value of housing can be defined as

$$p_t^F = \frac{p_{t+1} + \frac{\zeta}{1-\zeta} \frac{c_{h,t+1}^t}{h_{h,t+1}^t}}{\hat{R}_t}$$
$$= \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{h,t+\tau+1}^{t+\tau}}{h_{h,t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+T-1}}$$

where $\hat{R}_3 = \theta \frac{\lambda_1}{\lambda_2} + 1 - \theta$. Using the first-order condition of households,

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+\tau}} \frac{\lambda_1 p_t - \lambda_2 p_{t+1} - (\lambda_1 - \lambda_2 R_{t+1})(1-\theta) p_t}{\lambda_2}$$
$$+ \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t \ldots \hat{R}_{t+T-1}}$$

In the stationary equilibrium, $p_t = p_3^*$, $\hat{R}_3^* > 1$, $\lim_{T\to\infty} p_3^* \frac{1}{\left(\hat{R}_3^*\right)^T} = 0$

$$p^F = p_3^* \sum_{\tau=0}^{\infty} \frac{\hat{R}_3^* - 1}{\left(\hat{R}_3^*\right)^{\tau}} = p_3^*$$

Proof of Proposition 4

When $\tau = 0$, the total supply of credit by investors becomes $(1-\omega) \frac{\beta}{1+\beta} y$. The total credit demand from constrained households becomes $\omega \frac{1-\theta}{\theta} \frac{\beta}{\beta+1} y$. Note that both the supply and demand do not depend on interest rate. Therefore, bubble will arise iff

$$(1-\omega) \frac{\beta}{1+\beta} y > \omega \frac{1-\theta}{\theta} \frac{\beta}{\beta+1} y$$

which is equivalent to $\theta > \theta_L = \omega$. Therefore, if the economy stays at the case 1 of stationary equilibrium, where both investors and households are unconstrained, then the removal of pension system will not trigger a bubble equilibrium. If the economy stays at case 2 of stationary equilibrium, we have

$$\frac{p_2 H}{y} = \frac{1-\omega}{1-\theta} \left[1 - \tau - \frac{1}{1+\beta} \left(1 - \tau + \frac{\tau}{R_2}\right)\right]$$

In the bubble equilibrium, the housing wealth-to-GDP ratio is $\frac{\beta}{1+\beta}$. If $\tau > \frac{\theta-\omega}{1-\omega}$, then

$$\frac{p_2 H}{y} < \frac{(1-\omega)(1-\tau)}{1-\theta} \frac{\beta}{1+\beta} < \frac{\beta}{1+\beta}$$

Model extension

Investor's problem

The Lagrangian function is

$$\begin{aligned}
L &= \ln c_{i,t}^{t} + \beta \ln c_{i,t+1}^{t} \\
&+\lambda_1 \left[(1-\tau)\, y_t^t + p_t^r h_{i,t+1}^{R} - c_{i,t}^{t} - a_{i,t+1}^{t} - p_t h_{i,t+1}^{t} \right] \\
&+\lambda_2 \left[\tau\,(1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{i,t+1}^{t} + p_{t+1} h_{i,t+1}^{t} - \delta_r p_{t+1} h_{i,t+1}^{R} - c_{i,t+1}^{t} \right] \\
&+\mu_1 \left[a_{i,t+1}^{t} + (1-\theta)\, p_t h_{i,t+1}^{t} \right] \\
&+\mu_2 \left[h_{i,t+1}^{t} - h_{i,t+1}^{R} \right] \\
&+\nu_1 h_{i,t+1}^{t} \\
&+\nu_2 h_{i,t+1}^{R}
\end{aligned}$$

The FOCs become

$$\begin{aligned}
c_{i,t}^{t} &: \frac{1}{c_{i,t}^{t}} - \lambda_1 = 0 \\
c_{i,t+1}^{t} &: \frac{\beta}{c_{i,t+1}^{t}} - \lambda_2 = 0 \\
a_{i,t+1}^{t} &: -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\
h_{i,t+1}^{t} &: -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t + \mu_2 + \nu_1 = 0 \\
h_{i,t+1}^{R} &: \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 + \nu_2 = 0
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &\geq 0, \quad \text{if } a_{i,t+1}^{t} + (1-\theta)\, p_t h_{i,t+1}^{t} > 0, \text{then } \mu_1 = 0 \\
\mu_2 &\geq 0, \quad \text{if } h_{i,t+1}^{t} - h_{i,t+1}^{R} > 0,\ \text{then } \mu_2 = 0 \\
\nu_1 &\geq 0, \quad \text{if } h_{i,t+1}^{t} > 0, \text{then } \nu_1 = 0 \\
\nu_2 &\geq 0, \quad \text{if } h_{i,t+1}^{R} > 0,\ \text{then } \nu_2 = 0
\end{aligned}$$

The lifetime budget constraint for the investors is

$$\begin{aligned}
c_{i,t}^{t} + \frac{c_{i,t+1}^{t}}{R_{t+1}} &= (1-\tau)\, y_t^t + \frac{\tau\,(1+n)\, y_{t+1}^{t+1}}{R_{t+1}} \\
&\quad + \left(\frac{p_{t+1}}{R_{t+1}} - p_t \right) h_{i,t+1}^{t} + \left(p_t^r - \frac{\delta_r p_{t+1}}{R_{t+1}} \right) h_{i,t+1}^{R}
\end{aligned}$$

1. $a^t_{i,t+1} + (1-\theta)\, p_t h^t_{i,t+1} > 0$, $h^t_{i,t+1} - h^R_{i,t+1} > 0$, $h^t_{i,t+1} > 0$, $h^R_{i,t+1} > 0$, Then, $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. Plug them into the FOCs

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$-\lambda_1 p_t + \lambda_2 p_{t+1} = 0$$
$$\lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} = 0$$

The following equality holds

$$R_{t+1} = \frac{p_{t+1}}{p_t} = \frac{\delta_r p_{t+1}}{p^r_t} = \frac{(1-\delta_r)\, p_{t+1}}{p_t - p^r_t}$$

, and the optimal consumption rules are

$$c^t_{i,t} = \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y^t_t$$
$$c^t_{i,t+1} = \frac{\beta R_{t+1}}{1+\beta}\left[1 - \tau + \frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y^t_t$$

and the private credit, housing assets, and rental housing are jointly determined by

$$a^t_{i,t+1} + p_t h^t_{i,t+1} - p^r_t h^R_{i,t+1} = (1-\tau)\, y^t_t - c^t_{i,t}$$

Note that

$$\frac{\delta_r p_{t+1}}{p^r_t} = R_{t+1} = \frac{p_{t+1}}{p_t} = \frac{(1-\delta_r)\, p_{t+1}}{p_t - p^r_t}$$

Then,

$$R_{t+1} = \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t} = \frac{(1-\delta_r)\, p_{t+1} - R_{t+1}\,(1-\theta)\, p_t}{\theta p_t - p^r_t}$$

2. $a^t_{i,t+1} + (1-\theta)\, p_t h^t_{i,t+1} > 0$, $h^t_{i,t+1} - h^R_{i,t+1} > 0$, $h^t_{i,t+1} > 0$, $h^R_{i,t+1} = 0$, then $\mu_1 = \mu_2 = \nu_1 = 0$, $\nu_2 \geq 0$. Plug them into the FOCs,

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$-\lambda_1 p_t + \lambda_2 p_{t+1} = 0$$
$$\lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} + \nu_2 = 0$$

Hence,

$$R_{t+1} = \frac{p_{t+1}}{p_t} \leq \frac{\delta_r p_{t+1}}{p^r_t}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to the case 1.
(b) If $\mu_1 = \mu_2 = \nu_1 = 0$, $\nu_2 > 0$, then

$$\frac{\delta_r p_{t+1}}{p_t^r} > R_{t+1} = \frac{p_{t+1}}{p_t} > \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r}$$

and

$$a_{i,t+1}^t + p_t h_{i,t+1}^t = (1-\tau) y_t^t - c_{i,t}^t$$

Under this case, it is also true that

$$R_{t+1} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} > \frac{(1-\delta_r) p_{t+1} - R_{t+1}(1-\theta) p_t}{\theta p_t - p_t^r}$$

3. $a_{i,t+1}^t + (1-\theta) p_t h_{i,t+1}^t > 0$, $h_{t+1}^t - h_{i,t+1}^R = 0$, $h_{i,t+1}^t > 0$, $h_{i,t+1}^R > 0$, then $\mu_1 = \nu_1 = \nu_2 = 0$, $\mu_2 \geq 0$. Plug them into the FOCs,

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$-\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_2 = 0$$
$$\lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 = 0$$

Hence,

$$R_{t+1} \geq \frac{p_{t+1}}{p_t}$$
$$R_{t+1} \geq \frac{\delta_r p_{t+1}}{p_t^r}$$
$$R_{t+1} = \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to the case 1.
(b) If $\mu_1 = \nu_1 = \nu_2 = 0$, $\mu_2 > 0$, then

$$R_{t+1} = \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r} > \frac{p_{t+1}}{p_t} > \frac{\delta_r p_{t+1}}{p_t^r}$$

and

$$a_{i,t+1}^t + \left(p_t - p_t^r\right) h_{i,t+1}^t = (1-\tau) y_t^t - c_{i,t}^t$$
$$h_{i,t+1}^R = h_{i,t+1}^t$$

In this case, it is also true that

$$R_{t+1} = \frac{(1-\delta_r) p_{t+1} - R_{t+1}(1-\theta) p_t}{\theta p_t - p_t^r} > \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t}$$

4. $a^t_{i,t+1} + (1-\theta)\, p_t h^t_{i,t+1} > 0$, $h^t_{i,t+1} = h^R_{i,t+1} = 0$, then $\mu_1 = 0$, $\mu_2 \geq 0$, $\nu_1 \geq 0$, $\nu_2 \geq 0$. Plug them into the FOCs,

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$-\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_2 + \nu_1 = 0$$
$$\lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} - \mu_2 + \nu_2 = 0$$

Hence,

$$R_{t+1} \geq \frac{p_{t+1}}{p_t}$$
$$R_{t+1} \geq \frac{(1-\delta_r)\, p_{t+1}}{p_t - p^r_t}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to case 1
(b) If $\mu_1 = \mu_2 = \nu_1 = 0$, $\nu_2 > 0$, then we go back to case 2
(c) If $\mu_1 = \nu_1 = \nu_2 = 0$, $\mu_2 > 0$, then we go back to case 3
(d) If $\mu_1 = 0$, $\mu_2 + \nu_1 > 0$, $\nu_1 + \nu_2 > 0$, then $R_{t+1} > \frac{p_{t+1}}{p_t}$ and $R_{t+1} > \frac{(1-\delta_r)p_{t+1}}{p_t - p^r_t}$.

$$a^t_{i,t+1} = (1-\tau)\, y^t_t - c^t_{i,t}$$
$$h^R_{i,t+1} = h^t_{i,t+1} = 0$$

It is also true that

$$R_{t+1} > \frac{(1-\delta_r)\, p_{t+1} - R_{t+1}(1-\theta)\, p_t}{\theta p_t - p^r_t}$$
$$R_{t+1} > \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t}$$

5. $a^t_{i,t+1} + (1-\theta)\, p_t h^t_{i,t+1} = 0$, $h^t_{i,t+1} - h^R_{i,t+1} > 0$, $h^t_{i,t+1} > 0$, $h^R_{i,t+1} > 0$, then $\mu_1 \geq 0$, $\mu_2 = \nu_1 = \nu_2 = 0$. Plug them into the FOCs,

$$-\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0$$
$$-\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t = 0$$
$$\lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} = 0$$

Hence,

$$\frac{\lambda_1}{\lambda_2} \geq R_{t+1}$$
$$\frac{\lambda_1}{\lambda_2} \geq \frac{p_{t+1}}{p_t}$$
$$\frac{\lambda_1}{\lambda_2} = \frac{\delta_r p_{t+1}}{p^r_t}$$

Discussion:

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to case 1.

(b) If $\mu_1 > 0$, $\mu_2 = \nu_1 = \nu_2 = 0$, then

$$\frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t}$$

Use the equation $\frac{\lambda_1}{\lambda_2} = \frac{\delta_r p_{t+1}}{p_t^r}$ then we have an expression for R_{t+1}

$$R_{t+1} = \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta} < \frac{p_{t+1}}{p_t}$$

It follows that

$$R_{t+1}, \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} = \frac{\delta_r p_{t+1}}{p_t^r}$$

First of all, this suggests that the borrowing cost is smaller than the intertemporal rate of substitution Therefore, the investors must be borrowing-constrained. Secondly, the investors are indifferent between constrained-borrow-to-empty and constrained-borrow-to-rent, i.e.,

$$\frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} = \frac{(1-\delta_r) p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t - p_t^r}$$

Let $x \equiv \left(p_t h_{i,t+1}^t - \frac{p_t^r}{\theta} h_{i,t+1}^R\right)$ and $\gamma_{i,t} \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t}$. Rewrite the budget constraints as

$$c_{i,t}^t + \theta p_t h_{i,t+1}^t = (1-\tau) y_t + p_t^r h_{i,t+1}^R$$
$$c_{i,t+1}^t = \tau (1+n)(1+g) y_t^t + \left(p_t h_{i,t+1}^t - \frac{p_t^r}{\theta} h_{i,t+1}^R\right) \theta \gamma_{i,t}$$

Then,

$$c_{i,t}^t = (1-\tau) y_t^t - \theta x$$
$$c_{i,t+1}^t = \tau (1+n)(1+g) y_t^t + \theta \gamma_{i,t} x$$

Solve for x

$$x = \frac{\beta \gamma_{i,t} (1-\tau) y_t^t - \tau (1+n)(1+g) y_t^t}{\theta \gamma_{i,t} (\beta + 1)}$$

Therefore,

$$c_{i,t}^t = \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{\gamma_{i,t}}\right] y_t^t$$

$$
\begin{aligned}
c_{i,t+1}^t &= \frac{\beta \gamma_{i,t}}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{i,t}}\right] y_t^t \\
a_{i,t+1}^t &= -(1-\theta)\, p_t h_{i,t+1}^t \\
p_t h_{i,t+1}^t - \frac{p_t^r h_{i,t+1}^R}{\theta} &= \frac{\beta \gamma_{i,t}(1-\tau)-\tau(1+n)(1+g)}{\theta \gamma_{i,t}(\beta+1)} y_t^t
\end{aligned}
$$

6. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t = 0$, $h_{t+1}^t - h_{t+1}^R > 0$, $h_{t+1}^t > 0$, $h_{t+1}^R = 0$, then $\mu_1, \nu_2 \geq 0$, $\mu_2 = \nu_1 = 0$. Plug them into the FOCs,

$$
\begin{aligned}
-\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\
-\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t &= 0 \\
\lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} + \nu_2 &= 0
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\lambda_1}{\lambda_2} &\geq R_{t+1} \\
\frac{\lambda_1}{\lambda_2} &\geq \frac{p_{t+1}}{p_t} \\
\frac{\lambda_1}{\lambda_2} &\leq \frac{\delta_r p_{t+1}}{p_t^r}
\end{aligned}
$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to case 1
(b) If $\mu_1 > 0$, $\mu_2 = \nu_1 = \nu_2 = 0$, then we go back to case 5
(c) If $\mu_1 = \mu_2 = \nu_1 = 0$, $\nu_2 > 0$, then we go back to case 2
(d) If $\mu_1 > 0$, $\nu_2 > 0$, $\mu_2 = \nu_1 = 0$, then

$$
\frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t}
$$

Use the condition that $\frac{\lambda_1}{\lambda_2} < \frac{\delta_r p_{t+1}}{p_t^r}$, and the following inequality for R_{t+1} holds

$$
R_{t+1} > \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta}
$$

It turns out that $\frac{p_{t+1}-(1-\theta) R_{t+1} p_t}{\theta p_t} > \frac{p_{t+1}}{p_t}$ implies $\frac{p_{t+1}}{p_t} > R_{t+1}$. Therefore, it follows that

$$
R_{t+1}, \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} < \frac{\delta_r p_{t+1}}{p_t^r}
$$

First of all, this suggests that the borrowing cost is smaller than the intertemporal rate of substitution Therefore, the investors must be borrowing-constrained. Secondly, the investors prefer the constrained-borrow-to-empty to the constrained-borrow-to-rent, i.e.,

$$\frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} > \frac{(1-\delta_r) p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t - p_t^r}$$

Let $x \equiv p_t h_{t+1}^t$ and $\gamma_{i,t} \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t}$. Use the fact that

$$\begin{aligned} c_{i,t}^t &= (1-\tau) y_t^t - \theta p_t h_{i,t+1}^t \\ c_{i,t+1}^t &= \tau (1+n)(1+g) y_t^t - R_{t+1}(1-\theta) p_t h_{i,t+1}^t + p_{t+1} h_{i,t+1}^t \end{aligned}$$

Then,

$$\begin{aligned} c_{i,t}^t &= (1-\tau) y_t^t - \theta x \\ c_{i,t+1}^t &= \tau (1+n)(1+g) y_t^t + \theta \gamma_{i,t} x \end{aligned}$$

Solve for x

$$x = \frac{\beta \gamma_{i,t}(1-\tau) - \tau(1+n)(1+g)}{\theta \gamma_{i,t}(\beta+1)} y_t^t$$

Therefore,

$$\begin{aligned} c_{i,t}^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{i,t}}\right] y_t^t \\ c_{i,t+1}^t &= \frac{\beta\gamma_{i,t}}{1+\beta}\left[(1-\tau)+\frac{\tau(1+n)(1+g)}{\gamma_{i,t}}\right] y_{i,t}^t \\ a_{i,t+1}^t &= -(1-\theta) p_t h_{i,t+1}^t \\ p_t h_{i,t+1}^t &= \frac{\beta\gamma_{i,t}(1-\tau) - \tau(1+n)(1+g)}{\theta\gamma_{i,t}(\beta+1)} y_t^t \\ h_{i,t+1}^R &= 0 \end{aligned}$$

7. $a_{i,t+1}^t + (1-\theta) p_t h_{i,t+1}^t = 0$, i.e., the borrowing constraint of the investors is binding
$h_{i,t+1}^t - h_{i,t+1}^R = 0$, i.e., the investors rent all the houses out
$h_{i,t+1}^t > 0$, $h_{i,t+1}^R > 0$, i.e., the investors hold positive amount of housing
Therefore, $\mu_1, \mu_2 \geq 0$, $\nu_1 = \nu_2 = 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t + \mu_2 &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 &= 0 \end{aligned}$$

Hence,

$$\frac{\lambda_1}{\lambda_2} \geq R_{t+1}$$
$$\frac{\lambda_1}{\lambda_2} \geq \frac{p_{t+1}}{p_t}$$
$$\frac{\lambda_1}{\lambda_2} \geq \frac{\delta_r p_{t+1}}{p_t^r}$$

Use the fact that

$$-\lambda_1 p_t + \lambda_2 p_{t+1} + (\lambda_1 - \lambda_2 R_{t+1})(1-\theta) p_t + \mu_2 = 0$$
$$\lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 = 0$$

Solve for $\frac{\lambda_1}{\lambda_2}$

$$\frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r) p_{t+1} - R_{t+1}(1-\theta) p_t}{\theta p_t - p_t^r}$$

(a) If $\mu_1 = 0, \mu_2 = 0, \nu_1 = \nu_2 = 0$, then we go back to case 1.
(b) If $\mu_1 > 0, \mu_2 = 0, \nu_1 = \nu_2 = 0$, then we go back to case 5.
(c) If $\mu_1 = 0, \mu_2 > 0, \nu_1 = \nu_2 = 0$, then we go back to case 3.
(d) If $\mu_1 > 0, \mu_2 > 0, \nu_1 = \nu_2 = 0$, then we have

$$\frac{\lambda_1}{\lambda_2} > R_{t+1}$$
$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1}}{p_t}$$
$$\frac{\lambda_1}{\lambda_2} > \frac{\delta_r p_{t+1}}{p_t^r}$$

Use the expression $\frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r) p_{t+1} - R_{t+1}(1-\theta) p_t}{\theta p_t - p_t^r}$, the above three inequalities imply

$$R_{t+1} < \frac{(1-\delta_r) p_{t+1}}{p_t - p_t^r}$$
$$R_{t+1} < \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta}$$

where I use the assumption $\theta p_t - p_t^r > 0$. Therefore,

$$\frac{(1-\delta_r) p_{t+1} - R_{t+1}(1-\theta) p_t}{\theta p_t - p_t^r} = \frac{\lambda_1}{\lambda_2} > \frac{\delta_r p_{t+1}}{p_t^r}, \frac{p_{t+1}}{p_t}, R_{t+1}$$

It is also true that

$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1} - R_{t+1}(1-\theta)\, p_t}{\theta p_t}$$
$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r}$$

Recall that

$$c_{i,t}^t = (1-\tau)\, y_t^t + p_t^r h_{i,t+1}^R - \theta p_t h_{i,t+1}^t$$
$$c_{i,t+1}^t = (1+n)(1+g)\, y_t^t + R_{t+1} a_{i,t+1}^t + p_{t+1} h_{i,t+1}^t - \delta_r p_{t+1} h_{i,t+1}^R$$

Let $x \equiv \left(p_t - \frac{p_t^r}{\theta}\right) h_{i,t+1}^t$, $\gamma_{i,t} \equiv \frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r)\frac{p_{t+1}}{p_t} - R_{t+1}(1-\theta)}{\theta - \frac{p_t^r}{p_t}}$. Then, the above budget constraint becomes

$$c_{i,t}^t = (1-\tau)\, y_t^t - \theta x$$
$$c_{i,t+1}^t = (1+n)(1+g)\, y_t^t + \theta \gamma_{i,t} x$$

Solve for x

$$x = \frac{\beta \gamma_{i,t}(1-\tau)\, y_t^t - \tau_{t+1} y_{t+1}^t}{\theta \gamma_{i,t}(\beta+1)}$$

Therefore,

$$\left(p_t - \frac{p_t^r}{\theta}\right) h_{i,t+1}^t = \frac{\beta \gamma_{i,t}(1-\tau)\, y_t^t - (1+n)(1+g)\, y_t^t}{\theta \gamma_{i,t}(\beta+1)}$$
$$h_{i,t+1}^t = h_{t+1}^R$$
$$c_{i,t}^t = \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{\gamma_{i,t}}\right] y_t^t$$
$$c_{i,t+1}^t = \frac{\beta \gamma_{i,t}}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{\gamma_{i,t}}\right] y_t^t$$

8. $a_{i,t+1}^t + (1-\theta)\, p_t h_{i,t+1}^t = 0$, $h_{t+1}^t - h_{i,t+1}^R = 0$, $h_{i,t+1}^t = h_{i,t+1}^R = 0$, then $\mu_1, \mu_2, \upsilon_1, \upsilon_2 \geq 0$.

$$c_{i,t}^t = (1-\tau)\, y_t^t$$
$$c_{i,t+1}^t = \tau(1+n)(1+g)\, y_t^t$$

Household's problem

The Lagrangian function is

$$\begin{aligned} L \; &= \ln c^t_{h,t} + \beta\,(1-\zeta) \ln c^t_{h,t+1} + \beta\zeta \ln\left(h^r_{h,t+1} + h^t_{h,t+1}\right) \\ &+\lambda_1 \left[(1-\tau)\, y^t_t - p^r_t h^r_{h,t+1} - p_t h^t_{h,t+1} - c^t_{h,t} - a^t_{h,t+1}\right] \\ &+\lambda_2 \left[\tau\,(1+n)\,(1+g)\, y^t_t + R_{t+1} a^t_{h,t+1} + p_{t+1} h^t_{h,t+1} - c^t_{h,t+1}\right] \\ &+\mu_1 \left[a^t_{h,t+1} + (1-\theta)\, p_t h^t_{h,t+1}\right] \\ &+\nu_1 h^t_{h,t+1} \\ &+\nu_2 h^r_{h,t+1} \end{aligned}$$

The FOCs become

$$\begin{aligned} c^t_{h,t} &: \frac{1}{c^t_{h,t}} - \lambda_1 = 0 \\ c^t_{h,t+1} &: \frac{\beta\,(1-\zeta)}{c^t_{h,t+1}} - \lambda_2 = 0 \\ a^t_{h,t+1} &: -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\ h^t_{h,t+1} &: \frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1\,(1-\theta)\, p_t + \nu_1 = 0 \\ h^r_{h,t+1} &: \frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p^r_t + \nu_2 = 0 \end{aligned}$$

where

$$\begin{aligned} \mu_1 &\geq 0,\ \text{if } a^t_{h,t+1} + (1-\theta)\, p_t h^t_{t+1} > 0, \text{then } \mu_1 = 0 \\ \nu_1 &\geq 0,\ \text{if } h^t_{h,t+1} > 0, \text{then } \nu_1 = 0 \\ \nu_2 &\geq 0,\ \text{if } h^r_{h,t+1} > 0,\ \text{then } \nu_2 = 0 \end{aligned}$$

and the lifetime budget constraint is given by

$$c^t_{h,t} + \frac{c^t_{h,t+1}}{R_{t+1}} + p^r_t h^r_{h,t+1} + \left(p_t - \frac{p_{t+1}}{R_{t+1}}\right) h^t_{h,t+1} = (1-\tau)\, y^t_{h,t} + \frac{\tau\,(1+n)\,(1+g)\, y^t_t}{R_{t+1}}$$

1. $a^t_{h,t+1} + (1-\theta)\, p_t h^t_{h,t+1} > 0, h^t_{h,t+1} > 0, h^r_{h,t+1} > 0$, then $\mu_1 = \nu_1 = \nu_2 = 0$

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} &= 0 \\ \frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \\ \frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p^r_t &= 0 \end{aligned}$$

Hence,

$$\frac{\lambda_1}{\lambda_2} = R_{t+1} = \frac{p_{t+1}}{p_t - p_t^r} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t - p_t^r}$$

The optimal decision rules are

$$c_{h,t}^t = \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_{t+1}}\right] y_t^t$$
$$c_{h,t+1}^t = \frac{\beta (1-\zeta) R_{t+1}}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_{t+1}}\right] y_t^t$$
$$h_{h,t+1}^r + h_{h,t+1}^t = \frac{\beta\zeta}{p_t^r} c_{h,t}^t$$
$$\left(p_t - p_t^r\right) h_{h,t+1}^t + a_{h,t+1}^t = (1-\tau) y_t^t - (1+\beta\zeta) c_{h,t}^t$$

2. $a_{h,t+1}^t + (1-\theta) p_t h_{h,t+1}^t > 0$, $h_{h,t+1}^t > 0$, $h_{h,t+1}^r = 0$, then $\mu_1 = \nu_1 = 0$, $\nu_2 \geq 0$. If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to case 1. If $\mu_1 = \nu_1 = 0$, $\nu_2 > 0$,

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$\frac{\beta\zeta}{h_{h,t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} = 0$$
$$\frac{\beta\zeta}{h_{h,t+1}^t} - \lambda_1 p_t^r + \nu_2 = 0$$

Hence,

$$\frac{\lambda_1}{\lambda_2} = R_{t+1} < \frac{p_{t+1}}{p_t - p_t^r} < \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t - p_t^r}$$

This suggests that if the rental price is high enough, i.e., $p_t^r > p_t - \frac{p_{t+1}}{R_{t+1}}$, unconstrained workers will choose to own houses. The optimal policy rules are

$$c_{h,t}^t = \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_{t+1}}\right] y_t^t$$
$$c_{h,t+1}^t = \frac{\beta (1-\zeta) R_{t+1}}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_{t+1}}\right] y_t^t$$
$$h_{h,t+1}^t = \frac{\beta\zeta}{p_t - \frac{p_{t+1}}{R_{t+1}}} c_{h,t}^t$$
$$a_{h,t+1}^t = (1-\tau) y_t^t - \frac{(1+\beta\zeta) p_t - \frac{p_{t+1}}{R_{t+1}}}{p_t - \frac{p_{t+1}}{R_{t+1}}} c_{h,t}^t$$

3. $a^t_{h,t+1} + (1-\theta) p_t h^t_{h,t+1} > 0$, $h^t_{h,t+1} = 0$, $h^r_{h,t+1} > 0$, then $\mu_1 = 0$, $\nu_1 \geq 0$, $\nu_2 = 0$.If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to case 1. If $\mu_1 = \nu_2 = 0$, $\nu_1 > 0$

$$-\lambda_1 + \lambda_2 R_{t+1} = 0$$
$$\frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \nu_1 = 0$$
$$\frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p^r_t = 0$$

Hence,

$$\frac{\lambda_1}{\lambda_2} = R_{t+1} > \frac{p_{t+1}}{p_t - p^r_t} > \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t - p^r_t}$$

The optimal policy rules are

$$c^t_{h,t} = \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{R_{t+1}}\right] y^t_t$$
$$c^t_{h,t+1} = \frac{\beta}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{R_{t+1}}\right] y^t_t$$
$$p^r_t h^r_{h,t+1} = \frac{\beta\zeta}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{R_{t+1}}\right] y^t_t$$

4. $a^t_{h,t+1} + (1-\theta) p_t h^t_{h,t+1} = 0$, $h^t_{h,t+1} > 0$, $h^r_{h,t+1} > 0$, then $\mu_1 \geq 0$, $\nu_1 = \nu_2 = 0$. If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to case 1. If $\mu_1 > 0$, $\nu_1 = 0$, $\nu_2 = 0$

$$-\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0$$
$$\frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t = 0$$
$$\frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p^r_t = 0$$

Hence, the condition for R_{t+1} is

$$R_{t+1} < \frac{p_{t+1}}{p_t - p^r_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t - p^r_t}$$

Because

$$c^t_{h,t} = (1-\tau) y^t_t - \theta p_t h^t_{h,t+1} + p^r_t h^t_{h,t+1} - p^r_t \left(h^t_{h,t+1} + h^r_{h,t+1}\right)$$
$$c^t_{h,t+1} = \tau(1+n)(1+g) y^t_t + (p_{t+1} - R_{t+1}(1-\theta) p_t) h^t_{h,t+1}$$

Then, we have

$$1 + \beta\zeta = \lambda_1 (1-\tau) y^t_t - \lambda_1 h^t_{h,t+1} \left(\theta p_t - p^r_t\right)$$

and

$$\beta\left(1-\zeta\right)=\lambda_{2}\tau\left(1+n\right)\left(1+g\right)y_{t}^{t}+\lambda_{2}\left(p_{t+1}-R_{t+1}\left(1-\theta\right)p_{t}\right)h_{h,t+1}^{t}$$

Combine the above two equations and let $\frac{\lambda_1}{\lambda_2}\equiv\gamma_{h,t}$, then we have

$$\left(1+\beta\right)c_{t}^{t}=\frac{\tau\left(1+n\right)\left(1+g\right)y_{t}^{t}}{\gamma_{h,t}}+\left(1-\tau\right)y_{t}^{t}$$

If we know $\gamma_{h,t}$, then we can express $c_{h,t}^{t}$, $c_{h,t+1}^{t}$, $h_{h,t+1}^{t}$ in terms of $\gamma_{h,t}$

$$\begin{aligned}1+\beta\ &=\frac{\left(1-\tau\right)y_{t}^{t}}{\left(1-\tau\right)y_{t}^{t}-\theta p_{t}h_{h,t+1}^{t}-p_{t}^{r}h_{h,t+1}^{r}}\\&\quad+\beta\left(1-\zeta\right)\frac{\tau\left(1+n\right)\left(1+g\right)y_{t}^{t}}{\tau\left(1+n\right)\left(1+g\right)y_{t}^{t}+\left(p_{t+1}-R_{t+1}\left(1-\theta\right)p_{t}\right)h_{h,t+1}^{t}}\end{aligned}$$

Use $\frac{1+\beta\zeta}{(1-\tau)y_t^t-(\theta p_t-p_t^r)h_{h,t+1}^t}=\lambda_1=\frac{1}{c_{h,t}^t}$, the above equation can be simplified into

$$\begin{aligned}1+\beta&=\frac{\left(1-\tau\right)\left(1+\beta\zeta\right)y_{t}^{t}}{\left(1-\tau\right)y_{t}^{t}-\left(\theta p_{t}-p_{t}^{r}\right)h_{h,t+1}^{t}}\\&\quad+\beta\left(1-\zeta\right)\frac{\tau\left(1+n\right)\left(1+g\right)y_{t}^{t}}{\tau\left(1+n\right)\left(1+g\right)y_{t}^{t}+\left(p_{t+1}-R_{t+1}\left(1-\theta\right)p_{t}\right)h_{h,t+1}^{t}}\end{aligned}$$

This is a quadratic equation for $p_t h_{h,t+1}^t$. Let

$$\begin{aligned}x&=p_{t}h_{h,t+1}^{t}\\\hat{\theta}&=\theta-\frac{p_{t}^{r}}{p_{t}}\\\varphi&=\frac{p_{t+1}}{p_{t}}-\left(1-\theta\right)R_{t+1}\\a&=\left(1-\tau\right)y_{t}^{t}\\b&=\tau\left(1+n\right)\left(1+g\right)y_{t}^{t}\\1+\beta&=\frac{\left(1+\beta\zeta\right)a}{a-\hat{\theta}x}+\frac{\beta\left(1-\zeta\right)b}{b+\varphi x}\end{aligned}$$

with one solution is zero, the other solution is

$$x=\frac{a\varphi\beta\left(1-\zeta\right)-b\hat{\theta}\left(1+\beta\zeta\right)}{\hat{\theta}\varphi\left(1+\beta\right)}$$

We can still define $\gamma_{h,t}$

$$\gamma_{h,t} = \frac{\lambda_1}{\lambda_2} = \frac{c^t_{h,t+1}}{\beta(1-\zeta)c^t_{h,t}} = \frac{(b+\varphi x)(1+\beta\zeta)}{\beta(1-\zeta)\left(a-\hat{\theta}x\right)}$$
$$= \frac{\varphi}{\hat{\theta}} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t - p^r_t}$$

which gives

$$c^t_{h,t} = \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{h,t}}\right]y^t_t$$
$$c^t_{h,t+1} = \frac{\beta(1-\zeta)\gamma_{h,t}}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_{h,t}}\right]y^t_t$$
$$p_t h^t_{h,t+1} = \frac{p_t}{\theta p_t - p^r_t}\left[(1-\tau)y^t_t-(1+\beta\zeta)c^t_{h,t}\right]$$
$$h^r_{h,t+1} = \frac{(1-\tau)y^t_t - \theta p_t h^t_{h,t+1} - c^t_{h,t}}{p^r_t}$$
$$a^t_{h,t+1} = -(1-\theta)p_t h^t_{h,t+1}$$

5. $a^t_{h,t+1} + (1-\theta)p_t h^t_{h,t+1} = 0$, $h^t_{h,t+1} > 0$, $h^r_{h,t+1} = 0$, then $\mu_1 \geq 0, \nu_1 = 0, \nu_2 \geq 0$.If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to case 1. If $\mu_1 = 0, \nu_1 = 0, \nu_2 > 0$, then we go back to case 2. If $\mu_1 > 0, \nu_1 = 0, \nu_2 = 0$, then we go back to case 4. If $\mu_1 > 0, \nu_1 = 0, \nu_2 > 0$, then the solution is the same as the benchmark model without rental market.
6. $a^t_{h,t+1} + (1-\theta)p_t h^t_{h,t+1} = 0$, $h^t_{h,t+1} = 0$, $h^r_{h,t+1} > 0$, then $\mu_1 \geq 0, \nu_1 \geq 0, \nu_2 = 0$. If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to case 1. If $\mu_1 > 0, \nu_1 = 0, \nu_2 = 0$, then we go back to case 4. If $\mu_1 = 0, \nu_1 > 0, \nu_2 = 0$, then we go back to case 3. If $\mu_1 > 0, \nu_1 > 0, \nu_2 = 0$, then

$$-\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0$$
$$\frac{\beta\zeta}{h^r_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1(1-\theta)p_t + \nu_1 = 0$$
$$\frac{\beta\zeta}{h^r_{h,t+1}} - \lambda_1 p^r_t = 0$$

Either

$$\frac{\lambda_1}{\lambda_2} > R_{t+1} > \frac{p_{t+1}}{p_t} > \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t - p^r_t}$$

or

$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t - p^r_t} > \frac{p_{t+1}}{p_t} > R_{t+1}$$

$$
\begin{aligned}
a^t_{h,t+1} &= 0\\
h^t_{h,t+1} &= 0\\
c^t_{h,t+1} &= \tau\,(1+n)\,(1+g)\,y^t_t\\
c^t_{h,t} &= \frac{1}{1+\beta\zeta}\,(1-\tau)\,y^t_t\\
p^r_t h^r_{h,t+1} &= \frac{\beta\zeta}{1+\beta\zeta}\,(1-\tau)\,y^t_t
\end{aligned}
$$

Proof of Lemma 3

Suppose households are not borrowing-constrained. The Focs of households become

$$
\begin{aligned}
-\lambda_1 + \lambda_2 R_{t+1} &= 0\\
\frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + v_1 &= 0\\
\frac{\beta\zeta}{h^r_{h,t+1} + h^t_{h,t+1}} - \lambda_1 p^r_t + \nu_2 &= 0
\end{aligned}
$$

Suppose $h^r_{t+1} > 0$, then $v_2 = 0$,

$$\lambda_1 p^r_t - \lambda_1 p_t + \lambda_2 p_{t+1} + v_1 = 0$$

Therefore,

$$R_{t+1} = \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} + \frac{v_1}{\lambda_1}}{p_t - p^r_t} \geq \frac{p_{t+1}}{p_t - p^r_t} > \frac{p_{t+1}\,(1-\delta_r)}{p_t - p^r_t}$$

This suggests that investors would not hold housing assets because the return of investment in housing assets is strictly less than the return on consumption loans. Hence, $h^r_{h,t+1} = 0$ if households are borrowing-constrained. This is a contradiction. Therefore, $h^r_{h,t+1} = 0$ if households are unconstrained.

Proof of Proposition 5

Since our point of interest is to see whether frictional rental market can resolve the problem of vacant houses and prevent the rise of bubbles, I assume $\theta > \theta_L = \omega$, such that there exists a bubble after the pension reform when $\delta_r = 0$. From Lemma 10, we know that investors will hold housing assets only if households are borrowing-constrained. Therefore, I only consider the equilibrium where households are borrowing-constrained and investors lend to households.

When there is a housing bubble, $R^* = 1$. For the investors to be indifferent between holding empty houses and renting them out, it must be $p^r = \delta_r p$. For the households to rent positive amount of housing, the necessary condition is

$$R^* < \frac{p}{p - p^r} < \frac{\lambda_1}{\lambda_2} = \gamma_h = \frac{\theta}{\theta - \delta_r}$$

which is obviously satisfied when $R^* = 1$. The demand function for rental housing is given by

$$\begin{aligned} p^r h_h^r &= y - c - \theta p h_h \\ &= \frac{\beta}{1+\beta} y - \frac{\theta}{\theta - \delta_r} \frac{\beta(1-\zeta)}{1+\beta} y \end{aligned}$$

If $\delta_r \geq \theta\zeta$, then $p^r h_h^r < 0$. Households demand zero rental housing if the rental market friction $\delta_r \geq \theta\zeta$.

Housing bubble can still exist even with active rental market. The credit supply is given by

$$\int a_i \mathrm{d}\mu_i = (1-\omega)\left(1 - \frac{1}{1+\beta}\right) y + p^r \int h_i^R \mathrm{d}\mu_i - p \int h_i \mathrm{d}\mu_i$$

where $h_i \geq h^R$. Let us suppose $h_i = h_i^R + h_i^B$, where h_i^B is the amount of vacant houses.

$$\int a_i \mathrm{d}\mu_i = (1-\omega) \frac{\beta}{1+\beta} y + \left(p^r - p\right) \int h_i^R \mathrm{d}\mu_i - p \int h_i^B \mathrm{d}\mu_s$$

The credit demand function can be written as

$$\int a_h \mathrm{d}\mu_h = -\omega \frac{1-\theta}{\theta - \delta_r} \frac{\beta(1-\zeta)}{1+\beta} y$$

The credit market clearing condition requires that $\int a_i \mathrm{d}\mu_i + \int a_h d\mu_h = 0$. Hence,

$$\begin{aligned} &p \int h_i^B \mathrm{d}\mu_i \\ &\quad = (1-\omega) \frac{\beta}{1+\beta} y - \left(p - p^r\right) \int h_i^R \mathrm{d}\mu_s - \omega \frac{1-\theta}{\theta - \delta_r} \frac{\beta(1-\zeta)}{1+\beta} y \\ &\quad = \frac{\beta}{1+\beta} y \left(1 - \frac{\omega\zeta}{\delta_r}\right) \end{aligned}$$

where the second equality comes from the market clearing condition for rental market, $\int h_i^R \mathrm{d}\mu_i = \int h_h^r \mathrm{d}\mu_h$. If $\delta_r > \omega\zeta$, then $p \int h_i^B \mathrm{d}\mu_i > 0$, i.e., there are empty housing held by investors even through the rental market is active.

Proof of Proposition 6

In the equilibrium, if $R_{t+1} \equiv (1+n)(1+g)$, then $\frac{K_t}{A_t L_t} = \left(\frac{n+g+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$. We know that this is the lowest equilibrium interest rate. Hence, $K_{t+1} = \left(\frac{n+g+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} A_{t+1} L_{t+1}$

is maximum asset demand the production sector can absorb. If there exists bubble in the equilibrium, then the following condition holds

$$A_t (1-\omega) L_t \frac{\beta}{1+\beta} \left(\frac{K_t}{A_t L_t}\right)^{\alpha} > A_t \omega L_t \frac{1-\theta}{\theta} \frac{\beta}{1+\beta} \left(\frac{K_t}{A_t L_t}\right)^{\alpha} + K_{t+1}$$

Because $K_{t+1} = \frac{K_t}{A_t L_t} A_{t+1} L_{t+1}$, the above condition can be simplified as

$$(1-\omega) \frac{\beta}{1+\beta} \frac{n+g+\delta}{\alpha} > \omega \frac{1-\theta}{\theta} \frac{\beta}{1+\beta} \frac{n+g+\delta}{\alpha} + 1 + n + g$$

which implies

$$\theta > \omega \frac{1}{1-\alpha \frac{1+\beta}{\beta} \frac{n+g+1}{n+g+\delta}}.$$

Proof of Proposition 7

We know that households are constrained, and investor holds housing assets close to the neighborhood of new stationary equilibrium. From the financial market constraint, we can show that $K_{t+1} = (1-\omega) L_t a_{i,t+1} + \omega L_t a_{h,t+1}$. Because

$$a_{i,t+1} + p_t h_{i,t+1} = \frac{\beta}{1+\beta} w_t$$
$$a_{h,t+1} = -(1-\theta) p_t h_{h,t+1}$$

Plug them to the expression for K_{t+1}, we have

$$K_{t+1} = (1-\omega) L_t \frac{\beta}{1+\beta} w_t - (1-\theta) p_t h_{h,t+1} \omega L_t$$
$$= (1-\omega) L_t \frac{\beta}{1+\beta} w_t + \omega L_t \frac{\beta}{1+\beta} w_t - p_t H$$

Hence, $p_t H + K_{t+1} = L_t \frac{\beta}{1+\beta} w_t$. Because

$$w_t = (1-\alpha) A_t K_t^{\alpha} (A_t L_t)^{-\alpha} L_t$$

then

$$\tilde{p}_t H + \tilde{k}_{t+1} (1+n+g) = \frac{\beta}{1+\beta} (1-\alpha) \tilde{k}_t^{\alpha}$$

where $p_t = \tilde{p}_t A_t L_t$, $K_{t+1} = \tilde{k}_{t+1} A_{t+1} L_{t+1}$.

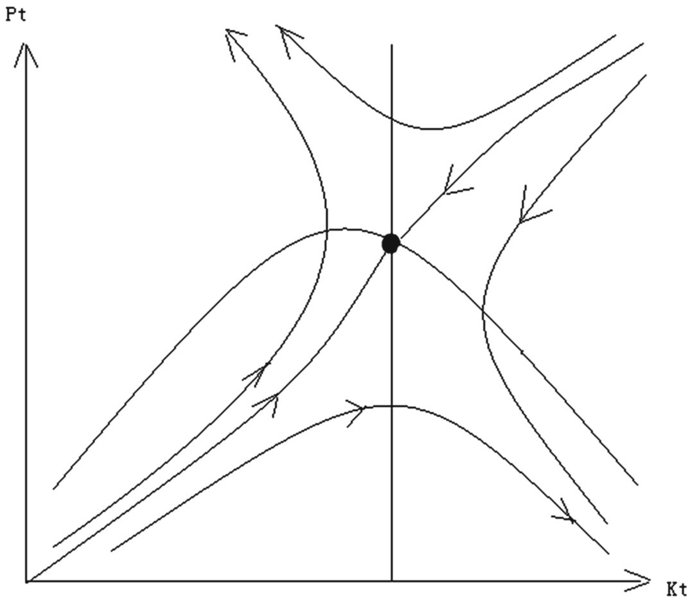

Fig. 12 Phase diagram for the transitional dynamics after the pension reform

When investor holds housing assets, we know that $p_{t+1}/p_t = R_{t+1}$, or equivalently,

$$\frac{\tilde{p}_{t+1}}{\tilde{p}_t} = \left(1 + \alpha \tilde{k}_{t+1}^{\alpha-1} - \delta\right) / (1 + n + g)$$

Therefore, those two equations determine an autonomous system of $\left(\tilde{p}_t, \tilde{k}_t\right)$ with $\tilde{p}_t > 0$ and $\tilde{k}_t > 0$. The phase diagram is shown by Fig. 12. Note that $\tilde{p}_t = 0$ cannot be a stationary equilibrium price because households will demand infinite amount.

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