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RESEARCH ARTICLE

Rationalizability in general situations

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Abstract The main purpose of this paper is to present an analytical framework that can be used to study rationalizable strategic behavior in general situations—i.e., arbitrary strategic games with various modes of behavior. We show that, under mild conditions, the notion of rationalizability defined in general situations has nice properties similar to those in finite games. The major features of this paper are (1) our approach does not require any kind of technical assumptions on the structure of the game, and (2) the analytical framework provides a unified treatment of players' general preferences,

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including expected utility as a special case. In this paper, we also investigate the relationship between rationalizability and Nash equilibrium in general games.

Keywords Strategic games · General preferences · Rationalizability · Common knowledge of rationality · Nash equilibrium

JEL Classification C70 · D81

1 Introduction

The notion of rationalizability proposed by Bernheim (1984) and Pearce (1984) is one of the most important and fundamental solution concepts in noncooperative game theory; see, e.g., Osborne and Rubinstein (1994, Chapter 4). The basic idea behind this notion is that rational behavior should be justified by "rational beliefs," and conversely, "rational beliefs" should be based on rational behavior. The notion of rationalizability captures the strategic implications of the assumption of "common knowledge of rationality" (see Tan and Werlang 1988), which is different from the assumption of "commonality of beliefs" or "correct conjectures" in an equilibrium (see Aumann and Brandenburger 1995).

In the literature, most of the studies of rationalizable strategic behavior have been restricted to finite games.[1] The main purpose of this paper is to extend the notion of rationalizability to general strategic games that can inherit properties of conventional rationalizability. Since many important models arising in economic applications are games with infinite strategy spaces and discontinuous payoff functions, e.g., models of price and spatial competition, auctions, and mechanism design,[2] it is clearly important and practically relevant to extend the notion of rationalizability to arbitrary strategic games.

In the definition of conventional rationalizability, each player is implicitly assumed to be Bayesian rational—i.e., each player maximizes the expected utility given his probabilistic belief about the opponents' strategy choices. However, at the individual level, though subjective expected utility maximization is undoubtedly the dominant model in economics, many economists would probably view axioms such as "transitivity" or "monotonicity" as more basic tenets of rationality than the sure-thing principle and other components of the Savage (1954) model. The Ellsberg paradox and related experimental evidence demonstrate that a decision maker may display

[1] Bernheim (1984, Proposition 3.2) and Tan and Werlang (1988) studied the properties of rationalizable strategies in compact (metric) and continuous games. There are also a few exceptional examples on infinite games such as Arieli's (2010) analysis of rationalizability in continuous games where every player's strategy set is a Polish space and the payoff function of each player is bounded and continuous and Zimper's (2006) discussions on a variant of "strong point-rationalizability" in games with metrizable strategy sets. See also Jara-Moroni (2012) and Yu (2010, 2014) for discussions on rationalizability in games with a continuum of players.

[2] See, e.g., Bergemann and Morris (2005a, 2005b), Bergemann et al. (2011), and Kunimoto and Serrano (2011). In particular, Bergemann et al. (2011) and Kunimoto and Serrano (2011) considered infinite mechanisms (game forms) for which transfinite rounds of deletion of never-best replies or dominated strategies are necessary.

an aversion to uncertainty or ambiguity and, thereby, motivate generalizations of the subjective expected utility model; see, e.g., Camerer and Weber (1992) and Gilboa and Marinacci (2013) for surveys on recent developments. The notion of "rationality" should, therefore, be extended to accommodate various modes of behavior discussed in economics: The notion of "rationality" can be defined as the maximization with respect to a preference ordering in alternative models of preferences, such as the probabilistic sophistication model (Machina and Schmeidler 1992), the multi-priors model (Gilboa and Schmeidler 1989), the Choquet expected utility model (Schmeidler 1989), the lexicographic preference model (Blume et al. 1991), the Knightian uncertainty model (Bewley 1986).[3] Another important motivation for extending the notion of rationalizability to the case of general preferences is that, in the mechanism design/implementation theory, it is rather natural and standard for us to consider the very general form of preference orderings for individuals in real-life environments; see, e.g., Osborne and Rubinstein (1994, Chapter 10). Subsequently, it leads to the question how to define and analyze the notion of rationalizability in general game situations with different modes of behavior.

Epstein (1997) extended the concept of rationalizability to a variety of general preference models by characterizing rationalizability and survival of iterated deletion of never-best response strategies as the (equivalent) implications of rationality and common knowledge of rationality. In his analysis Epstein offered a "model of preference" to allow for different categories of "regular" preferences such as subjective expected utility, probabilistically sophisticated preference, Choquet expected utility and the multi-priors model. However, from a technical point of view, Epstein's (1997) analysis relies on topological assumptions on the game structure and, in particular, his discussions on rationalizability are restricted to finite games. Apt (2007) relaxed the finite setup of games and studied rationalizability by an iterative procedure, but his analysis implicitly requires players' preferences to have a certain form of expected utility. In this paper we study rationalizable strategic behavior in general situations: arbitrary strategic games with various modes of behavior.

To define the notion of rationalizability, we need to consider a system of preferences/beliefs for restricted parts (or parings) of a game. By using Harsanyi's (1967-68) notion of type, we introduce the simple analytical framework—the "model of situation," which specifies a set of admissible and feasible types for each of players in every possible restriction of game situation. A player, endowed with a type, is able to make a choice decision over his own strategies. Our approach is topology-free and with no measure-theoretic assumptions, and it is applicable to any arbitrary strategic game with different modes of behavior.

In a related paper, Apt and Zvesper (2010) provided a broad and general approach to various forms of customary iterative solution concepts in arbitrary strategic games with a special emphasis on the role of monotonicity in "rationality." Our analysis of rationalizable strategic behavior is, in this respective, harmonious with Apt and Zvesper's (2010) approach. As we have emphasized, this paper focuses on how to extend the notion of rationalizability to general strategic games with various modes

[3] Eichberger and Kelsey (2011) showed that some experimental results which contradict Nash equilibrium can be explained by the hypothesis that subjects view their opponents' behavior as ambiguous.

of behavior that inherits the properties of conventional rationalizability, while Apt and Zvesper's (2010) paper focuses on examining and comparing, in the context of epistemic analysis with possibility correspondences, various forms of customary iterative solution concepts in arbitrary strategic games through the monotonic property of "rationality" behind the iterated dominance notions.

We offer a definition of rationalizability in general situations (Definition 1). Roughly speaking, a set of strategy profiles is regarded as "rationalizable" if every player's strategy in this set can be justified by some of the player's types associated with the set. We show that the set of all rationalizable strategy profiles is the largest (w.r.t. set inclusion) rationalizable set in product form (Proposition 1), which can be derived from an iterated elimination of never-best responses (IENBR). Moreover, IENBR is an order-independent procedure (Proposition 2). In addition, we study the epistemic foundation of rationalizability in general situations: We formulate and prove that rationalizability is the strategic implication of common knowledge of rationality (Proposition 5). We show an equivalence between the notion of rationalizability and the notion of a posteriori equilibrium in general settings (Proposition 6).

In this paper, we also investigate the relationship between rationalizability and Nash equilibrium. We demonstrate through an example that the IENBR procedure may generate spurious Nash equilibrium and, then, offer a necessary and sufficient condition for no spurious Nash equilibria (Proposition 3). In dominance-solvable games, the unique Nash equilibrium can be obtained by IENBR, and moreover, rationalizable strategic behavior in a wide range of preference models is observationally indistinguishable from Nash equilibrium behavior (Proposition 4). We show that, through examples, rationalizability neither implies nor is implied by iterated strict dominance defined by Chen et al. (2007) in general game situations. It is worthwhile to emphasize that one important feature of this paper is that, throughout this paper, we do not require any kind of technical assumptions on the structure of the game or particular strong behavioral assumptions on players' preferences; in particular, we do not require the compactness, convexity, continuity, and measure-theoretic conditions on strategy sets and payoff functions, and we do not even assume that players' preferences have utility function representations.

The rest of this paper is organized as follows. Section 2 is the setup. Sections 3 and 4 present the main results concerning rationalizability with IENBR and Nash equilibrium, respectively. Section 5 discusses the relationship between rationalizability and iterated strict dominance. Section 6 provides the epistemic foundation for rationalizability. Section 7 offers concluding remarks.

2 Setup

Consider an arbitrary strategic game:[4]

[4] We here adopt the conventional game-theory framework which includes the component of payoff functions for players; our analysis of this paper is applicable to a more general framework with players' preference orderings over consequences of the game. (In particular, we keep payoff functions/preference orderings in the framework only for the purpose of discussing the notion of Nash equilibrium in the usual way.) Throughout this paper, we consider only the sets which satisfy the ZFC axioms; see, e.g., Jech (2003, p. 3).

$$G \equiv (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}),$$

where N is an (in)finite set of players, S_i is an (in)finite set of player i's strategies, and $u_i : S \to \mathbb{R}$ is player i's arbitrary payoff function where $S \equiv \times_{i \in N} S_i$. For $s \in S$ let $s \equiv (s_i, s_{-i})$.

The notion of "type" due to Harsanyi (1967-68) is a simple and parsimonious description of the exhaustive uncertainty facing a player, including the player's knowledge, preferences/beliefs. While the notion of "type" was firstly proposed for games with incomplete information, the notion is equally useful to analyze uncertainty about the actual play of the complete-information games—i.e., each player faces uncertainty not only about the primitive uncertainty corresponding to the actual play of opponents, but also the opponents' "types" representing all the relevant characteristics; see, e.g., Brandenburger (2007, pp. 467–468) and Perea (2012, pp. 124–126) for discussions. For the purpose of this paper, we use a generalized notion of "type" (for each player i) that specifies a preference relation over i's strategies when i faces the comprehensive and exhaustive uncertainty in a game. That is, the player, endowed with a type, has one corresponding preference relation over his own strategies, according to which the player can make his choice decision. Formally, we introduce a *model of situation* for game G:

$$\top^G \equiv \{\top_i^G(\cdot)\}_{i \in N},$$

where $\top_i^G(\cdot)$ is defined for every subset $S' \subseteq S$ and every player $i \in N$. (We assume that $\top_i^G(\varnothing) = \varnothing$.) The set $\top_i^G(S')$ is interpreted as player i's *type space* in the reduced game $G|_{S'} \equiv (N, \{S'_i\}_{i \in N}, \{u_i|_{S'}\}_{i \in N})$, where $u_i|_{S'}$ is the payoff function u_i restricted on S'. In other words, $\top_i^G(S')$ is the set of all plausible types of player i when the player faces strategic uncertainty about the other players' actions in $S'_{-i} \equiv \{s_{-i} | (s_i, s_{-i}) \in S'\}$.

Each type $t_i \in \top_i^G(S')$ has a corresponding *preference relation (or binary relation)* $\succsim_{t_i}$ over player i 's strategies in S_i. (The indifference relation, $\sim_{t_i}$, is defined as usual, i.e., $s_i \sim_{t_i} s'_i$ iff $s_i \succsim_{t_i} s'_i$ and $s'_i \succsim_{t_i} s_i$.) For our purpose, we can interpret $\top_i^G(S')$ as the set of player i's plausible conditional preferences given that i's opponents' strategies lie in S'_{-i}. For instance, we may consider $\top_i^G(S')$ as a probability space or a preference space defined on S'.

The following three examples demonstrate that our analytical framework in this paper can be applied to games where the players have different kinds of preferences, including the standard subjective expected utility (SEU) as a special case.

Example 1 Consider a finite game G. Player i's belief about the strategies that the opponents play in the reduced game $G|_{S'}$ is defined as a probability distribution μ_i over S'_{-i}, i.e., $\mu_i \in \Delta\left(S'_{-i}\right)$ where $\Delta\left(S'_{-i}\right)$ is the set of probability distributions over S'_{-i}. For any μ_i, the expected payoff of s_i can be calculated by

$$U_i(s_i, \mu_i) = \sum_{s_{-i} \in S'_{-i}} \mu_i(s_{-i}) \cdot u_i(s_i, s_{-i})$$

where $\mu_i(s_{-i})$ is the probability assigned by μ_i to s_{-i}. That is, μ_i generates an SEU preference over S_i. For our purpose we define a model of situation (on G) as follows:

$$\top^G = \{\top_i^G(\cdot)\}_{i \in N},$$

where, for every player $i \in N$, $\top_i^G(S') = \Delta\left(S'_{-i}\right)$ for every subset $S' \subseteq S$. Note that the beliefs are "correlated" in the sense that a belief is represented by a joint probability distribution over the opponents' strategies. The model of situation allows representing player's beliefs as product (independent) or degenerated (point) probability distributions over opponents' strategies.[5]

Example 2 Consider a finite game G. An act on S'_{-i} is a real-valued function defined on S'_{-i}. Let $\mathcal{F}\left(S'_{-i}\right)$ denote the set of acts on S'_{-i}. Let $\mathcal{P}(\cdot)$ be a model of preference defined in Epstein (1997, pp. 6–7). That is, $\mathcal{P}(\cdot)$ is a correspondence mapping each S'_{-i} to a nonempty set of utility functions over $\mathcal{F}\left(S'_{-i}\right)$ such that for every $U_i \in \mathcal{P}\left(S'_{-i}\right)$, $U_i(r) = r$ for each constant act $r \in \mathbb{R}$ and $U_i(f) \geq U_i\left(f'\right)$ whenever $f \geq f'$. (With the topological assumption on the game structure: compact Hausdorff strategy sets, Epstein and Wang (1996) constructed a type space for a wide class of "regular" preferences, including the standard SEU preference as a special case.) Observe that each strategy s_i can be identified with an act $u_i(s_i, \cdot)$ on S'_{-i}. We define a model of situation (on G) as follows:

$$\top^G = \{\top_i^G(\cdot)\}_{i \in N},$$

where, for each player $i \in N$, $\top_i^G(S') = \mathcal{P}\left(S'_{-i}\right)$ for every subset $S' \subseteq S$. From this perspective, the standard assumption in game theory that players are SEU maximizers corresponds to the restriction that players' preferences lie in a suitable subset of $\mathcal{P}\left(S'_{-i}\right)$. Thus, our framework presented in this paper can be used for analyzing strategic behavior in games with various models of preference discussed in Epstein (1997).

Example 3 Consider a game G where each strategy set S_i is a measurable space with algebra $\mathcal{S}_i$ and u_i is a bounded measurable function with respect to the product algebra on S. Player i's belief about the strategies that the opponents play in the reduced game $G|_{S'}$ is defined as a probability measure μ_i over S'_{-i} where S'_{-i} is also endowed with the product algebra. For any μ_i, the expected payoff of s_i can be calculated by

$$U_i(s_i, \mu_i) = \int_{S'_{-i}} u_i(s_i, s_{-i})\, d\mu_i(s_{-i}).$$

That is, μ_i generates a topology-free SEU preference over S_i. [Heifetz and Samet (1998) showed that, only with the measure-theoretic assumption on strategy sets, a

5 In the game-theory literature, players are typically assumed to be Bayesian rational, that is, each player forms a prior over the space of states of the world and maximizes the expected value of some fixed vNM index on outcomes. The model of situation also allows representing player's beliefs as other forms of subjective expected utility preferences such as Borgers's (1993) ordinal expected utility and the state-dependent utility preferences discussed in Morris and Takahashi (2011).

type space can be explicitly constructed in such an environment.] We define a model of situation (on G) as follows:

$$\top^G = \{\top_i^G(\cdot)\}_{i \in N},$$

where, for each player $i \in N$, $\top_i^G(S') = ba\left(S'_{-i}\right)$ for every subset $S' \subseteq S$ where $ba\left(S'_{-i}\right)$ is the set of finitely additive probability measures over S'_{-i}. That is, the model of situation can be used to analyze games with topology-free SEU preferences.

In the framework of this paper, we impose no essential behavioral assumption on preferences; in particular, we do not assume that preferences have utility function representations. Our approach is applicable to most of preference models discussed in the literature such as the SEU model (Savage 1954), the OEU model (Borgers 1993), the probabilistic sophistication model (Machina and Schmeidler 1992), the multi-priors model (Gilboa and Schmeidler 1989), the Choquet expected utility model (Schmeidler 1989), the lexicographic preference model (Blume et al. 1991), the Knightian uncertainty model (Bewley 1986). For $t_i \in \top_i^G(S')$, a strategy $s_i \in S_i$ is *one of most preferred actions for* t_i if $s_i \succsim_{t_i} s_i'$ for all $s_i' \in S_i$. (Notice that even if a reduced game $G|_{S'}$ is concerned, any strategy of player i in the original game G can be a candidate for the most preferred choices.) Let

$$\beta(t_i) \equiv \left\{s_i \in S_i | s_i \succsim_{t_i} s_i' \quad \text{for all } s_i' \in S_i\right\}.$$

We first present a definition of rationalizability in a game G with the situation model $\top^G$. The spirit of this definition is that for every strategy in a rationalizable set, the player can always find some of types defined over this set to support his choice of strategy.[6]

Definition 1 A subset $R \subseteq S$ is *a rationalizable set in* $\top^G$ if $\forall i$ and $\forall s \in R$, there exists some $t_i \in \top_i^G(R)$ such that $s_i \in \beta(t_i)$. We denote the collection of all rationalizable sets in $\top^G$ as $\mathcal{R}\left(\top^G\right)$.

In Definition 1, a rationalizable set satisfies the "internally" consistent property: A rationalizable set consists of its best responses, but not all its best responses are required to be included in the rationalizable set. Proposition 5(1) in Section 6 shows that the assumption of "common knowledge of rationality" leads to a rationalizable set. We note that the concept of a rationalizable set is related to Basu and Weibull's (1991) concept of a "closed under rational behavior (curb)" set which, in contrast, satisfies an "externally" consistent property—i.e., a curb set must contain all its best responses, but the curb set may contain something that is not a best response.[7] For our discussions, we need the following condition on the model of situation $\top^G$.

[6] Definition 1 can be viewed as a generalization of the best response property; indeed, we may define $\beta(t_i)$ as a choice set for type t_i, so that it can be used to model and analyze different behavioral patterns and decision rules for t_i. Throughout this paper, for simplicity we focus on pure strategies; we can apply our analytical framework to the mixed extensions of finite games by allowing for using mixed strategies in finite games.

[7] We thank a referee for drawing our attention to Basu and Weibull's concept of the curb set. It is easy to verify that the largest rationalizable set R^* (in Proposition 1) is a curb set.

C1 (Monotonicity) $\forall i$, $\top_i^G(S') \subseteq \top_i^G(S'')$ if $S' \subseteq S''$.

The monotonicity condition requires that when one player faces a greater degree of strategic uncertainty, the player possesses more types available for resolving uncertainty. In the literature on information economics, a type of a player is interpreted as the initial private information, about all the uncertainty regarding the state of nature in a game situation, that player has. From this point of view, it is natural to assume that there are more types available if there is more strategic uncertainty about the choices of the players. Most of models discussed in the literature satisfy the monotonicity condition C1; in particular, it is easy to verify that the models of situation defined in Examples 1–3 satisfy C1. One prominent example that violates C1 is the iterated elimination of weakly dominated strategies (IEWDS): A weakly dominated strategy in a game may no longer be a weakly dominated strategy in the reduced game after eliminating some strategies.

We call a strategy profile rationalizable in $\top^G$ if this profile lies in a rationalizable set in $\top^G$. Let $R^* \equiv \cup_{R \in \mathcal{R}(\top^G)} R$ be the set of all rationalizable strategy profiles in $\top^G$. The following proposition asserts that, under C1, there is the largest (w.r.t. set inclusion) rationalizable set in product form, which consists of all the rationalizable strategy profiles.

Proposition 1 *Under C1, the set of all rationalizable strategy profiles in* $\top^G$ *is the largest (product form) rationalizable set in* $\top^G$.

Proof It suffices to show that $R^* \equiv \cup_{R \in \mathcal{R}(\top^G)} R$ is a rationalizable set in $\top^G$. Let $s \in R^*$. Then, there exists a rationalizable set R in $\top^G$ such that $s \in R$. Thus, for every player i, there exists some $t_i \in \top_i^G(R)$ such that $s_i \in \beta(t_i)$. Since $R \subseteq R^*$, by C1, $t_i \in \top_i^G(R^*)$ and $s_i \in \beta(t_i)$. Thus, R^* is a rationalizable set in $\top^G$.

Let $s \in \times_{i \in N} R_i^*$ where $R_i^* \equiv \{s_i | s \in R^*\}$. Then, for every player i, there exists $t_i \in \top_i^G(R^*)$ such that $s_i \in \beta(t_i)$. Since $R^* \subseteq \times_{i \in N} R_i^*$, again by C1, $t_i \in \top_i^G(\times_{i \in N} R_i^*)$ and $s_i \in \beta(t_i)$. That is, $\times_{i \in N} R_i^*$ is a rationalizable set in $\top^G$. Since $R^* \equiv \cup_{R \in \mathcal{R}(\top^G)} R$, it must be the case that $R^* = \times_{i \in N} R_i^*$. □

Remark For $Z \subseteq S$ let $\varphi(Z) = \times_{i \in N} \left\{ s_i | s_i \in \beta(t_i) \text{ for some } t_i \in \top_i^G(Z) \right\}$. Then, R^* is the largest fixed point of the monotonic operator $\varphi : 2^S \mapsto 2^S$. See Apt and Zvesper (2010) and Luo (2001, Sect. 4.1) for a general approach to rationalizability-like solution concepts by using Tarski's fixed point theorem on complete lattices; see also Brandenburger et al. (2011) for related discussions.

3 IENBR and rationalizability

In the literature, rationalizability can also be defined as the outcome of an iterated elimination of never-best responses. We employ a possibly transfinite elimination process that can be used for any arbitrary game.[8] Let λ^0 denote the first element in an

[8] Lipman (1994) demonstrated that, in infinite games, we may need the transfinite induction to analyze the strategic implication of "common knowledge of rationality." See also Chen et al.'s (2007) Example 1 for

ordinal Λ, and let $\lambda + 1$ denote the successor to λ in Λ. For $S'' \subseteq S' \subseteq S$, S' is said to be *reduced to* S'' (denoted by $S' \to S''$) if, $\forall s \in S' \backslash S''$, there exists some player i such that $s_i \notin \beta(t_i)$ for any $t_i \in \top_i^G(S')$. Note: $S' \to S'$ for any $S' \subseteq S$.

Definition 2 An *iterated elimination of never-best responses (IENBR)* is a family $\{R^\lambda\}_{\lambda \in \Lambda}$ such that

(a) $R^{\lambda^0} = S$,
(b) $R^\lambda \to R^{\lambda+1}$ (and $R^\lambda = \cap_{\lambda' < \lambda} R^{\lambda'}$ for a limit ordinal λ), and
(c) $R^\infty \equiv \cap_{\lambda \in \Lambda} R^\lambda \to S'$ (where $S' \subseteq S$) only if $S' = R^\infty$.

Definition 2(c) can be viewed as the "stopping" condition for the IENBR procedure: There is no element in the outcome R^∞ which can be eliminated for further consideration. Note that the definition of IENBR procedure does not require the elimination of *all* never-best response strategies in each round of elimination. This flexibility raises a question whether the IENBR procedure always results in a unique set of rationalizable outcomes. The following proposition asserts that, for any given arbitrary game, there exists an IENBR procedure defined in Definition 2, and furthermore, Definitions 1 and 2 are equivalent.

Proposition 2 *In a model of situation* $\top^G$*, there is an IENBR procedure, and moreover, under C1,* $R^\infty = R^*$ *for any IENBR procedure* $\{R^\lambda\}_{\lambda \in \Lambda}$*—i.e., IENBR is an order-independent procedure.*

Proof [9]We first show, for any arbitrary game G, that there is an IENBR procedure in $\top^G$. Let $S' \to^* S''$ denote "$S' \to S''$ and $S' \neq S''$." By using Transfinite Induction [see, e.g., Jech (2003, p. 21)], we define a quasi-procedure $\{Q^\lambda\}_{\lambda \leq \Lambda}$ as: There exists an ordinal Λ such that

(i) $Q^{\lambda^0} = S$, and
(ii) $Q^\lambda \to^* Q^{\lambda+1}$ (and $Q^\lambda = \cap_{\lambda' < \lambda} Q^{\lambda'}$ for a limit ordinal $\lambda \leq \Lambda$).

Let $\mathcal{Q}$ be the (nonempty) set of all the quasi-procedures.[10] We define a binary relation $\preccurlyeq$ on $\mathcal{Q}$ as:

$$\{Q^\lambda\}_{\lambda \leq \Lambda} \preccurlyeq \left\{\overline{Q}^\lambda\right\}_{\lambda \leq \overline{\Lambda}} \quad \text{if} \quad \Lambda \leq \overline{\Lambda} \text{ and } Q^\lambda = \overline{Q}^\lambda \quad \text{for all} \quad \lambda \leq \Lambda.$$

It is easy to verify that $(\mathcal{Q}, \preccurlyeq)$ is a partially ordered set. (For the transitivity, for example, assume that $\{Q^\lambda\}_{\lambda \leq \Lambda} \preccurlyeq \{\widetilde{Q}^\lambda\}_{\lambda \leq \widetilde{\Lambda}}$ and $\{\widetilde{Q}^\lambda\}_{\lambda \leq \widetilde{\Lambda}} \preccurlyeq \left\{\overline{Q}^\lambda\right\}_{\lambda \leq \overline{\Lambda}}$. Then,

Footnote 8 continued
the reason why we may need a transfinite process for iterated deletion of strictly dominated strategies in general games.

[9] In fact, we can alternatively construct a concrete IENBR procedure similar to one constructed in Chen et al. (2007). We thank a referee for providing us with useful comments and suggestions that lead to this proof for the existence of an IENBR procedure.

[10] Note that each quasi-procedure in $\mathcal{Q}$ can be viewed as an element, which satisfies the property (i)–(ii), in the power set of 2^S. By the Axiom Schema of Separation [see, e.g., Jech (2003, p. 3)], $\mathcal{Q}$ is a set in ZFC.

$\{Q^\lambda\}_{\lambda \leq \Lambda} \preccurlyeq \left\{\overline{Q}^\lambda\right\}_{\lambda \leq \overline{\Lambda}}$ since $\Lambda \leq \widetilde{\Lambda} \leq \overline{\Lambda}$ and $Q^\lambda = \widetilde{Q}^\lambda = \overline{Q}^\lambda$ for all $\lambda \leq \Lambda$). Note that, if $\{Q^\lambda\}_{\lambda \leq \Lambda}$ is a maximal element in $(\mathcal{Q}, \preccurlyeq)$, no element can be eliminated from Q^Λ, which implies that Definition 2(c) is satisfied. Therefore, it suffices to prove that there exists a maximal element in $(\mathcal{Q}, \preccurlyeq)$ for the existence of an IENBR procedure.

Consider a chain $\mathcal{C}$ in $(\mathcal{Q}, \preccurlyeq)$. Let $\Lambda_\mathcal{C} \equiv \left\{\Lambda | \{Q^\lambda\}_{\lambda \leq \Lambda} \in \mathcal{C}\right\}$ and $\Lambda^* \equiv \cup \Lambda_\mathcal{C}$. By Jech's (2003, p. 20) (2.4), $\Lambda^* = \sup \Lambda_\mathcal{C}$ is an ordinal. For arbitrary quasi-procedures $\{Q^\lambda\}_{\lambda \leq \Lambda}$ and $\{\widetilde{Q}^\lambda\}_{\lambda \leq \widetilde{\Lambda}}$ in $\mathcal{C}$, $Q^\lambda = \widetilde{Q}^\lambda \equiv \overline{Q}^\lambda$ if $\lambda \leq \Lambda$ and $\lambda \leq \widetilde{\Lambda}$, and hence, there is a unique $\overline{Q}^\lambda$ for any $\lambda \in \Lambda^*$. Moreover, if Λ^* is not a limit ordinal, then $\Lambda^* \in \Lambda_\mathcal{C}$ and $\overline{Q}^{\Lambda^*}$ is also determined in this way. Let $\overline{Q}^{\Lambda^*} \equiv \cap_{\lambda \in \Lambda^*} \overline{Q}^\lambda$ if Λ^* is a limit ordinal. Thus, we obtain a quasi-procedure $\left\{\overline{Q}^\lambda\right\}_{\lambda \leq \Lambda^*}$. Because $\Lambda \leq \Lambda^*$, $\{Q^\lambda\}_{\lambda \leq \Lambda} \preccurlyeq \left\{\overline{Q}^\lambda\right\}_{\lambda \leq \Lambda^*}$ for all $\{Q^\lambda\}_{\lambda \leq \Lambda} \in \mathcal{C}$. That is, we find an upper bound $\left\{\overline{Q}^\lambda\right\}_{\lambda \leq \Lambda^*} \in \mathcal{Q}$ for $\mathcal{C}$. By Zorn's lemma [see, e.g., Jech (2003, p. 49)], there exists a maximal element in $(\mathcal{Q}, \preccurlyeq)$, which generates an IENBR procedure in $\top^G$.

Now, we consider an arbitrary IENBR procedure $\{R^\lambda\}_{\lambda \in \Lambda}$. By Definition 2, $\forall s \in R^\infty$, every player i has some $t_i \in \top_i^G(R^\infty)$ such that $s_i \in \beta(t_i)$. So R^∞ is a rationalizable set, and hence, $R^\infty \subseteq R^*$. Under C1, by Proposition 1, R^* is a rationalizable set in $\top^G$ and, hence, survives every round of elimination in Definition 2. So $R^* \subseteq R^\infty$. That is, $R^\infty = R^*$ for any IENBR procedure $\{R^\lambda\}_{\lambda \in \Lambda}$. □

Remark It is worthwhile to note that if $\beta_i(t_i) = \varnothing$ for each type t_i, then Definition 2 implies that the IENBR procedure ends up with the empty set (as we assume that $\top_i^G(\varnothing) = \varnothing$). For example, if one player has a dominant strategy but some of the other players has no best response, Definition 2 implies that the set of rationalizable strategies is empty. On a conceptual level, a player's rationalizable strategy should be justified by the opponents' rationalizable strategies. That is, the definition of a player's rationalizable strategy must be dependent on the definition of the other players' rationalizable strategies. Therefore, there is no rationalizable strategies even if one player has an obvious action of play in this circumstance.[11]

4 Nash equilibrium and rationalizability

Recall that a strategy profile s^* in S is a *(pure) Nash equilibrium* in G if for every player i,

$$u_i\left(s^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \quad \forall s_i \in S_i.$$

To study the relationship between Nash equilibrium and rationalizability, we need a weak consistency requirement between payoff functions and the preferences of types.[12] For strategy profile $s \in S$, player i's *Dirac type* $\delta_i(s)$ is a type with the property:

[11] We thank a referee for pointing out this to us.

[12] In this paper, we impose no essential condition for the relationship between the preference relation $\succsim_{t_i}$ of a type t_i and the payoff function u_i—i.e., the only link between payoff functions and the preferences

$$\forall s_i', s_i'' \in S_i, u_i(s_i', s_{-i}) \geq u_i(s_i'', s_{-i}) \quad \text{iff} \quad s_i' \succsim_{\delta_i(s)} s_i''.$$

A Dirac type $\delta_i\,(s)$ is a degenerated type with which player i behaves as if he faces a certain play s_{-i} of the opponents; in probabilistic models, a Dirac type is a point mass that represents a point belief about the opponents' choices. Observe that s^* is a Nash equilibrium iff, for every player i, s_i^* is a best response to $\delta_i\,(s^*)$. The following condition requires that the only possible type for a deterministic case—i.e., a singleton of a certain play of the opponents—is a Dirac type. This condition is a rather natural requirement when strategic uncertainty is reduced to a special deterministic case of certainty.

C2 (Diracability) $\forall i, \top_i^G(\{s\}) = \{\delta_i\,(s)\}$ for all $s \in S$.

It is easy to verify that various models of situation defined in Examples 1–3 satisfy C2. C1 and C2 jointly imply that $\delta_i\,(s) \in \top_i^G(S')$ if $s \in S'$, i.e., the type space on S' contains all the possible Dirac types on S'. Observe that, under C2, every Nash equilibrium is a rationalizable strategy profile. Propositions 1 and 2 imply that, under C1 and C2, every Nash equilibrium survives IENBR and, if G admits a Nash equilibrium, the IENBR procedure yields a nonempty set of rationalizable outcomes: $R^\infty = R^*$. The following example taken from Chen et al. (2007) demonstrates that a Nash equilibrium in the reduced game after an IENBR procedure may be a spurious Nash equilibrium, i.e., it is not a Nash equilibrium in the original game.

Example 4 Consider a two-person symmetric game:[13] $G \equiv \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1, 2\}$, $S_1 = S_2 = [0, 1]$, and for all $s_i, s_j \in [0, 1]$, $i, j = 1, 2$, and $i \neq j$

$$u_i(s_i, s_j) = \begin{cases} 1, & \text{if } s_i \in [1/2, 1] \text{ and } s_j \in [1/2, 1]\,, \\ 1 + s_i, & \text{if } s_i \in [0, 1/2) \text{ and } s_j \in (2/3, 5/6)\,, \\ s_i, & \text{otherwise.} \end{cases}$$

We consider the standard SEU model for G. It is easily verified that $R^\infty = [1/2, 1] \times [1/2, 1]$ since every strategy $s_i \in [0, 1/2)$ is strictly dominated and hence never a best response. That is, IENBR leaves the reduced game $G|_{R^\infty} \equiv (N, \left\{R_i^\infty\right\}_{i\in N}, \{u_i|_{R^\infty}\}_{i\in N})$ that cannot be further reduced. Clearly, R^∞ is the set of Nash equilibria in the reduced game $G|_{R^\infty}$, but the set of Nash equilibria in game G is $\{s \in R^\infty | s_1, s_2 \notin (2/3, 5/6)\}$. Thus, IENBR generates spurious Nash equilibria $s \in R^\infty$ where some $s_i \in (2/3, 5/6)$. Observe that in G, $u_i\left(., s_j\right)$ has a maximizer for $s_j \notin (2/3, 5/6)$, but $u_i\left(., s_j\right)$ has no maximizer for $s_j \in (2/3, 5/6)$. That is, some player has no best response in such a spurious Nash equilibrium, while each player should have a best response in a (normal) Nash equilibrium.

Definition 3 $G \equiv (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ *has well-defined best responses on* $S' \subseteq S$ if $\forall i \in N$ and $\forall s \in S'$, $\beta\,(\delta_i\,(s)) \neq \varnothing$.

Let NE denote the set of Nash equilibria in G, and $NE|_{R^\infty}$ the set of Nash equilibria in the reduced game $G|_{R^\infty} \equiv (N, \{R_i^\infty\}_{i\in N}, \{u_i|_{R^\infty}\}_{i\in N})$. A sufficient and necessary

Footnote 12 continued
of types is given by the very weak Diracability condition (C2) in this section and the strong monotonicity condition (C3) in Sect. 5.

[13] The game of this example is in the class of Reny's (1999) better-reply secure games.

condition under which IENBR generates no spurious Nash equilibria is provided as follows.

Proposition 3 *Under C1 and C2,* $NE = NE|_{R^\infty}$ *iff G has well-defined best responses on* $NE|_{R^\infty}$.

Proof ("Only if" part.) Let $s^* \in NE|_{R^\infty}$. Since $NE|_{R^\infty} = NE$, $s_i^* \in \beta(\delta_i(s^*))\ \forall i$. Thus, $\beta(\delta_i(s^*)) \neq \varnothing$ for all i.

("If" part.) (i) Let $s^* \in NE$. Under C1 and C2, by Propositions 1 and 2, $s^* \in R^\infty$, and hence, $s^* \in NE|_{R^\infty}$. So $NE \subseteq NE|_{R^\infty}$. (ii) Let $s^* \in NE|_{R^\infty}$. Since G has well-defined best responses on $NE|_{R^\infty}$, for every player i there exists some $\hat{s}_i \in S_i$ such that $\hat{s}_i \in \beta(\delta_i(s^*))$, which implies that $\hat{s}_i \succsim_{\delta_i(s^*)} s_i^*$ and $(\hat{s}_i, s_{-i}^*) \in R^\infty$ under C1 and C2. Since $s^* \in NE|_{R^\infty}$, $s_i^* \succsim_{\delta_i(s^*)} \hat{s}_i$. Therefore, $s_i^* \sim_{\delta_i(s^*)} \hat{s}_i$, and hence, $s_i^* \in \beta(\delta_i(s^*))$. That is, $s^* \in NE$. So $NE|_{R^\infty} \subseteq NE$. □

This sufficient and necessary condition in Proposition 3 does not involve any topological assumption on the original or the reduced games. In Chen et al.'s (2007) Corollary 4, some classes of games with special topological structures were proved to "preserve Nash equilibria" for the iterated elimination of strictly dominated strategies. These results are also applicable to the IENBR procedure defined in this paper. If a game is solvable by an IENBR procedure, the following corollary asserts that the unique strategy profile surviving the procedure is the only Nash equilibrium.

Corollary 1 *Under C1 and C2,* $R^\infty = NE$ *if* $|R^\infty| = 1$.

Proof Let $R^\infty = \{s^*\}$. By C2, s_i^* is a best response to $\delta_i(s^*)$ for every player i. So $s^* \in NE$, and hence, $R^\infty \subseteq NE$. By Proposition 1, $NE \subseteq R^\infty$. □

5 Rationalizability and iterated strict dominance

In this section, we show that, through examples, rationalizability in general game situations neither implies nor is implied by iterated strict dominance defined by Chen et al. (2007). This is because, in the general environments, an undominated strategy need not be a best response in a model of situation, and conversely, a best response in a model of situation is not necessarily undominated, even in the case of (correlated) probabilistic models.

Since the model of situation can be applied to some particular class of probabilistic models such as the product (independent) probability model and the degenerated (point) probability model, it is easy to see that an undominated strategy may fail to be a best response in finite games with such restrictive types of probabilistic beliefs; see, e.g., Brandenburger and Dekel (1987, Sect. 3). Alternatively, the following example (due to Andrew Postlewaite), which appears in Bergemann and Morris (2005a, Footnote 8), shows that an undominated strategy need not be a best response in an infinite game with (correlated) probabilistic beliefs.[14]

[14] For simplicity, we here consider strategies dominated by pure strategies. Examples 5 and 6 are still valid if we allow mixed strategies.

Example 5 Consider a two-person symmetric game $G = \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1, 2\}$, and for $i = 1, 2$, $S_i = \{0, 1, 2, ...\}$ and

$$u_i(s_i, s_{-i}) = \begin{cases} 1, & \text{if } s_i = 0; \\ 2, & \text{if } s_i \geq 1 \text{ and } s_i > s_{-i}; \\ 0, & \text{if } s_i \geq 1 \text{ and } s_i \leq s_{-i}. \end{cases}$$

Let $\top^G$ be the model of situation generated by expected utility preferences with (countably additive) probability measures, i.e., $\top_i^G(S') = \Delta\left(S'_{-i}\right)$ for all $S' \subseteq S$. Clearly, $s_i = 0$ is not strictly dominated, because for any $s_i \geq 1$, $u_i(0, s_{-i}) = 1 > 0 = u_i(s_i, s_{-i})$ for $s_{-i} \geq s_i$. But, $s_i = 0$ cannot be a best response in $\top^G$. To see this, note that for any $\mu_i \in \Delta(S_{-i})$ and any $s_i > 0$,

$$\int u_i(s_i, s_{-i})\, d\mu_i(s_{-i}) = 2\mu_i(\{s_{-i} | s_{-i} < s_i\}) \to 2 \quad \text{as } s_i \to \infty.$$

Hence, there is some $s_i > 0$ such that $\int u_i(s_i, s_{-i})\, d\mu_i(s_{-i}) > 1 = \int u_i(0, s_{-i})\, d\mu_i(s_{-i})$.

The following example, which is modified from Stinchcombe (1997), shows that a strictly dominated strategy can be a best response in a game with "finitely additive" probabilistic beliefs.

Example 6 Consider a two-person game $G = \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1, 2\}$, $S_1 = \{0, 1\}$, $S_2 = \{1, 2, ...\}$, and $u_1(0, s_2) = 0$, $u_1(1, s_2) = 1/s_2$, $u_2(s_2, 0) = 0$, and $u_2(s_2, 1) = s_2$ for all $s_2 \in S_2$. Let the algebra on S_2 be the power set of S_2. Let $\top^G$ be the model of situation generated by expected utility preferences generated by *finitely additive* probability charges, i.e., $\top_i^G(S') = ba\left(S'_{-i}\right)$ for all $S' \subseteq S$, where $ba\left(S'_{-i}\right)$ is the space of finitely additive probability charges on S'_{-i}.

Clearly, $s_1 = 0$ is strictly dominated by $s_1 = 1$. However, $s_1 = 0$ can be a best response in $\top^G$. To see this, it suffices to show that $\int u_1(1, s_2)\, d\mu(s_2) = \int u_1(0, s_2)\, d\mu(s_2) = 0$ for some $\mu \in ba(S_2)$. To find such a μ, let $\mu_m \in ba(S_2)$ be the uniform distribution on $\{1, 2, ..., m\}$. By Alaoglu's Theorem [see, e.g., Royden (1968, Theorem 10.17)], there are $\mu \in ba(S_2)$ and a sequence of $\{\mu_m\}$ such that

$$\lim_{m\to\infty} \mu_m(E) = \mu(E) \quad \text{for each } E \subseteq \mathbb{N}. \tag{1}$$

Since $u_1(0, s_2) = 0$ for all s_2, it follows that $\int u_1(0, s_2)\, d\mu(s_2) = 0$. To see $\int u_1(1, s_2)\, d\mu(s_2) = 0$, observe that for every $K \geq 1$,

$$0 \leq \int u_1(1, s_2)\, d\mu(s_2) \leq \sum_{s_2=1}^{K} \mu(\{s_2\}) + \frac{1}{K}\mu(\{s_2 : s_2 > K\}). \tag{2}$$

Since $\mu_m(\{s_2\}) \to 0$ and $\mu_m(\{s_2 : s_2 > K\}) \to 1$, it follows from (1) that $\mu(\{s_2\}) = 0$ and $\mu(\{s_2 : s_2 > K\}) = 1$. Since (2) holds for all $K \geq 1$, $\int u_1(1, s_2)\, d\mu(s_2) = 0$.

A strategy $s_i \in S_i$ is said to be *dominated given* $S' \subseteq S$ if for some strategy $\widehat{s}_i \in S_i$, $u_i(\widehat{s}_i, s'_{-i}) > u_i(s_i, s'_{-i})$ for all $s'_{-i} \in S'_{-i}$. (This definition of strict dominance is applicable to the standard one used in finite games—i.e., a pure strategy is allowed to be strictly dominated by some randomized or mixed strategy—by considering the "mixed extensions" of finite games instead.) For any subsets $S', S'' \subseteq S$ where $S'' \subseteq S'$, we use the notation $S' \mapsto S''$ to signify that for any $s \in S' \backslash S''$, some s_i is dominated given S'. In general games, Chen et al. (2007) offered the well-defined order-independent iterated elimination of strictly dominated strategies. Let λ^0 denote the first element in an ordinal Λ, and let $\lambda + 1$ denote the successor to λ in Λ.

Definition 4 An *iterated elimination of strictly dominated strategies (IESDS*)* is defined as a family $\{\mathcal{D}^\lambda\}_{\lambda \in \Lambda}$ such that

(a) $\mathcal{D}^{\lambda^0} = S$,
(b) $\mathcal{D}^{\lambda} \mapsto \mathcal{D}^{\lambda+1}$ (and $\mathcal{D}^{\lambda} = \cap_{\lambda' < \lambda} \mathcal{D}^{\lambda'}$ for a limit ordinal λ), and
(c) $\mathcal{D} \equiv \cap_{\lambda \in \Lambda} \mathcal{D}^{\lambda} \mapsto S'$ (where $S' \subseteq S$) only if $S' = \mathcal{D}$.

Next, we present an equivalence result between rationalizability and IESDS* in the class of dominance-solvable games. We say that a game G is "dominance solvable" if the procedure of IESDS* leads to a unique strategy profile—i.e., by performing the procedure of iterated elimination of strictly dominated strategies, there is only one strategy left for each player; for example, the standard Cournot game (Moulin 1984), Bertrand oligopoly with differentiated products, and the arms-race games (Milgrom and Roberts 1990). We need the following condition on a situation model $\top^G$.

C3 (Strong Monotonicity) If a strategy $\widehat{s}_i \in S_i$ strictly dominates another strategy $s_i \in S_i$ given S'—i.e., $u_i\left(\widehat{s}_i, s'_{-i}\right) > u_i\left(s_i, s'_{-i}\right)\ \forall s'_{-i} \in S'_{-i}$, then $\widehat{s}_i \succ_{t_i} s_i$ for all $t_i \in \top_i^G(S')$.

The strong monotonicity requires that a strategy be strictly preferred to another strategy if the former strategy strictly payoff-dominates the latter one. This condition on $\top^G$ seems to be rather natural and is satisfied by most of preference models discussed in the literature, e.g., the SEU model, the OEU model, the probabilistic sophistication model, the multi-priors model, the Choquet expected utility model, the lexicographic preference model, the Knightian uncertainty model.[15] From a decision-theoretic point of view, the "transitivity" or "strong monotonicity" condition is considered to be more basic tenets of rationality than the sure-thing principle and other components of the standard Savage model; see Luce and Raiffa (1957, Chapter 13) and Epstein (1997) for more discussions. The following proposition asserts that in dominance-solvable games, the notion of rationalizability defined in any situation model $\top^G$ satisfying Diracability and strong monotonicity (but not necessarily satisfying monotonicity) is equivalent to the Nash equilibrium, which can be solved by IESDS*.

Proposition 4 *Suppose that G is a dominance-solvable game with a situation model $\top^G$ satisfying C2 and C3. Then, $\mathcal{D} = R^* = NE$.*

[15] Nevertheless, as demonstrated in Example 6, the expected utility preference model with a finitely additive probability charge may violate C3.

Proof Since G is dominance solvable, $\mathcal{D} = NE$. Let R be a rationalizable set in $\top^G$. Then, by C3, R is an undominated set—i.e., for every i, $s_i \in R_i$ is not dominated given R. Therefore, $R \subseteq \mathcal{D}^\lambda$ for all λ, and hence, $R^* \subseteq \mathcal{D} = NE$. By C2, the singleton of a Nash equilibrium is a rationalizable set in $\top^G$. Consequently, $R^* = \mathcal{D} = NE$. □

Remark Proposition 4 says that in dominance-solvable games, rationalizability defined in any situation model $\top^G$ satisfying C2 and C3 yields the unique set of outcomes of iterated strict dominance, which is consistent with the Nash equilibrium outcome. The result implies that the Nash equilibrium behavior is observationally indistinguishable from the rationalizable strategic behavior in such situation models.[16] Proposition 4 also implies that IESDS* generates no spurious Nash equilibria in dominance-solvable games.

6 Epistemic conditions of rationalizability

In this section we provide epistemic conditions for rationalizability in general games. In doing epistemic analysis, we need to extend the model of situation in Sect. 2 to the space of states. Consider a space Ω of *states*, with typical element $\omega \in \Omega$. A subset $E \subseteq \Omega$ is referred to as an *event*. A *model of situation on* Ω is given by

$$\top \equiv \{\top_i(\cdot)\}_{i \in N},$$

where $\top_i(\cdot)$ is defined over subsets $E \subseteq \Omega$. The set $\top_i(E)$ is player i's type space for given event E, which can be interpreted as player i's type space conditional on event E; each type $t_i \in \top_i(E)$ has a preference relation $\succsim_{t_i}$ defined on player i's strategies in S_i under which the complement of E is regarded as impossible. (As usual we assume that $\top_i(\varnothing) = \varnothing$.) For example, if $\top_i(E)$ is applied to the case of the probability measure space, $\top_i(E)$ can be considered as the space of probability measures conditional on subset E of Ω. The model of situation on Ω can also be viewed as a type structure used in epistemic game theory to model interactive beliefs in which a type of a player is a joint belief about the states of nature and the types of the other players [(see, e.g., Brandenburger (2007)].[17]

[16] Chen and Luo (2012) showed that rationalizability under general preferences can be indistinguishable from the outcome of the IESDS procedure for a class of (in)finite games where each player's strategy space is compact Hausdorff and each player's payoff function is continuous and "concave-like." The indistinguishability result in Proposition 4 does not rely on the structure of strategic game.

[17] We take a point of view that an epistemic model is a pragmatic and convenient framework to be used for doing epistemic analysis; see Aumann and Brandenburger (1995, Sect. 7a) for related discussions. There are many examples of well-defined type spaces: Mertens and Zamir (1985) constructed a compact Hausdorff type space, Brandenburger and Dekel (1993) constructed a Polish type space, Heifetz and Samet (1998) provided an alternative "topology-free" construction of type space, and Epstein and Wang (1996) constructed a compact Hausdorff (nonprobabilistic) type space in a setting of "regular" preferences. In this paper, we are mainly concerned with the analysis of the game-theoretic solution concept of rationalizability in general situations. In particular, we do not assume that preferences have utility function representations.

An *epistemic model for* $\top^G$ is defined by

$$\mathcal{M}\left(\top^G\right) \equiv (\Omega, \top, \{\mathbf{s}_i\}_{i\in N}, \{\mathbf{t}_i\}_{i\in N}),$$

where Ω is the space of states, $\top$ is a model of situation on Ω, $\mathbf{s}_i(\omega) \in S_i$ is player i's strategy at state ω, and $\mathbf{t}_i(\omega) \in \top_i(\Omega)$ is player i's type at state ω; see, e.g., Aumann (1999) and Osborne and Rubinstein (1994, Chapter 5). Denote by $\mathbf{s}(\omega)$ the strategy profile at ω and let

$$S^E \equiv \{\mathbf{s}(\omega) \,|\, \omega \in E\}.$$

Apparently, from an analyst's point of view, the model of situation, $\top$, defined on Ω should be consistent with the model of situation, $\top^G$, defined on G. For this purpose, in this paper we require the epistemic model $\mathcal{M}\left(\top^G\right)$ to satisfy the following consistency property:

[Consistency] For any event $E \subseteq \Omega$, $\top_i(E) = \top_i^G(S^E)\ \forall i$.

That is, the consistency property requires that the type space on an event be consistent with the type space on the strategies projected from the event, and thus, each player behaves in a natural way with respect to the marginalization in the epistemic model. This requirement is much in the same spirit of the notion of "coherence" imposed in the analysis of hierarchy of beliefs and preferences [see, e.g., Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Epstein and Wang (1996)]. We would like to point out that the consistency property is not a behavioral condition for the players in games, but it is made only for the (comprehensive) epistemic model adopted by an analyst to be harmonious with the (simple) analytical framework used in Sect. 2.

We say "*player i knows/believes an event E at ω*" if $\mathbf{t}_i(\omega) \in \top_i(E)$ since the complement of E is regarded as impossible under $\mathbf{t}_i(\omega) \in \top_i(E)$.[18] Let

$$K_i E \equiv \{\omega \in \Omega \,|\, i \text{ knows } E \text{ at } \omega\}.$$

An event $\boxed{E} \subseteq E$ is called a *common-knowledge/self-evident event (in E)*, if

$$\boxed{E} \subseteq K_i \boxed{E} \quad \text{for all } i \in N.$$

Say player i is "*rational at ω*" if $\mathbf{s}_i(\omega)$ is one of most preferred actions for $\mathbf{t}_i(\omega)$. Let

$$\mathfrak{R}_i \equiv \{\omega \in \Omega \,|\, i \text{ is rational at } \omega\}$$

and

$$\mathfrak{R} \equiv \cap_{i\in N} \mathfrak{R}_i.$$

[18] This formalism can be easily applied to Aumann's definition of knowledge by the possibility correspondence in a semantic framework and the notion of "belief with probability one" in a probabilistic model, for instance. Some readers may prefer the term "believes E" rather than "knows E." In this paper, we do not particularly distinguish between "knowledge" and "belief."

That is, $\mathfrak{R}$ is the event "everyone is rational." The following proposition provides an epistemic characterization for rationalizability: The notion of rationalizability is the strategic implication of common knowledge of rationality.

Proposition 5 *(1) In any epistemic model* $\mathcal{M}\left(\top^G\right)$, $S^{\boxed{\mathfrak{R}}}$ *is a rationalizable set in* $\top^G$. *(2) Suppose that* R *is a rationalizable set in* $\top^G$. *Then, there is an epistemic model* $\mathcal{M}\left(\top^G\right)$ *in which* $S^{\boxed{\mathfrak{R}}} = R$ *for some common-knowledge event* $\boxed{\mathfrak{R}}$.

Proof (1) Since $\boxed{\mathfrak{R}} \subseteq \mathfrak{R}$ is a common-knowledge event, for any $\omega \in \boxed{\mathfrak{R}}$, $\mathbf{s}_i(\omega) \in \beta(\mathbf{t}_i(\omega))$ and $\mathbf{t}_i(\omega) \in \top_i\left(\boxed{\mathfrak{R}}\right)$ for all i. By Consistency, $\mathbf{t}_i(\omega) \in \top_i^G\left(S^{\boxed{\mathfrak{R}}}\right)$. Therefore, $\forall i$ and $\forall s \in S^{\boxed{\mathfrak{R}}}$, there exists some $t_i \in \top_i^G\left(S^{\boxed{\mathfrak{R}}}\right)$ such that $s_i \in \beta(t_i)$. That is, $S^{\boxed{\mathfrak{R}}}$ is a rationalizable set in $\top^G$.

(2) Let R be a rationalizable set in $\top^G$. Define an epistemic model for $\top^G$:

$$\mathcal{M}\left(\top^G\right) \equiv \left(\Omega, \top, \{\mathbf{s}_i\}_{i\in N}, \{\mathbf{t}_i\}_{i\in N}\right),$$

such that

$$\text{(i) } \Omega = \left\{(s_i, t_i)_{i\in N} \mid t_i \in \top_i^G(R) \text{ and } s_i \in \beta(t_i) \cap R_i\right\};$$
$$\text{(ii) } \forall i, \top_i(E) = \top_i^G(S^E) \quad \text{if } E \subseteq \Omega;$$
$$\text{(iii) } \forall i, \mathbf{s}_i(\omega) = s_i \text{ and } \mathbf{t}_i(\omega) = t_i \quad \text{if } \omega = (s_i, t_i)_{i\in N}.$$

Clearly, every player i is rational across states in Ω. By Consistency, $\Omega \subseteq K\Omega$. Therefore, $\boxed{\mathfrak{R}} = \Omega$ is common-knowledge event satisfying $S^{\Omega} = R$. □

Remark In the standard semantic model of knowledge, it is well known that the above "fixed-point" definition of "common knowledge" is equivalent to the traditional "iterative" formalism of "common knowledge"; see, e.g., Aumann (1976, 1999) and Monderer and Samet (1989). In general cases, the "fixed-point" definition of "common knowledge" is a more fundamental notion. Under the "monotonic" information and knowledge structures, it can be shown that the "fixed-point" definition of "common knowledge" is equivalent to an "iterative" notion of "common knowledge" *possibly by using transfinite levels of mutual knowledge*; see Heifetz (1996, 1999) for more discussions. If, moreover, the information and knowledge structures satisfy a "limit closure" property: What happens at finite levels determines what happens at the limit, it can be shown that the "fixed-point" definition of "common knowledge" is equivalent to the traditional "iterative" definition *by using a countably infinite number of levels of mutual knowledge*; see Fagin et al. (1999).

Within the standard expected utility framework in finite games, Brandenburger and Dekel (1987) offered the notion of "a posteriori equilibrium," a strengthening of

Aumann's (1974) notion of subjective correlated equilibrium and showed an equivalence between rationalizability and a posteriori equilibrium. The equivalence implies that the assumption of common knowledge of rationality also provides a formal epistemic justification for this equilibrium notion. In finite models, Epstein (1997) extended this equivalence result to "regular" preferences including the subjective expected utility model. We end this section by presenting such an equivalence result for arbitrary games with various modes of behavior in the analytical framework used in this paper.

A strategy profile function $\mathbf{s} : \Omega \to S$ in an epistemic model $\mathcal{M}\left(\top^G\right)$ for game G is said to be an *a posteriori equilibrium in* $\mathcal{M}\left(\top^G\right)$ if for every player $i \in N$,

$$\forall \omega \in \Omega, \quad \mathbf{s}_i(\omega) \succsim_{\mathbf{t}_i(\omega)} s_i \; \forall s_i \in S_i,$$

i.e., $\mathbf{s}_i(\omega) \in \beta(\mathbf{t}_i(\omega))$.

Proposition 6 *The strategy profile s^* is rationalizable in $\top^G$ if and only if there exist an epistemic model $\mathcal{M}\left(\top^G\right)$ and an a posteriori equilibrium* $\mathbf{s}$ *in $\mathcal{M}\left(\top^G\right)$ such that $s^* = \mathbf{s}(\omega)$ for some $\omega \in \Omega$.*

Proof ("If" part.) Let $\mathbf{s}$ be an a posteriori equilibrium in an epistemic model $\mathcal{M}\left(\top^G\right)$. Then, for every player i and every $s \in S^\Omega$, $s_i \in \beta(t_i)$ for some $t_i \in \top_i(\Omega)$. By Consistency, for every player i and every $s \in S^\Omega$, $s_i \in \beta(t_i)$ for some $t_i \in \top_i^G(S^\Omega)$. That is, the set S^Ω is a rationalizable set in $\top^G$. Thus, the profile s^* is rationalizable in $\top^G$ if $s^* = \mathbf{s}(\omega)$ for some $\omega \in \Omega$.

("Only if" part.) Let s^* be a rationalizable strategy profile in $\top^G$. Then, there is a rationalizable set R in $\top^G$ which contains s^*. Thus, for every player i and every $s \in R$, there is $t_i \in \top_i^G(R)$ such that $s_i \in \beta(t_i)$. Define an epistemic model for G :

$$\mathcal{M}\left(\top^G\right) \equiv \left(\Omega, \top, \{\mathbf{s}_i\}_{i\in N}, \{\mathbf{t}_i\}_{i\in N}\right),$$

such that

$$\text{(i) } \Omega = \left\{(s_i, t_i)_{i\in N} \,|t_i \in \top_i^G(R) \text{ and } s_i \in \beta(t_i)\right\};$$
$$\text{(ii) } \forall i, \top_i(E) = \top_i^G(S^E) \text{ if } E \subseteq \Omega;$$
$$\text{(iii) } \forall i, \mathbf{s}_i(\omega) = s_i \text{ and } \mathbf{t}_i(\omega) = t_i \text{ if } \omega = (s_i, t_i)_{i\in N}.$$

Therefore, for every player i and every $\omega = (s_i, t_i)_{i\in N}$ in Ω, $\mathbf{s}_i(\omega) \in \beta(\mathbf{t}_i(\omega))$. That is, $\mathbf{s}$ is an a posteriori equilibrium in $\mathcal{M}\left(\top^G\right)$. Thus, for each rationalizable profile s^* in $\top^G$, we can find an a posteriori equilibrium $\mathbf{s}$ in $\mathcal{M}\left(\top^G\right)$ and a state $\omega^* = \left(s_i^*, t_i^*\right)_{i\in N}$ in Ω such that $s^* = \mathbf{s}(\omega^*)$. □

7 Concluding remarks

In this paper, we have presented a simple and unified framework for analyzing rationalizable strategic behavior in general environments—i.e., arbitrary strategic games with various modes of behavior; in particular, we have introduced the "model of situation" to define the notion of rationalizability in games with (in)finite players, arbitrary strategy spaces, and arbitrary payoff functions. In this paper, we have focused on the concept of rationalizability in strategic games and aimed at identifying the maximal domain of environments to which most of well-known properties of the rationalizability notion in finite games can be extended. We have shown that the notion of rationalizability possesses nice properties similar to those in finite games discussed in standard textbooks. Our approach in this paper is completely topology-free and with imposing no measure-theoretic assumption on the structure of the game and is applicable to any arbitrary strategic game.

We would like to emphasize that one important feature of this paper is that the framework allows the players to have different preferences which include the subjective expected utility as a special case. In the light of the analysis of this paper, we seek fairly natural and few behavioral assumptions on players' preference relations as weak as possible to make our analysis applicable to a wide range of strategic problems; in particular, we do not assume that preferences have utility function representations. The general analysis of this paper is applicable to any arbitrary strategic game with various modes of behavior.[19]

To close this paper, we would like to point out some possible extensions of this paper for future research. The extension of this paper to general games with incomplete information or with complex social and coalitional interactions is an important and intriguing subject for further research. The exploration of the notion of extensive-form rationalizability in general dynamic games is also an important research topic for further study [see, e.g., Greenberg et al. (2009) and Vannetelbosch (1999) for some related work in this direction].

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[19] Throughout this paper, C1 is perhaps the only essential behavioral assumption under which the rationalizability defined in general situations possesses nice properties as in the case of finite games. C2 can be removed if one does not care about its relationship with the Nash equilibrium. Morris and Takahashi (2011) did not impose this condition in their analysis. C3 is rather mild and innocuous, and the condition is satisfied by almost all preference models discussed in the literature.

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