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SYMPOSIUM

Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent and price-dependent preferences

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Abstract We generalize the classical equilibrium existence theorems by dispensing with the assumption of continuity of preferences. Our new existence results allow us to dispense with the interiority assumption on the initial endowments. Furthermore, we allow for non-ordered, interdependent and price-dependent preferences.

Keywords Continuous inclusion property · Abstract economy · Existence of Walrasian equilibria

JEL Classification C62 · D51

1 Introduction

The classical equilibrium existence theorems of Nash (1950), Debreu (1952), Arrow and Debreu (1954) and McKenzie (1954) were generalized to games/abstract economies where agents' preferences need not be transitive or complete and therefore need not be representable by utility functions (see for example, Mas-Colell 1974; Shafer and Sonnenschein 1975; Gale and Mas-Colell 1975; Borglin and Keiding 1976; Shafer 1976; Yannelis and Prabhakar 1983; Wu and Shen 1996 among others). The

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need to drop the transitivity assumption from equilibrium theory was motivated by behavioral/experimental works which demonstrated that consumers do not necessarily behave in a transitive way.

A different line of literature pioneered by Dasgupta and Maskin (1986) and Reny (1999) necessitated the need to drop the continuity assumption on the payoff function of each agent. Their works were motivated by many realistic applications (for example, Bertrand competition and auctions), and generalizations of the Nash–Debreu equilibrium existence theorems were obtained where payoff functions need not be continuous. In other words, a new literature emerged on equilibrium existence theorems with discontinuous payoffs.[1]

The first aim of this paper is to generalize the equilibrium existence theorems of Shafer and Sonnenschein (1975) and Yannelis and Prabhakar (1983) by dispensing with the continuity assumption of the preference correspondences. Although the proof of our equilibrium existence theorem in an abstract economy follows the approach of Yannelis and Prabhakar (1983), we cannot rely on continuous selections results, as it was the case in their work (and even earlier in Gale and Mas-Colell 1975). Indeed, the preference correspondence may not admit any continuous selection in our setting.[2]

Our second aim is to obtain the existence of Walrasian equilibria in an exchange economy where the preference correspondences could be discontinuous, nontransitive, incomplete, interdependent and price-dependent. An additional point we would like to emphasize is that contrary to the standard existence results in the literature, we do not impose the assumption that the initial endowment is an interior point of the consumption set.

The paper proceeds as follows. Section 2 collects notations and definitions. Section 3 provides a proof of the existence of equilibrium for an abstract economy, which extends the results of Shafer and Sonnenschein (1975) and Yannelis and Prabhakar (1983). The existence of Walrasian equilibrium with finite and infinite dimensional commodity spaces is proved and discussed in Sect. 4 .

2 Basics

Let X and Y be linear topological spaces, and let ψ be a correspondence from X to Y. Then ψ is said to be **lower hemicontinuous** if $\psi^l(V) = \{x \in X : \psi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y and **upper hemicontinuous** if $\psi^u(V) = \{x \in X : \psi(x) \subseteq V\}$ is open in X for every open subset V of Y. In addition, if the set $G = \{(x, y) \in X \times Y : y \in \psi(x)\}$ is open (resp. closed) in $X \times Y$, then we say that ψ has an **open (resp. closed) graph**. If $\psi^l(y)$ is open for each $y \in Y$, then ψ is said to have **open lower sections**.

At some $x \in X$, if there exists an open set O_x such that $x \in O_x$ and $\cap_{x' \in O_x} \psi(x') \neq \emptyset$, then we say ψ has the local intersection property. Furthermore, ψ is said to have the **local intersection property** if this property holds for every $x \in X$.

[1] See the symposium of Carmona (2011) for additional references.

[2] It should be mentioned that independently of our work, Reny (2013) has also obtained related results.

Clearly, every nonempty correspondence with open lower sections has the local intersection property. Yannelis and Prabhakar (1983) proved a continuous selection theorem and several fixed-point theorems by assuming that ψ has open lower sections. Based on the local intersection property, Wu and Shen (1996) generalized the results of Yannelis and Prabhakar (1983). Recently, Scalzo (2015) proposed the "local continuous selection property" and proved that this condition is necessary and sufficient for the existence of continuous selections.

Mappings with the local intersection property have found applications in mathematical economics and game theory [see Wu and Shen (1996) and Prokopovych (2011) among others].

We now introduce the "continuous inclusion property," which includes the above conditions as special cases.

Definition 1 A correspondence ψ from X to Y is said to have the **continuous inclusion property** at x if there exists an open neighborhood O_x of x and a nonempty correspondence $F_x : O_x \to 2^Y$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and $\text{co}F_x$[3] has a closed graph.[4]

The continuous inclusion property is motivated by the majorization idea in general equilibrium (see the KF-majorization in Borglin and Keiding 1976 and L-majorization in Yannelis and Prabhakar 1983), and also the "multiply security" condition of McLennan et al. (2011), the "continuous security" condition of Barelli and Meneghel (2013) and the "correspondence security" condition of Reny (2013) in the context of discontinuous games.

Remark 1 If the correspondence ψ from X to Y has the local intersection property at x, then F_x can be chosen as a constant correspondence which only contains a singe point of $\cap_{x' \in O_x} \psi(x')$, and hence ψ also has the continuous inclusion property at x. As a result, any nonempty correspondence with open lower sections has the continuous inclusion property.[5]

3 Equilibria in abstract economies

3.1 Results

In this section we prove the existence of equilibrium for an abstract economy with an infinite number of commodities and a countable number of agents.

An **abstract economy** is a set of ordered triples $\Gamma = \{(X_i, A_i, P_i) : i \in I\}$, where

- I is a countable set of **agents**.

[3] For a correspondence F, $\text{co}F$ denotes the convex hull of F.

[4] If the sub-correspondence F_x has a closed graph and X is finite dimensional, then $\text{co}F_x$ still has a closed graph since the convex hull of a closed set is closed in finite dimensional spaces. However, this may not be true if one works with infinite dimensional spaces. One can easily see that assuming the sub-correspondence F_x is convex valued and has a closed graph would suffice for our aim.

[5] Reny (2013) proposed a similar condition called "correspondence security" in the setting of discontinuous games and proved an equilibrium existence theorem for an abstract game.

- X_i is a nonempty set of **actions** for agent i. Set $X = \prod_{i \in I} X_i$.
- $A_i : X \to 2^{X_i}$ is the **constraint correspondence** of agent i.
- $P_i : X \to 2^{X_i}$ is the **preference correspondence** of agent i.

An **equilibrium** of Γ is a point $x^* \in X$ such that for each $i \in I$:

1. $x_i^* \in \overline{A_i}(x^*)$, where $\overline{A_i}$ denotes the closure of A_i, and
2. $P_i(x^*) \cap A_i(x^*) = \emptyset$.

If $A_i \equiv X_i$ for all $i \in I$, then the point x^* is called a **Nash equilibrium**.

For each $i \in I$, let $\psi_i(x) = A_i(x) \cap P_i(x)$ for all $x \in X$.

Theorem 1 *Let $\Gamma = \{(X_i, A_i, P_i) : i \in I\}$ be an abstract economy such that for each $i \in I$:*

i. *X_i is a nonempty, compact, convex and metrizable subset of a Hausdorff locally convex linear topological space;*
ii. *A_i is nonempty and convex valued;*
iii. *the correspondence $\overline{A_i}$ is upper hemicontinuous;*
iv. *ψ_i has the continuous inclusion property at each $x \in X$ with $\psi_i(x) \neq \emptyset$;*
v. *$x_i \notin co\psi_i(x)$ for all $x \in X$.*

Then Γ has an equilibrium.

Proof Fix $i \in I$. Let $U_i = \{x \in X : \psi_i(x) \neq \emptyset\}$.[6] Since ψ_i has the continuous inclusion property at each $x \in U_i$, there exist an open set $O_x^i \subseteq X$ such that $x \in O_x^i$ and a correspondence $F_x^i : O_x^i \to 2^{X_i}$ with nonempty values such that $F_x^i(z) \subseteq \psi_i(z)$ for any $z \in O_x^i$ and $\mathrm{co}F_x^i$ is closed. Then $O_x^i \subseteq U_i$, which implies that U_i is open. Since X is metrizable, U_i is paracompact [see Michael (1956, p. 831)]. Moreover, the collection $\mathscr{C}_i = \{O_x^i : x \in X\}$ is an open cover of U_i. There is a closed locally finite refinement $\mathcal{F}_i = \{E_k^i : k \in K\}$, where K is an index set and E_k^i is a closed set in X [see Michael (1953, Lemma 1)].

For each $k \in K$, choose $x_k \in X$ such that $E_k^i \subseteq O_{x_k}^i$. For each $x \in U_i$, let $I_i(x) = \{k \in K : x \in E_k^i\}$. Then $I_i(x)$ is finite for each $x \in U_i$. Let $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)\right)$ for $x \in U_i$. For each x and $k \in I_i(x)$, $F_{x_k}^i(x) \subseteq \psi_i(x)$. Thus, $\mathrm{co}F_{x_k}^i(x) \subseteq \mathrm{co}\psi_i(x)$, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x) \subseteq \mathrm{co}\psi_i(x)$. As a result, we have $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)\right) \subseteq \mathrm{co}\psi_i(x)$.

Define the correspondence

$$H_i(x) = \begin{cases} \phi_i(x) & x \in U_i; \\ \overline{A_i}(x) & \text{otherwise.} \end{cases}$$

Then it is obvious that H_i is nonempty and convex valued. Moreover, H_i is also compact valued [see Lemma 5.29 in Aliprantis and Border (2006)].

Since $\mathrm{co}F_{x_k}^i$ has a closed graph in E_k^i and E_k^i is a compact Hausdorff space, it is upper hemicontinuous. For each x, $I_i(x)$ is finite, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)$

[6] If $U_i = \emptyset$ for all i, then the correspondence $\overline{A} = \prod_{i \in I} \overline{A_i}$ is nonempty, convex valued and upper hemicontinuous. As a result, there exists a fixed-point x^* of $\overline{A}$ which is an equilibrium.

is the union of values for a finite family of upper hemicontinuous correspondences, and hence is upper hemicontinuous at the point x [see Aliprantis and Border (2006, Theorem 17.27)]. Then $\phi_i(x)$ is the convex hull of $\cup_{k \in I_i(x)} \mathrm{co} F^i_{x_k}(x)$, and it is compact for all $x \in U_i$; hence it is upper hemicontinuous on U_i [see Aliprantis and Border (2006, Theorem 17.35)]. Note that $H_i(x)$ is $\phi_i(x)$ when $x \in U_i$, and $\overline{A_i}(x)$ when $x \notin U_i$. Since U_i is open, analogous to the argument in Yannelis and Prabhakar (1983, Theorem 6.1), H_i is upper hemicontinuous on the whole space. Let $H = \prod_{i \in I} H_i$. Since H is nonempty, convex and closed valued, by the Fan-Glicksberg fixed-point theorem, there exists a point $x^* \in X$ such that $x^* \in H(x^*)$.

Since $\phi_i(x) \subseteq \overline{A_i}(x)$ for $x \in U_i$ and $H_i(x) \subseteq \overline{A_i}(x)$ for any x, which implies that $x^*_i \in \overline{A_i}(x^*)$. Note that if $x^* \in U_i$ for some $i \in I$, then $x^*_i \in \mathrm{co}\left(\cup_{k \in I_i(x^*)} \mathrm{co} F^i_{x_k}(x^*)\right) \subseteq \mathrm{co}\psi_i(x^*)$, a contradiction to assumption (v). Thus, we have $x^* \notin U_i$ for all $i \in I$. Therefore, $\psi_i(x^*) = \emptyset$, which implies that $A_i(x^*) \cap P_i(x^*) = \emptyset$. That is, x^* is an equilibrium for Γ. □

Remark 2 If in the above theorem, set $A_i \equiv X_i$, assume that ψ_i is convex valued and drop assumption (v), then one can obtain a generalization of the Gale and Mas-Colell (1975) fixed-point theorem (see He and Yannelis 2014). That is, let $\psi_i : X \to X_i$ be a convex valued correspondence with the continuous inclusion property at each x such that $\psi_i(x) \neq \emptyset$. Then there exists a point $x^* \in X$ such that for each i, either $x^*_i \in \psi_i(x^*)$ or $\psi_i(x^*) = \emptyset$.[7]

Below, we show that the theorem of Shafer and Sonnenschein (1975) and Theorem 6.1 of Yannelis and Prabhakar (1983) on the existence of equilibrium in an abstract economy can be obtained as corollaries. Note that in Shafer and Sonnenschein (1975) the correspondence A_i is compact valued for each $i \in I$, and therefore there is no need to work with the closure of A_i. That is, an equilibrium x^* should satisfy $x^*_i \in A_i(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$. In Yannelis and Prabhakar (1983), the equilibrium notion is the same as defined above.

Corollary 1 [Shafer and Sonnenschein (1975)] *Let $\Gamma = \{(X_i, A_i, P_i) : i \in I\}$ be an abstract economy such that for each $i \in I$:*

i. *X_i is a nonempty, compact and convex subset of $\mathbb{R}^l_+$;*
ii. *A_i is nonempty, convex and compact valued;*
iii. *A_i is a continuous correspondence;*
v. *P_i has an open graph;*
vi. *$x_i \notin co\psi_i(x)$ for all $x \in X$.*[8]

Then Γ has an equilibrium x^*; that is, for any $i \in I$, $x^*_i \in A_i(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$.

[7] For recent related results, see Prokopovych (2013, 2015).

[8] Shafer and Sonnenschein (1975) assume that $x_i \notin \mathrm{co} P_i(x)$ for all $x \in X$, but their proof still holds under this more general condition. The same comment is also valid for the existence theorem of Yannelis and Prabhakar (1983), see condition (vi) of Corollary 2 below.

Proof For each $i \in I$, define a mapping $U_i \colon \text{Gr}(A_i) \to \mathbb{R}$ by $U_i(y, x_i) = \text{dist}((y, x_i), \text{Gr}^C(P_i))$, where $\text{Gr}(A_i)$ is the graph of A_i, $\text{Gr}^C(P_i)$ denotes the complement of the graph of P_i, and $\text{dist}(\cdot, \cdot)$ denotes the usual distance on $\mathbb{R}^l_+$. Since P_i has an open graph, U_i is continuous. Let $m_i(x) = \max_{z \in A_i(x)} U_i(x, z)$ and $\phi_i(x) = \{z \in A_i(x) \colon U_i(x, z) = m_i(x)\}$ for each $x \in X$. Since A_i is continuous, by the Berge maximum theorem [see Aliprantis and Border (2006, Theorem 17.31)], ϕ_i is nonempty, compact valued and upper hemicontinuous. At any point x such that $\psi_i(x) = P_i(x) \cap A_i(x) \neq \emptyset$, we have $m_i(x) > 0$, and hence $\phi_i(x) \subseteq \psi_i(x)$. Thus, the continuous inclusion property holds, and by Theorem 1, there is an equilibrium. □

Corollary 2 (Yannelis and Prabhakar (1983, Theorem 6.1)) *Let* $\Gamma = \{(X_i, A_i, P_i) \colon i \in I\}$ *be an abstract economy such that for each* $i \in I$*:*

i. X_i *is a nonempty, compact, convex and metrizable subset of a Hausdorff locally convex linear topological space;*
ii. A_i *is nonempty and convex valued;*
iii. *the correspondence* $\overline{A_i}$ *is upper hemicontinuous;*
iv. A_i *has open lower section;*
v. P_i *has open lower section;*
vi. $x_i \notin co\psi_i(x)$ *for all* $x \in X$.

Then Γ *has an equilibrium* x^**; that is, for each* $i \in I$, $x_i^* \in \overline{A}_i(x^*)$ *and* $P_i(x^*) \cap A_i(x^*) = \emptyset$.

Proof By Fact 6.1 in Yannelis and Prabhakar (1983), ψ_i has open lower sections. As a result, ψ_i has the continuous inclusion property at each $x \in X$ when $\psi_i(x) \neq \emptyset$. Then the result follows from Theorem 1. □

Remark 3 Note that our Theorem 1 also covers Theorem 10 of Wu and Shen (1996). Wu and Shen (1996) did not impose the metrizability condition on X_i, but directly assumed that U_i is paracompact. Our proof still holds under this condition.

Remark 4 In condition (iv) of Theorem 1, we assume that ψ_i has the continuous inclusion property at each $x \in X$ with $\psi_i(x) \neq \emptyset$. It is natural to ask whether we can impose conditions on the correspondences P_i and A_i separately and then verify that their intersection ψ_i has the continuous inclusion property [for example, see conditions (iv) and (v) in Yannelis and Prabhakar (1983, Theorem 6.1)]. However, a simple example can be constructed to show that a combination of the following two conditions cannot guarantee our condition (iv):

1. P_i has the continuous inclusion property at x when $P_i(x) \neq \emptyset$;
2. A_i has an open graph.

Suppose that there is only one agent and $X = [0, 1]$, $A(x) = (0, 1]$ and

$$P(x) = \begin{cases} [0, 1], & x = 1; \\ \{0\}, & x \in [0, 1). \end{cases}$$

Then it is obvious that P has the continuous inclusion property and A has an open graph. However,

$$\psi(x) = \begin{cases} (0, 1], & x = 1, \\ \emptyset, & x \in [0, 1); \end{cases}$$

does not have the continuous inclusion property.

Note that if $A_i = X_i$ is a constant correspondence, we can assume that P_i has the continuous inclusion property at each $x \in X$ with $P_i(x) \neq \emptyset$, and the existence of Nash equilibrium follows as a corollary.

Corollary 3 *Let $\Gamma = \{(X_i, P_i) : i \in I\}$ be a game such that for each $i \in I$:*

i. *X_i is a nonempty, compact, convex and metrizable subset of a Hausdorff locally convex linear topological space;*
ii. *P_i has the continuous inclusion property at each $x \in X$ with $P_i(x) \neq \emptyset$;*
iii. *$x_i \notin coP_i(x)$ for all $x \in X$.*

Then Γ has a Nash equilibrium x^*; that is, for each $i \in I$, $P_i(x^*) = \emptyset$.

3.2 Relationship with Carmona and Podczeck (2015)

Subsequent to this paper, Carmona and Podczeck (2015) dropped the metrizability condition on X_i and generalized our conditions (4) and (5) as follows.

Let $I(x) = \{i \in I : \psi_i(x) \neq \emptyset\}$. For every $x \in X$ such that $I(x) \neq \emptyset$ and $x_i \in \overline{A}_i(x)$ for all $i \in I$, there is an agent $i \in I(x)$ such that,

1. ψ_i has the continuous inclusion property at x;
2. $x_i \notin \text{co}\psi_i(x)$.

Notice that our proof above still goes through under this condition by slightly modifying the definition of the set U_i as

$$\{x \in X : \psi_i \text{ has the continuous inclusion property at } x\}.$$

The metrizability condition in our Theorem 1 is not needed. Following a similar argument as in Borglin and Keiding (1976) and Toussaint (1984), we provide an alternative proof for Theorem 1 in which the set of agents can be any arbitrary (finite or infinite set) and X_i need not to be metrizable for each i.[9]

[9] It should be noted that using the existence of maximal element theorem for L-majorized correspondences (see Yannelis and Prabhakar 1983), it is known that the metrizability assumption is not needed. Indeed, the proof of Borglin and Keiding (1976) remains valid if one replaces the KF-majorization by L-majorization. The existence of maximal element theorem for correspondences having the continuous inclusion property can be used to show that the metrizability in our Theorem 1 is not needed, see Footnote 10.

Alternative proof of Theorem 1 For each $i \in I$, define a correspondence H_i from X to X_i as follows:

$$H_i(x) = \begin{cases} \psi_i(x), & x_i \in \overline{A}_i(x); \\ \overline{A}_i(x), & x_i \notin \overline{A}_i(x). \end{cases}$$

We will show that H_i has the continuous inclusion property at each x such that $H_i(x) \neq \emptyset$.

1. If $x_i \in \overline{A}_i(x)$, then $\psi_i(x) = H_i(x) \neq \emptyset$, which implies that there exists an open neighborhood O_x of x and a nonempty correspondence $F_x : O_x \to 2^{X_i}$ such that $F_x(z) \subseteq \psi_i(z)$ for any $z \in O_x$ and $\mathrm{co}F_x$ has a closed graph. For any $z \in O_x$, $F_x(z) \subseteq \psi_i(z) = H_i(z)$ if $z_i \in \overline{A}_i(z)$, and $F_x(z) \subseteq \psi_i(z) \subseteq \overline{A}_i(z) = H_i(z)$ if $z_i \notin \overline{A}_i(z)$.
2. Consider the case that $x_i \notin \overline{A}_i(x)$. Since the correspondence $\overline{A}_i$ is upper hemicontinuous and closed valued, it has a closed graph. As a result, one can find an open neighborhood O_x of x such that $z_i \notin \overline{A}_i(z)$ and hence $H_i(z) = \overline{A}_i(z)$ for any $z \in O_x$. As $\overline{A}_i$ is upper hemicontinuous, closed and convex valued, H_i has the continuous inclusion property.

Let $I(x) = \{i \in I : H_i(x) \neq \emptyset\}$. Define a correspondence $H : X \to 2^X$ as

$$H(x) = \begin{cases} \left(\prod_{i \in I(x)} H_i(x)\right) \times \left(\prod_{j \in I \setminus I(x)} X_j\right), & I(x) \neq \emptyset; \\ \emptyset, & I(x) = \emptyset. \end{cases}$$

It can be easily checked that $H(x)$ has the continuous inclusion property at each x such that $H(x) \neq \emptyset$.

In addition, one can easily show that $x \notin \mathrm{co}H(x)$ for any $x \in X$. Indeed, fix any $x \in X$. If $I(x) = \emptyset$, then $H(x) = \emptyset$, which implies that $x \notin \mathrm{co}H(x)$. If $I(x) \neq \emptyset$, then there exists an agent i such that $H_i(x) \neq \emptyset$. If $x_i \in \overline{A}_i(x)$, then $x_i \notin \mathrm{co}\psi(x) = \mathrm{co}H_i(x)$. If $x_i \notin \overline{A}_i(x)$, then $x_i \notin \mathrm{co}H_i(x)$ as $H_i(x) = \overline{A}_i(x)$ (since $\overline{A}_i(x)$ is convex). Hence, $x \notin \mathrm{co}H(x)$.

By Corollary 1 in He and Yannelis (2014),[10] there exists a point $x^* \in X$ such that $H(x^*) = \emptyset$, which implies that $I(x^*) = \emptyset$. That is, for any i, $H_i(x^*) = \emptyset$, which implies that $x_i^* \in \overline{A}_i(x^*)$ and $\psi_i(x^*) = H_i(x^*) = \emptyset$. □

Remark 5 The previous proof adapted in Theorem 1 seems to be suitable to cover the case where the set of agents is a measure space as in Yannelis (1987). It is not clear whether the above proof can be easily extended to a measure space of agents.

4 Existence of Walrasian equilibria

An **exchange economy** $\mathcal{E}$ is a set of triples $\{(X_i, P_i, e_i) : i \in I\}$, where

[10] Suppose that X is a compact and convex subset of a Hausdorff locally convex linear topological space. Let $P : X \to 2^X$ be a correspondence such that $x \notin \mathrm{co}P(x)$ for all $x \in X$. If P has the continuous inclusion property at each $x \in X$ such that $P(x) \neq \emptyset$, then there exists a point $x^* \in X$ such that $P(x^*) = \emptyset$.

- I is a finite set of agents;
- $X_i \subseteq \mathbb{R}^l_+$ is the **consumption set** of agent i, and $X = \prod_{i \in I} X_i$;
- $P_i : X \times \triangle \to 2^{X_i}$ is the **preference correspondence** of agent i, where $\triangle$ is the set of all possible prices;[11]
- $e_i \in X_i$ is the **initial endowment** of agent i, where $e = \sum_{i \in I} e_i \neq 0$.

Let $\triangle = \{p \in \mathbb{R}^l_+ : \sum_{k=1}^{l} p_k = 1\}$. Given a price $p \in \triangle$, the **budget set** of agent i is $B_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$. Let $\psi_i(p, x) = B_i(p) \cap P_i(x, p)$ for each $i \in I$, $x \in X$ and $p \in \triangle$. Then $\psi_i(p, x)$ is the set of all allocations in the budget set of agent i at price p that he prefers to x.

A **free disposal Walrasian equilibrium** for the exchange economy $\mathcal{E}$ is $(p^*, x^*) \in \triangle \times X$ such that

1. for each $i \in I$, $x_i^* \in B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$;
2. $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i$.

Theorem 2 *Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions: for each $i \in I$,*

1. *X_i is a nonempty compact convex subset of $\mathbb{R}^l_+$;*[12]
2. *ψ_i has the continuous inclusion property at each $(p, x) \in \triangle \times X$ with $\psi_i(p, x) \neq \emptyset$ and $x_i \notin co\psi_i(p, x)$.*

Then $\mathcal{E}$ has a free disposal Walrasian equilibrium.

Proof The proof follows the idea of Arrow and Debreu (1954), which introduces a fictitious player; see also Shafer (1976).

For each $i \in I$, $p \in \triangle$ and $x \in X$, let $A_i(p, x) = B_i(p)$. Define the correspondences $A_0(p, x) = \triangle$ and $P_0(p, x) = \{q \in \triangle : q \cdot \left(\sum_{i \in I}(x_i - e_i)\right) > p \cdot \left(\sum_{i \in I}(x_i - e_i)\right)\}$. Let $I_0 = I \cup \{0\}$. Then for any $i \in I_0$, A_i is nonempty, convex valued and upper hemicontinuous on $\triangle \times X$.

Note that $\psi_i(p, x) = A_i(p, x) \cap P_i(p, x)$ has the continuous inclusion property for each $i \in I$. Moreover, let $\psi_0(p, x) = A_0(p, x) \cap P_0(p, x) = P_0(p, x)$. Fix any $(p, x) \in \triangle \times X$ such that $\psi_0(p, x) \neq \emptyset$, pick $q \in \psi_0(p, x)$, then $(q - p) \cdot \left(\sum_{i \in I}(x_i - e_i)\right) > 0$. Since the left side of the inequality is continuous, there is an open neighborhood O of (p, x) such that for any $(p', x') \in O$, $(q - p') \cdot \left(\sum_{i \in I}(x_i' - e_i)\right) > 0$, which implies that the correspondence ψ_0 has the continuous inclusion property. In addition, it is obvious that ψ_0 is convex valued and $p \notin \psi_0(p, x)$ for any $(p, x) \in \triangle \times X$.

Thus, we can view the exchange economy $\mathcal{E}$ as an abstract economy $\Gamma = \{(X_i, A_i, P_i) : i \in I_0\}$ which satisfies all the conditions of Theorem 1. Therefore, there exists a point $(p^*, x^*) \in \triangle \times X$ such that

1. $x_i^* \in A_i(p^*, x^*) = B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$ for each $i \in I$ and

[11] We allow for very general preferences, which can be interdependent and price-dependent. See McKenzie (1955) and Shafer and Sonnenschein (1975) for more discussions. For agent i, $y_i \in P_i(x, p)$ means that y_i is strictly preferred to x_i provided that all other components are unchanged at the price $p \in \triangle$.

[12] The commodity space X_i can be sufficiently large. For example, we can let $X_i = \{x_i \in \mathbb{R}^l_+ : x_i \leq K \cdot \sum_{i \in I} e_i\}$, where K is an arbitrarily large positive number.

2. $P_0(p^*, x^*) = \psi_0(p^*, x^*) = \emptyset$.

Let $z = \sum_{i \in I}(x_i^* - e_i)$. Then (1) implies that $p^* \cdot z \leq 0$, and (2) implies that $q \cdot z \leq p^* \cdot z$ for any $q \in \triangle$, and hence $q \cdot z \leq p^* \cdot z \leq 0$. Suppose that $z \notin \mathbb{R}^l_-$. Thus, there exists some $k \in \{1, \ldots, l\}$ such that $z_k > 0$. Let $q' = \{q_j\}_{1 \leq j \leq l}$ such that $q_j = 0$ for any $j \neq k$ and $q_k = 1$. Then $q' \in \triangle$ and $q' \cdot z = z_k > 0$, a contradiction. Therefore, $z \in R^l_-$, which implies that $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i$.

Therefore, (p^*, x^*) is a free disposal Walrasian equilibrium. □

Remark 6 We have imposed the compactness condition on the consumption set. It is not clear to us at this stage whether this condition can be dispensed with. When agents' preferences are continuous, one can work with a sequence of economies with compact consumption sets, which are the truncations of the original consumption set. Then the existence of Walrasian equilibrium allocations and prices can be proved in each truncated economy. Since the set of feasible allocations and the price set are both compact, there exists a convergent point. By virtue of the continuity of preferences, one can show that this is indeed a Walrasian equilibrium of the original economy. The convergence argument fails in our setting as we do not require the continuity assumption on preferences. Consequently, relaxing the compactness assumption seems to be an open problem.[13]

We must add that the compactness assumption is not unreasonable at all. The world is finite, and the initial endowment for each good is also finite. Thus, by assuming that for each good, $\|x_i\| \leq K \cdot \sum_{i \in I} \|e_i\|$, where K is a sufficiently large number and I is the set of all agents in the world, no real restriction on the attainability of the consumption of each good is imposed.

Note that in Theorem 2 we allowed for free disposal. Below we prove the existence of a non-free disposal Walrasian equilibrium following the proof of Shafer (1976).

Hereafter we allow for negative prices: $\triangle' = \{p \in \mathbb{R}^l : \|p\| = \sum_{k=1}^{l} |p_k| \leq 1\}$ is the set of all possible prices. Let $B_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i + 1 - \|p\|\}$ and $\psi_i(p, x) = P_i(x, p) \cap B_i(p)$ for each $i \in I$, $x \in X$ and $p \in \triangle'$. Let $K = \{x : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$ and $\text{pr}_i : X \to X_i$ be the projection mapping for each $i \in I$.

A **(non-free disposal) Walrasian equilibrium** for the exchange economy $\mathcal{E}$ is $(p^*, x^*) \in \triangle' \times X$ such that

1. $\|p^*\| = 1$;
2. for each $i \in I$, $x_i^* \in B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$;
3. $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$.

If p^* is a Walrasian equilibrium price, then $\|p^*\| = 1$ and $B_i(p^*) = \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\}$, which is the standard budget set of agent i.

Theorem 3 *Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions: for each $i \in I$,*

[13] As suggested by an anonymous referee, one could allow $X_i = \mathbb{R}^l$ by assuming that if $x_i \in X_i$ and $x_i' \in P_i(x)$, then also $(1-\lambda)x_i + \lambda x_i' \in P_i(x)$ for all $0 < \lambda < 1$. With this assumption one needs to consider only one truncation of the consumption sets (any truncation which contains the feasible consumption points as interior points).

1. *X_i is a nonempty compact convex subset of $\mathbb{R}^l_+$;*
2. *ψ_i has the continuous inclusion property at each $(p, x) \in \triangle' \times X$ with $\psi_i(p, x) \neq \emptyset$, and $x_i \notin co\psi_i(p, x)$.*
3. *for each $x_i \in pr_i(K)$ and $p \in \triangle'$, $x_i \in bdP_i(x, p)$, where bd denotes boundary.*

Then $\mathcal{E}$ has a Walrasian equilibrium.

Proof Repeating the arguments in the first two paragraphs of the proof of Theorem 2, one could show that there exists a point $(p^*, x^*) \in \triangle' \times X$ such that

1. $x_i^* \in A_i(p^*, x^*) = B_i(p^*)$ for each $i \in I$, which implies that $p^* \cdot x_i^* \leq p^* \cdot e_i + 1 - \|p^*\|$;
2. $\psi_i(p^*, x^*) = \emptyset$ for each $i \in I$;
3. $P_0(p^*, x^*) = \psi_0(p^*, x^*) = \emptyset$.

Let $z = \sum_{i \in I}(x_i^* - e_i)$. We must show that $z = 0$. Suppose that $z \neq 0$. From (3), it follows that $q \cdot z \leq p^* \cdot z$ for any $q \in \triangle'$. Let $q = \frac{z}{\|z\|}$. Then $q \in \triangle'$ and $p^* \cdot z \geq q \cdot z > 0$. Let $q^* = \frac{p^*}{\|p^*\|}$. Since $\frac{p^*}{\|p^*\|} \cdot z \geq p^* \cdot z \geq q^* \cdot z$, it follows that $\|p^*\| = 1$. As a result, $p^* \cdot x_i^* \leq p^* \cdot e_i$ (since $x_i^* \in A_i(p^*, x^*)$), which implies that $p^* \cdot z = p^* \cdot \sum_{i \in I}(x_i^* - e_i) \leq 0$, a contradiction. Thus, $z = 0$; that is, $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$, $x^* \in K$.

Note that $x_i^* \in \mathrm{pr}_i(K)$ implies that $x_i^* \in \mathrm{bd}P_i(x^*, p^*)$. Since $x_i^* \in B_i(p^*)$ and $x_i^* \notin \mathrm{co}\psi_i(p^*, x^*)$, $x_i^* \notin P_i(x^*, p^*)$. If there exists some i such that $p^* \cdot x_i^* < p^* \cdot e_i + 1 - \|p^*\|$, then due to assumption (3), $x_i^* \in \mathrm{bd}P_i(x^*, p^*)$ implies that one can find a point $y_i \in P_i(x^*, p^*)$ such that x_i^* and y_i are sufficiently close, and $p^* \cdot y_i < p^* \cdot e_i + 1 - \|p^*\|$. Thus, $y_i \in \psi_i(p^*, x^*)$, which contradicts (2). Therefore, $p^* \cdot x_i^* = p^* \cdot e_i + 1 - \|p^*\|$ for each $i \in I$, and summing up over all i yields $\|p^*\| = 1$.

Therefore, (p^*, x^*) is a Walrasian equilibrium. □

Remark 7 Shafer (1976) proved the existence of non-free disposal Walrasian equilibrium based on the equilibrium existence result of Shafer and Sonnenschein (1975) (see Corollary 1 above). Thus, the main theorem of Shafer (1976) follows from our Corollary 1 and Theorem 3.

Below, we provide an alternative proof of the theorem of Shafer (1976) without invoking the norm of the price $\|p\|$ into the budget set. It requires the nonsatiation condition for one agent only. Furthermore, the proof below remains unchanged if the consumption set is a nonempty, norm compact and convex subset of a Hausdorff locally convex topological vector space. This is not the case in Shafer (1976)'s proof, since the norm of prices is part of the budget set. Recall that the price space $\triangle'$ is weak* compact by Alaoglu's theorem, and $\triangle'$ may not be metrizable unless the space of allocations is separable.

Theorem 4 *Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions:*

1. *for each $i \in I$, let X_i be a nonempty compact convex set of $\mathbb{R}^l_+$;*
2. *for each $i \in I$, ψ_i has the continuous inclusion property at each $(p, x) \in \triangle' \times X$ with $\psi_i(p, x) \neq \emptyset$, and for any $x_i \in X_i$, $x_i \notin co\psi_i(p, x)$;*

3. *for any* $p \in \triangle'$ *and* x *in the set of feasible allocations*

$$\mathcal{A} = \left\{ x \in X : \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i \right\},$$

there exists an agent $i \in I$ *such that* $P_i(x, p) \neq \emptyset$.

Then $\mathcal{E}$ *has a Walrasian equilibrium* (p^*, x^*)*; that is,*

1. $p^* \neq 0$*;*
2. *for each* $i \in I$, $x_i^* \in B_i(p^*)$ *and* $\psi_i(p^*, x^*) = \emptyset$*;*
3. $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$.

Most of the proof proceeds as in Theorem 2. We repeat the argument here for the sake of completeness.

Proof For each $i \in I$, $p \in \triangle'$ and $x \in X$, let $A_i(p, x) = B_i(p)$. Denote $X_0 = \triangle'$, and define the correspondences $A_0(p, x) \equiv \triangle'$ and $P_0(p, x) = \{q \in \triangle' : q\left(\sum_{i \in I}(x_i - e_i)\right) > p\left(\sum_{i \in I}(x_i - e_i)\right)\}$.[14] Let $I_0 = I \cup \{0\}$. Let $\psi_0(p, x) = A_0(p, x) \cap P_0(p, x) = P_0(p, x)$. As shown in the proof of Theorem 2, for each $i \in I_0$, the correspondence ψ_i is convex valued, $(p, x) \notin \psi_i(p, x)$ for any $(p, x) \in \triangle' \times X$, and has the continuous inclusion property.

We have constructed an abstract economy $\Gamma = \{(X_i, P_i, A_i) : i \in \{0\} \cup I\}$. By Theorem 1, there exists a point $(p^*, x^*) \in \triangle' \times X$ such that

1. $x_i^* \in A_i(p^*, x^*) = B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$ for each $i \in I$;
2. $P_0(p^*, x^*) = \psi_0(p^*, x^*) = \emptyset$.

Let $z = \sum_{i \in I}(x_i^* - e_i)$. Then (1) implies that $p^*(z) \leq 0$, and (2) implies that $q(z) \leq p^*(z)$ for any $q \in \triangle'$, and hence $q(z) \leq p^*(z) \leq 0$. As a result, $z = 0$;[15] that is, $x^* \in \mathcal{A}$. To complete the proof, we must show that $p^* \neq 0$. Suppose otherwise; that is, $p^* = 0$. Then $B_i(p^*) = X_i$ and $\psi_i(p^*, x^*) = P_i(x^*, p^*) = \emptyset$ for each $i \in I$, a contradiction to condition (3). Therefore, (p^*, x^*) is a Walrasian equilibrium. □

Remark 8 In Theorems 2, 3 and 4, the condition that ψ_i has the continuous inclusion property at each $(p, x) \in \triangle \times X$ with $\psi_i(p, x) \neq \emptyset$, and $x_i \notin \text{co}\psi_i(p, x)$ for each i can be weakened following the argument in Sect. 3.2. In particular, one can let $I(x) = \{i \in I : \psi_i(p, x) \neq \emptyset\}$ and assume that for every $x \in X$ such that $I(x) \neq \emptyset$ and $x_i \in A_i(p, x)$ for all $i \in I$, there is an agent $i \in I(x)$ such that,

1. ψ_i has the continuous inclusion property at (p, x);
2. $x_i \notin \text{co}\psi_i(p, x)$.

In other words, the continuous inclusion property is not required for all agents, but only for some agents. The proofs of Theorems 2, 3 and 4 can still go through under this new condition.[16] For pedagogical reasons, we work with condition (2) in Theorem 2.

[14] The function $q(x)$ is viewed as the inner product $q \cdot x$ when q is a price vector and x is an allocation.

[15] If $z \neq 0$, then there exists a point $q \in \triangle'$ such that $q(z) < 0$, which implies that $-q(z) > 0$. However, $-q \in \triangle'$, a contradiction.

[16] Such a remark has been also made by Carmona and Podczeck (2015).

5 Concluding remarks

Remark 9 Theorem 4 can be extended to a more general setting with an infinite dimensional commodity space. In particular, the commodity space can be any normed linear space whose positive cone may not have an interior point, and the set of prices is a subset of its dual space. If the consumption sets are nonempty, norm compact and convex, and the price space is weak* compact, then the proof of Theorem 4 remains unchanged.

Remark 10 To prove the existence of a Walrasian equilibrium in economies with infinite dimensional commodity spaces, Mas-Colell (1986) proposed the "uniform properness" condition when the preferences are transitive, complete and convex. Yannelis and Zame (1986) and Podczeck and Yannelis (2008) proved the existence result with non-ordered preferences using the "extreme desirability" condition. All the above results impose on the commodity space a lattice structure. Our Theorem 4 does not require the extreme desirability or uniform properness condition, and no ordering or lattice structure is needed on the commodity space. It should be noticed that the proof of our Theorem 4 requires that the evaluation map $(p, x_i) \to p(x_i)$ from $\triangle' \times X_i$ to $\mathbb{R}$ is continuous for $\triangle'$ with the weak* topology, while this joint continuity property of the evaluation map is not required in the papers above.

Mas-Colell (1986) provided an example of a single agent economy in which the preference is reflexive, transitive, complete, continuous, convex and monotone, but there is no quasi-equilibrium.[17] We show that his example does not satisfies our condition (2) of Theorem 4 when the commodity space is compact.

In the example of Mas-Colell (1986), the commodity space is the space of signed bounded countably additive measures $L = ca(K)$ with the bounded variation norm $\|\cdot\|_{BV}$, where $K = Z_+ \cup \{\infty\}$ is the compactification of the positive integers. Let $x_i = x(\{i\})$ for $x \in L$ and $i \in K$. For every $i \in K$, define a function $u_i : [0, \infty) \to [0, \infty)$ by

$$u_i(t) = \begin{cases} 2^i t & t \leq \frac{1}{2^{2i}}; \\ \frac{1}{2^i} - \frac{1}{2^{2i}} + t & t > \frac{1}{2^{2i}}. \end{cases}$$

The preference relation P is given by $U(x) = \sum_{i=1}^{i=\infty} u_i(x_i)$, which is concave, strictly monotone and weak* continuous.

Suppose that $X = \{x \in L_+ : \|x\|_{BV} \leq M\}$ for some sufficiently large positive integer M. Fix the initial endowment $e = (0, M, 0, \ldots, 0) \in X$ and the price $p_0 = 0$. Then $\psi(p_0, e) = B(p_0) \cap P(e) \neq \emptyset$, as $y = (M, 0, \ldots, 0) \in \psi(p_0, e)$. For each $i \in K$, let $w_i(\{j\}) = 1$ if $j = i$ and 0 otherwise. Fix a linear functional $p \in L'$ such that $p(w_2) = 0$ and $p(w_i) > 0$ for $i \neq 2$. Set $p_n = \frac{p}{n}$. Then $B(p_n) = \{0, m, 0, \ldots, 0\}$, where $0 \leq m \leq M$. However, for any $z \in B(p_n)$, $z \notin P(e)$. Consequently, $\psi(p_n, e) = \emptyset$. This implies that the correspondence ψ does not have the continuous inclusion property when the commodity space is compact, as $p_n \to 0$

[17] The pair (p^*, x^*) is called a free (non-free) disposal quasi-equilibrium if: (1) for each $i \in I$, $x_i^* \in B_i(p^*)$; (2) $x_i \in P_i(x^*, p^*)$ implies that $p^* \cdot x_i \geq p^* \cdot e_i$; (3) $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i$ ($\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$).

when $n \to \infty$. Therefore, the example of Mas-Colell (1986) violates condition (2) of our Theorem 4.

Remark 11 If we interpret the infinite dimensional commodity space as goods over an infinite time horizon, the weak, Mackey and weak* topologies on preferences imply that agents are impatient, because those topologies are generated by finitely many continuous linear functionals and they impose a form of "myopia" (i.e., tails do not matter, see for example Bewley (1972) and Araujo et al. (2011) among others). As our theorems drop the continuity assumption, it will be interesting to see if one can prove the existence theorem with patient agents relying on such discontinuous preferences.

Remark 12 Contrary to the standard existence results of Walrasian equilibrium, in the above theorems we do not impose the assumptions that the initial endowment is an interior point of the consumption set or the preference has an open graph/open lower sections. Below we give an example in which the preferences are discontinuous, and a Walrasian equilibrium exists. Notice that none of the classical existence theorems cover the example below.

Example 1 Consider the following 2-agent 2-good economy:

1. The set of available allocations for both agents is $X_1 = X_2 = [0, 1] \times [0, 1]$.
2. Agent 1's preference correspondence depends on $x_1 = (x_1^1, x_1^2)$ and $x_2 = (x_2^1, x_2^2)$:

$$P_1(x_1, x_2) = \left\{\left(y_1^1, y_1^2\right) \in X_1 : y_1^1 \cdot y_1^2 > x_1^1 \cdot x_1^2\right\} \setminus \left\{\left(y_1^1, y_1^2\right) \in X_1 : y_1^1 - x_1^1 = y_1^2 - x_1^2, y_1^1 < \frac{3}{2}x_1^1\right\}.$$

[18] The preference of agent 2 is defined similarly.
3. The initial endowments are given by $e_1 = (\frac{1}{3}, \frac{2}{3})$ and $e_2 = (\frac{2}{3}, \frac{1}{3})$.

Note that P_i does not have open lower sections for any $i = 1, 2$. For example,

$$P_i^l\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{\left(y_i^1, y_i^2\right) \in [0, 1] \times [0, 1] : y_i^1 \cdot y_i^2 < \frac{1}{4}, y_i^1 \neq y_i^2\right\} \cup \left\{(z, z) : 0 \leq z \leq \frac{1}{3}\right\}$$

which is neither open nor closed. As a result, P_i does not have an open graph.

We show that the conditions of Theorem 2 hold. Pick any point $(p, x) \in \triangle \times X$ such that $\psi_i(p, x) \neq \emptyset$, then there exists a point $y_i \in \psi_i(p, x) = B_i(p) \cap P_i(x)$. Since $y_i \in P_i(x)$, it follows that $y_i^1 \cdot y_i^2 > x_i^1 \cdot x_i^2$. Thus, one can pick a point $z_i = (z_i^1, z_i^2)$ such that $z_i^j < y_i^j$ for $j = 1, 2$ and z_i is an interior point of $P_i(x)$.[19] Consequently, there exists an open neighborhood O_i of x_i such that $(z_i^1, z_i^2) \in P(x_i', x_{-i})$ for any

[18] Given an allocation $x = (x_1, x_2) = ((x_1^1, x_1^2), (x_2^1, x_2^2))$ in the edgeworth box, the set of allocations which is preferred to x for agent 1 is the set of all points above the curve $y_1^1 \cdot y_1^2 = x_1^1 \cdot x_1^2$ such that the segment $\{(y_1^1, y_1^2) : y_1^1 - x_1^1 = y_1^2 - x_1^2, x_1^1 \leq y_1^1 < \frac{3}{2}x_1^1\}$ is removed.

[19] For example, one can choose the point $z_i = (y_i^1 - \epsilon, y_i^2 - 2\epsilon)$, where ϵ is a positive number. It is easy to see that if ϵ is sufficiently small, then z_i is an interior point of $P_i(x)$.

$x_i' \in O_i$ and $x_{-i} \in X_{-i}$. Furthermore, due to the fact that $z_i^j < y_i^j$ for $j = 1, 2$, we have $0 < p \cdot z_i < p \cdot y_i \leq p \cdot e_i$, which implies that there exists a neighborhood O_p of p, $z_i \in B_i(p')$ for any $p' \in O_p$. Define the correspondence $F_{(p,x)}$ as follows: $F_{(p,x)}(p', x') \equiv \{z_i\}$ for any $(p', x') \in O_p \times (O_i \times X_{-i})$.

Then we have:

1. $O_p \times (O_i \times X_{-i})$ is an open neighborhood of (p, x);
2. $F_{(p,x)}(p', x') \equiv \{z_i\} \subseteq \psi_i(p', x')$ for any $(p', x') \in O_p \times (O_i \times X_{-i})$;
3. $F_{(p,x)}$ is a single-valued constant correspondence and hence is closed.

Therefore, ψ has the continuous inclusion property at (p, x). In addition, it is easy to see that $x_i \notin \mathrm{co}\psi_i(p, x)$. By Theorem 2 above, there exists a Walrasian equilibrium. Indeed, it can be easily checked that (p^*, x^*) is a unique Walrasian equilibrium, where $p^* = (p_1^*, p_2^*) = (\frac{1}{2}, \frac{1}{2})$ and $x_1^* = x_2^* = (\frac{1}{2}, \frac{1}{2})$. Notice that even if the endowment is on the boundary $e_1 = (0, 1)$ and $e_2 = (1, 0)$, the equilibrium still remains the same.

Remark 13 A natural question that arises is whether or not the continuous inclusion property is easily verifiable for an economy. In the example above, we have demonstrated that it is easily verifiable, and it can be used to obtain the existence of a Walrasian equilibrium. Below we present another example in which one can easily check that the continuous inclusion property does not hold, and there is no Walrasian equilibrium. In this example, the preferences are continuous, and the initial endowment is not an interior point of the consumption set.

Example 2 There are two agents $I = \{1, 2\}$ and two goods x and y. The payoff functions are given by $u_1(x, y) = x + y$ and $u_2(x, y) = y$, which are continuous. The initial endowments are $e_1 = (\frac{1}{2}, 0)$ and $e_2 = (\frac{1}{2}, 1)$. The consumption sets for both agents are $[0, 2] \times [0, 2]$. In this example, one can easily see that there is no Walrasian equilibrium, but a quasi-equilibrium $((x^*, y^*), p^*)$ exists, where $(x^*, y^*) = (x_i^*, y_i^*)_{i \in I}$, and $(x_1^*, y_1^*) = (1, 0)$, $(x_2^*, y_2^*) = (0, 1)$, $p^* = (0, 1)$.

In this example, the continuity inclusion property does not hold. Consider agent 1 in the above quasi-equilibrium. Since $p^* \times e_1 = 0$, the budget set of agent 1 is $B_1(p^*) = \{(x_1, 0) \colon x_1 \in [0, 2]\}$. In addition, the set of allocations for agent 1 which are preferred to (x_1^*, y_1^*) is $P_1(x^*, y^*) = \{(x_1, y_1) \in [0, 2] \times [0, 2] \colon x_1 + y_1 > x_1^* + y_1^* = 1 + 0 = 1\}$. Thus, $\psi_1(p^*, (x^*, y^*)) = B_1(p^*) \cap P_1(x^*, y^*) = \{(x_1, 0) \colon x_1 \in (1, 2]\}$, which is nonempty.

However, if we slightly perturb the price p^* by assuming that it is $q = (\epsilon, 1 - \epsilon)$ for sufficiently small $0 < \epsilon < \frac{1}{4}$, then the budget set of agent 1 is $B_1(q) = \{(x_1, y_1) \in [0, 2] \times [0, 2] \colon x_1 \cdot \epsilon + y_1 \cdot (1 - \epsilon) \leq \frac{1}{2}\epsilon\}$, which implies that $x_1 \leq \frac{1}{2}$ and $y_1 \leq \frac{1}{2}\frac{\epsilon}{1-\epsilon} < \frac{1}{6}$. Thus, $x_1 + y_1 < \frac{1}{2} + \frac{1}{6} = \frac{2}{3} < 1$ for all $(x_1, y_1) \in B_1(q)$, which implies that $\psi_1(q, (x^*, y^*)) = B_1(q) \cap P_1(x^*, y^*) = \emptyset$.

Therefore, in any neighborhood O of $((x^*, y^*), p^*)$, there is a point $((x^*, y^*), q) \in O$ such that $\psi_1(q, (x^*, y^*)) = \emptyset$, which implies that the continuity inclusion property does not hold. It can be easily checked that the weaker condition discussed in Remark 8 still fails in this example.

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