

Edgeworth's conjecture and the number of agents and commodities

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Abstract We answer the question asked by Robert Aumann as to whether core equivalence depends on there being “many more agents than commodities.” We show that for a large class of commodity spaces, which might be infinite-dimensional and even non-separable, core equivalence is indeed equivalent to the presence of “many more agents than commodities” when allocations are Bochner integrable. By contrast, we show that in a classical model of an atomless economy with an infinite-dimensional commodity space, the model where the commodity space is $L_\infty(\mu)$ with the Mackey topology and allocations are Gelfand integrable, core equivalence holds in full generality, even when there are “many more commodities than agents.” The assumptions we make on economies are much weaker than what is commonly used in core equivalence results for infinite-dimensional commodity spaces and reduce to Aumann's original assumptions when there are finitely many commodities.

Keywords Core equivalence · Edgeworth's conjecture · Infinite-dimensional commodity space

JEL Classification C62 · C71 · D41 · D50

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1 Introduction

That in an economic environment with many insignificant agents all gains from trade are exhausted precisely at market outcomes is an idea going back to Edgeworth in the nineteenth century. The clearest and most natural formulation of this idea can be found in [Aumann \(1964\)](#). Taking an atomless measure space to represent the set of agents, so that in a mathematically precise sense there are many agents, each of them being negligible in the set of all agents, Aumann showed in his core equivalence theorem, under surprisingly weak assumptions, that the core allocations of an economy are exactly the Walrasian allocations. In Aumann's model, the number of commodities is finite. Of course, with finitely many commodities, an atomless measure space of agents does not just mean "many agents," but actually means "many more agents than commodities." One may see "many more agents than commodities" as an assumption on top of that of "many agents" and may ask whether it is important for core equivalence that the former assumption holds in addition to the latter. The issue was raised by Aumann himself. As Mertens writes in the 1991 reprint of his core equivalence paper from 1970: "If I remember correctly that conversation with R. Aumann, he was stressing the importance of going beyond the separable case [...] to check whether equivalence did not depend on there being (many) more traders than commodities."

It was first shown by [Tourky and Yannelis \(2001\)](#) that the "many more agents than commodities" aspect of Aumann's model indeed matters for core equivalence, even when the space of agents is atomless. Assuming the generalized continuum hypothesis, Tourky and Yannelis showed that, given a vector space E whose algebraic dimension is at least the cardinal of the continuum, there are a vector order and a norm on E , so that E becomes an ordered non-separable Hilbert space with positive cone having non-empty interior, and an economy with an atomless measure space of agents and E as commodity space such that—allocations taken to be Bochner integrable—all assumptions made in [Aumann \(1964\)](#) hold, but such that core equivalence fails. Extending the scope of the result in [Tourky and Yannelis \(2001\)](#), it was shown in [Podczeck \(2003\)](#) that core equivalence can fail whenever the commodity space is a non-separable Banach space.

Of course, the algebraic dimension of a linear space provides a natural way to extend the usual notion of "number of commodities" to infinite-dimensional commodity spaces. However, there is a problem. The algebraic dimension of every infinite-dimensional Banach space is at least the cardinal of the continuum, even without the continuum hypothesis.¹ On the other hand, there are (positive) core equivalence results for commodity spaces that are separable infinite-dimensional Banach spaces, e.g., [Rustichini and Yannelis \(1991\)](#). A main factor behind these results is that in a separable commodity space, trades among agents can be approximated by points in some countable subset, so that, with continuous preferences, all welfare gains through trading can be approximately achieved in a countable subset of the commodity space. From this perspective, a separable infinite-dimensional commodity space looks as if there were just countably many commodities. So, one may view an atomless economy

¹ This was first shown in [Mackey \(1945\)](#). An alternative proof can be found in [Lacey \(1973\)](#).

where the commodity space is an infinite-dimensional but separable Banach space as an economy in which there are “many more agents than commodities,” even though the algebraic dimension of the commodity space is not smaller than the cardinal of the continuum. In particular, if one wants to interpret core equivalence as a manifestation of “many more agents than commodities,” one cannot use the algebraic dimension of a commodity space as the notion of the number of commodities.

In this paper, we formalize the idea of “many more agents than commodities” as a condition that is satisfied in an atomless economy whenever the commodity space is separable; see Sect. 4. In Sect. 5, we then consider the case in which the commodity space is an ordered Banach space whose positive cone has non-empty interior, and allocations are taken to be Bochner integrable (the setting treated in [Tourky and Yannelis 2001](#); [Podczeck 2003](#)) and show that under assumptions on the primitives of an economy exactly as in [Aumann \(1964\)](#), core equivalence is indeed equivalent to the presence of “many more agents than commodities,” without any special set-theoretic assumptions in the background.

There are some subtleties, concerning the order of the commodity space and the fact that the assumptions in [Aumann \(1964\)](#) do not include transitivity of preferences. For a discussion of these points and for relevant results, see Sect. 5. In Sect. 6, we show that the identification of core equivalence with “many more agents than commodities” also holds for many economically relevant commodity spaces which are not covered by the previous results, in particular for the L_p -spaces, $1 \leq p < \infty$.

In Sect. 7, we revisit the setting of [Mertens' \(1970\)](#) core equivalence result. The commodity space is $L_\infty(\mu)$ with the Mackey topology and feasibility is defined in terms of the Gelfand integral, viewing $L_\infty(\mu)$ as the dual of $L_1(\mu)$. In particular, continuity of preferences is required with respect to the Mackey topology. It is assumed in [Mertens \(1970\)](#) that $L_\infty(\mu)$ with the Mackey topology is separable; thus, there are “many more agents than commodities.” We show, however, that core equivalence holds without this assumption. In fact, there might be many more commodities than agents. This also answers a question raised by [Mertens](#). By weakening some assumptions made by [Mertens](#), we actually get a core equivalence result that reduces to that of [Aumann \(1964\)](#) when the number of commodities is finite.

So, the notion of integral chosen to define feasibility of allocations matters for core equivalence. The crucial difference in this regard between the Gelfand and the Bochner integral is that with the latter integral, an allocation must be strongly measurable and therefore essentially separably valued. If the commodity space itself is not separable, this means that Bochner integrable allocations take values in thin subspaces of the commodity space, which, compared with the case of Gelfand integrable allocations, makes the blocking possibilities of any coalition rather limited.

The organization of this paper is as follows. The next section contains the principal definitions. Section 3 contains a restatement of [Aumann's \(1964\)](#) core equivalence theorem, which is the point of reference for our results, which are stated in Sects. 5–7, after a Sect. 4 where our formalization of the notion of “many more agents than commodities” may be found. The proofs of our results are given in Sects. 8–11. In Appendix 1, connections between some of the assumptions we use and assumptions known from the literature are established, and in Appendix 2, some mathematical background information is provided.

2 Notation and terminology

(1) Let E be a linear topological space. Assume that some notion of integrability for functions from measure spaces to E is given. Now an *economy* \mathcal{E} with commodity space E is given by a list $\mathcal{E} = [(T, \mathcal{T}, \nu), \langle X_t, \succ_t, e(t) \rangle_{t \in T}]$ where

- (T, \mathcal{T}, ν) is a complete probability space, representing the space of agents;
- $X_t \subseteq E$ is the consumption set of agent t ;
- $\succ_t \subseteq X_t \times X_t$ is the (strict) preference relation of agent t ;
- $e(t) \in E$ is the initial endowment of agent t ;
- the endowment map $e: T \rightarrow E$, given by $t \mapsto e(t)$, is integrable.

The economy \mathcal{E} is said to be *atomless* if the probability space (T, \mathcal{T}, ν) of agents is atomless.

An *allocation* in the economy \mathcal{E} is an integrable function $f: T \rightarrow E$ such that $f(t) \in X_t$ for almost all $t \in T$. An allocation f is said to be *feasible* if

$$\int_T f(t) \, d\nu(t) = \int_T e(t) \, d\nu(t).$$

Price systems are continuous linear functionals on the commodity space E . A *Walrasian equilibrium* for the economy \mathcal{E} is a pair (p, f) , where f is a feasible allocation and p is a price system, such that for almost every $t \in T$:

- (i) $pf(t) \leq pe(t)$ and
- (ii) if $x \succ_t f(t)$ then $px > pe(t)$.²

A feasible allocation f is said to be a *Walrasian allocation* if there is a price vector p such that the pair (p, f) is a Walrasian equilibrium. An allocation f is a *core allocation* if it is feasible and if it cannot be *blocked* by any non-negligible coalition, i.e., if there is no $S \in \mathcal{T}$ with $\nu(S) > 0$ and no allocation $g: T \rightarrow E$ such that

- (i) $\int_S g(t) \, d\nu(t) = \int_S e(t) \, d\nu(t)$, i.e., g is feasible for S , and
- (ii) $g(t) \succ_t f(t)$ for almost all $t \in S$.

The *core* is the set of all core allocations.

(2) We need to fix some additional notation and terminology.

(a) If E is a linear topological space, E^* denotes the (topological) dual space, i.e., the space of all continuous linear functionals on E .

(b) If E is an ordered linear space, we write \geq for the order of E , and E_+ for the positive cone; thus, $E_+ = \{x \in E: x \geq 0\}$. Elements of E_+ are called *positive*. We write $x > y$ to mean “ $x \geq y$ and $x \neq y$.”

(c) If E is an ordered linear topological space, i.e., an ordered linear space endowed with a linear space topology such that the positive cone E_+ is closed, then E^* is always viewed as being endowed with the dual order defined from E ; thus for any

² To avoid unnecessary clutter of parentheses, we frequently write qx , rather than $q(x)$, for the value of a linear functional q at a point x in its domain when no confusion can arise.

$p, q \in E^*, p \geq q$ means $px \geq qx$ for all $x \in E_+$, so that, in particular, $E_+^* = \{p \in E^* : px \geq 0 \text{ for all } x \in E_+\}$.

(d) Let E be an ordered linear topological space.

(i) An element $x \in E$ is called *strictly positive* if $x \geq 0$ and $px > 0$ for every $p \in E^*$ with $p > 0$. Of course, if $E = \mathbb{R}^\ell$ with the usual topology and order, then this definition says nothing else than that a vector is strictly positive if it is larger than zero in each coordinate. A more general fact is that if E is locally convex and E_+ has non-empty interior, then the strictly positive vectors in E are the interior points of E_+ (see Aliprantis and Tourky 2007, Lemma 2.17, p. 73).

(ii) A linear functional on E , in particular an element $p \in E^*$, is called *strictly positive* if $px > 0$ for every $x \in E$ with $x > 0$.

(iii) Note that if E_+ has non-empty interior, or if E is a Banach lattice, then any positive linear functional on E is continuous, i.e., belongs to E^* .

(e) A convex subset H of a cone Λ in a vector space E is a *base* of Λ if for each $x \in \Lambda \setminus \{0\}$, there is a unique $h \in H$ and a unique $\lambda > 0$ such that $x = \lambda h$; if E is a normed space, then such a set H is said to be a *bounded base* of the cone Λ if H is a norm-bounded subset of E .

Remark 1 Let E be an ordered normed space and suppose that E_+ has a base H . Then by Aliprantis and Tourky (2007, Theorem 1.47 and Exercise 2, p. 42), there is a strictly positive linear functional q on E such that $H = \{x \in E_+ : qx = 1\}$. If E_+ has an interior point, such a q must be continuous, i.e., must be an element of E_+^* . It follows from these two facts that if E_+ has a bounded base and an interior point, then the dual cone E_+^* must have non-empty interior for the dual norm.

Remark 2 The notion of “strictly positive” as stated in (d)(i) involves the topology of E in an essential way. For example, if (Ω, Σ, μ) is a σ -finite measure space, then for $L_\infty(\mu)$ with the Mackey topology defined from the duality with $L_1(\mu)$, an element $p \in L_\infty(\mu)$ is strictly positive if (for any of its versions) $p(\omega) > 0$ a.e. in Ω , while for $L_\infty(\mu)$ with the $\|\cdot\|_\infty$ -topology, a $p \in L_\infty(\mu)$ is strictly positive if and only if there is an $\varepsilon > 0$ such that $p(\omega) > \varepsilon$ a.e. in Ω .

3 Aumann's classical core equivalence result

In Aumann's (1964) model, the commodity space E is \mathbb{R}^ℓ (with the usual topology and order) and allocations are Lebesgue integrable functions. The following assumptions are made by Aumann (1964):

(P) (Positive consumption) For each $t \in T, X_t = E_+$.

(D) (Desirability) For each $t \in T, \succ_t$ is strictly monotone, i.e., if $x, y \in X_t$ and $x > y$ then $x \succ_t y$.

(C) (Continuity) For each $t \in T$ and each $x \in X_t$, the set $\{y \in X_t : y \succ_t x\}$ is open in X_t .

(AM) (Aumann measurability) If f, g are allocations, then $\{t \in T : f(t) \succ_t g(t)\}$ is a measurable set.

(EC) (Endowments are consumption vectors) For each $t \in T, e(t) \in X_t$.

(RA) (Resource availability) The aggregate endowment $\int_T e(t) \, d\nu(t)$ is strictly positive.

Here is Aumann's (1964) classical core equivalence result.

Theorem 1 *Let \mathcal{E} be an atomless economy with commodity space $E = \mathbb{R}^\ell$. If (P), (D), (C), (AM), (EC), and (RA) are satisfied, then the core of the economy \mathcal{E} coincides with the set of Walrasian allocations.*

A few remarks are in order. In Aumann (1964), the continuity assumption on preferences is stated in a form slightly stronger than (C); not only the upper sections of the preference relations are assumed to be open, but also the lower sections. However, in the proof given by Aumann, only the former assumption is used.

Instead of assuming (AM), several papers on core equivalence make a weaker measurability assumption in which only constant allocations are compared. We refer to Podczeck (2004) for a discussion of the relationship between these two measurability assumptions.

There is a growing literature on core equivalence in atomless economies with asymmetric information among agents; see, e.g., Einy et al. (2001), Angeloni and Martins-da Rocha (2009), and Hervés-Beloso et al. (2005). We will not investigate this context here.

4 Many more agents than commodities

In this section, we want to give the phrase “many more agents than commodities” a precise definition. For this, we need some more notation.

(a) We write \mathfrak{c} for the cardinal of the continuum, i.e., the cardinal of \mathbb{R} .

(b) If E is a linear space, we write $\dim^a(E)$ for the algebraic dimension of E , i.e., the least cardinal of any subset of E with a full linear span, and if E is actually a linear topological space, we write $\dim(E)$ for the least cardinal of any subset of E with a dense linear span. As noted in the introduction, $\dim^a(E) \geq \mathfrak{c}$ if E is an infinite-dimensional Banach space.

(c) Given an atomless probability space (T, \mathcal{T}, ν) , we write $\mathcal{N}(\nu)$ for the ideal of null sets in T , and $\text{add } \mathcal{N}(\nu)$ for the least cardinal of any family in $\mathcal{N}(\nu)$ whose union is not in $\mathcal{N}(\nu)$. Note that $\text{add } \mathcal{N}(\nu) \leq \mathfrak{c}$. (To see this, let $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$ be a sequence of partitions of T into measurable subsets such that for each n , $\nu(A) = 2^{-n}$ for each $A \in \mathcal{T}_n$. Then the non-empty sets of the form $\bigcap_n A_n$, where $A_n \in \mathcal{T}_n$ for each n , form a partition of T into no more than \mathfrak{c} null sets.) But of course, $\text{add } \mathcal{N}(\nu)$ is uncountable because the union of countably many null sets is a null set.

Now let (T, \mathcal{T}, ν) be an atomless probability space of agents, and E a linear topological space, taken to be the commodity space. In Tourky and Yannelis (2001), $\dim^a(E)$ is the notion of number of commodities, and “many more agents than commodities” is formally expressed by saying that $\text{add } \mathcal{N}(\nu) > \dim^a(E)$.³ However, as

³ Actually, to give the “number of agents” a formal expression, Tourky and Yannelis (2001) define a cardinal for a probability space, which they call the “weight” of this space. Now if a probability space (T, \mathcal{T}, ν) is

Footnote 3 continued

pointed out in the introduction, one may well view an atomless economy where the commodity space is a separable infinite-dimensional Banach space as an economy with “many more agents than commodities.” But this is incompatible with the formalization above, because $\text{add } \mathcal{N}(\nu) \leq \mathfrak{c}$ but $\text{dim}^a(E) \geq \mathfrak{c}$ if E is an infinite-dimensional Banach space.

A notion of the “number of commodities” which is more permissive than $\text{dim}^a(E)$ is provided by $\text{dim}(E)$. Informally, while $\text{dim}^a(E)$ amounts to a notion of the number of commodities in physical terms, $\text{dim}(E)$ amounts to a notion of the number of commodities in economic terms, incorporating certain substitutability relations among commodities. Of course, if $E = \mathbb{R}^\ell$, then $\text{dim}(E) = \text{dim}^a(E)$, and $\text{dim}(E)$ is just the number of commodities in the usual sense. Here is our formalization of “many more agents than commodities,” following that in [Tourky and Yannelis \(2001\)](#) concerning the side of the agents, but with $\text{dim}(E)$ in place of $\text{dim}^a(E)$ on the side of the commodities.

Definition 1 An atomless economy satisfies (MMATC) (has “many more agents than commodities”) if $\text{add } \mathcal{N}(\nu) > \text{dim}(E)$, where E is the commodity space of the economy and ν is the measure on the set of agents.

Now according to this definition, there are “many more agents than commodities” in an atomless economy whenever the commodity space is separable. But it should be noted that (MMATC) may not imply separability of the commodity space. As an example of what is conceivable, take the unit interval with Lebesgue measure for the space of agents, and recall that under Martin’s axiom, $\text{add } \mathcal{N}(\nu) = \mathfrak{c}$ if ν is Lebesgue measure, and that Martin’s axiom is compatible with \mathfrak{c} being an arbitrarily large cardinal (see [Jech 2003](#), Theorem 16.13 and Corollary 26.41, or Appendix 2). So, we can have plenty of uncountable cardinals κ for which (MMATC) holds with $\text{dim}(E) = \kappa$.

5 Core equivalence when the commodity space is an ordered Banach space whose positive cone has non-empty interior

In this section, we adopt the setting considered by [Tourky and Yannelis \(2001\)](#) to relate the core equivalence problem to the notion of “many more agents than commodities:” The commodity space E is an ordered Banach space with positive cone having non-empty interior, and allocations are taken to be Bochner integrable.

The first theorem in this section shows that, in this setting, Aumann’s classical core equivalence result carries over verbatim to an infinite-dimensional commodity space provided that there are “many more agents than commodities.” As this latter condition trivially holds if the commodity space is \mathbb{R}^ℓ , the theorem contains Aumann’s result as a special case. In particular, the theorem significantly strengthens the core equivalence result of [Tourky and Yannelis \(2001\)](#), showing that, as in Aumann’s result, transitivity and irreflexivity of preferences are not needed. Recall that (MMATC) may not imply that the commodity space is separable.

atomless, this cardinal is the same as $\text{add } \mathcal{N}(\nu)$. We will use this latter expression because it is standard terminology in measure theory and has a more compact definition.

Theorem 2 *Let E be an ordered Banach space whose positive cone E_+ has non-empty interior and let \mathcal{E} be an atomless economy with commodity space E such that (P), (D), (C), (AM), (EC), (RA), and (MMATC) are satisfied. Then the core of the economy \mathcal{E} coincides with the set of Walrasian allocations.*

In fact, in the context of Theorem 2, the assumption of “many more agents than commodities” is pivotal for core equivalence:

Theorem 3 *Let (T, \mathcal{T}, ν) be an atomless probability space, and E an ordered Banach space such that E_+ has non-empty interior and E^* has strictly positive elements. If $\text{add } \mathcal{N}(\nu) \leq \dim(E)$, then there is an economy \mathcal{E} with (T, \mathcal{T}, ν) as space of agents and E as commodity space such that (P), (D), (C), (AM), (EC), and (RA) are satisfied but such that there is a core allocation which is not Walrasian.*

This remains true if all individual endowments are required to be strictly positive.

Together, Theorems 2 and 3 show that (MMATC) is essentially equivalent to core equivalence. In the proof of Theorem 3, strictly positive prices are used to construct strictly monotone preferences. Of course, if the commodity space is such that there are no strictly positive prices, then there can be no economies with strictly monotone preferences and a Walrasian equilibrium, and the question addressed by Theorem 3 becomes pointless.

In the context of Theorems 2 and 3, preferences need not be transitive. Transitivity may be seen as a reasonable property of preferences. So, let us state this property formally as an assumption.

(Tr) For each $t \in T$, \succ_t is transitive.

The next theorem, together with the subsequent example, shows that if (Tr) is assumed, then, in the context of Theorem 2, “many more agents than commodities” need no longer be a prerequisite for core equivalence. As may be inferred from the proof, the reason is that the effects of strict monotonicity of preferences can be strengthened by transitivity, so that it becomes easier for any non-negligible group of agents to find common directions of improvement.

Theorem 4 *Let K be a compact Hausdorff space and let the commodity space E be $C(K)$ with the usual norm and order. Let \mathcal{E} be an atomless economy satisfying (P), (D), (C), (AM), (EC), (RA), and (Tr). Let $M_+^1(K)$ be the set of tight Borel probability measures on K . If $M_+^1(K)$ is first countable in the weak*-topology, then the core of the economy \mathcal{E} coincides with the set of Walrasian allocations.*

Example Let $E = C(I^\sharp)$ where I^\sharp is the split interval (or “double arrow space”), i.e., $I^\sharp = (0, 1] \times \{0\} \cup [0, 1) \times \{1\} \subseteq \mathbb{R}^2$, endowed with the lexicographical order topology; see, e.g., Engelking (1989, 3.10.C, p. 212). The space I^\sharp is a separable compact Hausdorff space, and as shown in Pol (1982), the space $M_+^1(I^\sharp)$ of tight Borel probability measures on I^\sharp is weak*-first countable. Separability of I^\sharp implies that $M_+^1(I^\sharp)$ has elements with full support. Thus, the dual of $C(I^\sharp)$ has strictly positive elements, so Assumptions (P) and (D) are simultaneously satisfiable for $C(I^\sharp)$ as commodity space. Thus, economies satisfying the assumptions in Theorem 4 for $C(I^\sharp)$ as commodity space do exist. Now the order of I^\sharp has continuum many “jumps,” so the

weight of I^\sharp is \mathfrak{c} . This implies that $\dim(C(I^\sharp)) = \mathfrak{c}$ (use Engelking 1989, 3.2.I, p. 147, together with Fact 1 in Sect. 8). Hence, as $\text{add } \mathcal{N}(v) \leq \mathfrak{c}$ for any atomless probability measure, (MMATC) fails for any atomless economy with commodity space $C(I^\sharp)$.

Now in the results on failure of core equivalence presented in Tourky and Yannelis (2001) and Podczeck (2003), preferences actually have continuous utility representations and are, in particular, transitive. So, one may ask for the relationship between these results and Theorem 4. The point is that in the core non-equivalence results in Tourky and Yannelis (2001) and Podczeck (2003), the positive cone of the commodity space has a bounded base [see 2(2)(e)] in addition to having a non-empty interior. On the other hand, in Theorem 4 the commodity space is Banach lattice. It is a well-known fact that no infinite-dimensional Banach lattice can have simultaneously a non-empty interior and a bounded base,⁴ and therefore there is no contradiction between Theorem 4 and the core non-equivalence results in Tourky and Yannelis (2001) and Podczeck (2003).

As noted in the introduction, in these non-equivalence results the continuum hypothesis is assumed.⁵ Of course, if the continuum hypothesis holds, then our condition (MMATC) is equivalent to the commodity space being separable. In view of this, the continuum hypothesis is not just an innocent assumption, but imposes severe restrictions on the scope of our notion of “many more agents than commodities.” It is therefore desirable to avoid the continuum hypothesis. Our next theorem shows that if the positive cone of the commodity space has a bounded base, as in the above mentioned results of Tourky and Yannelis (2001) and Podczeck (2003), then Theorem 3 remains true for preferences with continuous utility representations, without any need for the continuum hypothesis.

(UR) For each $t \in T$, \succ_t has a continuous utility representation.

Theorem 5 *Let the commodity space E be an ordered Banach space such that E_+ has non-empty interior. If E_+ has a bounded base, then given an atomless economy \mathcal{E} , conditions (P), (D), (C), (AM), (EC), (RA), and (UR) together imply core equivalence if and only if (MMATC) is satisfied.*

This remains true if all individual endowments are strictly positive.

To compare Theorems 4 and 5, recall that if E is an ordered Banach space such that E_+ has a bounded base and non-empty interior, then the dual cone E_+^* must have non-empty interior for the dual norm (see Remark 1), whereas if E is an infinite-dimensional Banach lattice such that E_+ has non-empty interior, then E_+^* has empty interior. Thus, if preferences are strictly monotone, so that the members of E_+^* reflect

⁴ To see this, use Remark 1 and the fact that if E is an infinite-dimensional Banach lattice such that E_+ has non-empty interior, then the dual cone E_+^* has empty norm-interior.

⁵ Actually, in Tourky and Yannelis (2001), GCH (the generalized continuum hypothesis) is assumed, and in addition, it is assumed that the algebraic dimension of the commodity space is a regular cardinal. But this excludes many possible commodity spaces from the analysis. For example, let A be a set such that $\#(A)$ is a singular cardinal with uncountable cofinality, such as ω_{ω_1} . Then $\#(A) \geq \mathfrak{c}$ by CH. Let $E = \ell_2(A)$. Then $\#(A) \leq \dim^a(E) \leq \#(E) \leq \mathfrak{c}^\omega \cdot \#(A)^\omega = \#(A)^\omega$. Since $\text{cf}(\#(A)) > \omega$, GCH implies that $\#(A)^\omega = \#(A)$, so $\dim^a(E) = \#(A)$, a singular cardinal. Note that it is relatively consistent that GCH holds and every uncountable limit cardinal is singular, i.e., is not weakly inaccessible (Kunen 2011, Corollary II.6.26).

the possible marginal rates of substitution, then the latter case may be seen as imposing more restrictions on the possible diversity of preferences than the former.

Remark 3 A vector order such that the positive cone has both a bounded base and a non-empty interior exists on every Banach space. For example, given a Banach space E , let Λ be an “ice cream cone” in E , i.e., a closed convex cone of the form

$$\Lambda = \{x \in E : qx \geq \varepsilon \|x\|\}$$

where $q \in E^*$ is of norm 1 and ε is a number with $0 < \varepsilon < 1$ (see [Aliprantis and Tourky 2007](#), p. 99). Setting $x \geq y$ whenever $x - y \in \Lambda$ defines a vector order on E such that $E_+ = \Lambda$. The fact that q is of norm 1 and $\varepsilon < 1$ implies that Λ has non-empty interior. Evidently the set $H = \{x \in \Lambda : qx = 1\}$ is a base of Λ . Moreover, if $x \in H$, then $\|x\| \leq 1/\varepsilon$, and thus H is bounded.

In view of this remark, we have the following corollary of [Theorem 5](#), which parallels the statement of the core non-equivalence results in [Tourky and Yannelis \(2001\)](#) and [Podczeck \(2003\)](#) (but without the continuum hypothesis).

Corollary *Let E be a Banach space and (T, \mathcal{T}, ν) an atomless probability space. Suppose $\dim(E) \geq \text{add } \mathcal{N}(\nu)$. Then there is a vector order on E , for which E is an ordered Banach space with $\text{int } E_+ \neq \emptyset$, and an economy \mathcal{E} , with (T, \mathcal{T}, ν) as space of agents and E as commodity space, such that (P), (D), (C), (AM), (EC), (RA), and (UR) hold, but such that there is a core allocation which is not Walrasian.*

This is the case even when all individual endowments are required to be strictly positive.

6 Core equivalence when the commodity space is an order-continuous Banach lattice

Infinite-dimensional commodity spaces arising in applications often require dealing with consumption sets having empty interior. Such contexts are not covered by the treatment in the previous section. In this section, we allow for consumption sets with empty interior. We will restrict attention to the framework where the commodity space is a Banach lattice with order-continuous norm (in short, an order-continuous Banach lattice). This framework is sufficiently general for the core equivalence problem to be of interest when consumption sets may have empty interior. Note that the L_p -spaces for $1 \leq p < \infty$ are covered.

It is well known that if the commodity space is infinite-dimensional and consumption sets have empty interior, then continuity of preferences does not guarantee appropriate bounds on marginal rates of substitution in order for preferred sets to have supporting price vectors. As a consequence, core equivalence may fail through marginal rates of substitution that are not properly bounded; see [Example 5.1](#) in [Rustichini and Yannelis \(1991\)](#). To avoid such a technical failure of core equivalence, which is unrelated to whether or not there are “many more agents than commodities,” we will employ the following assumption, which is a rephrasing of the notion of “extremely desirable commodity” used in the core equivalence result in [Rustichini and Yannelis \(1991\)](#).

(EDC) (Extremely desirable commodity) There are a $v \in E_+ \setminus \{0\}$ and a strictly positive $q \in E^*$ such that for each $t \in T$ and any number $\lambda > 0$, $x + \lambda(v + u) \succ_t x$ whenever $x \in X_t$ and $u \in E$ are such that $x + \lambda(v + u) \in X_t$ and $q|u| < 1$.

Here $|u|$ is the absolute value of u , i.e., the supremum of u and $-u$. Actually, in [Rustichini and Yannelis \(1991\)](#) the notion of “extremely desirable commodity” is stated for the case in which the consumption sets of the agents are equal to the positive cone of the commodity space. In [Appendix 1\(A\)](#), we show that, in this case, the notion of “extremely desirable commodity” in the statement of (EDC) is indeed equivalent to that in [Rustichini and Yannelis \(1991\)](#). For us, the version we use is easier to operate with.

Another assumption that has been used in the context of core equivalence to deal with consumption sets with empty interior is as follows.

(US) (Uniform substitutability) There are strictly positive linear functionals a and $b \in E^*$, with $a \leq b$, such that for each $t \in T$, $x + v - u \succ_t x$ whenever $x \in X_t$ and $u, v \in E_+$ are such that $x + v - u \in X_t$ and $av > bu$.

This condition was developed by [Zame \(1986\)](#). In (B) in [Appendix 1](#), we show that (US) implies (EDC). Observe that if $E = \mathbb{R}^\ell$, then (EDC) is automatically satisfied if (P) and (D) are. This is not true of (US).

As in the previous section, allocations are taken to be Bochner integrable in the following theorem.

Theorem 6 *Let the commodity space E be an order-continuous Banach lattice such that E_+ has strictly positive elements. Then given an atomless economy \mathcal{E} , Assumptions (P), (D), (C), (AM), (EC), (RA), (Tr), and (EDC) together imply core equivalence if and only if (MMATC) is satisfied.*

This remains true if all individual endowments are strictly positive and if (EDC) is strengthened to (US), and (Tr) to (UR).

Remark 4 The sufficiency part of [Theorem 6](#) remains true if the requirement that E be order-continuous is dropped; see the proof of this theorem. However, by [Theorem 4](#) and the example following the statement of that theorem, the necessity part does not hold in all Banach lattices.

7 Core equivalence when the commodity space is $L_\infty(\mu)$ with the Mackey topology

In this section, the commodity space E is taken to be $L_\infty(\mu)$ endowed with the Mackey topology defined from the usual pairing of $L_\infty(\mu)$ with $L_1(\mu)$, the measure μ being σ -finite, and allocations are taken to be Gelfand integrable, viewing $L_\infty(\mu)$ as the (norm) dual of $L_1(\mu)$. Recall that the usual pairing of $L_\infty(\mu)$ and $L_1(\mu)$ is given by integration, i.e., $px = \int_\Omega p \times x \, d\mu$, $p \in L_1(\mu)$, $x \in L_\infty(\mu)$ and that the Mackey topology of $L_\infty(\mu)$ for this pairing is the strongest locally convex topology on $L_\infty(\mu)$ such that $L_1(\mu)$ is the (topological) dual. In the sequel, we will refer to this topology simply as the Mackey topology of $L_\infty(\mu)$.

According to the definitions in Sect. 2, price vectors are taken to be elements of the topological dual of the commodity space, so, as in [Mertens \(1970\)](#), they are now required to be in $L_1(\mu)$. Furthermore, assumptions (C) and (RA), which involve topological notions, have now to be interpreted in the Mackey topology of $L_\infty(\mu)$. In particular, (RA) does not imply that the aggregate endowment of an economy is strictly positive for the $\|\cdot\|_\infty$ -topology of $L_\infty(\mu)$.

Remark 5 It has been argued in [Tourky and Yannelis \(2001, Remark10.1\)](#) that weak integrals such as the Gelfand or Pettis integral may be inappropriate to define feasibility of allocations, because there could be functions with integral equal to zero that are everywhere positive and nonzero. (E.g., let (T, \mathcal{T}, ν) be the unit interval with Lebesgue measure, let E be the Hilbert space $\ell_2([0, 1])$, and define $h: T \rightarrow E$ by setting $h(t) = 1_{\{t\}}$ for each $t \in T$.) Under Assumptions (P) and (D), adding such a function to a given allocation improves every agent without affecting feasibility. As a consequence of this kind of free lunch, the core would be empty and core equivalence holds trivially. This is not a problem in the setting defined above. As the measure μ is σ -finite, there is a strictly positive element $p \in L_1(\mu)$. Now for $x \in E_+$, $px = 0$ can happen only if $x = 0$. So, under Assumption (P), an allocation with Gelfand integral equal to zero must be zero almost everywhere in T . See (D) of [Appendix 1](#) for a more precise statement.

The following assumption, which is stronger than Assumption (RA) if $L_\infty(\mu)$ is infinite-dimensional, is implied by the assumptions made in [Mertens \(1970\)](#) on the endowments in an economy [see (C) in [Appendix 1](#)].

(TAE) (Thick aggregate endowment) There is a non-decreasing sequence (e_n) of $\|\cdot\|_\infty$ -Bochner integrable functions $e_n: T \rightarrow E_+$ such that $e_n(t) \rightarrow e(t)$ in the Mackey topology a.e. in T and such that $\int_T e_n(t) d\nu(t) \in \|\cdot\|_\infty$ -int E_+ for each n .⁶

Observe that (TAE) does not imply the initial allocation to be $\|\cdot\|_\infty$ -Bochner integrable. In fact, the initial allocation need not even be $\|\cdot\|_\infty$ -strongly measurable. But of course, (TAE) implies that the aggregate endowment of an economy belongs to the $\|\cdot\|_\infty$ -interior of E_+ and that there is a Mackey-separable subset of E_+ which contains the individual endowment of almost all agents. In these latter two aspects, (TAE) is stronger than (RA).

In view of the fact that the indefinite Bochner integrals of the functions e_n are ν -continuous, (TAE) says that small groups of agents can have no corner (i.e., monopoly power) on the market, in the sense that, for some $\varepsilon > 0$, if $T' \subseteq T$ is such that $\nu(T') < \varepsilon$, then $\int_{T \setminus T'} e(t) d\nu(t) \in \|\cdot\|_\infty$ -int E_+ , i.e., the aggregate endowment of the complementary group $T \setminus T'$ contains all commodities.

Note that (TAE) does not imply that the individual endowments $e(t)$ are in the $\|\cdot\|_\infty$ -interior of E_+ . In this aspect, (TAE) is actually more general than the assumptions on endowments made in [Mertens \(1970\)](#). In fact, when $E = \mathbb{R}^\ell$ and (EC) holds, then (TAE) is equivalent to (RA).

⁶ Bochner integrability of the maps e_n implies that they are also Gelfand integrable; moreover, the integrals according to these two notions of integrability agree. Thus, the statement of (TAE) is consistent with the present setting where allocations are taken to be Gelfand integrable.

In the following core equivalence result, (MMATC) does not play any role; the “size” of the commodity space may be arbitrarily large, without any relation to properties of the space of agents. Note in this regard that for any cardinal κ , there is a probability measure μ such that least cardinal of any subset of $L_\infty(\mu)$ whose linear span is dense in $L_\infty(\mu)$ for the Mackey topology is larger than κ . (Just look at the usual measure on $\{0, 1\}^\kappa$, where κ is an arbitrary cardinal.)

Theorem 7 *Let (Ω, Σ, μ) be a σ -finite measure space and let the commodity space E be $L_\infty(\mu)$ with the Mackey topology. Let \mathcal{E} be an atomless economy with commodity space E such that (P), (D), (C), (AM), and (TAE) are satisfied. Then the core of the economy \mathcal{E} coincides with the set of Walrasian allocations.*

As noted above, if $E = \mathbb{R}^\ell$ and (EC) is satisfied, then (TAE) is equivalent to (RA), so Theorem 7 includes Aumann's core equivalence result as a special case. In the core equivalence result of Mertens (1970), individual endowments are assumed to belong to $\|\cdot\|_\infty$ -int E_+ ; also, the desirability assumption includes a transitivity requirement. For these reasons, Merten's result does not include that of Aumann.

As already noted in the introduction, in Mertens (1970) the commodity space is assumed to be separable for the Mackey topology. However, Mertens asked whether this assumption may be dropped. Our result above gives an affirmative answer.

As also noted above, assumption (TAE) is stronger than (RA). The next theorem shows that if the assumptions on preferences are strengthened so as to include (EDC) and (Tr), then core equivalence holds again with the standard assumptions (RA) and (EC), and (TAE) is not needed.⁷

Theorem 8 *Let (Ω, Σ, μ) be a σ -finite measure space and let the commodity space E be $L_\infty(\mu)$ with the Mackey topology. Let \mathcal{E} be an atomless economy with commodity space E . If (P), (D), (C), (AM), (RA), (EC), (Tr), and (EDC) are satisfied, then the core of the economy \mathcal{E} coincides with the set of Walrasian allocations.*

Note that, given some price system, optimal consumptions of agents, and hence their demands, reflect marginal rates of substitution. Assumption (EDC) imposes bounds on marginal rates of substitution, uniformly across the agents in an economy. Thus, this assumption may be seen as implying that small groups of agents cannot have monopsony power in the market. As remarked above, (TAE) may be seen as an assumption implying that small groups of agents cannot have monopoly power in the market. Thus, what drives the two core equivalence results in this section seems to be that the assumptions imply that there is some kind of “thickness” on at least one side of the market.

8 Preliminaries for the proofs

In this section, we introduce further notation, record two basic facts, and make some general preparing remarks for the proofs.

⁷ In the statement of (EDC), the functional q is required to be in the topological dual of the commodity space, which now means that this functional must be an element of $L_1(\mu)$.

(1) (a) Given a measure space (T, \mathcal{T}, ν) , we write ν^* for the outer measure defined from ν .

(b) If Z is a topological space and $A \subseteq Z$, then $\text{int } A$ denotes the interior of A , and $\text{cl } A$ or \bar{A} the closure of A .

(c) If E is a linear topological space and $A \subseteq E$, then $\text{co } A$ denotes the convex hull of A , and $\overline{\text{co}} A$ the closed convex hull; further, $\text{span } A$ denotes the linear span of A , i.e., the set of all (finite) linear combinations of members of A .

(d) For elements x, y of a Riesz space X , the expressions $x^+, x^-, |x|, x \vee y, x \wedge y$, and $x \perp y$ have the usual lattice theoretic meaning; we refer to [Aliprantis and Tourky \(2007\)](#) for this as well as for the Riesz space-related facts that will be used in some of the proofs below.

(2) If Z is a topological space and $A \subseteq Z$, then $\text{dens}(A)$ denotes the density character of A , i.e., the least cardinal of any subset of A which is dense in A .

Recall that for a linear topological space E , $\text{dim}(E)$ is defined to be the least cardinal of any set $A \subseteq E$ such that $\text{span } A$ is dense in E . By the following fact, we may replace the cardinal $\text{dim}(E)$ in the definition of (MMATC) by $\text{dens}(E)$.⁸

Fact 1 *If E is a linear topological space, then $\text{dens}(E) = \text{dim}(E)$ if $\text{dim}(E)$ is infinite.*

(Indeed, it is clear that $\text{dim}(E) \leq \text{dens}(E)$. For the reverse inequality, note that if $A \subseteq E$ and $\text{span } A$ is dense in E , then the subset of $\text{span } A$ consisting of those linear combinations of members of A where the coefficients are rational is dense in E , too.)

(3) It will be convenient to have the following definition.

Definition 2 Given a measurable space (T, \mathcal{T}) and a linear space X , we say that a set A of functions from T to X is *decomposable* if whenever $f, g \in A$ and $S \in \mathcal{T}$ then also $1_S \times f + 1_{T \setminus S} \times g \in A$.

The following fact may be inferred from the proof of Theorem 6.2 in Yannelis (1991, p. 22).

Fact 2 *Let (T, \mathcal{T}, ν) be a totally finite measure space, and E a Banach space. Let A be a set of Bochner integrable functions from T to E and let $B = \{\int f : f \in A\}$. If A is decomposable and (T, \mathcal{T}, ν) is atomless, then the norm-closure of B is convex.*

(4) The theorems to be proved contain the implication that a Walrasian allocation is a core allocation. That this is true is a standard and well-known fact, so in the proofs given below we will not look at this implication.

(5) According to the definition of economy as stated in Sect. 2, the probability space (T, \mathcal{T}, ν) of agents is complete. We will invoke this fact frequently without explicit reference.

(6) In the proofs below, it is assumed without loss of generality that $E \neq \{0\}$ whenever this is needed for an argument but is not implied by the assumptions in force. Note that the assumption $E \neq \{0\}$ means in particular that 0 cannot be an interior point of the positive cone E_+ .

⁸ As the cardinal $\text{add } \mathcal{N}(\nu)$ is uncountable, this replacement leaves the condition in (MMATC) the same also when $\text{dim}(E)$ is finite because in this case $\text{dens}(E)$ is of course countable.

9 Proofs of Theorems 2–4

9.1 Proof of Theorem 2

Let f be a core allocation of the given economy and let

$$A = \{h: T \rightarrow E: \text{for some } S \in \mathcal{T}, h = 1_S \times g - 1_S \times e \text{ where } g: T \rightarrow E_+ \text{ is a measurable simple function with } g(t) \succ_t f(t) \text{ a.e. in } S\}.$$

Every $h \in A$ is $\|\cdot\|_\infty$ -Bochner integrable; write $B = \{\int h \, d\nu: h \in A\}$. Clearly, A is decomposable, so, as ν is atomless, Fact 2 implies that $\text{cl } B$ is convex. Moreover, $0 \in B$, so $\text{cl } B$ is non-empty.

Note that $(\text{cl } B) \cap \text{int}(-E_+) = \emptyset$. Otherwise, as $0 \notin \text{int } E_+$, there are a $v \in \text{int } E_+$ and an $S \in \mathcal{T}$ with $\nu(S) > 0$ such that $\int_S g \, d\nu - \int_S e \, d\nu = -v$ where g is as in the definition of A . As the measure ν is atomless and the indefinite Bochner integral of the function $g - e$ is ν -continuous, we may assume, shrinking the set S if necessary, that $\nu(T \setminus S) > 0$. For each $n \in \mathbb{N}$, set $T_n = \{t \in T \setminus S: e(t) + nv \geq f(t)\}$. Note that $T_n \in \mathcal{T}$ for each n because the map $t \mapsto e(t) + nv - f(t)$, being Bochner integrable, is Borel measurable. Now because v , belonging to $\text{int } E_+$, is an order unit of E , we have $\bigcup_{n=0}^\infty T_n = T \setminus S$. Thus, as $\nu(T \setminus S) > 0$, there is an $n_1 \in \mathbb{N}$ such that $\nu(T_{n_1}) > 0$. Choose $n_2 \in \mathbb{N}$ so that both $n_2 > n_1$ and $\nu(T_{n_1}) > 1/n_2$. By Assumptions (D) and (P), $e(t) + n_2v \succ_t f(t)$ for all $t \in T_{n_1}$. Because ν is atomless, we can choose an $F \subseteq T_{n_1}$ with $\nu(F) = 1/n_2$. Define a function $f': T \rightarrow E_+$ by setting $f'(t) = g(t)$ for $t \in S$, $f'(t) = e(t) + n_2v$ for $t \in F$, and $f'(t) = 0$ for $t \in T \setminus (F \cup S)$. Then $f'(t) \succ_t f(t)$ for almost all $t \in F \cup S$. Moreover, f' is Bochner integrable and we have

$$\begin{aligned} \int_{F \cup S} f'(t) \, d\nu(t) &= \int_F e(t) + n_2v \, d\nu(t) + \int_S g(t) \, d\nu(t) \\ &= \int_F e(t) \, d\nu(t) + v + \int_S e(t) \, d\nu(t) - v = \int_{F \cup S} e(t) \, d\nu(t). \end{aligned}$$

Thus, the coalition $F \cup S$ can block f via f' , contradicting the property of f being a core allocation. Thus, $(\text{cl } B) \cap \text{int}(-E_+) = \emptyset$.

Now, as $\text{int } E_+$ is non-empty, it follows from the separation theorem that there is a nonzero positive $p \in E^*$ such that $pz \geq 0$ for each $z \in B$. For each $x \in E_+$, let $N_x = \{t \in T: x \succ_t f(t), px < pe(t)\}$. From Assumptions (P) and (AM), together with the fact that the map $t \mapsto pe(t)$ is measurable, it follows that $N_x \in \mathcal{T}$ for each $x \in E_+$. Hence, the fact that $pB \geq 0$ implies that N_x is a null set for each $x \in E_+$.

By (MMATC), E_+ has a dense subset D such that $\text{add } \mathcal{N}(v) > \#(D)$.⁹ It follows that $N = \bigcup_{x \in D} N_x$ is a null set, and by Assumptions (P) and (C), and continuity of p :

$$(*) \text{ For all } t \in T \setminus N, \text{ if } x \succ_t f(t), \text{ then } px \geq pe(t).$$

It is now routine to verify that for almost all $t \in T$,

⁹ Recall that if Z is a metric space and $Y \subseteq Z$, then $\text{dens}(Y) \leq \text{dens}(Z)$.

- (i) $pf(t) \leq pe(t)$;
- (ii) if $x \succ_t f(t)$, then $px > pe(t)$.

Note first that by (P) and (D), (*) implies that $pf(t) \geq pe(t)$ a.e. in T . Consequently, since f is feasible, i.e., since $\int_T f(t) \, d\nu(t) = \int_T e(t) \, d\nu(t)$, (i) holds a.e. in T . Let $S = \{t \in T : pe(t) > 0\}$. By (C) and (P), (*) implies that (ii) holds a.e. in S . Now by (RA), the fact that p is nonzero and positive implies that $\nu(S) > 0$, so by (P) and (D) again, the fact that (ii) holds a.e. in S implies that p is strictly positive. Consequently, by (P) and (EC), for $t \in T \setminus S$ a failure of (ii) implies that $e(t) \succ_t f(t)$. By (AM), the set $\{t \in T : e(t) \succ_t f(t)\}$ is measurable. Because f is a core allocation, this set must be a null set, so (ii) holds also a.e. in $T \setminus S$. We may conclude that the pair (p, f) is a Walrasian equilibrium. □

9.2 Proof of Theorem 3

Fix an interior point \bar{e} of E_+ and a strictly positive $q \in E^*$ such that $q\bar{e} = 1$. Let $H = \{x \in E_+ : qx = 1\}$. Observe that $\text{dens}(H) = \text{dens}(E)$.

Write $\alpha = \text{add } \mathcal{N}(\nu)$ and choose a family $\langle N_\xi \rangle_{\xi < \alpha}$ of null sets in T such that $\bigcup_{\xi < \alpha} N_\xi$ is not a null set. Recall that in every metric space of density character κ , there is a disjoint family of non-empty open subsets which has cardinal κ (see, e.g., Engelking 1989, 4.1.H, or Hodel 1984, Theorem 8.1). As $\alpha \leq \text{dens}(H)$ by hypothesis, we can therefore find a family $\langle B_\xi \rangle_{\xi < \alpha}$ of open balls in E , each of them centered at some point $x_\xi \in H$, such that the family $\langle B_\xi \cap H \rangle_{\xi < \alpha}$ of intersections is disjoint. Using the fact that in a normed space the distance between the centers of two disjoint open balls cannot be smaller than the sum of the radii of these balls, we see that, in fact, the family $\langle B_\xi \rangle_{\xi < \alpha}$ is disjoint.

Now for each agent $t \in T$, let $X_t = E_+$ and $e(t) = \bar{e}$, so that (P) and (EC) are satisfied. In particular, the initial allocation $t \mapsto e(t)$ is Bochner integrable and we have $\int_T e(t) \, d\nu(t) = \bar{e} \in \text{int } E_+$, so (RA) is also satisfied.

As for preferences, for $\xi < \alpha$ and $t \in N_\xi$ let

$$\succ_t = \{(x, y) \in E_+ \times E_+ : qx > qy\} \cup ((B_\xi \cap E_+) \times \{\bar{e}\}),$$

and for $t \in T \setminus \bigcup_{\xi < \alpha} N_\xi$ let $\succ_t = \{(x, y) \in E_+ \times E_+ : qx > qy\}$. As q is strictly positive, (D) is satisfied, and as each $B_\xi \cap E_+$ is open in E_+ , (C) is satisfied, too.

Note that any separable subspace S of E can intersect only countably many members of the family $\langle B_\xi \rangle_{\xi < \alpha}$ (by the choice of this family). Hence, given any such S , we must have

$$\succ_t \cap S \times E \subseteq \{(x, y) \in E_+ \times E_+ : qx > qy\}$$

for almost all $t \in T$. As allocations are Bochner integrable, and Bochner integrable functions are essentially separably valued, it follows that given any two allocations $f, g : T \rightarrow E_+$, there is a null set $N \subseteq T$ such that

$$\{t \in T : f(t) \succ_t g(t)\} \setminus N = \{t \in T : qf(t) > qg(t)\} \setminus N.$$

This shows that (AM) is satisfied and also shows that the initial allocation is a core allocation.

Suppose there is a $p \in E^*$ for which the initial allocation is Walrasian. Then $q = \lambda p$ for some number λ . Otherwise, by a well-known fact from linear algebra, there is a $z \in E$ with $pz = 0$ and $qz > 0$, so that $p(\bar{e} + z) = p\bar{e}$ but $q(\bar{e} + z) > q\bar{e}$. As $\bar{e} \in \text{int } E_+$, we may assume that $\bar{e} + z \in E_+$. But then by the choice of preferences, we have $\bar{e} + z \succ_t \bar{e}$ for all $t \in T$, thus getting a contradiction to the assumption that the initial allocation is Walrasian for p . Now if $q = \lambda p$, then for the points x_ξ from above, we have $px_\xi = p\bar{e}$. But by the choice of preferences again, we have $x_\xi \succ_t \bar{e}$ for each $t \in N_\xi$ and each $\xi < \alpha$. Because $\bigcup_{\xi < \alpha} N_\xi$ is non-negligible, we get a contradiction as before, and it follows that the initial allocation is not Walrasian. \square

9.3 Proof of Theorem 4

Lemma 1 *Let E be an ordered Banach space whose positive cone E_+ has a non-empty interior. Let $p \in E_+^* \setminus \{0\}$, and let $G \subseteq E$ be such that whenever $q \in E_+^*$ satisfies $qx = px$ for all $x \in G$, then $q = p$. Then there is an $H \subseteq E_+$, with $\#(H) \leq \max\{\#(G), \omega\}$, such that whenever $x \in E_+$ and $c \in \mathbb{R}_+$ satisfy $px < c$, then there is a $y \in H$ with $y > x$ and $py < c$.*

Proof We may assume both that $G \cap \text{int } E_+ \neq \emptyset$ and that $pz = 1$ for all $z \in G$. Indeed, as $p \in E_+^* \setminus \{0\}$, we may pick an $e \in \text{int } E_+$ with $pe = 1$. Now for each $z \in G$, set $\beta_z = 1 - pz$. Let $G' = \{z + \beta_z e : z \in G\} \cup \{e\}$. Then $qz = pz$ for all $z \in G$ whenever $q \in E_+^*$ is such that $qy = py$ for all $y \in G'$. Note that $\max\{\#(G), \omega\} = \max\{\#(G'), \omega\}$. Thus, if necessary, we may replace G by G' .

Let K be the affine hull of G . In particular, K is convex and $pz = 1$ for all $z \in K$. Let $x \in E_+$ and suppose $px < 1$. There must be a $z \in K$ such that $z \in \{x\} + \text{int } E_+$. Otherwise, by the separation theorem, there is a nonzero $q \in E^*$ such that $qz \leq qy$ for all $z \in K$ and $y \in \{x\} + \text{int } E_+$. Now q must be positive, and because K is affine, there must be a number c such that $qz = c$ for all $z \in K$. As $K \cap \text{int } E_+ \neq \emptyset$, we must have $c > 0$, so replacing q by a suitable scalar multiple, we may assume $c = 1$. But then $qz = pz$ for all $z \in G$, and on the other hand, $qx \geq 1 > px$, contradicting the hypothesis on G .

Let L be the set of those members of K for which the coefficients in the representation as a linear combination of members of G are rational. Then L is dense in K , and we have $\#(L) \leq \max\{\#(G), \omega\}$. Since L is dense in K , and since for any $x \in E$, $K \cap (\{x\} + \text{int } E_+)$ is open in K , it follows from the previous paragraph that given $x \in E_+$ with $px < 1$, there is a $z \in L$ such that $z > x$.

Set $H = \{rz : z \in L, r \in \mathbb{Q}\} \cap E_+$ where \mathbb{Q} is the set of rational numbers. Then $\#(H) \leq \omega \cdot \#(L) \leq \max\{\#(G), \omega\}$. Suppose $x \in E_+$ and $c \in \mathbb{R}$ are such that $px < c$. Pick an $r \in \mathbb{Q}$ such that $px < r < c$. Now $p\frac{1}{r}x < 1$, so there is a $z \in L$ such that $z > \frac{1}{r}x$; clearly $rz > x$ and $px \leq prz = r < c$. Thus, H is as required. \square

Proof of Theorem 4 Let f be a core allocation of the given economy \mathcal{E} . As in the proof of Theorem 2, it follows that there is a nonzero positive $p \in E^*$ such that for each $x \in E_+$ the set $N_x = \{t \in T : x \succ_t f(t), px < pe(t)\}$ is a null set.

Write $v = 1_K \in C(K) \equiv E$ and let $\Delta = \{q \in E_+^* : qv = 1\}$. We may assume $p \in \Delta$. Since Δ may be identified with the set of all tight Borel probability measures on K , Δ is weak*-first countable by hypothesis. By the definition of the weak*-topology, this means there is a countable set $C \subseteq E$ such that whenever $q \in \Delta$ and $qx = px$ for all $x \in C$, then $q = p$. Of course, we may assume that v is in C , so that whenever $q \in E_+^*$ is such that $qx = px$ for all $q \in C$, then $q = p$.

By Lemma 1, it follows that there is a countable $H \subseteq E_+$ such that whenever $x \in E_+$ and $c \in \mathbb{R}_+$ satisfy $px < c$, there is a $y \in H$ with $y > x$ and $py < c$. As H is countable, $N = \bigcup_{x \in H} N_x$ is a null set. Now by Assumptions (P), (D), and (Tr), for any $t \in T$, if there is an $x \in E_+$ with $x \succ_t f(t)$ and $px < pe(t)$, then $t \in N$. Thus, by (P) again, if $t \in T \setminus N$ then $px \geq pe(t)$ whenever $x \succ_t f(t)$. As in the proof of Theorem 2, it follows that the pair (p, f) is a Walrasian equilibrium. \square

9.4 Proof of Theorem 5

In view of Theorem 2, the following statement needs to be proved.

Let E be an ordered Banach space such that E_+ has both a non-empty interior and a bounded base, and let (T, \mathcal{F}, ν) be an atomless probability space such that $\text{add } \mathcal{N}(\nu) \leq \text{dens}(E)$. Then there is an economy \mathcal{E} , with (T, \mathcal{F}, ν) as space of agents and E as commodity space, such that (P), (D), (C), (AM), (EC), (RA), and (UR) are satisfied, but such that there is a core allocation which is not Walrasian.

Let E and (T, \mathcal{F}, ν) be as in this statement. Let H be a bounded base of E_+ . By Remark 1, there is a strictly positive $q \in E^*$ such that $H = \{x \in E_+ : qx = 1\}$. Choose an interior point \bar{e} of E_+ with $\bar{e} \in H$.

Follow the proof of Theorem 3 down to the start of the construction of preferences. Now for $t \in T \setminus \bigcup_{\xi < \alpha} N_\xi$, define preferences exactly as in that proof (in particular, there is a continuous utility representation). For $t \in N_\xi$, $\xi < \alpha$, let preferences be given by the continuous utility function $u_t : E_+ \rightarrow \mathbb{R}$ defined by

$$u_t(x) = qx + k \text{ dist}(x - (qx - 1)\bar{e}, H \setminus B_\xi), \quad x \in E_+,$$

where $k > 0$ is real number, common for all $\xi < \alpha$ and all $t \in N_\xi$. Note that for the points x_ξ from the proof of Theorem 3, we have $u_t(x_\xi) > u_t(\bar{e})$ for each $t \in N_\xi$, $\xi < \alpha$. Below we will show that the hypothesis that E_+ has a bounded base implies that the number k can be chosen in such a way that the preferences given by the u_t 's are strictly monotone.

Supposing that has been done, we have, as in the proof of Theorem 3, an economy such that (P), (D), (C), (EC), and (RA) hold, and now also (UR) in addition. Observe that for any $\xi < \alpha$ and $t \in N_\xi$, and any $x \in E_+$, if $u_t(x) \neq qx$ then $x + \lambda\bar{e} \in B_\xi$ for some number $\lambda > 0$, and that if S is a separable subspace of E , then so is $S + \{\lambda\bar{e} : \lambda \in \mathbb{R}\}$. Consequently, the fact that allocations are Bochner integrable and therefore essentially separably valued implies that given any two allocations $f, g : T \rightarrow E_+$, there is a null set $N \subseteq T$ such that

$$\{t \in T : f(t) \succ_t g(t)\} \setminus N = \{t \in T : qf(t) > qg(t)\} \setminus N.$$

Thus, as in the proof of Theorem 3, (AM) is satisfied and the initial allocation is a core allocation. As in that proof, it follows that this allocation is not Walrasian.

Finally, to see that there is a number $k > 0$ such that the preferences given by the functions u_t are strictly monotone for all $t \in N_\xi$ and $\xi < \alpha$, note first that the fact that $H = \{x \in E_+ : qx = 1\}$ is a bounded base of E_+ means that there is a number $k' > 0$ such that $\|y\| < k'qy$ for all $y \in E_+ \setminus \{0\}$, which implies that

$$\|y - (qy)\bar{e}\| \leq \|y\| + (qy)\|\bar{e}\| < k'qy + (qy)\|\bar{e}\|$$

for any $y \in E_+ \setminus \{0\}$. Thus, setting $k = (k' + \|\bar{e}\|)^{-1}$, we have $qy > k\|y - (qy)\bar{e}\|$ for such y . Now for any $x, y \in E_+$, writing A for $H \setminus B_\xi$,

$$\text{dist}(x - (qx - 1)\bar{e}, A) \leq \|y - (qy)\bar{e}\| + \text{dist}(x + y - (q(x + y) - 1)\bar{e}, A),$$

and it follows that

$$k \text{ dist}(x - (qx - 1)\bar{e}, A) < qy + k \text{ dist}(x + y - (q(x + y) - 1)\bar{e}, A)$$

for any $x, y \in E_+$ with $y \neq 0$, showing that the preferences given by the u_t ’s are indeed strictly monotone for the above choice of k . □

10 Proof of Theorem 6

Lemma 2 *Let X be a Riesz space, with positive cone X_+ , let $v \in X_+$, and suppose that q is a strictly positive linear functional on X . Write $U = \{u \in X : q(|u|) < 1\}$. Let $x_i, i = 0, \dots, n$, and e be elements of X_+ , let $v_i > 0, i = 0, \dots, n$, be real numbers, and suppose $\sum_{i=0}^n v_i x_i + \lambda(v + u) = e$ for some $u \in U$ and some real number $\lambda > 0$. Then there exist real numbers $\lambda_i \geq 0$ and elements $u_i \in U, i = 0, \dots, n$, such that $x_i + \lambda_i(v + u_i) \in X_+$ for each $i = 0, \dots, n$ and such that $\sum_{i=0}^n v_i(x_i + \lambda_i(v + u_i)) = e$.*

Proof Scaling the element e and the numbers v_i by some common factor, if necessary, we may assume $\lambda = 1$, so that $v + u = e - \sum_{i=0}^n v_i x_i$.

Now if $v \geq u^-$, we are done by setting $u_i = u$ for all $i = 0, \dots, n, \lambda_0 = 1/v_0$, and $\lambda_i = 0$ for $i = 1, \dots, n$.

Otherwise, note that we must have $q(u^- - (v \wedge u^-)) > 0$ as q is strictly positive. Note also that $(v + u)^- = u^- - (v \wedge u^-)$. Thus we must have $u^- - (v \wedge u^-) \leq \sum_{i=0}^n v_i x_i$. By the Riesz decomposition theorem, we obtain elements $b_0, \dots, b_n \in X_+$ such that $\sum_{i=0}^n b_i = u^- - (v \wedge u^-)$ and $b_i \leq v_i x_i$ for each $i = 0, \dots, n$. Set $d_i = (1/v_i)b_i$, so that $x_i - d_i \in X_+$, and set

$$\lambda_i = \frac{qd_i}{q(u^- - (v \wedge u^-))}.$$

Note that $\sum_{i=0}^n v_i \lambda_i = 1$. Set $u_i = u^+ - (v \wedge u^-) - (1/\lambda_i)d_i$ if $\lambda_i > 0$, and $u_i = u$ otherwise. Then $x_i + \lambda_i(v + u_i) \in X_+$ for each $i = 0, \dots, n$, because $v - (v \wedge u^-) \geq 0$. If $\lambda_i > 0$, then by the triangle inequality for the absolute value,

$$\begin{aligned}
 |u^+ - (v \wedge u^-) - (1/\lambda_i)d_i| &\leq |u^+| + |v \wedge u^-| + |(1/\lambda_i)d_i| \\
 &= u^+ + (v \wedge u^-) + (1/\lambda_i)d_i,
 \end{aligned}$$

as the three summands involved are all in X_+ , so

$$q|u_i| \leq qu^+ + q(v \wedge u^-) + q(u^- - (v \wedge u^-)) = q|u|.$$

Hence $u_i \in U$ for each $i = 1, \dots, n$, because $u \in U$. Also, because $v_i d_i = b_i$ for each $i = 1, \dots, n$, and because $\sum_{i=0}^n v_i \lambda_i = 1$, we have

$$\begin{aligned}
 \sum_{i=0}^n v_i (\lambda_i (v + u_i)) &= v + u^+ - (v \wedge u^-) - \sum_{i=0}^n b_i \\
 &= v + u^+ - (v \wedge u^-) - (u^- - (v \wedge u^-)),
 \end{aligned}$$

so $\sum_{i=0}^n v_i (\lambda_i (v + u_i)) = v + u$, and it follows that $\sum_{i=0}^n v_i (x_i + \lambda_i (v + u_i)) = e$, because we have assumed $v + u = e - \sum_{i=0}^n v_i x_i$. □

Lemma 3 *Let E be an infinite-dimensional order-continuous Banach lattice such that E_+ has strictly positive elements. Then there is a strictly positive $p \in E^*$ such that given any cardinal $\kappa \leq \text{dens}(E)$, there is a family $\langle p_i \rangle_{i \in I}$ in the order interval $[-p, p]$, with $\#(I) = \kappa$, such that $p_i \neq 0$ for each $i \in I$ but such that on any separable subspace of E , p_i is zero for all but countably many $i \in I$.*

Proof By the representation theorem for order-continuous Banach lattices in Lindenstrauss and Tzafriri (1979, p. 25, Theorem 1.b.14), we may assume that, for some probability space (Ω, Σ, μ) , $L_\infty(\mu) \subseteq E \subseteq L_1(\mu)$ and $L_\infty(\mu) \subseteq E^* \subseteq L_1(\mu)$, the inclusions being continuous and order preserving; in particular, 1_Ω is a strictly positive element in E^* . [Recall for the reference that in a normed Riesz space, strictly positive elements are weak order units; see Aliprantis and Burkinshaw (1985, p. 266, Theorem 4.85, and p. 267(b)).]

Now by Fabian et al (2001, p. 364, Theorem 11.12), there is a family $\langle x_i, p_i \rangle_{i \in I}$ in $L_1(\mu) \times L_\infty(\mu)$, with $\#(I) = \text{dens}(L_1(\mu))$, such that

- (1) $p_i x_j \equiv \int_\Omega p_i \times x_j \, d\mu \neq 0$ if and only if $i = j$,
- (2) $\text{span}\{x_i : i \in I\}$ is dense in $L_1(\mu)$.

Scaling the elements p_i , if necessary, we may assume that all of them belong to $[-1_\Omega, 1_\Omega]$. Now (1) implies that if $x \in \text{span}\{x_i : i \in I\}$ then $p_i x = 0$ for all but finitely many $i \in I$, and hence (2) implies that for any $x \in L_1(\mu)$, $p_i x = 0$ for all but countably many $i \in I$ (consider a sequence in $\text{span}\{x_i : i \in I\}$ converging to x). It follows that if D is any countable subset of $L_1(\mu)$ then p_i is zero on D for all but countably many $i \in I$ (i.e., $\{i \in I : p_i d \neq 0 \text{ for some } d \in D\}$ is countable), and hence that if S is any separable subspace of E , then p_i is zero on S for all but countably many $i \in I$.

It remains to see that $\text{dens}(E) \leq \text{dens}(L_1(\mu))$. To this end, let $A \subseteq [-1_\Omega, 1_\Omega]$ be dense in $[-1_\Omega, 1_\Omega]$ for the weak topology of $L_1(\mu)$ and of minimal cardinal among the subsets of $[-1_\Omega, 1_\Omega]$ with this property. Note that since E is order-continuous, order

intervals in E are weakly compact, so, as $L_\infty(\mu) \subseteq E^*$, the weak topology of E agrees on $[-1_\Omega, 1_\Omega]$ with the weak topology of $L_1(\mu)$. Hence A is dense in $[-1_\Omega, 1_\Omega]$ also for the weak topology of E . Observe that $[-1_\Omega, 1_\Omega]$ separates the points of $L_1(\mu)$ and therefore also those of E^* . Thus A , being dense in $[-1_\Omega, 1_\Omega]$ for the weak topology of E , separates the points of E^* , so $\text{span } A$ is norm-dense in E . Let B be the subset of $\text{span } A$ consisting of those linear combinations of members of A in which the coefficients are rational. Then B is still norm-dense in E ; moreover, $\#(B) = \#(A)$. Thus $\text{dens}(E) \leq \#(A)$. Now by the choice of A , $\#(A) \leq \|\cdot\|_1\text{-dens}([-1_\Omega, 1_\Omega]) \leq \text{dens}(L_1(\mu))$, and we conclude that $\text{dens}(E) \leq \text{dens}(L_1(\mu))$. \square

Proof of Theorem 6 (a) Sufficiency Let E be as in the statement of the theorem, and let \mathcal{E} be an atomless economy with commodity space E such that all the assumption listed in the theorem, including (MMATC), are satisfied. Let f be a core allocation of the economy \mathcal{E} and let

$$A = \{h: T \rightarrow E: \text{ for some } S \in \mathcal{T}, h = 1_S \times g - 1_S \times e \text{ where } g: T \rightarrow E_+ \text{ is a measurable simple function with } g(t) \succ_t f(t) \text{ a.e. in } S\}.$$

Every $h \in A$ is Bochner integrable; write $B = \{\int h \, d\nu: h \in A\}$. As in the proof of Theorem 2, we see that $\text{cl } B$ is convex and non-empty.

Let $v \in E_+ \setminus \{0\}$ and $q \in E^*$ be chosen according to Assumption (EDC). Write $U = \{u \in E: q|u| < 1\}$ and note that U is convex. By the continuity of the lattice operations in E , the map $u \mapsto q|u|: E \rightarrow \mathbb{R}$ is continuous, so U is an open subset of E . Let

$$\Gamma = \{\lambda(v + u): u \in U, \lambda > 0\}.$$

As U is open in E and convex, so is Γ . We may assume that $0 \notin \Gamma$, replacing q by a sufficiently large multiple if necessary.

We claim that $(\text{cl } B) \cap -\Gamma = \emptyset$. Otherwise, $B \cap -\Gamma \neq \emptyset$ because Γ is open, and thus, as $0 \notin \Gamma$, there are an $S \in \mathcal{T}$ with $\nu(S) > 0$, a $u \in U$, and a $\lambda > 0$ such that $\int_S g \, d\nu + \lambda(v + u) = \int_S e \, d\nu$ where $g: T \rightarrow E_+$ is as in the definition of A . Now by Lemma 2, there are measurable simple functions $\tilde{\lambda}: T \rightarrow \mathbb{R}_+$ and $\tilde{u}: T \rightarrow U$ such that both $\int_S g(t) + \tilde{\lambda}(t)(v + \tilde{u}(t)) \, d\nu(t) = \int_S e \, d\nu$ and $g(t) + \tilde{\lambda}(t)(v + \tilde{u}(t)) \in E_+$ for every $t \in S$. Define $g': T \rightarrow E_+$ by setting $g'(t) = g(t) + \tilde{\lambda}(t)(v + \tilde{u}(t))$ if $t \in S$ and $g'(t) = 0$ otherwise. By Assumptions (P), (EDC), and (Tr), $g'(t) \succ_t f(t)$ a.e. in S , and we get a contradiction to the property of f being a core allocation.

By the separation theorem, it follows that there is a nonzero $p \in E^*$ such that $pB \geq 0$. As in the proof of Theorem 2, we see that there is a null set $N \subseteq T$ such that if $t \in T \setminus N$, then $px \geq pe(t)$ whenever $x \succ_t f(t)$. In particular, p must be positive. Indeed, pick any $a \in E_+ \setminus \{0\}$. By Assumptions (P) and (D), $f(t) + na \succ_t f(t)$ for each $t \in T$ and each $n \in \mathbb{N} \setminus \{0\}$, so $p(f(t) + na) \geq pe(t)$ for each $n \in \mathbb{N} \setminus \{0\}$ and $t \in T \setminus N$, and it follows that $pa \geq 0$.

Arguing as in the proof of Theorem 2, we can conclude that the pair (p, f) is a Walrasian equilibrium.

(b) Necessity Let the commodity space E be an order-continuous Banach lattice whose positive cone has strictly positive elements, and let (T, \mathcal{T}, ν) be an atomless probability space of agents such that $\text{add } \mathcal{N}(\nu) \leq \text{dens}(E)$. We will construct an economy such that (P), (D), (C), (AM), (EC), (RA), (UR), and (US) are satisfied, but such that there is a core allocation which is not Walrasian.

Fix a strictly positive element \bar{e} in E_+ . For each $t \in T$, let $X_t = E_+$ and $e(t) = \bar{e}$, so that (P), (EC), and (RA) are satisfied.

As in the proof of Theorem 3, write $\alpha = \text{add } \mathcal{N}(\nu)$ and choose a family $\langle N_\xi \rangle_{\xi < \alpha}$ of null sets in T such that $\bigcup_{\xi < \alpha} N_\xi$ is not a null set.

As $\alpha \leq \text{dens}(E)$, we may apply Lemma 3 to find a strictly positive $q \in E^*$ and a family $\langle q'_\xi \rangle_{\xi < \alpha}$ in the order interval $[-q, q]$ such that $q'_\xi \neq 0$ for each $\xi < \alpha$ but such that on any separable subspace of E , q'_ξ is zero for all but countably many $\xi < \alpha$. For each $\xi < \alpha$, set $q_\xi = q + (1/2)q'_\xi$, so that q_ξ is strictly positive. Clearly, $q_\xi \neq q$ for each $\xi < \alpha$, but on any separable subspace S of E , $q_\xi \upharpoonright S = q \upharpoonright S$ for all but countably many $\xi < \alpha$.

Now define preference relations \succ_t by setting

$$\succ_t = \{(x, y) \in E_+ \times E_+ : q_\xi x > q_\xi y\} \text{ for } t \in N_\xi, \xi < \alpha,$$

and $\succ_t = \{(x, y) \in E_+ \times E_t : qx > qy\}$ for $t \in T \setminus \bigcup_{\xi < \alpha} N_\xi$, so that (UR), (D), and (C) are satisfied. Set $a = (1/2)q$ and $b = (3/2)q$. Then for any $u, v \in E_+$ with $av > bu$, and any q' in the order interval $[a, b]$, we have $q'(v - u) \geq av - bu > 0$. Consequently, as q and each $q_\xi, \xi < \alpha$ are in $[a, b]$, (US) is also satisfied.

As allocations are Bochner integrable, and Bochner integrable functions are essentially separably valued, the choice of the preference relations implies that given any two allocations $f, g : T \rightarrow E_+$, there is a null set $N \subseteq T$ such that

$$\{t \in T : f(t) \succ_t g(t)\} \setminus N = \{t \in T : qf(t) > qg(t)\} \setminus N.$$

As in the proof of Theorem 3, we see from this that (AM) is satisfied and that the initial allocation is a core allocation.

Suppose there is a $p \in E^*$ for which the initial allocation $t \mapsto e(t) = \bar{e}$ is Walrasian. Write $I_{\bar{e}}$ for the order ideal in E generated by \bar{e} . Note that \bar{e} is an order unit of $I_{\bar{e}}$. Thus, if the equilibrium conditions hold for a $t \in N_\xi, \xi < \alpha$, we must have $q_\xi \upharpoonright I_{\bar{e}} = \lambda_\xi p \upharpoonright I_{\bar{e}}$ for some real number λ_ξ (see the end of the proof of Theorem 3). Since E is a Banach lattice, the fact that \bar{e} is strictly positive implies that $I_{\bar{e}}$ is dense in E (Aliprantis and Burkinshaw 1985, Theorem 4.85), so we have $q_\xi = \lambda_\xi p$ if $q_\xi \upharpoonright I_{\bar{e}} = \lambda_\xi p \upharpoonright I_{\bar{e}}$. Hence, as N_ξ is a null set for each $\xi < \alpha$, but $\bigcup_{\xi < \alpha} N_\xi$ is non-negligible, the assumption that the initial allocation is Walrasian for p implies that there is an uncountable set $H \subseteq \alpha$ such that $q_\xi \in \text{span}\{p\}$ for each $\xi \in H$. Now by construction, we have a $q \in E^*$ such that $q_\xi \neq q$ for all $\xi \in H$, but such that for any $z \in E$, $q_\xi z = qz$ for all but countably many $\xi \in H$. As H is uncountable, this implies that the set $\{q_\xi : \xi \in H\}$ cannot be included in a one-dimensional subspace of E^* . In particular, we cannot have $q_\xi \in \text{span}\{p\}$ for each $\xi \in H$. We thus obtain a contradiction, proving that the initial allocation is not Walrasian.

11 Proofs of Theorems 7 and 8

11.1 Basic facts

Fact 3 (Cf. Zame 1986, Sect. 9, p.1) *Let (Ω, Σ, μ) be a σ -finite measure space with $\mu(\Omega) = \infty$. Then there is a probability measure $\bar{\mu}$ on (Ω, Σ) such that:*

- (a) $L_\infty(\mu) = L_\infty(\bar{\mu})$, as Banach lattices, and in particular as sets, so that no distinction in notation is needed in (b) and (c) below.
- (b) The Mackey topology of $L_\infty(\mu)$ under the pairing with $L_1(\mu)$ is the same as that under the pairing with $L_1(\bar{\mu})$.
- (c) Whenever (T, \mathcal{T}, ν) is a totally finite measure space, a function $f : T \rightarrow L_\infty(\mu)$ is Gelfand integrable for the pairing of $L_\infty(\mu)$ with $L_1(\mu)$ if and only if it is Gelfand integrable for the pairing of $L_\infty(\mu)$ with $L_1(\bar{\mu})$, in which case both pairings yield the same Gelfand integral of f over any $S \in \mathcal{T}$.

Proof Let $\{B_n\}_{n \in \mathbb{N}}$ be a partition of Ω into measurable sets with $0 < \mu(B_n) < \infty$ for each $n \in \mathbb{N}$, and define $\bar{\mu} : \Sigma \rightarrow \mathbb{R}_+$ by $\bar{\mu}(A) = \sum_{n=0}^\infty 2^{-(n+1)} \mu(B_n)^{-1} \mu(A \cap B_n)$ for each $A \in \Sigma$. Then $\bar{\mu}$ is a probability measure which gives the same null sets in Ω as the measure μ and thus (a) holds.

Now for each $p \in L_1(\mu)$, set $\phi(p) = \sum_{n=0}^\infty 2^{n+1} \mu(B_n) 1_{B_n} \times p$. This defines a bijection ϕ from $L_1(\mu)$ to $L_1(\bar{\mu})$ such that for each $x \in L_\infty(\mu)$,

$$\int_\Omega x \times p \, d\mu = \int_\Omega x \times \phi(p) \, d\bar{\mu},$$

and thus (c) must hold. In fact, ϕ is an isomorphism for the norm-topologies of $L_1(\mu)$ and $L_1(\bar{\mu})$, therefore also for the weak topologies of these spaces, and thus (b) holds by the above equality because the Mackey topology of $L_\infty(\mu)$ is the topology of uniform convergence on weakly compact subsets of $L_1(\mu)$. □

Fact 4 *Let (Ω, Σ, μ) be a totally finite measure space, so that $L_\infty(\mu) \subseteq L_1(\mu)$. Then the Mackey topology of $L_\infty(\mu)$ and the $\|\cdot\|_1$ -topology of $L_1(\mu)$ agree on $\|\cdot\|_\infty$ -bounded subsets of $L_\infty(\mu)$. (See Zame 1986, Lemma A).*

Definition 3 Let (T, \mathcal{T}, ν) be a totally finite measure space, and (Ω, Σ, μ) a σ -finite measure space. A function $\eta : T \rightarrow L_\infty(\mu)$ is called a *weak*-null function* if for each $p \in L_1(\mu)$, $p\eta(t) = 0$ for almost all $t \in T$.

The point of this definition is that the exceptional set of measure zero is allowed to vary with $p \in L_1(\mu)$. Of course, if the measure space (Ω, Σ, μ) is separable, so that $L_1(\mu)$ has a countable subset separating the points of $L_\infty(\mu)$, then $\eta : T \rightarrow L_\infty(\mu)$ is a weak*-null function if and only if $\eta(t) = 0$ a.e. in T . In any case, given that μ is σ -finite, so that $L_1(\mu)$ has strictly positive elements, if $\eta : T \rightarrow L_\infty(\mu)$ is a weak*-null function, there can be no non-negligible $S \subseteq T$ such that $\eta(t) > 0$ for all $t \in S$.

Lemma 4 *Let (Ω, Σ, μ) be a σ -finite measure space. Then there is a $Q_+ \subseteq L_\infty(\mu)_+$ such that:*

- (a) Q_+ is dense in $L_\infty(\mu)_+$ for the Mackey topology.
- (b) If (T, \mathcal{T}, ν) is a totally finite measure space, S a subset of T , with $\nu^*(S) > 0$, and $h: S \rightarrow Q_+$ a function, then there are a set $S' \subseteq S$, with $\nu^*(S') > 0$, and an element $a \in L_\infty(\mu)_+$ such that $h(t) = a + \eta(t)$ for all $t \in S'$ where $\eta: T \rightarrow L_\infty(\mu)$ is a weak*-null function.

Proof In view of Fact 3, we may assume that μ is a probability measure, so that, in particular, $L_\infty(\mu) \subseteq L_1(\mu)$. Then by the proof of Lemma 9 in Podczeck (2004), there is a family $\langle x_i, p_i \rangle_{i \in I}$ of elements of $L_\infty(\mu) \times L_\infty(\mu)$ such that, writing Q for the set of all finite linear combinations of x_i 's with rational coefficients:

- (1) $p_i x_j \equiv \int_\Omega p_i \times x_j \, d\mu \neq 0$ if and only if $i = j$.
- (2) The set $\{p_i : i \in I\}$ separates the points of $L_\infty(\mu)$.
- (3) $Q \cap [0, 1_\Omega]$ is $\|\cdot\|_1$ -dense in $[0, 1_\Omega]$

(where $[0, 1_\Omega]$ is the order interval $\{x \in L_\infty(\mu) : 0 \leq x \leq 1_\Omega\}$).

By Fact 4, (3) implies that $Q \cap [0, 1_\Omega]$ is actually Mackey dense in $[0, 1_\Omega]$. As $L_\infty(\mu)_+ = \bigcup_{n=1}^\infty [0, n1_\Omega]$ and $nQ = Q$ for each $n \in \mathbb{N}$, it follows that $Q \cap L_\infty(\mu)_+$ is Mackey dense in $L_\infty(\mu)_+$. Set $Q_+ = Q \cap L_\infty(\mu)_+$.

Now let (T, \mathcal{T}, ν) , S , and h be as hypothesized in (b) of the lemma. For every $t \in T$, let $I_t = \{i \in I : p_i h(t) \neq 0\}$.

By (1) and the definition of Q_+ , I_t is finite for each $t \in S$. By the fact that $\nu^*(S) > 0$ and that the countable union of null set in S is a null set, we can choose an integer n and an $S_1 \subseteq S$ with $\nu^*(S_1) > 0$ such that $\#(I_t) = n$ for each $t \in S_1$. Note that if $F \subseteq I$ is such that $\#(F) = 0$, i.e., $F = \emptyset$, then $F \subseteq I_t$ for each $t \in S_1$. There is therefore an integer $\bar{k} \leq n$ which is maximal among the integers $k \leq n$ such that there is an $F \subseteq I$ with $\#(F) = k$ and with the property that there is an $S_2 \subseteq S_1$ such that $\nu^*(S_2) > 0$ and $F \subseteq I_t$ for each $t \in S_2$. Choose and fix F and S_2 corresponding in this sense to \bar{k} . Note that for each $i \in I \setminus F, \{t \in S_2 : p_i h(t) \neq 0\}$ is a null set.

As F is finite, the product of $\#(F)$ copies of the set of rational numbers is countable. Thus, as $\nu^*(S_2) > 0$, there are rational numbers $r_i, i \in F$, and a set $S' \subseteq S_2$ with $\nu^*(S') > 0$ such that for each $t \in S'$ and each $i \in F$ the coefficient of x_i in $h(t)$ as a member of Q is r_i . Set $a = \sum_{i \in F} r_i x_i$ (where $a = 0$ if $F = \emptyset$). Then (1) implies that $p_i h(t) = p_i a$ for each $t \in S'$ and each $i \in F$, and that $p_i a = 0$ for $i \in I \setminus F$. Define $\eta: T \rightarrow L_\infty(\mu)$ by setting $\eta(t) = h(t) - a$ if $t \in S'$ and $\eta(t) = 0$ otherwise. As $S' \subseteq S_2, \{t \in S' : p_i h(t) \neq 0\}$ is a null set for each $i \in I \setminus F$, and it follows that $\{t \in T : p_i \eta(t) \neq 0\}$ is a null set for each $i \in I$.

For each $p \in L_1(\mu)$, let $T_p = \{t \in T : p\eta(t) \neq 0\}$. Then T_{p_i} is a null set in T for each $i \in I$. If $p \in \text{span}\{p_i : i \in I\}$, then $T_p \subseteq \bigcup_{i \in I_p} T_{p_i}$ for some finite $I_p \subseteq I$, and it follows that T_p is a null set. Let p be an arbitrary element of $L_1(\mu)$. As we have $L_1(\mu)^* = L_\infty(\mu) \subseteq L_1(\mu)$, (2) implies that $\text{span}\{p_i : i \in I\}$ is dense in $L_1(\mu)$ by the Hahn–Banach theorem, so there is a sequence $\langle p_n \rangle$ in $\text{span}\{p_i : i \in I\}$ with $p_n \rightarrow p$. Now $T_p \subseteq \bigcup_{n \in \mathbb{N}} T_{p_n}$, and thus T_p is a null set. As $p \in L_1(\mu)$ was arbitrary, η is a weak*-null function.

It now also follows that the element $a \in L_\infty(\mu)$ defined above is positive. Indeed, otherwise there is a $p \in L_1(\mu)_+$ with $pa < 0$. As $a = h(t) - \eta(t)$ for $t \in S'$, and η is a weak*-null function, it follows that $ph(t) < 0$ for almost all $t \in S'$. But this is impossible because $h(t) \in Q_+ \subseteq L_\infty(\mu)_+$ for all $t \in S'$ and $v^*(S') > 0$. Thus we must have $a \in L_\infty(\mu)_+$. \square

11.2 Proof of Theorem 7

Let f be a core allocation of the given economy and let

- $A = \{h: T \rightarrow L_\infty(\mu): \text{ for some } S \in \mathcal{T}, h = 1_S \times g - 1_S \times e' \text{ where}$
- (a) $e': T \rightarrow L_\infty(\mu)_+$ is Bochner integrable with $e'(t) \leq e(t)$ a.e. in T ,
 - (b) $g: T \rightarrow L_\infty(\mu)_+$ is a measurable simple function such that for some weak*-null function $\eta: T \rightarrow L_\infty(\mu)$, $g(t) + \eta(t) \succ_t f(t)$ a.e. in $S\}$.

Then every $h \in A$ is $\|\cdot\|_\infty$ -Bochner integrable; write $B = \{\int h \, d\nu: h \in A\}$ where $\int h \, d\nu$ is the Bochner integral of h . Evidently, A is decomposable. Thus, as ν is atomless, it follows from Fact 2 that $\|\cdot\|_\infty$ -cℓ B is a convex subset of $L_\infty(\mu)$. Moreover, as $0 \in B$, $\|\cdot\|_\infty$ -cℓ B is non-empty.

Observe that $(\|\cdot\|_\infty$ -cℓ $B) \cap \|\cdot\|_\infty$ -int($-L_\infty(\mu)_+$) = \emptyset . Indeed, otherwise, because $0 \notin \|\cdot\|_\infty$ -int($-L_\infty(\mu)_+$), there are an $S \in \mathcal{T}$ with $\nu(S) > 0$ and a $v' \in \|\cdot\|_\infty$ -int $L_\infty(\mu)_+$ such that $\int_S g \, d\nu - \int_S e' \, d\nu = -v'$ where g and e' are as in the definition of A . Because the measure ν is atomless and the indefinite Bochner integral of the function $g - e'$ is ν -continuous with respect to $\|\cdot\|_\infty$, we may assume, shrinking the set S , if necessary, that $\nu(T \setminus S) > 0$. Now set $v = -(\int_S g \, d\nu - \int_S e' \, d\nu)$. As $e'(t) \leq e(t)$ a.e. in T , we have $v \geq v'$, and therefore $v \in \|\cdot\|_\infty$ -int $L_\infty(\mu)_+$. For each $n \in \mathbb{N}$, set $T_n = \{t \in T \setminus S: e(t) + nv \geq f(t)\}$. Then $\bigcup_{n=0}^\infty T_n = T \setminus S$. Hence, as $\nu(T \setminus S) > 0$, there must be an $n_1 \in \mathbb{N}$ such that $\nu^*(T_{n_1}) > 0$.¹⁰ Choose $n_2 \in \mathbb{N}$ so that both $n_2 > n_1$ and $\nu^*(T_{n_1}) > 1/n_2$. By Assumptions (P) and (D), $e(t) + n_2v \succ_t f(t)$ for all $t \in T_{n_1}$, and by (P) and (AM), $\{t \in T: e(t) + n_2v \succ_t f(t)\} \in \mathcal{T}$. Therefore, if G is a measurable envelope of T_{n_1} , then $e(t) + n_2v \succ_t f(t)$ for almost all $t \in G$ and we have $\nu(G) = \nu^*(T_{n_1}) > 1/n_2$. As ν is atomless, we can choose an $F \subseteq G$ with $\nu(F) = 1/n_2$. Clearly we may assume that $F \cap S = \emptyset$. Now according to (b) in the definition of A , there is a weak*-null function $\eta: T \rightarrow L_\infty(\mu)$ such that $g(t) + \eta(t) \succ_t f(t)$ for almost all $t \in S$. Define $f': T \rightarrow L_\infty(\mu)_+$ by setting $f'(t) = g(t) + \eta(t)$ for $t \in S$, $f'(t) = e(t) + n_2v$ for $t \in F$, and $f'(t) = 0$ for $t \in T \setminus (F \cup S)$. By construction, $f'(t) \succ_t f(t)$ a.e. in $F \cup S$. Also, being the sum of a Bochner integrable function and a weak*-null function, f' is Gelfand integrable and we have

¹⁰ Actually, the sets T_n can be shown to be measurable, but we don't need this property here.

$$\begin{aligned} \int_{F \cup S} f'(t) \, dv(t) &= \int_F e(t) + n_2 v \, dv(t) + \int_S g(t) + \eta(t) \, dv(t) \\ &= \int_F e(t) \, dv(t) + v + \int_S g(t) \, dv(t) \\ &= \int_F e(t) \, dv(t) + v + \int_S e(t) \, dv(t) - v = \int_{F \cup S} e(t) \, dv(t). \end{aligned}$$

Thus, the coalition $F \cup S$ can block f via f' , contradicting the property of f being a core allocation.

As $\|\cdot\|_\infty\text{-int } L_\infty(\mu)_+$ is non-empty, the separation theorem implies that there is a nonzero positive linear functional π on $L_\infty(\mu)$ such that $\pi z \geq 0$ for each $z \in B$. Now π can be identified with a finitely additive measure on Σ so that $\pi x = \int x \, d\pi$, $x \in L_\infty(\mu)$; in particular, $\pi(E) = 0$ whenever $\mu(E) = 0$. Let $\pi = \pi_c + \pi_f$ be the Yosida–Hewitt decomposition of π , where $\pi_c \geq 0$ is countably additive and $\pi_f \geq 0$ is purely finitely additive, and choose a sequence $\langle C_k \rangle_{k \in \mathbb{N}}$ in Σ with $\mu(\Omega \setminus C_k) \rightarrow 0$ such that $\pi_f(C_k) = 0$ for each $k \in \mathbb{N}$ (Yosida and Hewitt 1952, Theorems 1.23 and 1.22). We may view π_c as an element of $L_1(\mu)$.

Now $\pi_c \neq 0$. To see this, fix a $\|\cdot\|_\infty$ -Bochner integrable function $e' : T \rightarrow L_\infty(\mu)_+$ with $\int_T e'(t) \, dv(t) \in \text{int } L_\infty(\mu)_+$ such that $e'(t) \leq e(t)$ for almost all $t \in T$, as is possible by Assumption (TAE). The indefinite Bochner integral of e' is v -continuous, so there is an $\bar{\varepsilon}$ such that $\int_{T'} e'(t) \, dv(t) \in \text{int } L_\infty(\mu)_+$ whenever $v(T \setminus T') < \bar{\varepsilon}$. Now for each $n \in \mathbb{N}$ let $T_n = \{t \in T : n1_\Omega \succ_t f(t)\}$. For each $t \in T$, $n1_\Omega \succ f(t)$ if n is sufficiently large, so by Assumptions (P) and (D), we have $\bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} T_n = T$. By Assumptions (P) and (AM), T_n is measurable for each n , and it follows that there is an integer \bar{n} such that $v(T \setminus T_{\bar{n}}) < \bar{\varepsilon}$.

Consider the sequence $\langle \bar{n}1_{C_k} \rangle_{k \in \mathbb{N}}$ in $L_\infty(\mu)_+$, where the sets $C_k \in \Sigma$ are those chosen above. By (P) and (AM) again, the set $T_k = \{t \in T_{\bar{n}} : \bar{n}1_{C_k} \succ_t f(t)\}$ is measurable for each k . Note that because $\mu(\Omega \setminus C_k) \rightarrow 0$, $\bar{n}1_{C_k} \rightarrow \bar{n}1_\Omega$ in the Mackey topology of $L_\infty(\mu)$. Hence, by Assumption (C), we have $\bigcup_{l \in \mathbb{N}} \bigcap_{k \geq l} T_k = T_{\bar{n}}$ and thus $v(T_{\bar{n}} \setminus T_k) \rightarrow 0$ as $k \rightarrow \infty$. As $v(T \setminus T_{\bar{n}}) < \bar{\varepsilon}$, it follows that there is a \bar{k} such that $v(T \setminus T_{\bar{k}}) < \bar{\varepsilon}$. By the choice of $\bar{\varepsilon}$, $\int_{T_{\bar{k}}} e'(t) \, dv(t) \in \text{int } L_\infty(\mu)_+$, and thus we have $\pi \int_{T_{\bar{k}}} e'(t) \, dv(t) > 0$, as π is nonzero and positive. On the other hand, consider the allocation $g : T \rightarrow L_\infty(\mu)_+$ given by setting $g(t) = \bar{n}1_{C_{\bar{k}}}$ for each $t \in T$. Note that $\int_{T_{\bar{k}}} (g - e') \, dv \in B$ and that $\int_{T_{\bar{k}}} g \, dv = v(T_{\bar{k}})\bar{n}1_{C_{\bar{k}}}$. Thus would we have $\pi = \pi_f$, then $\pi \int_{T_{\bar{k}}} g \, dv = v(T_{\bar{k}})\bar{n}\pi_f(C_{\bar{k}}) = 0$, and we would get a contradiction to the fact that $\pi z \geq 0$ for each $z \in B$. Thus $\pi_c \neq 0$ must be true.

Note that $\pi_c z \geq 0$ for each $z \in B$. Otherwise, for some $S \in \mathcal{T}$ with $v(S) > 0$, and functions e', g , and η as in the definition of the set A , we have

$$\pi_c \int_S g \, dv < \pi_c \int_S e' \, dv \leq \pi \int_S e' \, dv$$

but $g(t) + \eta(t) \succ_t f(t)$ for almost all $t \in S$. Set $f' = 1_S \times g + 1_S \times \eta$ and note that by Assumption (P), f' takes values in $L_\infty(\mu)_+$ a.e. in T . For every $k \in \mathbb{N}$, define $g_k : T \rightarrow$

$L_\infty(\mu)_+$ and $\eta_k : T \rightarrow L_\infty(\mu)$ by setting $g_k(t) = 1_{C_k} \times g(t)$ and $\eta_k(t) = 1_{C_k} \times \eta(t)$ if $t \in S$, the sets C_k being as before, and $g_k(t) = \eta_k(t) = 0$ if $t \in T \setminus S$. Then for each k , g_k is a measurable simple function and η_k is a weak*-null function. (For any $q \in L_1(\mu)$, $q\eta_k(t) = q(1_{C_k} \times \eta(t)) = (1_{C_k} \times q)\eta(t)$ for all $t \in S$. Thus, since η is weak*-null function, and since $1_{C_k} \times q \in L_1(\mu)$ whenever $q \in L_1(\mu)$, η_k is a weak*-null function as well.) For each $k \in \mathbb{N}$, set $f_k = g_k + \eta_k$. Note that $f_k(t) = 1_{C_k} \times f'(t)$ for each $t \in T$. Thus, for each k , f_k takes values in $L_\infty(\mu)_+$ a.e. in T , because f' does. Moreover, for each k , f_k is Gelfand integrable, being the sum of a measurable simple function and a weak*-null function. Hence, by Assumptions (P) and (AM), for each k the set $S_k = \{t \in S : f_k(t) \succ_t f(t)\}$ belongs to \mathcal{T} . Arguing similarly as above, we see that $\nu(S_k) \rightarrow \nu(S)$ as $k \rightarrow \infty$, so $\int_{S_k} \pi_c e'(t) \, d\nu(t) \rightarrow \int_S \pi_c e'(t) \, d\nu(t)$. There must therefore be a \bar{k} such that $\pi_c \int_{S_{\bar{k}}} e' \, d\nu > \pi_c \int_S g \, d\nu$. Now because $\pi_f(C_{\bar{k}}) = 0$ (and $\pi_f \geq 0$), we have

$$\begin{aligned} \pi \int_{S_{\bar{k}}} g_{\bar{k}}(t) \, d\nu(t) &= \pi_c \int_{S_{\bar{k}}} g_{\bar{k}}(t) \, d\nu(t) \\ &\leq \pi_c \int_{S_{\bar{k}}} g(t) \, d\nu(t) < \pi_c \int_{S_{\bar{k}}} e'(t) \, d\nu(t) \leq \pi \int_{S_{\bar{k}}} e'(t) \, d\nu(t). \end{aligned}$$

But by construction, $\int_{S_{\bar{k}}} g_{\bar{k}} \, d\nu - \int_{S_{\bar{k}}} e' \, d\nu$ belongs to B , and we get a contradiction to the fact that $\pi z \geq 0$ for all $z \in B$. Thus, as claimed, $\pi_c z \geq 0$ for all $z \in B$.

Now by Assumption (TAE), there is a non-decreasing sequence $\langle e_n \rangle$ of Bochner integrable functions $e_n : T \rightarrow L_\infty(\mu)_+$ such that $e_n(t) \rightarrow e(t)$ in the Mackey topology for almost all $t \in T$. As $\pi_c \in L_1(\mu)$, we must have $\pi_c e_n(t) \rightarrow \pi_c e(t)$ for almost all $t \in T$. Consequently, for any $S \in \mathcal{T}$, $\int_S \pi_c e_n(t) \, d\nu(t) \rightarrow \int_S \pi_c e(t) \, d\nu(t)$, just by the monotone convergence theorem, and thus $\pi_c \int_S e_n(t) \, d\nu(t) \rightarrow \pi_c \int_S e(t) \, d\nu(t)$. It follows that whenever $S \in \mathcal{T}$ and $g : T \rightarrow L_\infty(\mu)_+$ are such that the conditions in (b) of the definition of the set A are satisfied, so that $\int_S g(t) \, d\nu(t) - \int_S e_n(t) \, d\nu(t) \in B$ for each n , then $\pi_c \int_S g(t) \, d\nu(t) \geq \pi_c \int_S e(t) \, d\nu(t)$.

Write $p = \pi_c$. We claim now that a.e. in T we have $px \geq pe(t)$ whenever $x \succ_t f(t)$. Otherwise, by Assumption (P), there is a map $h : S \rightarrow L_\infty(\mu)_+$ where $S \subseteq T$ with $\nu^*(S) > 0$ such that for all $t \in S$, $h(t) \succ_t f(t)$ but $ph(t) < pe(t)$. Let $Q_+ \subseteq L_\infty(\mu)_+$ be chosen according to Lemma 4. In particular, Q_+ is Mackey dense in $L_\infty(\mu)_+$. Therefore, as $p \in L_1(\mu)$, Assumptions (P) and (C) imply that we may assume that $h(t) \in Q_+$ for all $t \in S$. Now by Lemma 4(b), there are an $S_1 \subseteq S$, with $\nu^*(S_1) > 0$, an $a \in L_\infty(\mu)_+$, and a weak*-null function $\eta : T \rightarrow L_\infty(\mu)$ such that $h(t) = a + \eta(t)$ for all $t \in S_1$. Define $h' : T \rightarrow L_\infty(\mu)_+$ by setting $h'(t) = a + \eta(t)$ if $t \in S_1$, and $h'(t) = a$ otherwise. Note that h' is Gelfand integrable. Let $S_2 = \{t \in T : h'(t) \succ_t f(t)\}$. By Assumptions (P) and (AM), $S_2 \in \mathcal{T}$. Let $S_3 = \{t \in S_2 : ph'(t) < pe(t)\}$. Then also $S_3 \in \mathcal{T}$. Note that $S_1 \subseteq S_3$. Therefore, as $\nu^*(S_1) > 0$, we have $\nu(S_3) > 0$. Define $g : T \rightarrow L_\infty(\mu)_+$ by setting $g(t) = a$ for all $t \in T$. Observe that $h' - g$ is a weak*-null function. Hence, relative to the set S_3 , g satisfies the conditions in (b) of the definition of A , because $h'(t) \succ_t f(t)$ for all $t \in S_3$. On the other hand, we must have $pg(t) = ph'(t)$ for almost all $t \in T$. By the definition of S_3 , it follows that $p \int_{S_3} g(t) \, d\nu(t) < p \int_{S_3} e(t) \, d\nu(t)$, and we get a contradiction to what has been established in the previous paragraph.

Arguing as in the proof of Theorem 2, we can conclude that the pair (p, f) is a Walrasian equilibrium, observing that (TAE) implies that $p \int_T e(t) \, d\nu(t) > 0$, because p is a positive and nonzero element of $L_1(\mu)$. \square

11.3 Additional facts for the proof of Theorem 8

Fact 5 *Let (Ω, Σ, μ) be a totally finite measure space, so that $L_\infty(\mu) \subseteq L_1(\mu)$. Let (T, \mathcal{T}, ν) be a totally finite measure space, and $f: T \rightarrow L_\infty(\mu)$ a function; write f_1 for f viewed as a function from T to $L_1(\mu)$.*

- (a) *If f is Gelfand integrable, then f_1 is Pettis integrable, and for any $S \in \mathcal{T}$, the Pettis integral of f_1 over S agrees with the Gelfand integral of f over S .*
- (b) *If f_1 is Pettis integrable and $\|\cdot\|_\infty$ -bounded, then f is Gelfand integrable and for any $S \in \mathcal{T}$, the Pettis integral of f_1 over S agrees with the Gelfand integral of f over S .*

Proof By hypothesis, $L_1(\mu)^* = L_\infty(\mu) \subseteq L_1(\mu)$ and thus (a) holds. Now suppose f_1 is Pettis integrable, so that, in particular, the map $t \mapsto pf_1(t)$ is measurable for each $p \in L_\infty(\mu)$. But then $t \mapsto pf(t)$ must be measurable for each $p \in L_1(\mu)$, because $L_\infty(\mu)$ is sequentially dense in $L_1(\mu)$. Moreover, $|pf(t)| \leq \|p\|_1 \|f(t)\|_\infty$ for all $t \in T$ and all $p \in L_1(\mu)$. Hence, if f is $\|\cdot\|_\infty$ -bounded, then $t \mapsto |pf(t)|$ is integrable for each $p \in L_1(\mu)$ because (T, \mathcal{T}, ν) is totally finite, so f is Gelfand integrable, and by (a), for any $S \in \mathcal{T}$, the Pettis integral of f_1 over S agrees with the Gelfand integral of f over S . \square

Fact 6 *Let (Ω, Σ, μ) be a σ -finite measure space, (T, \mathcal{T}, ν) a totally finite measure space, and A a set of Gelfand integrable functions from T to $L_\infty(\mu)$. Suppose that (T, \mathcal{T}, ν) is atomless and that A is decomposable. Write $B = \{\int f: f \in A\}$ and suppose that B is $\|\cdot\|_\infty$ -bounded, or, equivalently, that B is included in an order interval of $L_\infty(\mu)$. Then the Mackey closure of B is convex.*

Proof Note first that, μ being σ -finite, $L_1(\mu)$ is weakly compactly generated and therefore measure compact.¹¹ By Fact 3 we may assume that μ is actually totally finite, so that $L_\infty(\mu) \subseteq L_1(\mu)$. Now, as $L_1(\mu)$ is measure compact, it follows from Podczeck (2004, proof of Lemma 6) together with Fact 5(a) that the $\|\cdot\|_1$ -closure of B is a convex subset of $L_1(\mu)$. Observe that, μ being totally finite, an order interval of $L_\infty(\mu)$ is both a closed subset of $L_1(\mu)$ and a Mackey closed subset of $L_\infty(\mu)$. As B is included in an order interval of $L_\infty(\mu)$, we now see from Fact 4 that the Mackey closure of B in $L_\infty(\mu)$ is convex. \square

Fact 7 *Let (T, \mathcal{T}, ν) be a totally finite measure space, and (Ω, Σ, μ) a σ -finite measure space. Let $f: T \rightarrow L_\infty(\mu)_+$ be Gelfand integrable and let $x \in L_\infty(\mu)$ be such that $0 \leq x \leq \int_T f$. Suppose $\mathcal{T} = 2^T$ and that f is $\|\cdot\|_\infty$ -bounded. Then there is*

¹¹ See Edgar (1979) for the definition of “measure compact” and the fact that a weakly compactly generated Banach space has this property.

a Gelfand integrable function $h : T \rightarrow L_\infty(\mu)_+$ such that $\int_T h = x$ and $h(t) \leq f(t)$ for all $t \in T$.¹²

Proof By Fact 3, we may assume that $L_\infty(\mu) \subseteq L_1(\mu)$. Then x may be viewed as element of $L_1(\mu)_+$, and by Fact 5(a), f may be viewed as a Pettis integrable function from T to $L_1(\mu)_+$, with Pettis integral equal to the Gelfand integral; moreover, the assumption that f is $\|\cdot\|_\infty$ -bounded implies that f is $\|\cdot\|_1$ -bounded as well. Now $L_1(\mu)$ is an order-continuous Banach lattice which, μ being totally finite, is weakly compactly generated and thus has the PIP.¹³ Hence by Podczeck (2004, Lemma 11), the hypothesis that $\mathcal{T} = 2^T$ and the facts that $0 \leq x \leq \int f \, d\nu$ and that f is $\|\cdot\|_1$ -bounded and has positive values imply the existence of a Pettis integrable $h : T \rightarrow L_1(\mu)_+$ with Pettis integral equal to x such that $h(t) \leq f(t)$ for all $t \in T$. As f is $\|\cdot\|_\infty$ -bounded and $0 \leq h(t) \leq f(t)$ for all $t \in T$, h actually takes values in $L_\infty(\mu)$ and is $\|\cdot\|_\infty$ -bounded. An appeal to Fact 5(b) concludes the proof. \square

Lemma 5 *Let (Ω, Σ, μ) be a σ -finite measure space, let $v \in L_\infty(\mu)$, and let q be a strictly positive element of $L_1(\mu)$. Write $U = \{u \in L_\infty(\mu) : q(|u|) < 1\}$. Let (T, \mathcal{T}, ν) be a totally finite measure space with $\mathcal{T} = 2^T$, let $h \rightarrow L_\infty(\mu)_+$ be Gelfand integrable, and let $e \in L_\infty(\mu)_+$. Suppose h is $\|\cdot\|_\infty$ -bounded and that for some $u \in U$ and some number $\lambda > 0$, $\int_T h(t) \, d\nu(t) + \lambda(v + u) = e$. Then there are an integrable function $\tilde{\lambda} : T \rightarrow \mathbb{R}_+$ and a function $\tilde{u} : T \rightarrow U$ such that $h(t) + \tilde{\lambda}(t)(v + \tilde{u}(t)) \in L_\infty(\mu)_+$ for almost all $t \in T$ and such that the function $t \mapsto h(t) + \tilde{\lambda}(t)(v + \tilde{u}(t))$ is Gelfand integrable with $\int_T h(t) + \tilde{\lambda}(t)(v + \tilde{u}(t)) \, d\nu(t) = e$.*

Proof Scaling e and the function h , if necessary, we may assume $\lambda = 1$.

Suppose first that $v \geq u^-$. In this case, set $\tilde{u}(t) = u$ and $\tilde{\lambda}(t) = 1/(v(T))$ for all $t \in T$, and we are done (clearly, we may assume $v(T) > 0$).

Otherwise, note that $q(u^- - (v \wedge u^-)) > 0$. Similarly as in the proof of Lemma 2, we see that $u^- - (v \wedge u^-) \leq \int_T h(t) \, d\nu(t)$. By Fact 7, there is a Gelfand integrable function $b : T \rightarrow L_\infty(\mu)_+$ such that $\int_T b(t) \, d\nu(t) = u^- - (v \wedge u^-)$ and $b(t) \leq h(t)$ for almost all $t \in T$. Define $\tilde{\lambda} : T \rightarrow \mathbb{R}_+$ by setting

$$\tilde{\lambda}(t) = \frac{qb(t)}{q(u^- - (v \wedge u^-))}$$

and note that $\int_T \tilde{\lambda}(t) \, d\nu(t) = 1$. Now define the function $\tilde{u} : T \rightarrow L_\infty(\mu)_+$ by setting $\tilde{u}(t) = u^+ - (v \wedge u^-) - (1/\tilde{\lambda}(t))b(t)$ if $\tilde{\lambda}(t) > 0$ and $\tilde{u}(t) = u$ otherwise. As $b(t) \leq h(t)$ for almost all $t \in T$ and $v - (v \wedge u^-) \geq 0$, we have $h(t) + \tilde{\lambda}(t)(v + \tilde{u}(t)) \in L_\infty(\mu)_+$ for almost all $t \in T$, and similarly as in the proof of Lemma 2, it follows that $\tilde{u}(t) \in U$ for almost all $t \in T$.

¹² This fact and Lemma 5 below are needed only if there exists an atomless probability space with no non-measurable subset. Recall that it is (relatively) consistent with ZFC that no such measure space exists. However, it is not known whether the existence of such a measure space is inconsistent with ZFC, and for this reason, we do not want to exclude such a measure space. See the brief discussion in Appendix 2(B).

¹³ See Edgar (1979) for the definition of ‘‘PIP’’ (Pettis integral property) and the fact that a weakly compactly generated Banach space has this property.

Note that $\tilde{\lambda}(t)\tilde{u}(t) = \tilde{\lambda}(t)(u^+ - (v \wedge u^-)) - b(t)$ for each $t \in T$. Consequently, the map $t \mapsto h(t) + \tilde{\lambda}(t)(v + \tilde{u}(t))$ is Gelfand integrable (because the maps b and h and because the map $\tilde{\lambda}$ is integrable), and because $\int_T b(t) \, d\nu(t) = u^- - (v \wedge u^-)$ and $\int_T \tilde{\lambda}(t) \, d\nu(t) = 1$, we get

$$\begin{aligned} & \int_T h(t) + \tilde{\lambda}(t)(v + \tilde{u}(t)) \, d\nu(t) \\ &= \int_T h(t) \, d\nu(t) + v + u^+ - (v \wedge u^-) - \int_T b(t) \, d\nu(t) \\ &= \int_T h(t) \, d\nu(t) + v + u^+ - (v \wedge u^-) - (u^- - (v \wedge u^-)) \\ &= \int_T h(t) \, d\nu(t) + v + u = e. \end{aligned}$$

□

11.4 Proof of Theorem 8

Let f be a core allocation of the given economy. Recall for the following arguments that the probability space (T, \mathcal{T}, ν) of agents is complete.

(a) To start the proof that f is Walrasian, let $v \in L_\infty(\mu)_+ \setminus \{0\}$ and $q \in L_1(\mu)$ be chosen according to Assumption (EDC). Write $U = \{u \in L_\infty(\mu) : q|u| < 1\}$. Because the lattice operations in $L_\infty(\mu)$ are continuous for the Mackey topology, the map $u \mapsto q|u| : L_\infty(\mu) \rightarrow \mathbb{R}$ is Mackey continuous, and thus U is an open neighborhood of 0 in $L_\infty(\mu)$ for the Mackey topology. Let

$$\Gamma = \{\lambda(v + u) : u \in U, \lambda > 0\}.$$

As U is Mackey open, so is Γ . As in the proof of Theorem 6, Γ is convex and we may assume that $0 \notin \Gamma$. Set

$$K = \{p \in L_1(\mu) : p\Gamma \geq 0, pv = 1\},$$

so that $|pU| \leq 1$ whenever $p \in K$. As the polars of the weakly compact subsets of $L_1(\mu)$ form a local base at zero in $L_\infty(\mu)$ for the Mackey topology, it follows from the bipolar theorem that K is a weakly compact set in $L_1(\mu)$.

(b) Let

$$\bar{r} = \sup\{r \in \mathbb{R} : \text{there is an } S \in \mathcal{T} \text{ with } \nu(S) \geq r \text{ and } 2^S \subseteq \mathcal{T}\}.$$

Because $2^\emptyset = \{\emptyset\} \subseteq \mathcal{T}$, there is an $S_n \in \mathcal{T}$ with $2^{S_n} \subseteq \mathcal{T}$ and $\nu(S_n) > \bar{r} - (1/n)$ for each integer $n > 0$. Set $T_1 = T \setminus \bigcup_{n>0} S_n$ and $T_2 = \bigcup_{n>0} S_n$. Then T_1 and T_2 belong to \mathcal{T} and $2^{T_2} \subseteq \mathcal{T}$; moreover, $\nu(T_2) = \bar{r}$, so every $S \subseteq T_1$ with $S \in \mathcal{T}$ and $\nu(S) > 0$ has a non-measurable subset.

Note that if $S \in \mathcal{T}$ and $g : S \cap T_2 \rightarrow L_\infty(\mu)$ is any $\|\cdot\|_\infty$ -bounded function, then for every $p \in L_1(\mu)$ the integral $\int_T p((1_{S \cap T_2} \times g)(t)) \, dv(t)$ is defined. Thus, for any such g , the map $1_{S \cap T_2} \times g : T \rightarrow L_\infty(\mu)$ is Gelfand integrable.

(c) For each $n \in \mathbb{N}$, let

$$A_n = \{h : T \rightarrow L_\infty(\mu) : \text{for some } S \in \mathcal{T}, h = 1_S \times g - 1_S \times e \text{ where}$$

- (i) $g : T \rightarrow L_\infty(\mu)_+$ is a function with $0 \leq g \leq n1_\Omega$ such that $g(t) \succ_t f(t)$ a.e. in S ,
- (ii) $1_{T_1} \times g$ is a measurable simple function

$$\}.$$

For each n , every $h \in A_n$ is Gelfand integrable; let $B_n = \{\int h \, dv : h \in A_n\}$ where $\int h \, dv$ is the Gelfand integral of h . Note that A_n is decomposable for each n and that B_n is included in the order interval $[-\int e \, dv, n1_\Omega]$. Hence, as ν is atomless, Fact 6 implies that the Mackey closure of B_n is convex for each n . Note that $0 \in B_n$ for each n , so the Mackey closure of B_n is non-empty for each n .

Now $B_n \cap -\Gamma = \emptyset$ for every n . Otherwise, as $0 \notin \Gamma$, for some n there are an $S \in \mathcal{T}$ with $\nu(S) > 0$, a $u \in U$, and a $\lambda > 0$ such that $\int_S g \, dv + \lambda(v + u) = \int_S e \, dv$ where $g : T \rightarrow L_\infty(\mu)_+$ is as in the definition of A_n . Write $g_1 = g \upharpoonright T_1$, $g_2 = g \upharpoonright T_2$, and set $S_1 = S \cap T_1$ and $S_2 = S \cap T_2$, so that $\int_S g \, dv = \int_{S_1} g_1 \, dv + \int_{S_2} g_2 \, dv$. By Lemma 2, there are points $u_1, u_2 \in U$ and $e_1, e_2 \in L_\infty(\mu)_+$, and numbers $\lambda_1, \lambda_2 \geq 0$, such that $\int_{S_i} g_i \, dv + \lambda_i(v + u_i) = e_i, i = 1, 2$, and $e_1 + e_2 = \int_S e \, dv$. Now, g_1 being a measurable simple function, another application of Lemma 2 yields measurable simple functions $\tilde{\lambda}_1 : T_1 \rightarrow \mathbb{R}_+$ and $\tilde{u}_1 : T_1 \rightarrow U$ such that $\int_{S_1} g_1(t) + \tilde{\lambda}_1(t)(v + \tilde{u}_1(t)) \, dv(t) = e_1$ and $g_1(t) + \tilde{\lambda}_1(t)(v + \tilde{u}_1(t)) \in L_\infty(\mu)_+$ for $t \in T_1$. Applying Lemma 5 (with S_2 substituted for T), we can find an integrable function $\tilde{\lambda}_2 : T_2 \rightarrow \mathbb{R}_+$ and a function $\tilde{u}_2 : T_2 \rightarrow U$ with the result that the function $t \mapsto g_2(t) + \tilde{\lambda}_2(t)(v + \tilde{u}_2(t))$ is Gelfand integrable, with $\int_{S_2} g_2(t) + \tilde{\lambda}_2(t)(v + \tilde{u}_2(t)) \, dv(t) = e_2$, and $g_2(t) + \tilde{\lambda}_2(t)(v + \tilde{u}_2(t)) \in L_\infty(\mu)_+$ for all $t \in T_2$. Now define $g' : T \rightarrow L_\infty(\mu)_+$ by setting $g'(t) = g_1(t) + \tilde{\lambda}_1(t)(v + \tilde{u}_1(t))$ if $t \in T_1$, and $g'(t) = g_2(t) + \tilde{\lambda}_2(t)(v + \tilde{u}_2(t))$ if $t \in T_2$. Then g' is Gelfand integrable and $\int_S g' \, dv = e_1 + e_2 = \int_S e \, dv$. By Assumptions (P), (EDC), and (Tr), $g'(t) \succ_t f(t)$ a.e. in S , and we get a contradiction to the fact that f is a core allocation.

Note that since Γ is Mackey open, the fact that $B_n \cap -\Gamma = \emptyset$ implies that also the Mackey closure of B_n does not intersect $-\Gamma$.

(d) Using the separation theorem, it follows that for each $n \in \mathbb{N}$, there is a nonzero $p_n \in L_1(\mu)$ such that $p_n B_n \geq 0 \geq p_n(-\Gamma)$. Note that we must have $p_n v > 0$ for each n . Hence, by the facts noted in (b), we may assume that there is a nonzero $p \in L_1(\mu)$ such that $p_n \rightarrow p$ weakly in $L_1(\mu)$ as $n \rightarrow \infty$ (recall that in any Banach space, “weakly compact” implies “weakly sequentially compact” by the Eberlein–Šmulian theorem). As the sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ is non-decreasing, it follows that $p B_n \geq 0$ for each n .

Note that because the map $t \mapsto pe(t) : T \rightarrow \mathbb{R}$ is measurable, Assumptions (P) and (AM) imply that the set $\{t \in T_1 : x \succ_t f(t), px < pe(t)\}$ is a measurable set in T for each $x \in L_\infty(\mu)_+$. Therefore, the fact that $p B_n \geq 0$ for each n implies the following:

(i) For every $x \in L_\infty(\mu)_+$, the set $\{t \in T_1 : x \succ_t f(t), px < pe(t)\}$ is a null set.

As for T_2 , note that the fact that $pB_n \geq 0$ for each n implies that if $g : T \rightarrow L_\infty(\mu)_+$ is $\|\cdot\|_\infty$ -bounded and there is a non-negligible $S \subseteq T_2$ such that $g(t) \succ_t f(t)$ for almost all $t \in S$, then there is a non-negligible $S' \subseteq S$ such that $pg(t) \geq e(t)$ for all $t \in S'$. Note also that if $S \subseteq T_2$ is non-negligible and $g : T \rightarrow L_\infty(\mu)_+$ is any function, there is a non-negligible $S' \subseteq S$ such that g is $\|\cdot\|_\infty$ -bounded on S' . Consequently:

(ii) If $S \subseteq T_2$ is non-negligible and $g : S \rightarrow L_\infty(\mu)_+$ satisfies $g(t) \succ_t f(t)$ a.e. in S , then there is a non-negligible $S' \subseteq S$ such that $pg(t) \geq pe(t)$ for all $t \in S'$.

(e) Suppose there is an $S \subseteq T$ with $v^*(S) > 0$ and a function $h : S \rightarrow L_\infty(\mu)_+$ such that for all $t \in S$, $h(t) \succ_t f(t)$ and $ph(t) < pe(t)$. Note that by (d)(ii), $S \cap T_2$ is a null set. We may therefore assume that $S \subseteq T_1$. Arguing as in the proof of Theorem 7, we may assume that there are an $S_1 \subseteq S$ with $v^*(S_1) > 0$, an $a \in L_\infty(\mu)_+$, and a weak*-null function $\eta : T \rightarrow L_\infty(\mu)$ such that for every $t \in S_1$, $h(t) = a + \eta(t)$. According to (b), there is a non-measurable set $S_2 \subseteq S_1$. Define $h' : T \rightarrow L_\infty(\mu)_+$ by setting $h'(t) = a + \eta(t)$ if $t \in S_2$ and $h'(t) = a$ otherwise. Then h' is Gelfand integrable. By Assumptions (P) and (AM), the set $\{t \in T : h'(t) \succ_t f(t)\}$ is measurable, and hence so is the set

$$S_3 = \{t \in T : h'(t) \succ_t f(t), ph'(t) < pe(t)\}.$$

Let $S_4 = \{t \in T : a \succ_t f(t), pa < pe(t)\}$. Observe that $S_3 = S_2 \cup S_4$. By (d)(i) and (d)(ii), S_4 is a null set. As S_2 is a non-measurable, it follows that S_3 is non-measurable, and we get a contradiction.

(f) By Assumption (P), it follows that for almost all $t \in T$, $px \geq pe(t)$ whenever $x \succ_t f(t)$. In particular, p must be positive. Indeed, consider any $a \in L_\infty(\mu)_+ \setminus \{0\}$. By Assumptions (P) and (D), $f(t) + na \succ_t f(t)$ for each $t \in T$ and each $n \in \mathbb{N} \setminus \{0\}$. It follows that $p(f(t) + na) \geq pe(t)$ for each $n \in \mathbb{N} \setminus \{0\}$ and almost all $t \in T$ and hence that $pa \geq 0$. Arguing as in the proof of Theorem 2, we can conclude that the pair (p, f) is a Walrasian equilibrium. □

Appendix 1

(A) Let E be a Banach lattice and let \mathcal{E} be an economy with commodity space E . Suppose the consumption set of every agent is E_+ . Now the statement of the notion of “extremely desirable commodity” in Rustichini and Yannelis (1991) says that there are a $v \in E_+ \setminus \{0\}$ and a convex solid open neighborhood U of zero in E such that

- (a) for each $t \in T$ and any number $\lambda > 0$, $x + \lambda(u + v) \succ_t x$ whenever $x \in E_+$ and $u \in U$ are such that $x + \lambda(u + v) \in E_+$;
- (b) whenever $\delta_1, \dots, \delta_n$ are positive real numbers and x_1, \dots, x_n are elements of E_+ such that $\sum_{i=1}^n \delta_i = 1$ and $x_i \notin \delta_i U, i = 1, \dots, n$, then $\sum_{i=1}^n x_i \notin U$.¹⁴

¹⁴ Solidity of U is assumed by Rustichini and Yannelis (1991) in the proof of their Theorem 6.1.

Suppose this condition holds. Let ρ be the gauge of U . By the properties of U , ρ is a Riesz seminorm on E , and $U = \{x \in E : \rho(x) < 1\}$. Pick any $x, y \in E_+$, and set $\alpha = \rho(x)$ and $\beta = \rho(y)$. If $\alpha = 0$, then $\rho(x) + \rho(y) = \rho(y) \leq \rho(x + y)$, since ρ is a Riesz seminorm. Similarly, if $\beta = 0$, then $\rho(x) + \rho(y) \leq \rho(x + y)$. Assume that α and β are both larger than 0, and set $x_1 = \frac{1}{\alpha+\beta}x$ and $y_1 = \frac{1}{\alpha+\beta}y$. Then $x_1 \notin \frac{\alpha}{\alpha+\beta}U$ and $y_1 \notin \frac{\beta}{\alpha+\beta}U$. Now by (b), $x_1 + y_1 \notin U$, so $\rho(x + y) \geq \alpha + \beta$, and we see again that $\rho(x) + \rho(y) \leq \rho(x + y)$. Thus the seminorm ρ is additive on E_+ . There is therefore a positive linear functional q_1 on E which agrees with ρ on E_+ . By the fact that U is solid, we see that $U = \{u \in E : q_1|u| < 1\}$. Moreover, since E is a Banach lattice, the fact that the linear functional q_1 is positive implies that q_1 is continuous, i.e., $q_1 \in E^*$. Now it is assumed in Rustichini and Yannelis (1991) that E is separable, so E^* has a strictly positive element, q_2 say (see, e.g., Lindenstrauss and Tzafriri 1979, p. 25). Set $q = q_1 + q_2$. Then q is strictly positive, and $q_1|u| < 1$ whenever $u \in E$ is such that $q|u| < 1$. Hence, by (a), (EDC) must hold.

The other direction, i.e., that (EDC) implies the notion of “extremely desirable commodity” in Rustichini and Yannelis (1991) is immediate.

(B) Let E be a Banach lattice, and let a and b be as in the statement of (US). Choose any $v \in E_+$ with $av > 1$. Let $x \in X_t, u \in E$, and a number $\lambda > 0$ be given so that $b|u| < 1$ and $x + \lambda(v + u) \in X_t$. We may write

$$x + \lambda(v + u) = x - \lambda u^- + \lambda(v + u^+).$$

Now as a and b are positive, we have

$$a(\lambda(v + u^+)) > \lambda > \lambda b|u| \geq b(\lambda u^-).$$

Thus, (US) implies (EDC).

(C) Let (Ω, Σ, μ) be a σ -finite measure space, and let the commodity space be $L_\infty(\mu)$ with the Mackey topology. Let \mathcal{E} be an economy with probability space of agents (T, \mathcal{T}, ν) (and recall that the measure ν is complete according to our definition of economy). The assumption in Mertens (1970) on endowments is that the endowment map $e : T \rightarrow L_\infty(\mu)_+$ is Gelfand integrable (as in (TAE)) and that

- (i) $e(t) \in \|\cdot\|_\infty$ -int $L_\infty(\mu)_+$ for almost all T ;
- (ii) there is a non-decreasing sequence $\langle e_n \rangle$ of measurable countably valued functions $e_n : T \rightarrow L_\infty(\mu)_+$ such that $e_n(t) \rightarrow e(t)$ in the Mackey topology for almost all $t \in T$.

We will show now that this assumption implies (TAE). Note first that as (T, \mathcal{T}, ν) is a probability space, the property of e_n being measurable and countably valued implies that we can find a set $S_n \in \mathcal{T}$ with $\nu(T \setminus S_n) < 2^{-n}$ such that $1_{S_n} \times e_n$ is a simple function, i.e., takes only finitely many values. Set $T_n = \bigcap_{m \geq n} S_m$ for each n , so that the sequence $\langle T_n \rangle$ is non-decreasing with $\nu(T_n) \rightarrow \nu(T)$. Now for each n , $1_{T_n} \times e_n$ is a measurable simple function, and the sequence $\langle 1_{T_n} \times e_n \rangle$ is non-decreasing such that $(1_{T_n} \times e_n)(t) \rightarrow e(t)$ in the Mackey topology a.e. in T .

We may therefore assume that each e_n is a simple function. By Fact 3, we may also assume that (Ω, Σ, μ) is a probability space, so that $L_\infty(\mu) \subseteq L_1(\mu)$. Observe

that for almost all $t \in T$, $e_n(t)$ is an element of the order interval $[0, e(t)]$ for all n . Hence, by Fact 4, we have $e_n(t) \rightarrow e(t)$ in $L_1(\mu)$ for almost all $t \in T$. Thus, as a map from T to $L_1(\mu)$, e is strongly measurable, therefore Borel measurable as (T, \mathcal{T}, ν) is complete. Now for each number r , the set $\{x \in L_1(\mu) : x \geq r1_\Omega\}$ is a closed subset of $L_1(\mu)$, and it follows that the set $\{t \in T : e(t) \geq r1_\Omega\}$ is a measurable subset of T for each number r .

Note that (i) means that for almost every $t \in T$ there is an integer $n > 0$ such that $e(t) \geq (1/n)1_\Omega$. By the previous paragraph, we can therefore find a number $r > 0$ and a set $H \in \mathcal{T}$ with $\nu(H) \geq r$ such that $e(t) \geq r1_\Omega$ for each $t \in H$. For each $n \in \mathbb{N}$, define $e'_n : T \rightarrow L_\infty(\mu)$ by setting $e'_n(t) = e_n(t) \vee r1_\Omega$ if $t \in H$, and $e'_n(t) = e_n(t)$ if $t \in T \setminus H$. Then for each n , e'_n is a measurable simple function; in particular, e'_n is Bochner integrable. Observe that $\int_T e'_n(t) \, d\nu(t) \geq \nu(H)r1_\Omega$. Thus $\int_T e'_n(t) \, d\nu(t) \in \|\cdot\|_\infty\text{-int } L_\infty(\mu)_+$ for each n . Moreover, the sequence $\langle e'_n \rangle$ is non-decreasing, and using the fact that the lattice operations in $L_\infty(\mu)$ are continuous for the Mackey topology, we see that $e'_n(t) \rightarrow e(t)$ in the Mackey topology for almost all $t \in T$. Thus (TAE) holds.

(D) The following two lemmata show that in the context of Sect. 7, the Gelfand integral does not exhibit pathological features. With $S = T$ and $B = \Omega$, Lemma 6 below amounts to a translation of Remark 5 into formal language. Lemma 7 shows that if Assumption (P) holds, and an allocation f is feasible for a coalition $S \in \mathcal{T}$, i.e., $\int_S f(t) \, d\nu(t) = \int_S e(t) \, d\nu(t)$, then the agents belonging to S cannot get commodities that are not available in the aggregate endowment of S .

Lemma 6 *Let (T, \mathcal{T}, ν) be a probability space, (Ω, Σ, μ) a σ -finite measure space, and $f : T \rightarrow L_\infty(\mu)_+$ Gelfand integrable. Let $S \in \mathcal{T}$ and write $v = \int_S f(t) \, d\nu(t)$ for the Gelfand integral of f over S . If a set $B \in \Sigma$ is such that $1_B \times v = 0$, then the set $N = \{t \in S : 1_B \times f(t) \neq 0\}$ is a null set.*

Proof Choose a strictly positive $q \in L_1(\mu)$ (as is possible because μ is σ -finite). Note that

$$\int_S (1_B \times q)(f(t)) \, d\nu(t) = (1_B \times q) \left(\int_S f(t) \, d\nu(t) \right) = q \left(1_B \times \int_S f(t) \, d\nu(t) \right) = 0.$$

As $(1_B \times q)(f(t)) \geq 0$ for all $t \in S$, it follows that $(1_B \times q)(f(t)) = 0$ for almost all $t \in S$. Consequently, as $(1_B \times q)(f(t)) = q(1_B \times f(t))$ and q is strictly positive, we must have $1_B \times f(t) = 0$ for almost all $t \in S$. □

Lemma 7 *Let (T, \mathcal{T}, ν) , (Ω, Σ, μ) , f , S , and, v be as in the previous lemma. Let $C = \{\omega \in \Omega : v(\omega) > 0\}$, identifying v with any of its versions. Then there is a null set $N \subseteq S$ such that $f(t) = 1_C \times f(t)$ for all $t \in S \setminus N$.*

Proof Set $B = \Omega \setminus C$, so that $1_B \times v = 0$. By the previous lemma, there is a null set $N \subseteq S$ such that $1_B \times f(t) = 0$ for all $t \in S \setminus N$, which implies that $f(t) = 1_C \times f(t)$ for all $t \in S \setminus N$, because $f(t) = 1_B \times f(t) + 1_C \times f(t)$. □

Appendix 2

In this appendix, we provide some mathematical background information, collecting some basics on vector integrals and on set theory.

(A) Let (T, \mathcal{T}, ν) be a nontrivial complete and totally finite measure space, and E a Banach space, with norm $\| \cdot \|$. A measurable function $f: T \rightarrow E$ is called a *simple function* if $f(T)$ is finite. The integral of a simple function f is given by

$$\int_T f \, d\nu = \sum_{x \in E} \nu \circ f^{-1}(\{x\})x.$$

A function $f: T \rightarrow E$ is *strongly measurable* if there exists a sequence $\langle f_n \rangle$ of simple functions such that

$$\lim_{n \rightarrow \infty} \|f(t) - f_n(t)\| = 0$$

for ν -almost all t . Every strongly measurable function is Borel measurable. Since for ν -almost all t , $f(t) \in \text{cl}(\bigcup_n f_n(T))$, a strongly measurable function takes values in a separable subspace of E outside a ν -null set. A strongly measurable function $f: T \rightarrow E$ is *Bochner integrable* if there exists a sequence of simple functions $\langle f_n \rangle$ such that

$$\lim_{n \rightarrow \infty} \int_T \|f - f_n\| \, d\nu = 0.$$

In that case, the *Bochner integral* of f is

$$\int_T f \, d\nu = \lim_{n \rightarrow \infty} \int_T f_n \, d\nu.$$

This limit exists and is independent of the particular approximating sequence $\langle f_n \rangle$. A strongly measurable function f is Bochner integrable if and only if the function $t \mapsto \|f(t)\|$ is Lebesgue integrable (Diestel and Uhl 1977, Theorem II.2.2). Bochner integrals inherit many convenient properties of the Lebesgue integral. If f is Bochner integrable, we call the function from \mathcal{T} to E given by

$$A \mapsto \int_A f \, d\nu = \int_T 1_A \times f \, d\nu$$

the *indefinite Bochner integral*. The indefinite Bochner integral is a ν -continuous countably additive vector measure; ν -continuity meaning that if $\lim_{n \rightarrow \infty} \nu(A_n) = 0$, then $\lim_{n \rightarrow \infty} \|\int_{A_n} f \, d\nu\| = 0$ (Diestel and Uhl 1977, Theorem II.2.4(i)).

In order to be able to integrate functions with values that do not essentially lie in a separable subspace, one can use weak integrals. Let E^* be the topological dual of E . A function $f: T \rightarrow E$ is *weakly measurable* if $t \mapsto x^* f(t)$ is measurable for all $x^* \in E^*$. A function $f: T \rightarrow E^*$ is *weak*-measurable* if $t \mapsto x f(t)$ is

measurable for all $x \in E$. By Pettis' measurability theorem, a function is strongly measurable exactly when it is weakly measurable and there is a separable subspace of E containing almost all values (Diestel and Uhl 1977, Theorem II.1.2). A weak*-measurable function need not be weakly measurable and a weakly measurable function need not be strongly measurable (Diestel and Uhl 1977, Examples II.1.5 and II.1.6). Let $f: T \rightarrow E$ be weakly measurable. If for each $A \in \mathcal{T}$, there is an $x_A \in E$ such that $\int_A x^* f \, d\nu = x^* x_A$ for all $x^* \in E^*$, f is called *Pettis integrable* and

$$\int_A f \, d\nu = x_A$$

the *Pettis integral* of f over A . It agrees with the Bochner integral if the latter is well defined, so this notation is unambiguous. It is possible for a weakly measurable function that $t \mapsto x^* f(t)$ is integrable for all $x^* \in E^*$ without f being Pettis integrable (Diestel and Uhl 1977, Example II.3.3). Better behaved in that respect is the Gelfand integral. If $f: T \rightarrow E^*$ is weak*-measurable and $t \mapsto x f(t)$ is integrable for all $x \in E$, then f is *Gelfand integrable* and there is a unique $x^* \in E^*$ such that $\int_T x f \, d\nu = x^* x$ for all $x \in E$ (Diestel and Uhl 1977, Lemma II.3.1). We call

$$\int_T f \, d\nu = x^*$$

the *Gelfand integral* of f . It follows that whenever f is Gelfand integrable and $A \in \mathcal{T}$, there is an $x_A^* \in E^*$ such that $\int_A x f \, d\nu = x_A^* x$ for all $x \in E$.

(B) We now collect some set-theoretic results. Many of the results will not be used in proofs, but in discussing how certain results fit into the literature. All results of this paper are derivable from the usual axioms of set theory, i.e., Zermelo–Fraenkel set theory with the axiom of choice (ZFC).

Recall that a partially ordered set is *well-ordered* if every non-empty subset has a minimum. An *ordinal* is a set well-ordered by the relation “ \in or =” and such that every element of the set is also a subset. If α and γ are ordinals, we write $\alpha \leq \gamma$ if $\alpha \in \gamma$ or $\alpha = \gamma$. Every set of ordinals is well-ordered by \leq and each ordinal equals the set of strictly smaller ordinals. In particular, we can use notation such as $\langle x_\xi \rangle_{\xi < \alpha}$ to denote a transfinite sequence indexed by the ordinal α . There is no set containing all ordinals.

For each set X , there is a smallest ordinal κ such that there is a bijection from κ onto X . We call κ the *cardinal* of X and write $\kappa = \#(X)$. Finite cardinals can be identified with the natural numbers $0, 1, 2, 3, \dots$. In particular, $2 = \{0, 1\}$. The set of all finite cardinals is a cardinal itself, the first infinite cardinal ω . The next larger cardinal is ω_1 , the first uncountable cardinal. The cardinal of \mathbb{R} is of special importance and is denoted by \mathfrak{c} and called *the cardinal of the continuum*. If κ is a cardinal, we let κ^+ be the smallest cardinal strictly larger than κ . For example, $\omega^+ = \omega_1$. A cardinal of the form κ^+ is a *successor cardinal* and every other cardinal is a *limit cardinal*. The *continuum hypothesis* (CH) says that $\omega_1 = \mathfrak{c}$.

If κ and λ are cardinals, we let κ^λ be the cardinal of the set of functions from λ to κ . Since one can identify subsets with indicator functions, 2^κ is the cardinal of the set

of all subsets of κ . For every cardinal κ , we have $\kappa < 2^\kappa$. We have $\mathfrak{c} = 2^\omega$. CH can be written as $\omega^+ = 2^\omega$. The *generalized continuum hypothesis* (GCH) says that $\kappa^+ = 2^\kappa$ for every cardinal κ . There are other operations one can do with the cardinals κ and λ . We let $\kappa + \lambda$ be the cardinal of the disjoint union of κ and λ and $\kappa \cdot \lambda$ be the cardinal of their Cartesian product. If κ or λ is infinite and both are nonzero, these operations are trivial and one has $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.

An axiom A is *relatively consistent* with ZFC if every proof of a contradiction from ZFC together with A can be turned into a proof of a contradiction from ZFC alone. Clearly, everything is relatively consistent with ZFC if there is a contradiction provable from ZFC. So, we assert our faith in mathematics and assume this is not possible. The generalized continuum hypothesis, and therefore also the continuum hypothesis, is relatively consistent with ZFC (Kunen 2011, Theorem II.6.24). A weakening of the continuum hypothesis is provided by *Martin's axiom*. The original statement of Martin's axiom is slightly intricate, but Martin's axiom is equivalent to the following statement: If X is a compact Hausdorff topological space in which every disjoint family of non-empty open subsets is countable, then the intersection of less than \mathfrak{c} open dense subsets is non-empty (Kunen 2011, Lemma III.3.17). A consequence of Martin's axiom is that the additivity of Lebesgue measure is \mathfrak{c} , that is, the union of less than \mathfrak{c} Lebesgue null sets is again a null set (Kunen 2011, Lemma III.3.28). Baire's category theorem for compact Hausdorff spaces shows that Martin's axiom in its topological version is implied by the continuum hypothesis. However, Martin's axiom is much weaker; in fact, there is a precise sense in which Martin's axiom is consistent with the cardinal of the continuum being arbitrarily large (Kunen 2011, Theorem V.4.1).

There are axioms that are widely used and widely taken to be consistent even though their relative consistency cannot be established. For example, one cannot prove the relative consistency of the existence of an atomless probability space in which every subset is measurable. By a result of Ulam, if such a probability space exists, a so-called weakly inaccessible cardinal must exist too (Jech 2003, Theorem 10.1). Such weakly inaccessible cardinals are known to imply the consistency of ZFC (Kunen 2011, Corollary II.6.26 and Theorem II.6.23). But by Gödel's second incompleteness theorem, an axiom system strong enough to prove the consistency of ZFC is not relatively consistent with ZFC.

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