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RESEARCH ARTICLE

Rationalizability in large games

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Abstract This paper characterizes both *point-rationalizability* and *rationalizability* in large games when societal responses are formulated as distributions or averages of individual actions. The sets of point-rationalizable and rationalizable societal responses are defined and shown to be convex, compact and equivalent to those outcomes that survive iterative elimination of never best responses, under point-beliefs and probabilistic beliefs, respectively. Given the introspection and mentalizing that rationalizability notions presuppose, one motivation behind the work is to examine their viability in situations where the terms *rationality* and *full information* can be given a more parsimonious, and thereby a more analytically viable, expression.

Keywords Large games · Rationalizability · Point-rationalizability · Closed under rational behavior (CURB) · Societal response

JEL Classification C72 · D80 · C65

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1 Introduction

Games with a finite number of players are not able to adequately represent many economic environments. In order to capture the numerical negligibility of individuals in the economy, many papers employ a continuum of players. A large number of such papers have recently used the idea of "rationalizability," which amounts to common knowledge of rationality of the agents. For example, Guesnerie (1992) introduces the concept of strong rationality in a specific market with a continuum of producers, and Evans and Guesnerie (2003) study the concept of strong rationality in macroeconomics. These works use an intuitive or model-specific definition of the idea of rationalizability. The goal of this paper is to provide a formal characterization of rationalizability in general game environments with a continuum of players.

Unlike in general equilibrium structures, as fashioned in the mid-1960s by Aumann (1964) and others following him, a continuum of players is now studied in a context where interdependence is made explicit and rendered analytically tractable. In the theory of large games, a player's payoff depends, in addition to his or her own actions, on some statistical summary[1] of the game. In contrast to games with finitely many players, each agent in a large game is strategically negligible. The *other* is no longer a player or a fully delineated group of players, but rather the society or the collective that is the formalized subject of the game. A player's actions, then, are influenced by how he or she conceives and conceptualizes the society of which he or she is a part, rather than how he or she conceptualizes specific individuals. Certain results such as the existence results of Nash equilibria have been well established in the theory of large games.[2] The objective of this paper is to examine notions of rationalizability in the context of large games.

The concept of rationalizability has been introduced into game theory independently by Bernheim (1984) and Pearce (1984).[3] They assert that agents only use strategies that are best responses to their forecasts and therefore, as in a Nash equilibrium, some strategies in the action set will never be played. Unlike Nash equilibrium, however, rationalizability does not assume that players correctly predict the actions of the other players. Instead, rationalizability only assumes common knowledge of rationality of the players. In this context, rationality means that all players are payoff maximizers, and common knowledge of rationality means that every player knows all other players are rational, and every player knows all other players know that everyone else is rational and so on.[4] While rationalizability assumes that a player will not choose a strategy which is not expected utility maximizing given any subjective belief about opponents' strategies, Bernheim (1984) also analyzes another concept, *point-rationalizability,* which assumes rationality under point-beliefs; this is to say, no player will choose a strategy which is not a best response to any strategy profile of his or her opponents.

[1] In the sequel, "societal response" is used interchangeably with "statistical summary," and "agent" with "player."

[2] See the survey and the references in Khan and Sun (2002).

[3] It is less known, but common knowledge of (or common belief in) rationality has already been studied in Böge and Eisele (1979). Their contribution is well explained in Section 7 of Perea (2013).

[4] Aumann (1976) offers a formal definition of common knowledge.

The set of rationalizable strategy profiles can be defined as the maximal subset of strategy profiles, such that any strategy of an individual player is a best response to his or her forecast of opponents' strategies within the subset. This characterization is related to Basu and Weibull (1991) where a set of strategy profiles is said to be *closed under rational behavior* (henceforth, *CURB*) if the set contains all its best responses. The set of rationalizable strategy profiles is thus the maximal CURB set. Or, equivalently, the set of rationalizable strategy profiles can be characterized as the set that survives the *iterated elimination*[5] of strategies which are never best responses. Bernheim (1984) shows that these characterizations are well defined and equivalent.

However, Bernheim (1984), Pearce (1984) and the subsequent literature[6] restrict attention to games with *a finite number of players*, except that Jara-Moroni (2012) focuses on a specific type of large games which is discussed in the sequel. This paper combines the literature on large games with the literature on rationalizability and characterize rationalizability in large games. To reiterate, this initiative is motivated by the fact that the idea of rationalizability appears in applied work with a continuum of players, despite the fact that a formal characterization of rationalizability in general game models is lacking. Specifically, the characterizations in Bernheim (1984) are adapted to large games. The set of rationalizable societal responses is demonstrated to be well defined as the maximum CURB set of societal responses. Moreover, similar to (Bernheim 1984), it is shown that the set is also related to the set of societal responses that survive iterative elimination of strategy profiles that are not best responses to forecasts of the society. This captures the ideas of the characterization in Bernheim (1984) as well as that of Pearce (1984). Thus, by focusing on rationalizability within a large game framework, this solution set can be justified by the epistemic assumption that it is common knowledge that only best responses are ever chosen. This epistemic interpretation of rationalizability may also provide a plausible justification for equilibrium play in large games. For example, if the set of rationalizable societal responses is a singleton, then any best response profile to that societal response forms a Nash equilibrium.

In this paper, rationalizability is studied in large games with several different formulations of societal responses. The first model takes distributions of individual actions as societal responses: the payoff to each player depends on both his and her own action and the proportion of other players who play each action. For example, in voting games,[7] the payoff of each voter depends on the electoral outcomes which in turn depends on the proportion of the electorate that votes for each candidate. The second model formulates averages of all players' actions as societal responses: each player's payoff depends on both own action and the averages of all actions. For example, in the rational expectations model of Muth,[8] the profit of each farmer depends on his or her own supply and the aggregate supply of all farmers. In terms of averages, averages

[5] This process is exactly the one used by Pearce (1984) to define rationalizability. But Pearce (1984) uses mixed strategies to eliminate the never best responses.

[6] See, for example, Brandenburger and Dekel (1987), Dekel et al. (2007), Chen and Luo (2012) and their references.

[7] See, for example, Banks and Duggan (2006).

[8] See, for example, Guesnerie (1992). There are some other rational expectations models, see the discussion in Khan (2008, p. 76).

of transformed actions are also discussed briefly to formulate transformed summary statistics as societal responses: players' payoffs depend on their own actions, and the mean of individual plays under a general transformation. For example, in monopolistic competition models with summary statistics,[9] the payoff of each firm depends on its own action, and some summary statistics of the aggregate strategy profile.

This paper is most closely related to the work of Jara-Moroni (2012) who characterizes rationalizability similarly, but within the framework of large games considered in Rath (1992), in which the action set is a compact subset of a finite-dimensional Euclidean space and societal responses are averages of individual actions. However, unlike Jara-Moroni (2012), the first formulation of societal responses in this paper works on distributions as considered in Khan and Sun (1995, 1999), Keisler and Sun (2009), Carmona and Podczeck (2009) and Noguchi (2009). The second formulation addresses this issue for the characterization of rationalizability in games with averages as societal responses, but with an infinite-dimensional action set. Moreover, note that the tools for the analysis in Jara-Moroni (2012) are based on the standard integration theory of correspondences in the finite-dimensional setting[10] that are no longer valid in an infinite-dimensional setting, whereas at the same time, the infinite-dimensional action set is widely used in the literature for large games and mathematical economics.

The rest of the paper is organized as follows: Sect. 2 presents the model and results on large games with societal responses formulated as distributions of individual actions. Section 3 shows the corresponding results when societal responses are formulated as averages of individual actions or of transformed actions when the action set is allowed to be infinite dimensional. Section 4 concludes the paper. Relevant results on correspondences are stated in Appendix A, and relegated proofs are provided in Appendix B.

2 Games with distributions as societal responses

A game with a continuum of players has two basic objects: an abstract atomless probability space[11] $(I, \mathcal{I}, \lambda)$ representing the space of player names, and a nonempty compact metric space A representing a common action space. Each player $i \in I$ chooses his or her own action from the action set A. Let $\mathcal{M}(A)$ be the space of probability measures on A endowed with the weak topology.[12] $\mathcal{M}(A)$ represents the collection of all societal responses which are distributions of possible plays in the game. The space of players' payoffs $\mathcal{U}_A$ is then given by the space of all continuous functions on the product space $A \times \mathcal{M}(A)$ endowed with its sup-norm topology and with $\mathcal{B}(\mathcal{U}_A)$, its Borel σ-algebra.

9 See, for example, Rauh (1997, 2003).

10 One can refer to Hildenbrand (1974) for the theory of correspondences in the finite-dimensional setting.

11 A probability space $(I, \mathcal{I}, \lambda)$ is *atomless* if for any $S \in \mathcal{I}$ with $\lambda(S) > 0$, there exists a $S' \in \mathcal{I}$, such that $S' \subseteq S$ and $0 < \lambda(S') < \lambda(S)$.

12 It is standard to forgo referring to this as the narrow topology, the topology of convergence in distribution, or the weak*-topology, the formally correct designation. Throughout this paper, for any metrizable topological space X, $\mathcal{M}(X)$ is used to denote the space of probability measures on X endowed with this weak topology. It is known that if X is a compact metric space, the space $\mathcal{M}(X)$ is also a compact metric space.

A *large game* $\mathcal{G}$ is a measurable function from I to $\mathcal{U}_A$. And a *strategy profile* of $\mathcal{G}$ is a measurable function $f : I \to A$ which specifies a strategy for each player. A *Nash equilibrium* of a game $\mathcal{G}$ is a strategy profile f^* of $\mathcal{G}$ such that for λ-almost all $i \in I$,

$$u_i(f^*(i), \lambda f^{*-1}) \geq u_i(a, \lambda f^{*-1})$$

for all $a \in A$, where $u_i \equiv \mathcal{G}(i)$.

It is well known, see Khan and Sun (1995) for example, that if the action set A of a large game is a countable compact metric space, there is a Nash equilibrium. In order to develop general results on the existence of a Nash equilibrium (and the characterization of point-rationalizability and rationalizability in the sequel) in a large game with a general compact metric space as its action set, the notion of a saturated probability space[13] is used to formalize the space of player names. A probability space is said to be *almost countably generated,* if its σ-algebra can be generated by a countable number of subsets together with the null sets. The basic definition of a saturated probability space is as follows.

Definition 1 A probability space $(I, \mathcal{I}, \lambda)$ is *saturated* if there is no subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, such that the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is almost countably generated, where $\mathcal{I}^S \equiv \{S' \cap S : S' \in \mathcal{I}\}$ and λ^S is the probability measure re-scaled from the restriction of λ to $\mathcal{I}^S$.

From the above definition, it follows that saturated probability spaces must be atomless.[14] Furthermore, the Lebesgue unit interval is not a saturated space, and any atomless Loeb probability space is saturated. The Lebesgue unit interval however can be extended into a saturated probability space.[15]

The following antecedent result is on the existence of Nash equilibria in a large game.[16]

Proposition 1 *A game $\mathcal{G} : (I, \mathcal{I}, \lambda) \longrightarrow \mathcal{U}_A$ has a Nash equilibrium if any of the following two (sufficient) conditions hold: (1) A is countable; (2) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.*

2.1 Point-rationalizability

In a large game $\mathcal{G}$, when a Nash equilibrium is used as a prediction of the outcome, it requires not only that each player knows the equilibrium of the game but also that

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[13] There are different terminologies related to the various equivalent definitions of a saturated probability space: it is called "$\aleph_1$-atomless" and "a probability space to have the saturation property" in Hoover and Keisler (1984), "nowhere separable" in Džamonja and Kunen (1995), "a probability space with a set of uncountable cardinals as its Maharam spectrum" in Fajardo and Keisler (2002),"super-atomless" in Podczeck (2008), "nowhere countably generated" in Loeb and Sun (2009) and "rich" in Noguchi (2009).

[14] See Podczeck (2008, Fact) for example.

[15] See, for example, Kakutani (1944), Podczeck (2008) and Sun and Zhang (2009).

[16] This result is summarized from Khan and Sun (1995, Theorem 10), Keisler and Sun (2009, Theorem 4.6), Carmona and Podczeck (2009, Theorem 2) and Noguchi (2009, Theorem 2). Throughout the paper, results previously available in the literature are called "Propositions."

each player correctly expects the societal response induced by the Nash equilibrium. But in general, it is unclear whether such requirements are satisfied. The implications of the assumption that rationality is common knowledge are now explored.

Unlike games with a finite number of players in which one could work on strategy profiles directly, in large games, to create their forecasts, all players will use intuition to access societal responses they face. Thus, a player is rational if the player uses only those strategies which are best responses to some point or probabilistic belief about societal responses. The remainder of this section starts with an analysis of reasonable societal responses in $\mathcal{G}$ in which forecasts of all players are assumed to be points in the set of all societal responses $\mathcal{M}(A)$. For any $\tau \in \mathcal{M}(A)$ and any $i \in I$, let $B(i, \tau)$ be player i's best response when he or she faces τ. That is,

$$B(i, \tau) = \arg\max_{a \in A} u_i(a, \tau).$$

Then, facing a set of societal responses $\mathcal{D} \subseteq \mathcal{M}(A)$ and being rational, player i will only choose a strategy from $B(i, \mathcal{D})$, where

$$B(i, \mathcal{D}) = \bigcup_{\tau \in \mathcal{D}} B(i, \tau)$$

is the image of $\mathcal{D}$ under $B(i, \cdot)$. If all players forecast the same set of societal responses $\mathcal{D} \subseteq \mathcal{M}(A)$, then a strategy profile that satisfies rationality will be a measurable selection chosen from the correspondence[17] $B(\cdot, \mathcal{D}) : \mathcal{I} \twoheadrightarrow A$.

Let $Pr : 2^{\mathcal{M}(A)} \to 2^{\mathcal{M}(A)}$ be a mapping, such that for any subset $\mathcal{D} \subseteq \mathcal{M}(A)$,

$$Pr(\mathcal{D}) = \left\{ \lambda f^{-1} : f \text{ is a measurable selection of } B(\cdot, \mathcal{D}) \right\}.$$

It is clear that $Pr(\mathcal{D})$ is the set of societal responses that are induced by all strategy profiles that are rationally played with respect to $\mathcal{D}$ under point-beliefs when all players are rational. To show $Pr(\mathcal{D})$ is well defined, one needs to show that the measurable selection of $B(\cdot, \mathcal{D})$ is not vacuous. Thus, it is necessary to check the measurability of the correspondence $B(\cdot, \mathcal{D})$ for some subset $\mathcal{D}$ of $\mathcal{M}(A)$ to ensure that a measurable selection exists. The following result provides a sufficient condition.[18]

Lemma 1 *For any nonempty closed set $\mathcal{D}$ of $\mathcal{M}(A)$, $B(\cdot, \mathcal{D}) : I \twoheadrightarrow A$ is measurable and has nonempty closed values.*[19]

Beginning from the set of all action distributions $\mathcal{M}(A)$, the assumption that player $i \in I$ is rational and knows that the other players are rational implies that reasonable

[17] A correspondence F from a set X to Y is a relation which assigns to each $x \in X$ a subset $F(x)$ of Y. $F : X \twoheadrightarrow Y$ is used to distinguish a correspondence from a function from X to Y. One can refer to Aliprantis and Border (2006, Chapters 17 and 18) for standard notions and results on correspondences. For sake of reference, a brief summary of results on correspondences that are used in this paper is provided in Appendix A.

[18] Proofs of all lemmas are provided in Appendix B.

[19] Hence, $B(\cdot, \mathcal{D})$ is also compact-valued since A is compact.

forecasts of societal responses should be from the set of all distributions induced by measurable selections from the correspondence $B(\cdot, \mathcal{M}(A))$. With the assumption of common knowledge of rationality, by iterating this logic, one obtains the following iterative elimination process:

$$Pr^0(\mathcal{M}(A)) = \mathcal{M}(A)$$
$$Pr^{t+1}(\mathcal{M}(A)) = Pr(Pr^t(\mathcal{M}(A))), \quad \text{for all } t \geq 1.$$

This elimination process, which keeps iteratively eliminating strategies which are never best responses under some point-belief until there is no unreasonable societal response, is called *Iterated Elimination of Never Best Responses* (henceforth, IENBR) under point-beliefs. Infinite repetition of this process[20] generates the set $\mathbb{P}^e_{\mathcal{G}}$, where $\mathbb{P}^e_{\mathcal{G}} = \bigcap_{t=0}^{\infty} Pr^t(\mathcal{M}(A))$.

Definition 2 An action distribution $\tau \in \mathcal{M}(A)$ survives IENBR under point-beliefs in $\mathcal{G}$, if it is an element of $\mathbb{P}^e_{\mathcal{G}}$.

Note that any societal response induced by any Nash equilibrium can never be eliminated by this process of IENBR under point-beliefs. Thus, that $\mathbb{P}^e_{\mathcal{G}}$ is not empty follows from the fact that a Nash equilibrium exists in $\mathcal{G}$, given that (1) or (2) in Proposition 1 holds. Besides nonemptiness, the next result shows that $\mathbb{P}^e_{\mathcal{G}}$, the set of all action distributions that survive IENBR under point-beliefs, has some additional properties.

Theorem 1 *In a large game $\mathcal{G} : (\mathcal{I}, \mathcal{I}, \lambda) \longrightarrow \mathcal{U}_A$, ${\mathbb{P}_{\mathcal{G}}}^e$ is a nonempty, compact and convex set if either of the following two conditions hold: (1) A is countable; (2) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.*

Proof Suppose (1) or (2) holds. Let $\{F^t\}$ be a sequence of correspondences such that $F^t : I \twoheadrightarrow A, t \geq 0$ is given by

$$F^0(i) = A, \quad \text{for all } i \in I$$
$$F^t(i) = B\left(i, Pr^{t-1}(\mathcal{M}(A))\right), \quad \text{for all } i \in I, \text{ if } t \geq 1,$$

where

$$Pr^t(\mathcal{M}(A)) = \{\lambda f^{-1} : f \text{ is a measurable selection of } F^t\}.$$

Fix $i \in I$. By Berge's maximum theorem, see, for example, Aliprantis and Border (2006, Theorem 17.31), the joint continuity of u_i on $A \times \mathcal{M}(A)$ implies that $B(i, \cdot)$ is upper hemicontinuous and compact-valued on $\mathcal{M}(A)$. Thus, by Proposition 2 (P2) in Appendix A, $B(i, \mathcal{M}(A))$ is a compact metric space because both A and $\mathcal{M}(A)$ are compact metric spaces. By Lemma 1, F^1 is measurable, nonempty-valued and compact-valued. By induction over t, together with Proposition 4 in Appendix A,

[20] This idea is referred to as an "eductive process" in Guesnerie (1992) and used to characterize a standard market with a continuum of producers.

$Pr^{t-1}(\mathcal{M}(A))$ is also compact and convex for all $t \geq 1$. Therefore, being the intersection of the nested family of nonempty compact convex sets $\{Pr^t(\mathcal{M}(A))\}$, $\mathbb{P}^e_{\mathcal{G}}$ is nonempty, convex and compact. □

Although $\mathbb{P}^e_{\mathcal{G}}$, the set of all action distributions that survive IENBR under point-beliefs, is a natural construct to represent rationality and common knowledge of rationality, point-rationalizability should be such that the set of point-rationalizable societal responses is also a fixed point of Pr. However, it is unclear whether the outcome that survives IENBR under point-beliefs satisfies this or not. Thus, to capture the epistemic definition of point-rationalizability in $\mathcal{G}$, a CURB set of societal responses under point-beliefs is proposed. A set of societal responses $\mathcal{D}$ is a CURB set under point-beliefs in $\mathcal{G}$ if it contains all action distributions that are induced by strategy profiles which are selections from $B(\cdot, \mathcal{D})$. More formally, a set of action distributions $\mathcal{D}$ is said to be CURB under point-beliefs in $\mathcal{G}$ if $\mathcal{D} \subseteq Pr(\mathcal{D})$. A CURB set of distributions under point-beliefs $\mathcal{D}$ is *tight* if $\mathcal{D} = Pr(\mathcal{D})$. Thus, point-rationalizability in $\mathcal{G}$ can be defined as follows.

Definition 3 The set of point-rationalizable action distributions of $\mathcal{G}$, $\mathbb{P}_{\mathcal{G}}$, is $\mathcal{G}$'s maximal tight CURB set of action distributions under point-beliefs.

Bernheim (1984) shows that in a game with a finite number of players, when all players have continuous payoffs and compact strategy sets, the set of point-rationalizable strategy profiles coincides with the set of strategy profiles that survive IENBR under point-beliefs. To connect the point-rationalizability to the process of IENBR under point-beliefs in large games, the next result establishes that $\mathbb{P}^e_{\mathcal{G}}$, the set of all societal responses that survive IENBR under point-beliefs, is equivalent to $\mathbb{P}_{\mathcal{G}}$, the set of point-rationalizable societal responses under some general conditions. Thus, under those conditions, $\mathbb{P}_{\mathcal{G}}$ naturally has all properties that $\mathbb{P}^e_{\mathcal{G}}$ has in a large game $\mathcal{G}$.

Theorem 2 *An action distribution of a large game $\mathcal{G} : (\mathcal{I}, \mathcal{I}, \lambda) \longrightarrow \mathcal{U}_A$ is point-rationalizable if and only if it survives IENBR under point-beliefs, i.e., $\mathbb{P}_{\mathcal{G}} = \mathbb{P}^e_{\mathcal{G}}$, provided one of the following two conditions holds: (1) A is countable; (2) $(I, \mathcal{I}, \lambda)$ is saturated.*

Proof Throughout this proof, the notations are as in the proof of Theorem 1. Suppose (1) or (2) holds. It needs to be shown that $\mathbb{P}^e_{\mathcal{G}} = Pr(\mathbb{P}^e_{\mathcal{G}})$ and $\mathbb{P}^e_{\mathcal{G}}$ is also the maximal tight CURB set of distributions under point-beliefs.

First, it is shown that $Pr(\mathbb{P}^e_{\mathcal{G}}) \subseteq \mathbb{P}^e_{\mathcal{G}}$. Suppose that $\tau \in Pr(\mathbb{P}^e_{\mathcal{G}})$. Then, $\tau \in Pr^0(\mathcal{M}(A))$, and by definition, there exists a measurable selection f of $B(\cdot, \mathbb{P}^e_{\mathcal{G}})$, such that $\tau = \lambda f^{-1}$. As $B(i, \mathbb{P}^e_{\mathcal{G}}) \subseteq F^t(i)$ for all $t \geq 0$ and all $i \in I$, f is also a selection of F^t. Consequently, $\tau \in Pr^t(\mathcal{M}(A))$ for all $t \geq 1$. Therefore, $Pr(\mathbb{P}^e_{\mathcal{G}}) \subseteq \mathbb{P}^e_{\mathcal{G}}$.

The next step is to show that $\mathbb{P}^e_{\mathcal{G}} \subseteq Pr(\mathbb{P}^e_{\mathcal{G}})$. Let $F : I \twoheadrightarrow A$ be a correspondence such that for all $i \in I$, $F(i) = cl\text{-Lim}\,\{F^t(i)\}$.[21] Take any point τ from $\mathbb{P}^e_{\mathcal{G}}$. It is clear

[21] For any sequence $\{x_t\}$ in a topological space, let cl-Lim $\{x_t\}$ be the set of its limit points. For any sequence of sets $\{A_t\}$ in a topology space, let cl-Lim $\{A_t\}$ be the union of all such cl-Lim $\{x_t\}$ with $x_t \in A_t$ for all t.

that $\tau \in Pr^t(\mathcal{M}(A))$ for all t. Hence, there exists a sequence of functions $\{f^t\}$, such that f^t is a measurable selection of F^t and $\lambda(f^t)^{-1} = \tau$, for all $t \geq 0$. It is trivial that $\{\lambda(f^t)^{-1}\}$ converges weakly to τ. According to Proposition 4 (P1) in Appendix A, there exists a measurable selection f^* of cl-Lim $\{f^t\}$ such that $\tau = \lambda f^{*-1}$. Therefore, $\tau \in \mathcal{D}_F$ where $\mathcal{D}_F = \{\lambda f^{-1} : f$ is a measurable selection *of* $F\}$.

Thus, to establish that $\mathbb{P}^e_{\mathcal{G}} \subseteq Pr(\mathbb{P}^e_{\mathcal{G}})$, it suffices to show that $\mathcal{D}_F \subseteq Pr(\mathbb{P}^e_{\mathcal{G}})$, i.e., any measurable selection of F is also a measurable selection of $B(\cdot, \mathbb{P}^e_{\mathcal{G}})$. One only needs to show that $F(i) \subseteq B(i, \mathbb{P}^e_{\mathcal{G}})$ for all $i \in I$. Toward this end, fix $i \in I$ and choose any $a \in F(i)$. According to the construction of F, one can find $a^t \in F^t(i)$ for each $t \geq 0$, such that a is the limit point of $\{a^t\}$. That is to say, there is a sequence $\{\tau^t\}$ such that $\tau^t \in Pr^{t-1}(\mathcal{M}(A))$ and $a^t \in B(i, \tau^t)$ for all t. Because $\mathbb{P}^e_{\mathcal{G}}$ is the intersection of a nested family of compact sets, the sequence $\{\tau^t\}$ admits a limit point $\tau^0 \in \mathbb{P}^e_{\mathcal{G}}$ and a subsequence $\{\tau^{t_k}\}$ converges weakly to τ^0. Furthermore, given that $B(i, \cdot)$ is compact-valued and upper hemicontinuous, one obtains $a \in cl$-Lim $\{B(i, \tau^{t_k})\} \subseteq B(i, \tau^0) \subseteq B(i, \mathbb{P}^e_{\mathcal{G}})$. Hence, $F(i) \subseteq B(i, \mathbb{P}^e_{\mathcal{G}})$ for all $i \in I$.

It has now been shown that $\mathbb{P}^e_{\mathcal{G}}$ is a tight CURB set of distributions under point-beliefs of $\mathcal{G}$ (i.e., $\mathbb{P}^e_{\mathcal{G}} = Pr(\mathbb{P}^e_{\mathcal{G}})$). The final step is to show that $\mathbb{P}^e_{\mathcal{G}}$ is the maximal one. Suppose not. Then there exists some $\mathcal{D} \subseteq \mathcal{M}(A)$ for which $Pr(\mathcal{D}) = \mathcal{D}$, and some number k, such that $\mathcal{D} \cap Pr^k(\mathcal{M}(A)) = \mathcal{D}$, i.e., $\mathcal{D} \cup Pr^k(\mathcal{M}(A)) = Pr^k(\mathcal{M}(A))$, but $\mathcal{D} \cap Pr^{k+1}(\mathcal{M}(A)) \subset \mathcal{D}$. Note that for any two sets, $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{M}(A)$, $Pr(\mathcal{D}_1) \cup Pr(\mathcal{D}_2) \subseteq Pr(\mathcal{D}_1 \cup \mathcal{D}_2)$. So,

$$Pr(\mathcal{D}) \cup Pr^{k+1}(\mathcal{M}(A)) \subseteq Pr(\mathcal{D} \cup Pr^k(\mathcal{M}(A))) = Pr^{k+1}(\mathcal{M}(A)).$$

Intersecting left and right sides with $\mathcal{D}$, one obtains $\mathcal{D} \subseteq \mathcal{D} \cap Pr^{k+1}(\mathcal{M}(A))$, which is a contradiction. This completes the proof. □

2.2 Rationalizability

Instead of being points in the set of all societal responses $\mathcal{M}(A)$, forecasts of any player under probabilistic beliefs are probability distributions on $\mathcal{M}(A)$. Rationalizability in this setting is defined and characterized in the remainder of this section.

Let $\mathcal{M}(\mathcal{M}(A))$ be the space of probability measures on $\mathcal{M}(A)$ endowed with the weak topology. For any $\mu \in \mathcal{M}(\mathcal{M}(A))$ and any $i \in I$, let $\hat{B}(i, \mu)$ be player i's best response when he or she faces μ. That is, $\hat{B}(i, \mu) = \arg\max_{a \in A} U_i(a, \mu)$, where $U_i(a, \mu) = \int_{\mathcal{M}(A)} u_i(a, \tau) d\mu(\tau)$ is the expected payoff of player i if a is played, and μ is player i's belief of societal responses. Thus, if player i facing the set of societal responses $\mathcal{D} \subseteq \mathcal{M}(A)$, being rational, player i will only choose a strategy from $\hat{B}(i, \mathcal{M}(\mathcal{D}))$ where

$$\hat{B}(i, \mathcal{M}(\mathcal{D})) = \bigcup_{\mu \in \mathcal{M}(\mathcal{D})} \hat{B}(i, \mu)$$

is the image of $\mathcal{M}(\mathcal{D})$[22] under $\hat{B}(i, \cdot)$.

[22] Note that for any nonempty $\mathcal{D} \subseteq \mathcal{M}(A)$, $\mathcal{M}(\mathcal{D}) = \{\mu \in \mathcal{M}(\mathcal{M}(A)) : \text{supp}\, \mu \subseteq \mathcal{D}\}$.

Let $\hat{Pr} : 2^{\mathcal{M}(A)} \rightarrow 2^{\mathcal{M}(A)}$ be a mapping such that for any subset $\mathcal{D} \subseteq \mathcal{M}(A)$,

$$\hat{Pr}(\mathcal{D}) = \{\lambda \hat{f}^{-1} : \hat{f} \text{ is a measurable selection of } \hat{B}(\cdot, \mathcal{M}(\mathcal{D}))\}.$$

Note that for any $\mathcal{D} \subseteq \mathcal{M}(A)$, $B(i, \mathcal{D}) \subseteq \hat{B}(i, \mathcal{M}(\mathcal{D}))$. Thus, when $\mathcal{D}$ is nonempty and closed, $\hat{Pr}(\mathcal{D})$ is nonempty, as Lemma 1 guarantees that there is a measurable selection of $B(\cdot, \mathcal{D})$, and such a selection is also a measurable selection of $\hat{B}(\cdot, \mathcal{M}(\mathcal{D}))$.

Thus, a set of action distributions $\mathcal{D}$ is a *CURB* set in $\mathcal{G}$ if $\mathcal{D} \subseteq \hat{Pr}(\mathcal{D})$. A CURB set of action distributions $\mathcal{D}$ is *tight* if $\mathcal{D} = \hat{Pr}(\mathcal{D})$. Similar to point-rationalizability, the set of rationalizable distributions can be defined as the maximum fixed point of $\hat{Pr}$.

Definition 4 The set of *rationalizable* action distributions of $\mathcal{G}$, $\mathbb{R}_{\mathcal{G}}$, is $\mathcal{G}$'s maximal tight CURB set of action distributions.

To characterize rationalizability in $\mathcal{G}$, one can now consider $\mathbb{R}^e_{\mathcal{G}}$, the set of all distributions that survive IENBR under probabilistic beliefs in $\mathcal{G}$. Starting from all societal responses $\mathcal{M}(A)$, by common knowledge of rationality under probabilistic beliefs,

$$\mathbb{R}^e_{\mathcal{G}} = \bigcap_{t=0}^{\infty} \hat{Pr}^t(\mathcal{M}(A)),$$

where $\hat{Pr}^0(\mathcal{M}(A)) = \mathcal{M}(A)$ and $\hat{Pr}^{t+1}(\mathcal{M}(A)) = \hat{Pr}(\hat{Pr}^t(\mathcal{M}(A)))$, for all $t \geq 1$.

The last result in this section is now ready to be presented.

Theorem 3 *In a large game $\mathcal{G} : (\mathcal{I}, \mathcal{I}, \lambda) \longrightarrow \mathcal{U}_A$, the set of rationalizable action distributions, $\mathbb{R}_{\mathcal{G}}$, is nonempty, compact and convex, and it is the same as $\mathbb{R}^e_{\mathcal{G}}$, if one of the following two conditions holds: (1) A is countable; (2) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.*

Proof The proof is a modification of proofs of Theorems 1 and 2 above. Let $\{\hat{F}^t\}$ be a sequence of correspondences such that $\hat{F}^t : I \twoheadrightarrow A, t \geq 0$ is given by

$$\hat{F}^0(i) = A, \quad \text{for all } i \in I$$
$$\hat{F}^t(i) = \hat{B}\left(i, \mathcal{M}\left(\hat{Pr}^{t-1}(\mathcal{M}(A))\right)\right), \quad \text{for all } i \in I, \text{ if } t \geq 1,$$

where

$$\hat{Pr}^t(\mathcal{M}(A)) = \{\lambda \hat{f}^{-1} : \hat{f} \text{ is a measurable selection of } \hat{F}^t\}.$$

For any $i \in I$, the continuity of u_i on $A \times \mathcal{M}(A)$ implies that U_i is continuous on $A \times \mathcal{M}(\mathcal{M}(A))$ by Proposition 7. Therefore, similar to the proof of Theorem 1, it is clear that $\mathbb{R}^e_{\mathcal{G}}$ is nonempty, closed and convex. Furthermore, let $\hat{F} : I \twoheadrightarrow A$ be a correspondence, such that for all $i \in I$,

$$\hat{F}(i) = cl\text{-Lim}\{\hat{F}^t(i)\}.$$

By arguments similar to the proof of Theorem 2, one can show that $\mathbb{R}^e_{\mathcal{G}}$ is also the maximum tight CURB set of action distributions in $\mathcal{G}$. The proof is now complete. □

The idea of finding tight CURB sets by iterating a mapping is implicit in the statement and proof of Berge (1963, Theorem 8, p.113); also see the proof of Jungbauer and Ritzberger (2011, Theorem 2) for a recent treatment.[23] However, the main technical challenges in the proofs of Theorems 1 to 3 are to identify the relevant convergence concept for the elimination process and to ensure some measurable selection from best responses in each step of elimination. This difficulty does not arise in the case of finite player games considered in Bernheim (1984). Nor is the solution covered by the game of a continuum of players considered in Jara-Moroni (2012) when the action space is finite dimensional, and the societal responses are seen as a convex combinations of actions. The following four remarks conclude this section.

Remark 1 An interesting situation arises when the set of rationalizable societal response $\mathbb{R}_{\mathcal{G}}$ of a game $\mathcal{G}$ is a singleton. As shown in Proposition 1, there always exists a Nash equilibrium in the games described in this section under (1) or (2). If $\mathbb{R}_{\mathcal{G}}$ is a singleton, the rationalizable action distribution must be the equilibrium distribution induced by its Nash equilibrium. This observation is also true for games in the next section.

Remark 2 The framework in this section is in terms of distributions. The most relevant mathematical tools revolve around distribution of a correspondence from an atomless probability space to a metric space. Recall that action set A is assumed to be a compact metric space in a game. When A is countable, the theory is developed in Khan and Sun (1995). If A is uncountable, one has to impose the saturation property upon the name space, so that one can use the distribution of correspondences in Keisler and Sun (2009).

Remark 3 The compactness assumption of the action set is essential to characterize rationalizability in this section as well as in that of Bernheim (1984) and Pearce (1984). In games with a finite number of players and general Polish action spaces, Arieli (2010) shows that the set of strategies which survive IENBR may not be the set of rationalizable strategies, and that ω_1 (the first uncountable ordinal) rounds might be necessary to get to the CURB set.[24] Thus, it is of interest to ask how one could characterize rationalizability in a large game with general action space without the assumption of compactness. It is possible that one can characterize rationalizability in way similar to this section when the compactness of the action set is relaxed by assuming instead that (a) the common action space is a complete metric space, and

[23] The author thanks the referee for these references.

[24] The author is thankful to the referee for pointing this out.

that (b) each player is permitted to choose his or her actions from a compact subset of this space.[25] However, for a large game with a general Polish space as its action set, it is unclear whether the set of rationalizable societal responses coincides with the set of societal responses that survive IENBR. Future work may address this question.

Remark 4 In a large game, the set of point-rationalizable societal responses is contained in the set of rationalizable societal responses. Even with finite actions, the inclusion can be proper. This follows from Example 4 in Jara-Moroni (2012).

3 Games with averages as societal responses

It is also of interest to consider large games where societal responses are formulated as averages of individual responses or transformed individual responses. This section is devoted to discussing both point-rationalizability and rationalizability in such settings, in particular, games with an infinite set of strategies which may not be expressed as a linear combination of a finite number of elements.[26] In Sect. 3.1, societal responses are averages of individual responses, whereas in Sect. 3.2, societal responses are averages of transformed individual responses. Throughout this section, the space of players is still assumed to be an atomless probability space $(I, \mathcal{I}, \lambda)$.

3.1 Point-rationalizable and rationalizable averages of actions

Note that any compact metric space can be isometrically embedded in a separable Banach space.[27] Thus, let the common action set A be a nonempty weakly compact set in a separable (infinite-dimensional) Banach space[28] $(X, \|\cdot\|)$. Let $\overline{con}\,(A)$ be the closed convex hull of A. As A is weakly compact, $\overline{con}\,(A)$ is weakly compact.[29] Let $\mathcal{C}_A$ be the space of weakly continuous real-valued functions on $A \times \overline{con}\,(A)$ endowed with the sup-norm topology with its Borel σ-algebra $\mathcal{B}(\mathcal{C}_A)$.

A *large game* $\mathcal{G}^C$ is a measurable function from I to $\mathcal{C}_A$. And a *strategy profile* of such a game is a measurable function $f : I \to A$. A *Nash equilibrium* of a game $\mathcal{G}^C$ is a $f^* : I \to A$, such that for λ-almost all $i \in I$,

25 See Yu and Zhang (2007) for details of such a large game setting.

26 Note that in Jara-Moroni (2012), the analysis for rationalizability is given for societal responses formulated as averages but with the action set being a compact subset of a finite-dimensional space.

27 In fact, for a compact metric space X, this separable Banach space is $\mathcal{C}(X)$, the space of continuous real-valued functions on X with the sup-norm topology with its Borel σ-algebra $\mathcal{B}(\mathcal{C}(X))$; see Aliprantis and Border (2006, Lemma 3.23 and Theorem 9.14) for example.

28 Measure spaces of agents and infinite-dimensional Banach spaces are widely used in the economics literature; see Rustichini and Yannelis (1991) for the existence of Nash equilibria of atomless games with infinite-dimensional action spaces and for the existence of competitive equilibria in models with an atomless measure space of agents and an infinite-dimensional commodity space, where the algebraic dimension of $L^{\infty}(E)$ is bigger than the algebraic dimension of the underlying strategy/commodity space for each non-null subset E of agents; also see Yannelis (2009) and Khan (2012) for recent developments.

29 This is implied by the Krein-Smulian Theorem; see Aliprantis and Border (2006, Theorem 6.35).

$$u_i(f^*(i), \int_I f^* d\lambda) \geq u_i(a, \int_I f^* d\lambda) \quad \text{for all } a \in A,$$

where $u_i \equiv \mathcal{G}^C(i)$ and $\int_I f^* d\lambda$ is the Bochner integral of f^* over I.

The following lemma is on the existence of Nash equilibria in $\mathcal{G}^C$.

Lemma 2 *A large game $\mathcal{G}^C : (I, \mathcal{I}, \lambda) \longrightarrow \mathcal{C}_A$ has a Nash equilibrium if one of the following two conditions holds: (1) A is countable; (2) $(I, \mathcal{I}, \lambda)$ is saturated.*

To discuss point-rationalizability and rationalizability in such a setting, note that a societal response in a game is now the Bochner integral of a strategy profile over I. Let $\mathcal{A} \equiv \overline{con}\,(A)$, the set of all societal responses. According to rationality, for any $\iota \in \mathcal{A}$ with which player $i \in I$ perceives as a societal response under some point-beliefs, player i will only choose action from $B(i, \iota)$ where

$$B(i, \iota) = \arg\max_{a \in A} u_i(a, \iota).$$

And, for any $\mu \in \mathcal{M}(\mathcal{A})$ with which player i perceives as a probability distribution over societal responses, player i will only choose action from $\hat{B}(i, \mu)$ where

$$\hat{B}(i, \mu) = \arg\max_{a \in A} \int_{\mathcal{A}} u_i(a, \iota) d\mu(\iota).$$

Thus, the elimination processes under point-beliefs and probabilistic beliefs are mappings $Pr : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ and $\hat{Pr} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$, respectively, such that for any subset $X \subseteq \mathcal{A}$,

$$Pr(X) = \left\{ \int_I f di : f \text{ is a measurable selection of } B(\cdot, X) \right\},$$

and

$$\hat{Pr}(X) = \left\{ \int_I \hat{f} di : \hat{f} \text{ is a measurable selection of } \hat{B}(\cdot, \mathcal{M}(X)) \right\},$$

where $B(i, X)$ is the image of X under $B(i, \cdot)$ and $\hat{B}(i, \mathcal{M}(X))$ is the image of $\mathcal{M}(X)$ under $\hat{B}(i, \cdot)$ for any $i \in I$.

Now consider iterative elimination of societal responses from $\mathcal{A}$. Eliminating from all societal responses $\mathcal{A}$, by common knowledge of rationality, in each step, players would only use forecasts based upon those averages that are generated from best responses to players' forecasts in turn based upon the set of averages in the previous step, under both point-beliefs and probabilistic beliefs. Formally,

$$Pr^0(\mathcal{A}) = \mathcal{A}, \quad \text{and} \quad Pr^{t+1}(\mathcal{A}) = Pr(Pr^t(\mathcal{A})), \quad \text{for all} \quad t \geq 1.$$
$$\hat{Pr}^0(\mathcal{A}) = \mathcal{A}, \quad \text{and} \quad \hat{Pr}^{t+1}(\mathcal{A}) = \hat{Pr}(\hat{Pr}^t(\mathcal{A})), \quad \text{for all} \quad t \geq 1.$$

As a result, in $\mathcal{G}^C$, the sets of all societal responses that survive IENBR, under point-beliefs and probabilistic beliefs, respectively, are

$$\mathbb{P}^e_{\mathcal{G}^C} = \bigcap_{t=0}^{\infty} Pr^t(\mathcal{A}) \quad \text{and} \quad \mathbb{R}^e_{\mathcal{G}^C} = \bigcap_{t=0}^{\infty} \hat{Pr}^t(\mathcal{A}).$$

Similarly, one can consider the fixed point property of these elimination processes to define point-rationalizability and rationalizability. A set of averages $X \subseteq \mathcal{A}$ is a *CURB* set if $X \subseteq \hat{Pr}(X)$ (or a CURB set under point-beliefs if $X \subseteq Pr(X)$.) A CURB set of averages X is *tight* if $X = \hat{Pr}(X)$. With these preparations, point-rationalizability and rationalizability in $\mathcal{G}^C$ are defined naturally as below.

Definition 5 In a large game $\mathcal{G}^C$, the set of *point-rationalizable* averages, $\mathbb{P}_{\mathcal{G}^C}$, is the maximal tight CURB set of averages under point-beliefs of $\mathcal{G}^C$, and the set of *rationalizable* averages, $\mathbb{R}_{\mathcal{G}^C}$, is the maximal tight CURB set of averages of $\mathcal{G}^C$.

The next theorem establishes the equivalence between rationalizability and the outcome of IENBR in a large game with averages of individual actions as societal responses where the action set is allowed to be infinite dimensional.

Theorem 4 *In a large game $\mathcal{G}^C : (I, \mathcal{I}, \lambda) \longrightarrow \mathcal{C}_A$, the set of point-rationalizable averages $\mathbb{P}_{\mathcal{G}^C} = \mathbb{P}^e_{\mathcal{G}^C}$, the set of rationalizable averages $\mathbb{R}_{\mathcal{G}^C} = \mathbb{R}^e_{\mathcal{G}^C}$, and these sets are nonempty, convex and weakly compact, if one of the following two conditions holds: (1) A is countable; (2) $(I, \mathcal{I}, \lambda)$ is saturated.*

The proof of Theorem 4 is similar to proofs of Theorems 1 to 3, and thus, it is relegated to Appendix B.

Now, note that norm continuous functions on norm compact sets are weakly continuous too. Together with the fact that if A is norm compact, $\overline{con}\,(A)$ is also norm compact, one can consider the space of payoffs as $\mathcal{C}^N_A$, the space of norm continuous real-valued functions on $A \times \overline{con}\,(A)$ endowed with the sup-norm topology with its Borel σ-algebra by this topology. Then, the following result is a direct corollary of Lemma 2 and Theorem 4.

Corollary 1 *Let a large game be a measurable function $\mathcal{G}^C : (I, \mathcal{I}, \lambda) \longrightarrow \mathcal{C}^N_A$. In $\mathcal{G}^C$, a Nash equilibrium exits, the set of point-rationalizable averages $\mathbb{P}_{\mathcal{G}^C} = \mathbb{P}^e_{\mathcal{G}^C}$, and the set of rationalizable averages $\mathbb{R}_{\mathcal{G}^C} = \mathbb{R}^e_{\mathcal{G}^C}$, if one of the following two conditions hold: (1) A is countable; (2) $(I, \mathcal{I}, \lambda)$ is saturated. Moreover, under such conditions, $\mathbb{P}_{\mathcal{G}^C}$ and $\mathbb{R}_{\mathcal{G}^C}$ are nonempty, convex and norm compact.*

One could also address point-rationalizability and rationalizability in large games under some alternative definitions of measurability and integrability of averages of individual responses. Consider the dual X^* of a separable Banach space X. Let the common action set A be a weak* compact subset of X^* and B the weak* closed convex hull of A. Note that B is still weak* compact. Let $\mathcal{C}_A^{B*}$ be the space of weak* continuous real-valued functions on $A \times B$, endowed with the sup-norm topology and its Borel σ-algebra $\mathcal{B}(\mathcal{C}_A^{B*})$ generated by this topology. Thus, a large game is simply a measurable function from $(I, \mathcal{I}, \lambda)$ to $\mathcal{C}_A^{B*}$. Nash equilibria, the process of IENBR and rationalizable averages can be defined as earlier in $\mathcal{G}^C$ but with all integrals interpreted as Gel′fand integrals. Under the condition that (1) A is countable, or (2) $(I,\mathcal{I},\lambda)$ is a saturated probability space, one can show that a Nash equilibrium always exists, the set of point-rationalizable averages is equivalent to the set of averages that survive IENBR under point-beliefs and the set of rationalizable averages is the same as the set of averages that survive IENBR. Furthermore, sets of point-rationalizable averages and rationalizable averages are non-empty, compact and convex. All one needs is Proposition 6. Note that Corollary 1 also covers the case of the norm topology in X^*, provided that the Bochner integral is used.

3.2 Point-rationalizable and rationalizable transformed summary statistics

There are some games in the literature in which societal responses are averages of transformed individual responses.[30] Let the common action set A be a nonempty compact metric space. As before, a *strategy profile* is a measurable function $f : I \to A$, which specifies a strategy for each player. Let s be a continuous function from A to the n-dimensional Euclidean space $\mathbb{R}^n$, and C the range of s. The continuity of s and compactness of A imply that C is also compact. Let Σ be the convex hull of C. Thus, for any given strategy profile f, let $\sigma_f = \int_I (s \circ f) d\lambda \in \Sigma$. The mean σ_f of $s \circ f$ is a transformed summary statistics of the society which the players can observe. Let $\mathcal{P}$ denote the space of all continuous payoff functions on $A \times \Sigma$ with the sup-norm. A *large game*[31] is thus a measurable function $\Gamma : I \to \mathcal{P}$, and its *equilibrium* is a strategy profile $f^* : I \to A$, such that each player plays a best response against the induced vector of summary statistics, i.e.,

$$\Gamma(i)(f^*(i), \sigma_{f^*}) \geq \Gamma(i)(a, \sigma_{f^*})$$

for λ-almost all $i \in I$ and $a \in A$.

Point-rationalizability and rationalizability in such a game Γ can be formulated as before with some subtle technical points. Focusing on point-rationalizability, note that as now societal responses in Γ are transformed summary statistics, the elimination

[30] See, Rauh (1997), for example, where the context of monopolistic competitions is considered. It is assumed there that the players' payoffs depend on their own action and the transformed summary statistics of the aggregate strategy profiles in terms of the moments of the distributions of players' actions. The existence of Nash equilibria has been shown in Rauh (2003) and Yu and Zhu (2005) for such games where societal responses are formulated as transformed summary statistics.

[31] The model used to characterize rationalizability in Jara-Moroni (2012) is a special case of the model considered here. To cover the case considered in Jara-Moroni (2012), one can take A as a subset of $\mathbb{R}^n$ and s as the identity map.

process under rationality and common knowledge of rationality under point-beliefs is a mapping $Pr : 2^{\Sigma} \to 2^{\Sigma}$ so that for any subset $X \in \Sigma$,

$$Pr(X) = \left\{ \int_I s \circ f(i) d\lambda,\ f \text{ is a measurable selection of } B(\cdot, X) \right\}.$$

Because $s : A \to C$ is continuous, the correspondence $s^{-1} : C \twoheadrightarrow A$, such that $s^{-1}(c) = \{a \in A : s(a) = c\}$, is a weakly measurable correspondence with non-empty closed values from the measurable space C with Borel σ-algebra to the compact metric space A. By the standard integration theory of correspondences in the finite-dimensional setting and the Kuratowski–Ryll–Nardzewski selection theorem, which is reported as Proposition 3 (P2) in Appendix A, one could characterize point-rationalizability in Γ as in a large game $\mathcal{G}^C$ that is studied earlier. Similarly, one can also formalize rationalizability in Γ.

All these characterizations can be carried out to a large game with summary statistics in infinite-dimensional Banach space instead of n-dimensional Euclidean space $\mathbb{R}^n$, provided that (1) the player space is modeled by a saturated probability space, or (2) C is countable. Moreover, the existence of Nash equilibria in the corresponding framework can be obtained as well. This is guaranteed by the properties of the integration of correspondences in infinite-dimensional spaces together with the Kuratowski–Ryll–Nardzewski selection theorem. In fact, if the player space in Γ is modeled by a saturated probability space, one could also rely on the model that is discussed in Sect. 2 directly: in any game Γ, all players' payoffs depend continuously on summary statistics, and therefore, they also depend continuously on the distribution of actions in the weak topology.[32] To see this, let $\mathcal{G}(i)(a, \tau) = \Gamma(i)(a, \int_A s d\tau)$ for all $a \in A$, $\tau \in \mathcal{M}(A)$ and $i \in I$. One can check that $\mathcal{G}$ is a large game as defined in Sect. 2, so that the characterization of rationalizability in $\mathcal{G}$ is well defined. Therefore, as in Γ, players's forecasts are based on summary statistics, and one can then take the equivalent class of rationalizable action distributions in $\mathcal{G}$ with respect to $\int_A s d\tau$ to get rationalizable summary statistics in Γ.

4 Conclusions

In this paper, point-rationalizability and rationalizability have been examined in several different settings emphasized in the theory of large games. In all cases, the set of rationalizable societal responses which is the maximum tight CURB set has been defined, respectively. In addition, it is shown to be equivalent to the set of all societal responses which survive IENBR of the same game.

The concept of rationalizability can be justified by the epistemic assumption that it is common knowledge that only best responses are ever chosen. Thus, a natural question to ask is under what general conditions one can relate equilibrium to rationalizability, so that one can provide a plausible justification for reaching equilibria. Moving beyond

[32] The author is thankful to the referee for this observation.

the questions of analysis, this is to ask how to apply the notion of rationalizability, as in Guesnerie and Jara-Moroni (2011), to the equilibrium outcomes in a broad spectrum of situations: monopolistic competition, as in Rauh (1997), financial markets, as in Angeletos et al. (2007), restaurant pricing and the economics of "social influences," as in Karni and Levin (1994), and many other scenarios that deal with a continuum of players.

The concepts that are developed in this paper are characterizations of "correlated" rationalizability.[33] Thus, another question of interest is whether one could also characterize independent rationalizability. This is a subtle question due to the fact that when one works with a process with a continuum of independent random variables, the sample realizations as well as the process itself are usually not measurable.[34] It is hoped that these questions can be addressed in subsequent work.

Appendix A

The first two propositions are on correspondences, whose details can be found in Aliprantis and Border (2006, Chapters 17 and 18).

Let S and X be topological spaces and F a correspondence from S to X. $F : S \twoheadrightarrow X$ *has closed values* or *is closed-valued* if $F(s)$ is a closed set for each $s \in S$. The terms *compact-valued, convex-valued and nonempty-valued* are similarly defined. Let $F^l(E) \equiv \{s \in S : F(s) \bigcap E \neq \emptyset\}$ for any subset E of X. If $F^l(C)$ is closed for each closed subset C of Y, $F : S \twoheadrightarrow X$ is *upper hemicontinuous*. F has closed graph is its graph $\{(s, x) \in S \times Y : x \in F(s)\}$ is a closed subset of $S \times X$.

Proposition 2 *$P1$: A correspondence with compact Hausdorff range space has closed graph if and only if it is upper hemicontinuous and closed-valued.*

$P2$: The image of a compact set under a compact-valued upper hemicontinuous correspondence is compact.

Let (S, Σ) be a measurable space and X a topological space. A correspondence $F : S \twoheadrightarrow X$ is said to be measurable, if for each closed subset C of X, $F^l(C) \in \Sigma$, then F is said to be measurable; F is said to be weakly measurable, if for each open subset O of X, $F^l(O) \in \Sigma$. A function f is said to be a selection of F if $f(s) \in F(s)$ for all $s \in S$. With this notation, the next proposition is a collection of measurability results of correspondence and its selections.

Proposition 3 *Let (S, Σ) be a measurable space and X a Polish space. Let F, F_1, F_2 be correspondences from S to X.*

- *$P1$: If F has nonempty compact values, then F is measurable if and only if it is weakly measurable.*
- *$P2$ (Kuratowski–Ryll–Nardzewski Selection Theorem): If F is weakly measurable and has nonempty closed values, then it admits a measurable selection.*
- *$P3$: F has closed graph if and only if it is upper hemicontinuous and closed-valued.*

[33] See Brandenburger and Dekel (1987) for the discussion on correlated and independent rationalizability.

[34] See Sun (2006) and its references.

P4: If F_1 and F_2 are closed-valued and measurable, then their intersection correspondence G, where G is such that for all $s \in S$, $G(s) = F_1(s) \bigcap F_2(s)$, is measurable and closed-valued.

Let X be a Polish space and $(\Omega, \mathcal{A}, P)$ an atomless probability space. A measurable function $f : \Omega \to X$ is a measurable selection of a correspondence $F : \Omega \twoheadrightarrow X$ if $f(\omega) \in F(\omega)$ for P-almost all $\omega \in \Omega$. The next proposition is culled from the corresponding results on the distribution of correspondences in Khan and Sun (1995) and Keisler and Sun (2009).

Proposition 4 *Let X be a compact metric space and $(\Omega, \mathcal{A}, P)$ an atomless probability space. Then the following results are valid if, in addition, (1) X is a countable, or (2) $(\Omega, \mathcal{A}, P)$ is a saturated probability space.*

P1: Let $\{f_n\}$ be a sequence of measurable functions from Ω to X, such that $\tau_n = P f_n{}^{-1}$ converges weakly to $\tau \in \mathcal{M}(X)$ as $n \to \infty$. Let $D(\omega) = cl\text{-}Lim\{f_n(\omega)\}$. Then, $D(\omega)$ is nonempty for almost all ω, and there exists a measurable selection f of D, such that $Pf^{-1} = \tau$.
P2: For any correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F = \{Pf^{-1} : f$ is a measurable selection of $F\}$ is convex.
P3: For any compact-valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is compact.

The next two propositions are modified from Khan and Sun (1996), Sun (1997), Podczeck (2008) and Sun and Yannelis (2008). Proposition 5 is based on Bochner integral, whereas Proposition 6 is based on the Gel′fand integral. See Yannelis (1991) and his references for an earlier treatment on the integration of correspondences over atomless probability spaces.

Let $(\Omega, \mathcal{A}, P)$ be a finite measure space and X a Banach space. If $f : \Omega \to X$ is a Bochner integrable function, $\int_E f dP$ is the *Bochner integral* of f over E then for $E \in \mathcal{F}$. If $F : \Omega \twoheadrightarrow X$ is a correspondence, the *Bochner integral of F* is, denoted by $\int_\Omega F dP$,

$$\int_T F dP = \left\{ \int_I f dP : f \text{ is a Bochner integrable selection of } F \right\}.$$

The next proposition deals with the Bochner integration of correspondences.

Proposition 5 *Let $(\Omega, \mathcal{A}, P)$ be an atomless probability space and X a separable Banach space. The following results are valid if, either (1) X is a countable or (2) $(\Omega, \mathcal{A}, P)$ is a saturated probability space,*

P1: Let $\{f_n\}$ be a sequence of measurable functions from Ω to X, such that $\int f_n dP$ converges to ι as $n \to \infty$. Let $D(\omega) = w\text{-}cl\text{-}Lim\{f_n(\omega)\}$. Then, $D(\omega)$ is nonempty for almost all ω, and there exists a measurable selection f of D such that $\int f dP = \iota$.
P2: For any correspondence F from $(\Omega, \mathcal{A}, P)$ into X, $\int_\Omega F dP$ is convex.
P3: For any integrably bounded, weakly compact-valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\int_\Omega F dP$ is weakly compact.

P4: Let Y be a metric space and F a correspondence from $\Omega \times Y$ to X, such that for each fixed $y \in Y$, $F(\cdot, y)$ is a measurable, weakly compact-valued correspondence from Ω to X. If $F(\omega, y)$ is upper hemicontinuous on Y for each fixed i, then $\int_\Omega F(\omega, y)dP$ is weakly upper hemicontinuous on Y.

Let $(\Omega, \mathcal{A}, P)$ be an atomless probability space and X the dual of a separable Banach space. If $f : \Omega \rightarrow X$ is a Gel′fand integrable function, $\int_E f dP$ is the *Gel′fand integral* of f over E then for $E \in \mathcal{F}$. If $F : \Omega \twoheadrightarrow X$ is a correspondence, the *Gel′ fand integral of F* is, denoted by $\int_\Omega F dP$,

$$\int_T F dP = \left\{ \int_I f dP : f \text{ is a Gel}'\text{fand integrable selection of } F \right\}.$$

The last proposition in this appendix is on the Gel′fand integration of correspondences.

Proposition 6 *Let $(\Omega, \mathcal{A}, P)$ be an atomless probability space and X the dual of a separable Banach space. The following results are valid if, either (1) X is a countable or (2) $(\Omega, \mathcal{A}, P)$ is a saturated probability space.*

P1: Let $\{f_n\}$ be a sequence of measurable functions from Ω to X, such that $\int f_n dP$ converges to ι as $n \rightarrow \infty$. Let $D(\omega) = w^*\text{-}cl\text{-}Lim\{f_n(\omega)\}$. Then, $D(\omega)$ is nonempty for almost all ω, and there exists a measurable selection f of D, such that $\int f dP = \iota$.

P2: For any correspondence F from $(\Omega, \mathcal{A}, P)$ into X, $\int_\Omega F dP$ is convex.

P3: For any integrably bounded, weak compact-valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\int_\Omega F dP$ is weak* compact.*

P4: Let Y be a metric space and F a correspondence from $\Omega \times Y$ to X, such that for any fixed $y \in Y$, $F(\cdot, y)$ is a measurable, weak compact-valued correspondence from Ω to X. If $F(\omega, y)$ is upper hemicontinuous on Y for each fixed i, then $\int_\Omega F(\omega, y)dP$ is weakly upper hemicontinuous on Y.*[35]

Appendix B

One can now prove the following lemma which is essential in the proof of Lemma 1.

Lemma 3 *Let X, Y be two nonempty compact metric spaces. Let F be a measurable correspondence with nonempty closed values from an atomless probability space $(I, \mathcal{I}, \lambda)$ to Y, and for each $i \in I$ consider a upper hemicontinuous and closed-valued correspondence $M(i, \cdot) : Y \twoheadrightarrow X$. Let $G : I \twoheadrightarrow Y \times X$ be such that $G(i)$ is the graph of $M(i, \cdot)$ for all $i \in I$. If G is measurable, then the correspondence $M(\cdot, F(\cdot)) : I \twoheadrightarrow X$ is measurable and closed-valued.*

[35] Part (1) of Propositions 5 and 6 is shown in Khan and Sun (1996). Part (2) of Propositions 5 and 6 is shown for the case of atomless Loeb measures in Sun (1997). Based on some advanced functional analytic methods, Podczeck (2008) proves P2 and P3 in Part (2) of Propositions 5 and 6. As shown in Sun and Yannelis (2008)), Part (2) of Propositions 5 and 6 follows easily from the corresponding properties for integration in the case of Loeb spaces in Sun (1997) and for distribution in the case of saturated spaces in Keisler and Sun (2009).

Proof Let $\phi : I \twoheadrightarrow Y \times X$ be a correspondence, such that $\phi(i) = F(i) \times X$ for all $i \in I$. For any open set O of $Y \times X$, consider $\phi^l(O) = \{i \in I : \phi(i) \bigcap O \neq \emptyset\}$. Let P_X and P_Y be the projection mappings from $Y \times X$ to X and Y, respectively. If $P_X(O) \neq X$, $\phi^l(O) = \emptyset \in \mathcal{I}$. If $P_X(O) = X$, $\phi^l(O) = \{i \in I : F(i) \bigcap P_Y(O) \neq \emptyset\} = F^l(P_Y(O)) \in \mathcal{I}$ too, because F is measurable and P_Y is continuous. Thus, ϕ is weakly measurable. By Proposition 3 (P1) in Appendix A, ϕ is also measurable because by construction ϕ has nonempty compact values. Let ϕ^G be such that $\phi^G(i) = \phi(i) \bigcap G(i)$ for all $i \in I$. For any fixed $i \in I$, since $M(i, \cdot)$ is upper hemicontinuous and closed-valued, $G(i)$ is closed by Proposition 3 (P3). Therefore, ϕ^G is closed-valued and measurable. Furthermore, note that by construction of ϕ^G, $M(i, S(i)) = P_X(\phi^G(i))$ for all $i \in I$. Therefore, the continuity of P_X implies that $M(\cdot, S(\cdot)) : I \twoheadrightarrow X$ is measurable and closed-valued. □

One can prove Lemma 1 now.

Proof of Lemma 1 For any fixed nonempty closed set $\mathcal{D}$ of $\mathcal{M}(A)$, let $\mathcal{D}(i) = \mathcal{D}$ for all $i \in I$. It is clear that $\mathcal{D}(\cdot)$ is a measurable and closed-valued correspondence. By Berge's maximum theorem, for each $i \in I$, the joint continuity of u_i on $A \times \mathcal{M}(A)$ implies that $B(i, \cdot)$ is upper hemicontinuous and has nonempty compact values on $\mathcal{M}(A)$. Thus, by Lemma 3, in order to show that $B(\cdot, \mathcal{D})$ is measurable and closed-valued, it suffices to show $G : I \twoheadrightarrow \mathcal{M}(A) \times A$ is measurable where $G(i)$ is the graph of $B(i, \cdot)$ for all $i \in I$.

Toward this end, for any given closed subset C of $A \times \mathcal{M}(A)$, let $U_C = \{f \in \mathcal{U}_A$: there exists$(a, \tau) \in C$, such that, $f(a, \tau) \geq f(a', \tau)$, for all $a' \in A\}$. It is clear that $\mathcal{G}^{-1}(U_C) = G^l(C)$ where $G^l(C) = \{i \in I : C \cap G(i) \neq \emptyset\}$. Thus, given $\mathcal{G}$ is measurable, it is sufficient to prove that U_C is closed. Let $\{f^n\}$ be a sequence in U_C that converges uniformly to f^*. As the uniform limit of a net of continuous real functions is still continuous (see, e.g., Aliprantis and Border (2006, Theorem 2.65)), $f^* \in \mathcal{U}_A$. So, for any $\varepsilon > 0$, the continuity of f^* implies that for any convergent sequence $\{(a^n, \tau^n)\} \to (a^*, \tau^*)$, there exists some $N_1 \in \mathbb{N}$ such that for any $n > N_1$, $|f^*(a^n, \tau^n) - f^*(a^*, \tau^*)| < \varepsilon/2$. The uniform convergence of g^n implies that there exists some $N_2 \in \mathbb{N}$, such that for any $n > N_2$, and for all $(a, \tau) \in A \times \mathcal{U}_A$, $|f^n(a, \tau) - f^*(a, \tau)| < \varepsilon/2$. Therefore, for any $n > N$, where $N = \max\{N_1, N_2\}$,

$$|f^n(a^n, \tau^n) - f^*(a^*, \tau^*)| \leq |f^n(a^n, \tau^n) - f^*(a^n, \tau^n)| \\ +|f^*(a^n, \tau^n) - f^*(a^*, \tau^*)| < \varepsilon.$$

Thus, $f^n(a^n, \tau^n) \to f^*(a^*, \tau^*)$. I now show that there exists some element (a, τ) in C such that $f^*(a, \tau) \geq f^*(a', \tau)$, for all $a' \in A$. For each $n \in \mathbb{N}$, since $f^n \in U_C$, there exists $(a^n, \tau^n) \in C$, such that

$$f^n(a^n, \tau^n) \geq f^n(a', \tau^n), \quad \text{for all } a' \in A.$$

Furthermore, the closeness of C implies that there is a subsequence of (a^n, τ^n) that converges to some $(\hat{a}, \hat{\tau}) \in C$. Without loss of generality, let the subsequence be the sequence itself. It is now clear that $(\hat{a}, \hat{\tau})$ satisfies $f^*(\hat{a}, \hat{\tau}) \geq f^*(a', \hat{\tau})$ for any $a' \in A$. Hence, $f^* \in U_C$. The proof is complete. □

The next result is used in the proof of Theorem 3 to characterize expected payoffs under probabilistic beliefs. One can check that it holds, along the lines of the proof of Jara-Moroni (2012, Lemma 3.7).

Proposition 7 *Let Y and X be compact metric spaces and u a continuous real-valued function on $Y \times X$. Let $U : Y \times \mathcal{M}(X)$ be a function such that for any $(y, \mu) \in Y \times \mathcal{M}(X)$,*

$$U(y, \mu) = \int_X u(y, x) d\mu.$$

U is continuous on $Y \times \mathcal{M}(X)$.

Proofs of Lemma 2 and Theorem 4 are as follows.

Proof of Lemma 2 Let $\phi : \overline{con}(A) \twoheadrightarrow \overline{con}(A)$ be the Bochner integral correspondence such that for $\iota \in \overline{con}(A)$, $\phi(\iota) = \int_I B(i, \iota) d\lambda$. By Berge's maximum theorem and Proposition 2 (P2) in Appendix A, ϕ has nonempty values. Then, together with the convexity and upper hemicontinuity results in Proposition 5 in Appendix A, one can appeal to the Fan-Glicksberg fixed point theorem to guarantee the existence of a Nash equilibrium. □

Proof of Theorem 4 First, note that one can show that for any closed subset $X \subseteq \mathcal{A}$, $B(\cdot, X)$ is measurable and has nonempty closed values by similar arguments in the proof of Lemma 1. Let $\{F^t\}$ be a sequence of correspondences, such that $F^t : I \twoheadrightarrow A, t \geq 0$ is given by

$$F^0(i) = A, \text{ for all } i \in I \quad \text{and} \quad F^t(i) = B\left(i, Pr^{t-1}(\mathcal{A})\right), \text{ for all } i \in I, \text{ if } t \geq 1,$$

where $Pr^t(\mathcal{A}) = \int_I F^t d\lambda$. Let $F : I \twoheadrightarrow A$ be a correspondence, such that for all $i \in I$, $F(i) = cl\text{-Lim}\{F^t(i)\}$. Then, one can now appeal to Proposition 5 (P1–P3) in Appendix A to complete the proof by arguments similar to proofs of Theorems 1, 2 and 3. □

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