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RESEARCH ARTICLE

Revealed preferences and aspirations in warm glow theory

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Abstract In warm glow models, an agent may prefer one alternative but aspire to choose another. The agent chooses her aspiration if she gets a sufficiently large warm glow payoff for acting as she aspires. This basic framework is widely used in models of turnout in elections and contributions to public goods, but is often criticized for being ad hoc. In this paper, we provide choice-theoretic foundations for warm glow theory. We characterize the empirical content of warm glow theory, show how to infer the core elements of the model from data and show that it is possible to predict behavior even when preferences and aspirations are not revealed. Our results provide support for assumptions often made in the literature and suggest new applications for warm glow models.

Keywords Warm glow · Revealed preferences · Aspirations · Choice

JEL Classification D01 · D63 · D71 · D74

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1 Introduction

In *warm glow* models an agent may *prefer* one alternative but *aspire* to choose another. An agent's aspiration is often understood as the alternative she thinks she ought to choose on ethical grounds. The agent receives a warm glow payoff for acting in accordance with her aspirations. If her warm glow payoff is sufficiently large, the agent may act as she aspires even if the action taken is costly and near inconsequential.

Warm glow models are used to accommodate a wide range of behavior including voting in large elections where the impact of a single vote is negligible. Agents motivated by warm glow payoffs vote because they think they should and not because they think a single vote may plausibly change the outcome. Warm glow models have also been used extensively in public good provision models. The model captures the idea that people may be motivated to act in socially beneficial ways (such as helping others, making philanthropic contributions, punishing socially undesirable behavior) at a private cost to themselves.

While warm glow models are used widely, they remain unaxiomatized. The contribution of this paper is to provide choice-theoretic foundations for warm glow theory, to show how to identify core elements of the model from data and to suggest new applications for the theory.

The standard formulation of warm glow theory combines instrumental and warm glow payoffs. For example, the overall payoff for choosing x is

$$U(x) = \begin{cases} u(x) + D & \text{if } x \text{ is an aspiration} \\ u(x) & \text{otherwise} \end{cases} \tag{1}$$

where $u(x)$ is the instrumental payoff associated with the choice of x and $D > 0$ is the warm glow payoff received when the agent aspires to choose x.

A well-known example of this basic structure is found in Riker and Ordeshook (1968). Their model reduces to a payoff for voting given by the formula

$$p\Delta u - c + D$$

where p is the probability an agent's vote is pivotal, Δu is the difference in payoff between the favored candidate and his opponent being elected, c is the cost of voting and D is the warm glow payoff received by voting for the favored candidate.[1] An agent votes if and only if the payoff for voting is positive. Riker and Ordeshook find some empirical support for their model, but left open the question of how to identify the core elements of the theory from data. In particular, it remains unclear whether votes reflects preferences or aspirations.

[1] Andreoni 1989 writes a more complex warm glow utility function as follows:

$$U = U(y, Y, g)$$

where y is the agent's consumption of private goods, Y is the total supply of the public good, and g is the warm glow the agent experiences by virtue of giving.

The need for choice-theoretic foundations for warm glow theory may be easily seen in the context of contribution to public goods problems. Consider a decision maker who aspires to donate as much as solicited. She is asked to make a small donation (s) and she does so. That is, s is chosen over n (no donation). However, when she is asked to make either a small donation or a large donation (l), then she chooses not to donate at all. So, n is chosen over s and l.

In effect, the introduction of the option to donate at a higher level reduces donations. Similar behavior is observed in the field study of Berger and Smith (1997). More important from the perspective of standard economic theory such choices violate WARP (the weak axiom of revealed preference). Warm glow theory easily accommodates this behavior. It suffices to assume that warm glow payoffs are large enough to compensate for the cost of the small donation but not for the cost of a large one.[2]

This example illustrates that the standard choice-theoretic foundations for utility functions do not apply either to overall payoff functions (U) or to instrumental payoffs (u). To see this, note that U is not a cardinal representation of an order over alternatives and u is not equated with choice (because $u(x)$ greater than $u(y)$ is not equivalent to x chosen over y).

One objective of this paper is to provide proper choice-theoretic foundations for warm glow theory. We axiomatize the warm glow model and show how to make suitable inferences about agents' preferences under the warm glow framework. Because the standard approach equating choice with preferences does not apply in warm glow models, there is a need to show how to deduce preferences from choice in these models. It should be noted that we consider the simple warm glow model in (1) and not more complex ones (e.g., Feddersen and Sandroni 2006 and Andreoni 1989). While the model in (1) is a simplification, it allows us to get directly at the crucial feature of warm glow models: the idea that aspiration is chosen only when the required sacrifice in utility is not too large. We want to demonstrate that this central feature of warm glow models is observationally meaningful.

It will be helpful for our purposes to define an ordinal warm glow model in the tradition of the revealed preference literature. In this model, an agent makes choices from subsets of alternatives called *issues*. The agent's choice is determined by a preference relation (associated with the utility u), an *aspiration function* and a *tolerance function*. The aspiration function determines which actions deliver the warm glow payoff (i.e., the aspiration). The tolerance function determines which actions, within each issue, are sufficiently costly so that they will not be chosen even if they are an aspiration. So, the agent chooses her aspiration if and only if she can tolerate it. This ordinal model of warm glow is observationally equivalent to warm glow model in (1). In cardinal warm glow models, Dee is willing to sacrifice utility in order to act as she aspires, provided that this sacrifice is not too large (i.e., smaller than D utiles). The ordinal model of warm glow is a first step in the determination of choice-theoretic foundations to this idea of preference intensity and sacrifice of a limited quantity of utility in order to satisfy an aspiration.

[2] It is critical here that the action that Dee aspires may be issue dependent. Thus, Dee may get a warm glow payoff D for different actions in different issues. In this example, Dee gets a warm glow payoff for a small donation only when this is the highest solicited donation.

If no structure is imposed on a nonobservable aspiration function, then any choice function can be accommodated by warm glow models. It suffices to assume that all actions are tolerable and the decision maker' choice is her aspiration. Clearly, such a model has no empirical content. To make progress, we must assume either that aspirations are observed or that they have some logical structure (or both).

Let us start with the assumption that aspirations are observed. Here, the main difficulty is that if the decision maker chooses her aspiration, then there is no direct way to make inferences about preferences (because it is unclear whether her choices are motivated by her preferences, by her aspirations or by both). However, if an agent aspires to y but chooses x, then we can directly infer that y is not tolerated when x is available. This reveals an intolerance relation. That is, Dee's utility for x is greater than her utility for y plus D and so x is so much better than y that y will never be chosen when x is available (even if y is an aspiration). In addition, x must be preferred to all other alternative she might have chosen instead (i.e., if the chosen x is not her aspiration, then it must be her preferred choice). This reveals a preference. These are simple inferences over intolerances and preferences that can be made directly. A key point in this paper is that there are more complex and indirect inferences that can be made by chaining together direct inferences. These indirect inferences can, in turn, be chained together to produce even further inferences. In spite of this difficulty, we provide an explicit formula that takes, as input, data in the form of arbitrary choices and aspirations and returns, as output, (1) a way to determine whether the warm glow model can accommodate the data, and (2) all possible inferences over preferences and tolerances. In addition, if aspirations are ordered, then the empirical content of warm glow theory can be fully characterized by two simple and elegant axioms.

So far, our results characterize the inferences of warm glow theory when both choices and aspirations are used as input. This is the traditional approach in applied warm glow models where aspirations are typically assumed to be commonly perceived ethical actions such as voting, contributing to public goods and performing activities often deemed to be praiseworthy. Our results can also be used to make predictions on observable behavior that follow under the null hypothesis that the warm glow model holds and aspirations are exogenously determined. Naturally, these predictions may not come about. This shows a combined test of the warm glow model and assumptions about aspirations.

In several situations, there may be doubts about whether it is possible to make legitimate assumptions about aspirations. In this case, aspirations must be assumed to be unobserved. Then, as argued above, it is necessary to assume some logical structure on aspirations: We assume that they are ordered and show that, in this case, the model is falsifiable and must satisfy (at least) the following property: If there is a set of issues that each contain an alternative x that is never chosen, then x cannot be chosen in the union of those issues. For example, if x is not chosen in the sets $\{x, y\}$ and $\{x, z\}$, then x cannot be chosen in the set $\{x, y, z\}$. We can make this prediction even though it is not possible to determine why x is rejected: It might be, for example, that x is most preferred but tolerates aspirations y and z, or it might be that x is the aspiration but it is not tolerated. Therefore, if aspirations are assumed to be ordered, it is possible to make predictions over behavior without being able to infer motivations. A full characterization of the empirical content of the warm glow model when aspirations are ordered and unobserved is still an open (and, we believe, hard) question.

The paper proceeds as follows. In Sect. 2, we provide a literature review that links our approach to the literature on warm glow decision making and a broader class of behavioral decision models. In Sect. 3, we present the formal model and prove that it is observationally equivalent to the standard warm glow model. In Sect. 4, we state the formal results in the case when aspirations are observable. In Sect. 5, we explore the model when aspirations are not observable. Section 6 considers possible extensions of our model. Section 7 concludes.

2 Literature review

In political science, the leading example of warm glow theory is Riker and Ordeshook (1968). Feddersen and Sandroni (2006) build upon their model. They endogenize aspirations and exploit the predictability of aspirations to generate comparative statics. Coate and Conlin (2004) find support for the Feddersen and Sandroni model in the field. Feddersen et al. (2009) find support for a particular form of the ethical voter model in laboratory experiments (see also Shayo and Harel 2012).

Andreoni (1989) surveys the literature on warm glow giving and develops a warm glow model where agents may aspire to contribute to public good. He shows that the warm glow payoff may help explain why government spending does not crowd out private donations as predicted in standard economic models. Andreoni (2006, pp. 1222–1223) argues that putting a warm glow motive is "an admittedly *ad hoc* fix," but "the experimental data is overwhelming in its support of warm glow." Most notably, Andreoni (1993), Andreoni (1995), Palfrey and Prisbrey (1996), Palfrey and Prisbrey (1997) and Andreoni and Miller (2002) find clear evidence of warm glow motives.

Recent work by Levine and Palfrey (2007) finds that ethical voter models are unnecessary to explain behavior in some laboratory voting experiments. Given the large set of models that are consistent with the same behavior it is important to develop an empirically grounded methodology that will not only allow selection among alternative functional forms of warm glow but also allow an assessment of whether warm glow is a useful theory compared to, say, standard economic models.

The warm glow model can be understood as a dual-self model and, therefore, is related to a growing literature in decision theory on multiple selves. Kalai et al. (2002) consider a basic model of multiple selves, where choice is optimal according to one of the selves. A literature review on multiple-self models can also be found in Cherepanov et al. (2012) and Ambrus and Rozen (2008).

Our approach is closely related to a few lines of research. First, consider models on status quo bias (see, among many contributions, Masatlioglu and Ok 2005; Sagi 2006; Salant and Rubinstein 2008). Under the assumption of observed but possibly unordered aspirations, our warm glow model could be reinterpreted as a model of status quo bias. For a given issue, one need only relabel an aspiration as the status quo. Dee departs from the status quo only if the utility gain is sufficiently large. We consider the case of unobserved aspirations and impose the logical structure of ordered aspirations to obtain empirical content. Under the reinterpretation of aspiration as status quo, the assumption of ordered aspirations is problematic because there is no reason for status quos to be ordered.

In our warm glow model, the most that can be observed for a given issue is a choice and an aspiration. Aspirations are an idiosyncratic feature of a decision maker and not an object that might be subject to experimental manipulation. In contrast, it is entirely sensible in status quo models to assume that one can observe, for a given issue, different choices as a result of different status quos. That is, unlike our model, the status quo literature assumes that the choice $C(B, x)$ may be observed for every issue B and every conceivable status quo $x \in B$. So, the input used in this paper to infer the core elements of warm glow theory is far more limited than the data used in status quo models. A variation of the Limited WARP axiom also holds in status quo models, but we are not aware of any clear counterpart to our warm glow axiom or to our main results.

Our model could also be reinterpreted as a model of temptation and self-control (see, among many contributions, Dillenberger and Sadowski 2012; Gul and Pesendorfer 2001, and Noor and Takeoka 2010). In this reinterpretation, Dee is tempted to take the action she prefers, but she receives a penalty D unless she takes the action she aspires (e.g., Dee may aspire to eat healthy foods, but prefers to eat unhealthy foods. So, she is tempted to eat unhealthy foods, but receives a psychological penalty D unless she eats as she aspires). This simple model of temptation would differ significantly from the existing literature. Even though one can find similarities between the warm glow payoff and the utility function in Gul and Pesendorfer (2001), the connection between these two models is more apparent than real. To see this, it may suffice to note that the Gul and Pesendorfer (2001) does not accommodate violations of WARP. The model of Dillenberger and Sadowski (2012) can accommodate behavioral anomalies at the level of menu choices, and the general temptation model of Noor and Takeoka (2010) does not accommodate violations of WARP once the choice of menu is fixed. Unlike most models of temptation, we do not use choices over menus as input, and, hence, one possible way to interpret our model is as model of choice with a fixed menu. The nonuse of choices over menus as data also leads to an axiomatic foundation that is mostly unrelated to the existing literature. Moreover, a critical contribution in this paper is the methodology showing how to identify Dee's preferences and tolerances from data. This methodology is both novel and significant given that all multiple-self models are prone to identification problems because it is unclear which self produced the choice. Finally, we point out that the model of Segal and Sobel (2007) can be construed as a warm glow model, although they do not describe it this way, which does not accommodate violations of WARP. Shayo and Harel (2012) apply the Segal and Sobel (2007) model to turnout problems. A general equilibrium model with warm glow preferences can be found in Allouch (2012).

3 Warm glow theory

3.1 Basic concepts

A decision maker, Dee, faces a set of choices over subsets of a finite set of alternatives X. A nonempty subset of alternatives $B \subseteq X$ is called an *issue*. Let $\mathcal{B}$ be the set of all issues with at least two alternatives. A*choice function* is a mapping $C : \mathcal{B} \to X$ such that $C(B) \in B$ for every $B \in \mathcal{B}$.

An *aspiration function* is a mapping $A : \mathcal{B} \to X$ such that $A(B) \in B$ for every $B \in \mathcal{B}$. Dee's actual choice may differ from her aspiration, so we call the choice function C Dee's *actual choice* function. Given an issue B and aspiration function A, let $1^{A,B} : B \to \{0, 1\}$ be an indicator function such that $1^{A,B}(x) = 1$ iff $x = A(B)$. That is, $1^{A,B}$ indicates Dee's aspiration in B.

We consider *utility functions* $u : X \to \Re$ such that $u(x) \neq u(y)$ if $x \neq y$. So, indifference is ruled out. Given an issue B and aspiration function A, let

$$U^{A,B}(x) = u(x) + D \cdot 1^{A,B}(x)$$

be Dee's utility function plus a warm glow payoff D for acting as she aspires.

Definition 1 A choice and aspiration function $(C, A) : \mathcal{B} \to X \times X$ is a warm glow (choice and aspiration) function if there exists a utility function u and a scalar $D \geq 0$ such that for every issue $B \in B$,

$$U^{A,B}(C(B)) > U^{A,B}(x) \quad \text{for every } x \in B, x \neq C(B). \tag{2}$$

That is, warm glow choice functions are produced by optimization of utility plus a warm glow payoff for acting as aspired. In the context of choice with an ethical component, the main assumption is that Dee aspires to act ethically and receives a payoff D if she does so. That is, as long as Dee's aspirations do not require her to sacrifice utility greater than D, she acts as she aspires. These sacrifices are not random or arbitrary. They are motivated by Dee's desire to act in harmony with her aspirations.

While (2) is the most basic model of warm glow, it is easy to think of generalizations. For example, perhaps in some issues, Dee has no aspirations or more than one aspiration. It would be useful to produce choice-theoretic foundations in a variety of models of warm glow, but the natural starting point is the benchmark model in (2) because this is the simplest and the most common model of warm glow used in the literature. Some generalizations (e.g., nonconstant warm glow payoffs and issues without aspirations) are considered, below, in this paper.

3.2 Warm glow theory and preferences

In this section, we introduce an ordinal model of choice as a first step in the determination of choice-theoretic foundations of warm glow theory. Below we show that this model is observationally equivalent to the cardinal warm glow model. Hence, the model in this subsection can be seen not as a new model, but rather as a convenient (for our purposes) reformulation of the warm glow model in (2).

As above, Dee is endowed with an aspiration function A. Dee is also endowed with a *preference order* R which is an asymmetric, transitive and complete binary relation. By standard convention, $x\,R\,y$ denotes that x is R-preferred to y. We say that x *R-optimizes* B and denote this by $x = R(B)$, if $x\,R\,b$ for every $b \in B$, $b \neq x$. If

$C(B) = R(B)$, then issue B is *resolved* by preference order R. The preference order R can be seen as the ordinal counterpart of the utility function u.

For any given issue, Dee acts as she aspires only if her aspirations are not too costly. So, Dee compares her aspiration to her preference and chooses her aspiration if and only if her aspiration is tolerable to her. If her aspiration is intolerable, then she chooses as she prefers. In the cardinal model of warm glow in (2), tolerance means a sacrifice of utility no greater than D. However, to formalize the idea of tolerance in an ordinal sense, we endow Dee with a *tolerance function* $\tau : X \to X$ that maps every alternative a into another alternative $\tau(a)$ that we call Dee's *tolerance limit*. The alternative $\tau(a)$ itself and every other alternative that Dee prefers to $\tau(a)$ are tolerable when a is preferred, but any alternative that is R-worse than $\tau(a)$ is intolerable.

For any binary relation R, let $R_=$ be the binary relation such that $x\ R_=\ y$ if and only if either $x\ R\ y$ or $x = y$. So, if

$$A(B)\ R_=\ \tau(R(B))$$

then Dee's aspiration is tolerable. If

$$\tau(R(B))\ R\ A(B)$$

then Dee's aspiration is intolerable.

The tolerance limit $\tau(a)$ of an alternative a cannot be R-preferred to a because $\tau(a)$ marks the least attractive option that Dee can tolerate when a is available. In addition, if b is intolerable in the presence of a, then it should remain intolerable in the presence of an even better alternative. Therefore, Dee's tolerance function τ must satisfy

$$\begin{array}{l}\text{(a) } a\ R_=\ \tau(a), \text{ and} \\ \text{(b) if } a'\ R\ a \text{ then } \tau(a')\ R_=\ \tau(a) \text{ (weak monotonicity)}\end{array} \tag{3}$$

Weak monotonicity is a restrictive condition that is maintained because it is critical to equate this model with the basic warm glow model in (2). We now define a warm glow choice and aspiration function in terms of a preference order and tolerance function.

Definition 2 A choice and aspiration function $(C, A) : B \to X \times X$ is a warm glow (choice and aspiration) function if there exists a preference order R, and a tolerance function τ that satisfies (3) such that for any issue $B \in B$,

$$C(B) = A(B) \quad \text{if } A(B)\ R_=\ \tau(R(B)) \tag{4}$$

$$C(B) = R(B) \quad \text{if } \tau(R(B))\ R\ A(B) \tag{5}$$

That is, Dee chooses as she aspires when her aspiration is tolerable and chooses as she prefers otherwise. We now show that this new definition is observationally equivalent to Definition 1.

Preliminary result 1 *A choice and aspiration function (C, A) satisfies Definition 1 if and only if it also satisfies Definition 2.*

The intuition underlying the preliminary result is simple. In the warm glow model of Definition 1, Dee maximizes $u(x) + D \cdot 1^{A,B}(x)$. If the utility of Dee's preferred choice exceeds, by D, the utility of her aspiration $A(B)$, then her aspiration is *too costly* and she chooses her most preferred alternative. On the other hand, if Dee's preferred choice does not exceed, by D, the utility of her aspiration $A(B)$, then her aspiration is tolerable and she chooses as she aspires. All formal proofs are found in the "Appendix."

The preliminary result demonstrates that warm glow theory can be defined using the ordinal concept of preference (and aspiration and tolerance functions). The utility function u is associated with the preference order R in the usual manner: High R-ranking is associated with higher utility from u. While D is assumed to be constant in the cardinal model of Definition 1, it should be noted that this assumption can be relaxed. The preliminary result shows that warm glow models with nonconstant warm glow payoffs may also be observationally equivalent to our ordinal model of warm glow. Our focus now shifts to delivering choice-theoretic foundation to warm glow theory.

4 Observed aspirations

In this section, we analyze the basic model under the assumptions that choices and aspirations are observable. The general case is notationally involved, and so, we start with the special case where aspirations are also assumed to be ordered to convey intuitions in a simple and direct way.

4.1 Ordered and observed aspirations

An aspiration function A is *ordered* if for any two pairs of issues B and B' such that $B \subseteq B'$, $A(B') \in B$ implies $A(B) = A(B')$. So, an ordered aspiration function must satisfy WARP.

We now characterize the empirical content of warm glow theory under the assumption of ordered and observed aspirations.

Definition 3 Let $\mathcal{B}^s$ be a set of issues such that choice and aspiration differ: $\mathcal{B}^s = \{B \in \mathcal{B} \text{ s.t. } C(B) \neq A(B)\}$.

By definition, if an issue is in $\mathcal{B}^s$, then choice differs from aspiration. As an example, consider the fact while the majority of respondents in surveys say that they intend to buy carbon-offsets, few actually do it. Thus, under the assumption that people aspire to buy carbon-offsets, the choice not to buy offsets reveals that respondents prefer not to buy offsets (because if Dee prefers to act as she aspires then she would). Formally, if $B \in \mathcal{B}^s$, then Dee must prefer her choice $C(B)$ to all feasible alternatives in B. Thus, we can define the *directly observed preference* relation $\succ^d$ as follows:

$$x \succ^d y \Leftrightarrow x \neq y \text{ and there is an issue } B \in \mathcal{B}^s \text{ s.t. } y \in B, x = C(B). \tag{6}$$

So, warm glow theory is falsifiable and must satisfy at least the following axiom.

Limited WARP: Let $B \in \mathcal{B}^s$ and $B' \in \mathcal{B}^s$ be a pair of issues such that $B \subseteq B'$. Then,

$$C(B') \in B \Longrightarrow C(B) = C(B'). \tag{7}$$

The Limited WARP (LWARP) axiom requires that WARP holds on $\mathcal{B}^s$. It holds under warm glow theory because, for all issues in $\mathcal{B}^s$, Dee's chooses as she prefers. However, Limited WARP does not fully characterize the empirical content of warm glow theory because, as the example below shows, indirect inferences about preferences can also be made.

Consider the example mentioned in the introduction. Suppose that Dee must decide how much money to donate to a charity. Suppose that Dee aspires to contribute as much as requested, but she only makes small contributions. So, given the choice between a small donation (s) and no donation (n), Dee aspires to a small donation and chooses it. However, between no donation and a large donation (l), Dee aspires to a large donation but chooses no donation. It follows that Dee prefers a small donation over a large one. Too see this assume, by contradiction, that Dee prefers l over s. Dee's choice of a small donation implies that Dee can tolerate choosing her aspiration s over n. So, Dee must be able to tolerate choosing l (as an aspiration) over n. But this contradicts the choice of n when l is the aspiration. Thus, Dee prefers s to l. This inference is indirect (i.e., beyond $\succ^d$) because it is not based on any issue in $\mathcal{B}^s$ such that l is available and s is the choice.[3]

In general, we can indirectly infer that Dee prefers y to z ($y \succ^{\text{ind}} z$) if there exists an alternative x and issues $B' \in \mathcal{B},\ B \in \mathcal{B}^s$ such that

$$x \in B',\ y = C\left(B'\right) \text{ and } x = C\left(B\right),\ z = A\left(B\right). \tag{8}$$

The indirect revealed preference $\succ^{\text{ind}}$ follows by the same argument given in the example above. If Dee prefers z to y, then whenever z is her aspiration she can tolerate it in the presence of x (because her choice in B' shows that she can tolerate y when x is available, and so she can also tolerate an even better alternative z when x is available). But this contradicts her choice of x in B when z was her aspiration. This indirect way to infer Dee's preferences leads to the following axiom.

Warm Glow Axiom: If $B \in \mathcal{B}^s$, $C\left(B\right) \in B'$ and either $A\left(B\right) \succ^d A\left(B'\right)$ or $A\left(B\right) = A\left(B'\right)$, then $B' \in \mathcal{B}^s$.

The warm glow (WG) axiom states that if her aspiration in B is too costly (so that she cannot tolerate it), then her aspiration in B' must remain too costly provided that (1) the choice in B is available B' and (2) her aspiration in B' is either the same as in B or directly revealed to be less preferred. The intuition behind the warm glow axiom is simple: If Dee's aspiration $A(B)$ is too costly when $C(B)$ is available, then an even more costly aspiration $A\left(B'\right)$ should remain too costly when $C(B)$ is still available.

[3] An alternative way to see this is as follows: The choice of n over l when l is as aspiration implies that $u(n) \geq u(l) + D$. The choice of s over n when s is as aspiration implies that $u(s) + D \geq u(n)$. So, $u(s) \geq u(l)$.

The warm glow axiom is stated in terms of observable data (because the relation $\succ^d$ is defined from choices and aspirations alone). This axiom ensures that revealed preferences obtained directly and indirectly do not contradict each other. To see this, consider a violation of the warm glow axiom. Let $x = C(B)$, $z = A(B)$, $y = C(B') = A(B')$. By 8, $y \succ^{\text{ind}} z$. So, we can indirectly infer that Dee prefers y to z. By assumption, either $y = z$ or $z \succ^d y$. So, we can directly infer that Dee prefers z to y.

The following result states that the empirical content of the warm glow model with ordered aspirations is fully characterized by these two axioms.

Theorem 1 *Let (C, A) be a choice and aspiration function such that A is ordered. (C, A) is a warm glow function if and only if the LWARP and WG axioms are satisfied.*

Theorem 1 demarcates the empirical scope of the warm glow model with observed and ordered aspiration functions. This characterization provides choice-theoretic foundations for the warm glow model.

4.2 Predicting behavior

Theorem 1 provides a general characterization of the predictions that follow from warm glow models with observed and ordered aspirations (i.e., violations of Limited WARP and the warm glow axiom will not be observed). However, special cases of these predictions are also of interest.

Suppose there are two issues B and B' such that x and y are available in both issues and Dee chooses x in B and y in B'. Now assume that Dee is given the choice between x and y. Standard theory makes no prediction about what Dee will choose because her previous choices of x and y violate WARP. In contrast, warm glow theory predicts that Dee will choose as she aspires. To see this, assume that Dee does not choose her aspiration (e.g., assume that her aspiration is x and her choice is y). Then, she cannot tolerate x in the presence of y. This contradicts her choice of x in B. So, warm glow theory not only accommodates some violations of WARP but can exploit behavioral anomalies to predict behavior.

Now suppose that $x = C(B)$ is the choice in B. Consider a subissue $B' \subset B$ such that x is the aspiration in B' (i.e., $x = A(B')$). Then, we can predict that x will also be chosen in B'. This is another simple prediction that follows from the warm glow model. To see this, suppose that another alternative $y \neq x$ is chosen in B'. Then, Dee cannot tolerate choosing x rather than y. But then Dee could not have chosen x in B since y was also available in B. Thus, if Dee does not choose x in B', we must reject either the warm glow model or the assumption that she aspires to x in B'.

Two points of interest emerge. First, even though we can predict Dee's choice of x in B', we may not be able to determine her motivations for that choice. That is, we may not be able to say whether Dee prefers x over alternatives in B' or she chooses x because she aspires to and can tolerate it.

Second, suppose that the behavior predicted above is not satisfied (i.e., $x = C(B) = A(B')$, $B' \subset B$ and $x \neq C(B')$). If $x \neq A(B)$, then we have a violation of LWARP, and if $x = A(B)$, then we have a violation of the WG axiom. However, these choices (i.e., $x = C(B) = A(B'),\ B' \subset B$ and $x \neq C(B')$) violate warm glow theory even if aspirations are not required to be orders. To see this, simply note that in the argument above, we did not use the assumption of ordered aspirations. Therefore, warm glow theory is falsifiable even if aspirations are observed but not necessarily ordered.

However, the LWARP and WG axioms are not sufficient to characterize warm glow theory if aspirations are not necessarily ordered. We illustrate this point with Example 2 (in the "Appendix"). This example provides a choice and unordered aspiration function satisfying the LWARP and WG axioms, but is not a warm glow function. We now turn to the general case of observed aspirations.

4.3 Observed aspirations: the general case

In this subsection, we eliminate the assumption that aspirations are ordered and work out the choice-theoretic foundation of warm glow theory. We provide two results. Our first result generalizes Theorem 1 and characterizes the empirical scope of the warm glow model with arbitrary (but observed) aspiration functions. Our second result determines all inferences that can be made about preferences and tolerances from data.

As mentioned in Sect. 2, all multiple-self models are prone to identification problems. As we show below, if Dee always acts as she aspires then, it is impossible to determine whether her choices reflect her preferences or aspirations. However, we deliver a closed-form formula that, given any choice and aspiration function, characterizes all valid inferences about preferences and tolerances in warm glow theory.

4.3.1 Definition of revealed preferences and tolerance relations

In this subsection, we formally define revealed preferences, tolerances and intolerances. A *warm glow pair* (R, τ) is a preference order R and a tolerance function τ that satisfies (3). A warm glow pair *underlies* a choice and aspiration function (C, A) if (4) and (5) hold for every issue $B \in \mathcal{B}$. Given a choice and aspiration function (C, A), let $\mathcal{P}_{C,A}$ be the set of all warm glow pairs that underlie (C, A), and let $\mathcal{R}_{C,A}$ be the set of all orders R such that $(R, \tau) \in \mathcal{P}_{C,A}$ for some tolerance function τ.

The binary relation $\succ^{\text{rev}}$ captures the *revealed preferences* implied by the observed choice and aspiration functions.

Definition 4 Given a choice and aspiration function (C, A), let $\succ^{\text{rev}}$ be the binary relation such that for any two alternatives x and y,

$$x \succ^{\text{rev}} y \Leftrightarrow x\ R\ y \quad \text{foreveryorder } R \in \mathcal{R}_{C,A}. \tag{9}$$

We say x is *revealed to be preferred* to y if x is R -preferred to y in every warm glow pair that underlies the choice and aspiration function. If x is not revealed to be preferred to y, then there is a warm glow pair that underlies the observed choice and aspiration function such that $y\ R\ x$. We also define *revealed intolerance* ($\succ_{+}^{\text{rev}}$) and *revealed tolerance* ($\vdash^{\text{rev}}$) relations.

Definition 5 Given a choice and aspiration function (C, A), let $\succ_{+}^{\text{rev}}$ be the binary relation such that for any two alternatives x and y,

$$x \succ_{+}^{\text{rev}} y \Leftrightarrow \tau(x)\ R\ y \quad \text{foreverywarmglowpair} \ (R, \tau) \in \mathcal{P}_{C,A}. \tag{10}$$

We say x is *revealed to not tolerate* y if y is R-ranked below the tolerance threshold for x in every warm glow pair that underlies the choice and aspiration function.

Definition 6 Let $\vdash^{\text{rev}}$ be the binary relation such that for any two alternatives x and y,

$$x \vdash^{\text{rev}} y \Leftrightarrow x\ R_{=}\ \tau(y) \quad \text{foreverywarmglowpair} \ (R, \tau) \in \mathcal{P}_{C,A}. \tag{11}$$

We say x is *revealed to be tolerated by* y if x is equal to or R-ranked above the tolerance threshold for y in every warm glow pair that underlies the choice and aspiration function.

These three revealed relations capture all binary relations (preference, intolerance and tolerance) that must hold. An intuitive descriptions of these concepts may help. If x is revealed to not tolerate y, then we know that Dee finds x to be "much better" than y (i.e., $u(x)$ must exceed $u(y)$ by at least D utiles). So, even if y is an aspiration, Dee finds it to costly to chose y in the presence of x. If x is revealed preferred to y, then we know that Dee finds x to be better than y (i.e., $u(x)$ must exceed $u(y)$). If x is revealed to be tolerated by y, then we know that Dee does not find x to be "much worse" than y (i.e., $u(x)$ must exceed $u(y) - D$). Hence, we have an hierarchy from "much better" to "better" to "not much worse" which reflects what we can infer about Dee's preference relations and some features of the intensities of her preferences. This can be easily seen formally. By definition and (3), $x \succ_{+}^{\text{rev}} y \Longrightarrow x \succ^{\text{rev}} y \Longrightarrow x \vdash^{\text{rev}} y$. Note also that $\succ_{+}^{\text{rev}}$ and $\succ^{\text{rev}}$ are necessarily transitive while $\vdash^{\text{rev}}$ is not.

By assumption, preference and tolerance relations persist across issues or when new alternatives are introduced. This allows predictions on behavior. As we mentioned above, if x is revealed to not tolerate y, $x \succ_{+}^{\text{rev}} y$, then y will never be chosen in the presence of x. If x is revealed to be tolerated by y, $x \vdash^{\text{rev}} y$, and x is the aspiration in some issue B such that $y \in B$, then y will not be chosen in B. Furthermore, if x is revealed to be tolerated by every alternative in B, then x will be chosen in B.

4.3.2 Directly revealed preference and tolerance relations

We now show relations directly revealed from choice. First note that Dee's actual choice is tolerated by every other feasible alternative. To see this, observe that if her actual choice is not her aspiration, then it must be most preferred and, hence, tolerable. On the other hand, if her actual choice is her aspiration, then her aspiration must be tolerated. We define the *directly observed tolerance* relation $\vdash$ as follows:

$$x \vdash y \Leftrightarrow x = y \text{ or there is an issue } B \text{ such that } x, y \in B,\ x = C(B) \tag{12}$$

When Dee's actual choice differs from her aspiration, she must prefer her actual choice to all other alternatives in the issue and find her aspiration intolerable. We recall

Table 1 Revealed relations implied by three alternatives

	$y \vdash^{\text{rev}} z$	$y \succ^{\text{rev}} z$	$y \succ_{+}^{\text{rev}} z$
$x \vdash^{\text{rev}} y$		$x \vdash^{\text{rev}} z$	$x \succ^{\text{rev}} z$
$x \succ^{\text{rev}} y$	$x \vdash^{\text{rev}} z$	$x \succ^{\text{rev}} z$	$x \succ_{+}^{\text{rev}} z$
$x \succ_{+}^{\text{rev}} y$	$x \succ^{\text{rev}} z$	$x \succ_{+}^{\text{rev}} z$	$x \succ_{+}^{\text{rev}} z$

that the *directly observed preference* relation $\succ^d$ is defined in 6. We also define the *directly observed intolerance* relation $\succ_+$ as follows:

$$x \succ_+ y \Leftrightarrow x \neq y \text{ and there is an issue } B \text{ s.t. } x, y \in B,\ x = C(B),\ y = A(B) \quad (13)$$

Lemma 1 *Suppose* (C, A) *is a warm glow choice and aspiration function, then:* $x \vdash y \Longrightarrow x \vdash^{\text{rev}} y$*;* $x \succ^d y \Longrightarrow x \succ^{\text{rev}} y$*; and* $x \succ_+ y \Longrightarrow x \succ_+^{\text{rev}} y$*.*

The lemma follows almost immediately from the definitions. If, for example, x is directly observed to not tolerate y, then there is an issue B s.t. $x, y \in B$, $x = C(B)$, $y = A(B)$, and for any warm glow pair (R, τ) that underlies (C, A), it must be that $x = R(B)$ and $\tau(x)\ R\ y$; otherwise, x cannot be chosen. Lemma 1 shows that we can (partially) reveal preferences and tolerances by making direct inferences from observed choice and aspirations.

A simple application of these results is as follows: Consider the example of Riker and Ordeshook (1968) discussed in the introduction. Sigelman (1982) found that 13 % of survey respondents said they voted when they actually abstained while 1 % said they abstained when they actually voted. So, assuming that those who said they voted, but did not, aspire to vote, we can reveal preferences for 14 % of the respondents. We find support for Riker and Ordeshook: 93 % (i.e., 13/14) of directly revealed preferences are consistent with their assumption that people prefer to abstain over voting. This result supports the long-held intuition that voting is largely a product of a sense of civic duty.

4.3.3 Indirectly revealed preference and tolerance relations

We have shown, in the previous subsection, that directly observed relations may reveal additional relations indirectly. Table 1 below summarizes the revealed relations that must hold between two alternatives x and z on the basis of their revealed relations to a third alternative y. These relations are extremely intuitive. For example, if Dee finds x to be better than y and y to be much better than z, then Dee must find x to be much better than z. This is very easy to prove. If $x \succ^{\text{rev}} y$ and $y \succ_+^{\text{rev}} z$, then $x\ R\ y$ and $\tau(y)\ R\ z$. By (3), $\tau(x)\ R_=\ \tau(y)\ R\ z$, and, by transitivity of R, it follows that $\tau(x)\ R\ z$. Thus, $x \succ_+^{\text{rev}} z$.[4]

Using Table 1, we can reveal new relations as a consequence of the relations directly revealed by Lemma 1. More generally, we may arrange alternatives into *chains* in which every pair of successive alternatives are connected by a directly revealed relation. We can then use the rules in Table 1 to reveal new relations. Revealed tolerance

[4] A complete proof of all relations is given as a part of the proof of Lemma 2 in the "Appendix."

and intolerance relations imply a revealed preference relation. Similarly, revealed preference relations can be combined with revealed tolerance or intolerance relations. Depending upon the number of intolerance and tolerance relations the chain includes, this process will either imply a new relation between the initial and terminal alternative or be indeterminate. Formally,

Definition 7 A chain (x, ρ) is an ordered sequence of alternatives $x = (\mathbf{x}_0, \ldots, \mathbf{x}_n)$ and directly observed relations $\rho = (\rho_1, \ldots, \rho_n)$ with $n \geq 1$ such that $\rho_i \in \{\succ_+, \succ^d, \vdash\}$ and $x_{i-1}\ \rho_i\ x_i$ holds for every $i = 1 \ldots n$. Alternative x_0 (x_n) is the initial (terminal) alternative.

So, a chain is a sequence of alternatives that links the initial and the terminal alternative. Each link in the chain consists of a direct revelation relation such a directly revealed preference relation, a directly tolerance relation or a directly revealed intolerance relation. We now define the central property of chains.

Definition 8 The characteristic $\chi(x, \rho)$ of the chain (x, ρ) is the difference between the number of times $\rho_i = \succ_+$ and the number of times $\rho_i = \vdash$ for $i = 1 \ldots n$.

The significance of the characteristic of the chain may not be clear at first. However, it captures a straightforward idea. Consider a chain from $\mathbf{x}_0$ to $\mathbf{x}_n$. Each link with a $\succ_+$ intuitively indicates an utility increase of at least D utiles, whereas each link with a $\vdash$ intuitively indicates that if utility is reduced at all, the reduction is no greater than D utiles (and $\succ^d$implies no reduction in utility). So, if we add how many times in the chain D utiles were gained and how many times D utiles were not lost, we can determine whether $\mathbf{x}_0$ is much better, better or merely not much worse than $\mathbf{x}_n$. The remaining subsection delivers a demonstration that this intuitive way of adding and subtracting utiles makes formal sense and is observationally meaningful. As pointed out, we must start by determining whether the chain has characteristic greater or equal to minus 1, zero or 1.

Definition 9 The following relations $\vdash^i$, $\succ^i$ and $\succ^i_+$ are implied by chains:

$$
\begin{aligned}
&x \vdash^i y \Leftrightarrow x = y \textit{ or there is a chain } (\mathbf{x}, \boldsymbol{\rho})\ s.t. \mathbf{x}_0 = x, \mathbf{x}_n = y \textit{ and } \chi(\mathbf{x}, \boldsymbol{\rho}) \geq -1,\\
&x \succ^i y \Leftrightarrow \textit{there is a chain } (\mathbf{x}, \boldsymbol{\rho}) \textit{ s.t. } \mathbf{x}_0 = x,\ \mathbf{x}_n = y \textit{ and } \chi(\mathbf{x}, \boldsymbol{\rho}) \geq 0, \text{and}\\
&x \succ^i_+ y \Leftrightarrow \textit{there is a chain } (\mathbf{x}, \boldsymbol{\rho}) \textit{ s.t. } \mathbf{x}_0 = x,\ \mathbf{x}_n = y \textit{ and } \chi(\mathbf{x}, \boldsymbol{\rho}) \geq 1.
\end{aligned} \tag{14}
$$

That is, by definition, $x \succ^i y$ iff x can be linked to y with a chain of characteristic greater or equal to 0, $x \vdash^i y$ iff x can be linked to y with a chain of characteristic greater or equal to -1, and $x \succ^i_+ y$ iff x can be linked to y with a chain of characteristic greater or equal to 1. Our central result in this section (Theorem 2 below) equates the observable relations implied by chains $\vdash^i$, $\succ^i$ and $\succ^i_+$ with all revealed tolerance, preference, and intolerance relations $\vdash^{\text{rev}}$, $\succ^{\text{rev}}$and $\succ^{\text{rev}}_+$. To obtain some additional intuition, note that all directly observed relations are also implied by chains. For example, if $x \succ^d y$, then the chain $((x, y), (\succ^d))$ has the characteristic 0 and so, $x \succ^i y$. The same applies to directly revealed tolerances and intolerances. That is, by definition, $x \vdash y \Longrightarrow x \vdash^i y$ and $x \succ_+ y \Longrightarrow x \succ^i_+ y$. But there are many more relations implied by chains including those that can be inferred indirectly using Table 1. In the following lemma, we show that all relations implied by chains are revealed.

Lemma 2 *Suppose* (C, A) *is a warm glow choice and aspiration function, then:* $x \vdash^i y \Longrightarrow x \vdash^{\text{rev}} y$; $x \succ^i y \Longrightarrow x \succ^{\text{rev}} y$; *and* $x \succ^i_+ y \Longrightarrow x \succ^{\text{rev}}_+ y$.

This lemma is based upon the fact that, by Lemma 1, all directly observed relations are revealed relations. The proof shows that the relations implied by chains are relations implied by the iterative application of rules in Table 1.

In order to obtain even more intuition for the use of chains, recall the simple indirectly revealed preference relation $\succ^{\text{ind}}$ defined in Sect. 4.1. By definition, $y \succ^{\text{ind}} z$ iff there exists an alternative x such that $y \vdash x \succ_+ z$. So, $y \succ^{\text{ind}} z$ implies that a chain of characteristic zero links y to z. Thus, $y \succ^{\text{ind}} z \Longrightarrow y \succ^i z$. Intuitively, if y is not much worse than x and x is much better than z, then y is better than z (because much better means gaining D utiles and not much worse means not loosing D utiles).

Besides the fact that all relations implied by chains are revealed, Lemma 2 also tells us that if (C, A) is a warm glow function, then the preference relations implied by chains, $\succ^i$, must be irreflexive. This follows immediately from the lemma and the fact that the preference relation R in any warm glow pair (R, τ) underlying (C, A) is irreflexive. In the following theorem, we characterize warm glow theory with observed aspirations. We show that (C, A) is a warm glow function if and only if the preference relation implied by chains $\succ^i$ is irreflexive. We also show that *all* revealed relations are implied by chains. That is, the deeper part of the theorem tells us that everything we can learn about the agent's motivations (preferences and tolerances) on the basis of data is obtained through chains. Because chains are defined in terms of directly observed relations, the theorem provides a simple and direct way to infer motivations and predict behavior.

Theorem 2 (C, A) *is a warm glow function if and only if there is no* x *such that* $x \succ^i x$. *If* (C, A) *is a warm glow function then*

$$
\begin{aligned}
& x \vdash^{\text{rev}} y \Leftrightarrow x \vdash^i y, \\
& x \succ^{rev} y \Leftrightarrow x \succ^i y \text{ and} \\
& x \succ^{\text{rev}}_+ y \Leftrightarrow x \succ^i_+ y.
\end{aligned}
$$

We argued above that (C, A) is a warm glow function only if preferences implied by chains are irreflexive. In the first part of the theorem, we show that this condition is also sufficient. We also show that all revealed preference relations are implied by chains. The proof is by construction (see Lemma 3 in the "Appendix"). Suppose that $\succ^i$ is irreflexive, and no preference relation between x and y is implied. We show that it is possible to construct warm glow pairs (R, τ) and (R', τ') underlying (C, A) such that $x\ R\ y$ and $y\ R'\ x$. We take all preferences implied by chains and add a single new relation, for example $x\ R\ y$. The addition of the new relation generates a new set of chains implying additional relations. We show that the resulting preference relations are still irreflexive, and the process can be repeated until a complete preference order has been defined. We then demonstrate that we can pair this complete order with a tolerance function τ such that (R, τ) is a warm glow pair underlying (C, A).

In the second part of the proof, we show that the only tolerance and intolerance relations that can be revealed are those that are implied by chains. Suppose, for example, that for a given warm glow function (C, A), x is revealed not to tolerate y ($x \succ_{+}^{\text{rev}} y$). Then, for every warm glow pair (R, τ) underlying (C, A) it is the case that $\tau(x)\ R\ y$. We show that there is a warm glow pair (R, τ) underlying (C, A) with a set of some technical properties. We use these properties in the main part of the proof to show that there must be an issue B for which an intolerance relation between $C(B)$ and $A(B)$ is directly observed and $x \succ^i C(B)$, $A(B) \succ^i y$. We then construct a chain $(\mathbf{x}, \rho)$ with initial alternative x and terminal alternative y such that $\chi(\mathbf{x}, \rho) > 0$. That is, any revealed intolerance relation must be implied by chains. Using a similar procedure, we show the same is true for tolerance relations.

The main theorem also implies that if there are no issues such that aspiration and choice are different, that is, $C(B) = A(B)$ for every issue B, then (C, A) is a warm glow function, and there are no revealed preference (or intolerance) relations. This is stated formally in the following corollary.

Corollary 1 *For any choice and aspiration function* (C, A) *such that* $C = A$,

(a) (C, A) *is a warm glow function;*
(b) $\mathcal{P}_{C,A}$ *contains all orders, that is, for every order* R *on* X*, there exists a tolerance function* τ *s.t.* $(R, \tau) \in \mathcal{R}_{C,A}$;[5]
(c) the sets of revealed preference and intolerance relations are both empty.

The proof is simple. If $A{=}C$, then there are no directly observed preference or intolerance relations. Any indirectly revealed relation would require a chain with at least three alternatives and characteristic greater or equal to -1. No such chains exist.

So, even if aspirations are observed, we may not reveal preferences and tolerances (even if the choices violate WARP). There must be at least one instance in which the agent is observed to aspire to one thing and choose another. On the other hand, even a single observation of intolerance can reveal a great deal of information about preferences and allow a variety of predictions. In some examples, a single issue in which Dee does not act as she aspires implies a complete revelation of her preference and tolerance relations (see Example 1 in the "Appendix").

So far, we have considered a model in which Dee's aspirations are observed. In the next section, we consider the case in which it is legitimate to make assumptions about Dee's aspirations in some issues but not in others.

4.4 Partially observed aspirations

Let an *observed aspiration function* be a function $\tilde{A} : \tilde{\mathcal{B}} \longrightarrow X$, $\tilde{\mathcal{B}} \subseteq \mathcal{B}$, such that $\tilde{A}(B) \in B$ for every $B \in \tilde{\mathcal{B}}$. We say the aspiration function A *extends* $\tilde{A}$ if $A(\mathcal{B}) = \tilde{A}(\mathcal{B})$ for every issue $B \in \tilde{\mathcal{B}}$.

Definition 10 $(C, \tilde{A})$ is a warm glow (choice and observed aspiration) function if there exists an aspiration function A such that A extends $\tilde{A}$ and (C, A) is a warm glow function.

[5] Just take $\tau(x) = a$ s.t. $zR_{=}a$ for all $z \in X$.

Now we define revealed preferences and tolerance relations for the case with partially known aspirations. Let $\mathcal{A}_{C,\tilde{A}}$ be a set of all aspiration functions A such that A extends $\tilde{A}$ and (C, A) is a warm glow function. Then, we may define revealed preference and tolerance relations for $(C, \tilde{A})$ as follows:

$$x \succ^{\text{rev}} y \text{ by } (C, \tilde{A}) \text{ iff } x \succ^{\text{rev}} y \text{ by } (C, A) \text{ for every } A \in \mathcal{A}_{C,\tilde{A}},$$
$$x \succ^{\text{rev}}_{+} y \text{ by } (C, \tilde{A}) \text{ iff } x \succ^{\text{rev}}_{+} y \text{ by } (C, A) \text{ for every } A \in \mathcal{A}_{C,\tilde{A}}, \text{ and}$$
$$x \vdash^{\text{rev}} y \text{ by } (C, \tilde{A}) \text{ iff } x \vdash^{\text{rev}} y \text{ by } (C, A) \text{ for every } A \in \mathcal{A}_{C,\tilde{A}}.$$

In other words, x is revealed to be preferred to y if and only if it is revealed to be preferred for any possible extension of the observed aspirations.

Definition 11 For a given observed aspiration function $\tilde{A}$ and choice function C let A^* be an aspiration function that extends $\tilde{A}$ and $A^*(B) = C(B)$ for all those issues such that an aspiration is not observed.

The following proposition shows that whenever aspirations are unknown and not necessarily ordered we may assume without loss of generality that they are identical to the actual choice.

Proposition 1 $(C, \tilde{A})$ *is a warm glow function if and only if* (C, A^*) *is a warm glow function. Moreover, the revealed preference and tolerance relations for* $(C, \tilde{A})$ *and* (C, A^*) *are the same, that is,*

$$x \succ^{\text{rev}} y \, by \, (C, \tilde{A}) \, iff \, x \succ^{\text{rev}} y \, by \, (C, A^*),$$
$$x \succ^{\text{rev}}_{+} y \, by \, (C, \tilde{A}) \, iff \, x \succ^{\text{rev}}_{+} y \, by \, (C, A^*), \, and$$
$$x \vdash^{\text{rev}} y \, by \, (C, \tilde{A}) \, iff \, x \vdash^{\text{rev}} y \, by \, (C, A^*).$$

The result follows directly from Theorem 2. We first show that if the preferences revealed by the extension A^* are not irreflexive, then the preferences revealed by any other extension $A \in \mathcal{A}_{C,\tilde{A}}$ are not irreflexive either and $(C, \tilde{A})$ cannot be a warm glow function. The particular extension A^* assumes that every unobserved aspiration is the same as observed choice. Thus, A^* implies no additional directly observed intolerance or preference relations beyond those directly observed in $\tilde{A}$. Intuitively, we get a minimum number of new relations implied by chains. Therefore, if the preferences revealed by A^* are reflexive, then preferences generated by any extension in $\mathcal{A}_{C,\tilde{A}}$ are as well. In addition, since relations revealed for $(C, \tilde{A})$ must hold for any extension including A^*, it follows that these are the only relations that are revealed.

The key implication of the proposition is that if we make an incorrect assumption that Dee's aspiration in an issue B is, say, alternative x (her actual aspiration is y) and her choice is also x, then we will not make incorrect inferences about her preferences and tolerances, and all our predictions about her behavior are still valid. However, if we make an incorrect assumption that Dee's aspiration in an issue B is x and her choice is $z \neq x$, then we will make some incorrect inferences about her motivations and we also make incorrect predictions about her behavior. This emphasizes the fact that observed aspirations are important only when they differ from choice. In particular, the case where Dee has no aspirations in some, but not all, issues can be accommodated by our simpler model.

We can now formally state an important result. Warm glow theory has no empirical content when aspirations are unknown and unordered.

Corollary 2 *For any choice function C, there is an aspiration function A such that (C, A) is a warm glow function.*

The proof follows immediately from Proposition 1 and Theorem 2. Proposition 1 tells us that if no aspirations are observed, then there is a function (C, A) that is a warm glow function if and only if $(C, A{=}C)$ is a warm glow function. From Theorem 2, it follows that no preferences are revealed for the function $(C, A{=}C)$ and, since no alternative is revealed to be preferred to itself, it is a warm glow function.

5 Unobserved and ordered aspirations

In Sect. 4, we show that warm glow theory has no empirical content when aspirations are unobserved and not necessarily ordered. In this section, we show that if aspirations are unknown but ordered, then warm glow model has empirical content. A full characterization of empirical content of warm glow theory in this case is still an open question.

When aspirations are ordered, an alternative x may be chosen in the union of a set of issues containing x only if it is chosen in at least one of the issues. We call this property the negative expansion axiom.

> Negative expansion axiom: If for every $i \in 1, n$ it is the case that $x \in B_i$, $x \neq C(B_i)$ then $x \neq C(\cup_i B_i)$.[6]

To see that the negative expansion axiom (NE) must hold under warm glow theory with ordered aspirations, observe that x can be chosen in the union of a set of issues containing it only if it is either the most preferred choice or the aspiration. If x is the aspiration (i.e., $x = A(\cup_i B_i)$), then, under the assumption of ordered aspirations, $x = A(B_i)$ for every i. Moreover, x must be tolerable by every other alternative in $\cup_i B_i$, and, therefore, x must be chosen in *every* issue B_i. On the other hand, if $x \neq A(\cup_i B_i)$, then x must be the most preferred alternative (i.e., $R(\cup_i B_i)$) and x must not tolerate $A(\cup_i B_i)$. Since $A(\cup_i B_i)$ is also the aspiration choice for every B_j that contains it, x must be chosen in every B_j that contains $A(\cup_i B_i)$.

The NE axiom demonstrates that warm glow theory has empirical content when aspirations are unknown but ordered. Suppose Dee makes a small donation s in $\{n, s\}$ but chooses not to donate n in $\{n, s, l\}$. Since $\{n, s, l\} = \{n, s\} \cup \{n, l\}$, $n \in \{n, s\} \cap \{n, l\}$ and n is chosen in $\{n, s, l\}$ but not in $\{n, s\}$,, the NE axiom implies that n must be chosen in $\{n, l\}$. We can also infer motivations. Dee's choice of n in $\{n, s, l\}$ must be motivated by her preference (if n is her aspiration in $\{n, s, l\}$, then she must choose

[6] The expansion axiom (see Manzini and Mariotti 2007) requires that if for every $i \in 1, n$ it is the case that $x \in B_i$, $x = C(B_i)$, then $x = C(\cup_i B_i)$.

n in $\{n, s\}$ as well). Therefore, her choice of s in $\{n, s\}$ must be motivated by her aspiration.

In some situations, it is possible to predict behavior even when it is not possible to infer motivations. For example, if we observe that y is chosen in $\{x, y\}$ and z is chosen in $\{x, z\}$, we cannot tell whether these choices are motivated by preferences or aspirations. However, NEA implies that x cannot be chosen in $\{x, y, z\}$.

6 Extensions

The main purpose of this paper is to provide choice-theoretic foundations for warm glow theory. The need for choice-theoretic foundations can be easily seen by the fact that even simple warm glow models can accommodate violations of WARP (e.g., the contribution to public goods example mentioned in the introduction). The ability of warm glow theory to accommodate behavioral anomalies comes from the fact that aspirations are issue dependent. This shows that standard choice-theoretic foundations do not apply to warm glow models. However, the traditional motivation for warm glow theory in applied work was not to accommodate violations of WARP. For example, the motivation of Riker and Ordeshook for introducing warm glow is that, without it, we would draw the unreasonable inference that people prefer voting to abstention even though the only real difference between the two alternatives is that voting is costly (in addition, in standard economics, agents would only vote if their utility for voting were arbitrarily large to compensate for vanishing pivot probabilities).

The difficulties in applying standard theory of choice to political science can also seen in the following example. Suppose that given the issue $\{a, h\}$ Dee's hypothetical choice is h (i.e., Dee is merely asked in a survey whether she would choose a or h and her answer is h), but her actual choice between a and h is a. If we apply standard theory, we infer a preference for a over h because standard theory disregards hypothetical choice as relevant for inferences over preferences. Now, if expected utility theory applies and Dee is given the choice between two lotteries $La = (pa, (1-p)z)$ and $Lh = (ph, (1-p)z)$ such that $p \in (0, 1)$, Dee must choose La. However, if the probability p is small, the choice between La and Lh becomes a near-hypothetical choice between a and h because z is implemented with high probability in both lotteries. In the context of a voting experiment, Feddersen et al. (2009) show that Dee may choose Lh, when h is morally appealing and a is monetarily appealing for Dee (see also Shayo and Harel 2012). This behavior is not a violation of WARP, but it violates expected utility theory and several related models of choice ordinarily used in economics and political science. This provides prima facie evidence that standard theory is not applicable when hypothetical and actual choice differs.

If lottery La is interpreted as "voting for a" and lottery Lh is interpreted as "voting for h" and z is the event where Dee's vote is not pivotal, then the question is why does Dee vote for the alternative she prefers the least (and why does, in a hypothetical situation, Dee select h over a). Warm glow theory provides a simple, logical and therefore compelling interpretation. Dee indeed prefers a over h because a is instrumentally beneficial. However, we assume that she aspires to choose h (or Lh in the case of lotteries) because these choices are commonly perceived to be ethical. As the probability p decreases, the chances that her choice is consequential decreases, and at some point, she can tolerate her aspiration and, hence, chooses it. Our simple model of warm glow can only accommodate lotteries if probabilities are restricted to take finitely many values. This is often analytically inconvenient and so extending warm glow theory to accommodate lotteries may be a valuable exercise (see Shayo and Harel 2012 for results in this direction).

7 Conclusion

In warm glow theory, an agent may prefer one alternative but aspire to choose another. She chooses her aspiration only if she can tolerate choosing it instead of her preferred choice. We provide choice-theoretic foundation for warm glow theory and a characterization of how to infer motivations. Our findings show that ad hoc assumptions used in the warm glow literature can be tested. In addition, warm glow theory generates predictions on behavior even when motivations cannot be inferred and standard theory does not apply.

Warm glow theory may be appropriate in settings where aspirations and actual choice may differ. This setting may be problematic for standard theory, but it is precisely the choices that differ from aspirations that deliver the critical data required to estimate the core elements of warm glow models.

8 Appendix

8.1 Examples

Our first example shows that a single issue where Dee does not act as she aspires can deliver a complete revelation of her preference and tolerance relations.

Example 1 There are three alternatives, x, y and z. Assume that $C(x, y) = x$ $A(x, y) = y$; $A(x, z) = C(x, z) = z$; $A(y, z) = C(y, z) = y$.

In this example, except for the binary choice between x and y, Dee acts as she aspires. Yet, her preference order is completely revealed from pairs of directly observed relations: $z \vdash x \succ_{+} y \vdash z$, and she must prefer x to z to y. It also follows that $\tau(x) = z$, $\tau(z) = y$ and $\tau(y) = y$.

Our second example shows observed choices and nonordered aspirations that violate the warm glow model of Definition 2 without violating the LWARP and the WG axiom.

Example 2 There are four different alternatives x, y, z and w. Assume that $C(x, y, w) = x$ $A(x, y, w) = w$; $C(x, y, z) = y$ $A(x, y, z) = z$; and for all other issues, both aspirations and actual choices are resolved by order $\bar{R}$ such that $x \bar{R} y \bar{R} z \bar{R} w$.

Note that $\{x, y, w\} \in \mathcal{B}^s$ and $\{x, y, z\} \in \mathcal{B}^s$, and all other issues do not belong to $\mathcal{B}^s$. LWARP is satisfied: There is no pair of nested issues in $\mathcal{B}^s$. WG is also satisfied because there is no issue B such that $A(B) = w$ or $w \succ^d A(B)$ (apart from $\{x, y, w\} \in \mathcal{B}^s$) and the only issue B such that $A(B) = z$ or $z \succ^d A(B)$ (apart from $\{x, y, z\} \in \mathcal{B}^s$)

is $\{z, w\}$. However, $C(x, y, z) = y \notin \{z, w\}$. At the same time, $\{x, y, w\} \in \mathcal{B}^s$ and $C(x, y, w) = x$ implies that $x \succ^d y$, while $\{x, y, z\} \in \mathcal{B}^s$ and $C(x, y, z) = y$ implies that $y \succ^d x$.

8.2 Proof of the preliminary result

We show that Definition 1 of warm glow function is equivalent to Definition 2.

Let (C, A) be a warm glow choice and aspiration function. Let utility function u and scalar $D \geq 0$ be such that property (2) of Definition 1 holds. Let R be a preference order associated with u. For any given alternative $a \in X$, let $\tau(a) \in X$ be the lowest R-ranked alternative such that $u(a) - D \leq u(\tau(a))$. So, $u(a) - D > u(b)$ for any alternative $b \in X$ such that $u(b) < u(\tau(a))$. We show that property (3) holds for (R, τ), and (R, τ) underlies (C, A) , that is, properties (4) and (5) of Definition 2 hold.

$u(a) \geq u(\tau(a))$ because $u(a) - D \leq u(a)$. In addition, if $u(a') > u(a)$, then $u(\tau(a')) \geq u\left(a'\right) - D > u\left(a\right) - D$, and, therefore, $u\left(\tau\left(a'\right)\right) \geq u\left(\tau\left(a\right)\right)$. Hence, (3) holds. Suppose for some issue B, $C\left(B\right) = A\left(B\right)$. Then, by (2), $U^{A,B}(C(B)) = u(A(B)) + D \geq U^{A,B}(R(B)) \geq u(R(B))$, and, therefore, $A\left(B\right)\ R_{=}\ \tau\left(R\left(B\right)\right)$. Now suppose that $C(B) \neq A(B)$. Then, $C(B) = R(B)$ because otherwise, by the definition of $R(B)$, $U^{A,B}(C(B)) = u(C(B)) < u(R(B)) \leq U^{A,B}(R(B))$ contradicting (2). Hence, by (2), $u(C(B)) = u\left(R\left(B\right)\right) > u(A(B)) + D$, and, therefore, $\tau\left(R\left(B\right)\right)\ R\ A\left(B\right)$. So, (4) and (5) hold.

Now, let (C, A) be a choice and aspiration function that satisfies (4) and (5) for some preference order R and tolerance function τ that satisfies (3). Let $D = 1$. We now show that there exists a utility function u such that (2) holds, and, in addition, u is associated with preference R, and for all $a \in X : u\left(a\right) < u\left(\tau\left(a\right)\right) + 1$. The proof is by induction on the size of X.

Assume that X has only two alternatives, that is, $|X| = 2$. So, let $X = \{x, y\}$, $x \neq y$, and $x = R\left(X\right)$, that is, $x\ R\ y$. We define $u\left(y\right) = 0$, and $u(x) = 0.5$ if $\tau(x) = y$ and $u(x) = 2$ if $\tau(x) = x$. By definition, u is associated with R and $u(a) < u(\tau(a)) + 1$ for $a \in X$ (note that, by (3), $\tau(y) = y$). In addition, (2) holds because, if $A\left(X\right) = x = R\left(X\right)$, then $C\left(X\right) = x$, and $U^{A,X}\left(x\right) = u\left(x\right) + 1 > 0 = u\left(y\right) = U^{A,X}\left(y\right)$. If $A\left(X\right) = y \neq R\left(X\right) = x$, then $C\left(X\right) = x$ if $\tau\left(x\right) = x$, and $U^{A,X}\left(x\right) = u\left(x\right) = 2 > 1 = u\left(y\right) + 1 = U^{A,X}\left(y\right)$, and $C\left(X\right) = y$ if $\tau\left(x\right) = y$, and $U^{A,X}\left(x\right) = u\left(x\right) = 0.5 < 1 = u\left(y\right) + 1 = U^{A,X}\left(y\right)$.

The induction assumption is that whenever $|X| = n$, there exists a utility function u associated with R such that $u(a) < u(\tau(a)) + 1$ for all $a \in X$ and (2) holds. Now assume that $|X| = n + 1$. Let $\bar{a} \in X$ be the highest R-ranked alternative. So, $\bar{a}\ R\ a$ for every $a \neq \bar{a}$. Let $\mathbf{a} \in X$ be the second highest R-ranked alternative. So, $\mathbf{a}\ R\ a$ for every $a \notin \{\bar{a}, \mathbf{a}\}$. Let $\tilde{X}$ be $X \backslash \{\bar{a}\}$. $|\tilde{X}| = n$ and, by the induction assumption, there exists a utility function $\tilde{u} : \tilde{X} \longrightarrow \Re$ associated with R on $\tilde{X}$ such that $\tilde{u}(a) < \tilde{u}(\tau(a)) + 1$ for all $a \in \tilde{X}$ and (2) holds for any issue $B \subseteq \tilde{X}$. If $\tau(\bar{a})$ is not R-ranked lowest, then let $\hat{a} \in X$ be the highest R-ranked option such that $\tau(\bar{a})\ R\ \hat{a}$.

Let $u : X \longrightarrow \Re$ be such that $u(a) = \tilde{u}(a)$ for any $a \in \tilde{X}$, and $u(\bar{a}) \in (\max\{\tilde{u}(\mathbf{a}), \tilde{u}\left(\hat{a}\right) + 1\}, \tilde{u}(\tau(\bar{a})) + 1)$ if $\tau(\bar{a}) \neq \bar{a}$ and $\tau(\bar{a})$ is not R-ranked lowest, $u(\bar{a}) > \tilde{u}(\mathbf{a}) + 1$ if $\tau(\bar{a}) = \bar{a}$, and $u(\bar{a}) \in (\tilde{u}(\mathbf{a}),\ \tilde{u}(\tau(\bar{a})) + 1)$ if $\tau(\bar{a})$ is R-ranked lowest.

We first show that u is well defined. If $\tau(\bar{a}) \neq \bar{a}$ then, by (3), $\tau(\bar{a})\ R_=\ \tau(\mathbf{a})$, and, by induction assumption, $\tilde{u}(\mathbf{a}) < \tilde{u}(\tau(\mathbf{a})) + 1 \leq \tilde{u}(\tau(\bar{a})) + 1$. In addition, if $\tau(\bar{a})$ is not R-ranked lowest, then, by definition, $\tau(\bar{a})\ R\ \hat{a}$, and, by induction assumption, $\tilde{u}\left(\hat{a}\right) < \tilde{u}(\tau(\bar{a}))$. Hence, u is well defined.

By definition, $u(\bar{a}) < u(\tau(\bar{a})) + 1$ and $u(\bar{a}) > u(\mathbf{a})$. So, by induction assumption, $u(a) < u(\tau(a)) + 1$ for all $a \in X$ and u is associated with R.

We now show that (2) holds. Let B be an issue such that $\bar{a} \in B$. So, by definition, $R(B) = \bar{a}$. Assume that $A(B)\ R_=\ \tau(\bar{a})$. Then, by (4), $C(B) = A(B)$. It follows that $U^{A,B}(C(B)) = u(C(B)) + 1 \geq u(\tau(\bar{a})) + 1 > u(\bar{a}) \geq u(x) = U^{A,B}(x)$ for every $x \in B$, $x \neq C(B)$. Now assume that $\tau(\bar{a})\ R\ A(B)$. Then, by (5), $C(B) = R(B) = \bar{a}\ R\ A(B)$ (and $\tau(\bar{a})$ is not R-ranked lowest). If $\tau(\bar{a}) \neq \bar{a}$, then $U^{A,B}(C(B)) = u(\bar{a}) > u\left(\hat{a}\right) + 1 \geq u(A(B)) + 1 = U^{A,B}(A(B))$. In addition, $U^{A,B}(C(B)) = u(\bar{a}) > u(x) = U^{A,B}(x)$ for all $x \in B$, $x \notin \{A(B), C(B)\}$. If $\tau(\bar{a}) = \bar{a}$, then $U^{A,B}(C(B)) = u(\bar{a}) > \tilde{u}(\mathbf{a}) + 1 \geq u(x) + 1 \geq U^{A,B}(x)$ for all $x \neq \bar{a}$. □

8.3 Proof of Lemma 1

Let (C, A) be a warm glow choice and aspiration function, and suppose for x and y, there exists issue B such that $x, y \in B$, $x = C(B)$, that is, $x \vdash y$. If $x \neq y$ and $x \neq A(B)$, then $x \succ^d y$, and if, in addition, $y = A(B)$, then $x \succ_+ y$. Let (R, τ) be any warm glow pair that underlies (C, A). In what follows, we use transitivity of R and properties (3) of warm glow pair (R, τ). Since (R, τ) underlies (C, A) either $x = R(B)$, $\tau(x)\ R\ A(B)$ or $x = A(B)\ R_=\ \tau(R(B))$ must hold. If $x = A(B)\ R_=\ \tau(R(B))$, then $R(B)\ R_=\ y$ implies $\tau(R(B))\ R_=\ \tau(y)$, and $x\ R_=\ \tau(y)$ must hold. If $x \neq A(B)$, then $x = R(B)\ R_=\ y\ R_=\ \tau(y)$. In either case, $x\ R_=\ \tau(y)$ holds, and since (R, τ) was chosen arbitrarily, $x \vdash^{\mathrm{rev}} y$. Now, if $x \neq y$ and $x \neq A(B)$ then $x = R(B)\ R\ y$ and $x \succ^{\mathrm{rev}} y$. If, in addition, $y = A(B)$, then $\tau(x)\ R\ y$ and $x \succ_+^{\mathrm{rev}} y$. □

8.4 Proof of Lemma 2

Let (R, τ) be any warm glow pair underlying (C, A).

Step 1. We, first, prove that $a \succ^i b$ implies $a \succ^{\mathrm{rev}} b$. For a given integer k, we define binary relations $\succ^k$ as follows:

- if $k = 0$, let $x \succ^0 y$ iff $x\ R\ y$;
- if $k > 0$, let $x \succ^k y$ iff there exists a sequence of alternatives $x = x_0, x_1, \ldots, x_{k-1}, x_k = y$ such that for all $i = 1, k : \tau(x_{i-1})\ R\ x_i$;
- if $k < 0$, let $x \succ^k y$ iff there exists a sequence of alternatives $x = x_0, x_1, \ldots, x_{-k-1}, x_{-k} = y$ such that for all $j = 1, -k : x_{j-1}\ R_=\ \tau\left(x_j\right)$.

We show now that $x \succ^m y \succ^n z$ implies $x \succ^{m+n} z$.

(1) $x \succ^0 y \succ^k z$ implies $x \succ^k z$. If $k = 0$, then, by transitivity of R, $x\ R\ y\ R\ z$ implies $x\ R\ z$. If $k > 0$, then, by (3), $x\ R\ y$ implies $\tau(x)\ R_=\ \tau(y)$, and

$\tau(x)\ R_= \tau(y)\ R\ y_1$ implies $\tau(x)\ R\ y_1$. If $k<0$, then $x\ R\ y\ R_= \tau(y_1)$ implies $x\ R_= \tau(y_1)$.

(2) $x \succ^k y \succ^0 z$ implies $x \succ^k z$. If $k>0$, then $\tau(x_{k-1})\ R\ y\ R\ z$ implies $\tau(x_{k-1})\ R\ z$. If $k<0$, then, by (3), $y\ R\ z$ implies $\tau(y)\ R_= \tau(z)$, and $x_{-k-1}\ R_= \tau(y)\ R_= \tau(z)$ implies $x_{-k-1}\ R_= \tau(z)$.

(3) $x \succ^m y \succ^n z$, $m\cdot n>0$, implies $x \succ^{m+n} z$. The types of both sequences are the same, and the combined sequence implies $x \succ^{m+n} z$.

(4) $x \succ^1 y \succ^{-1} z$ implies $x \succ^0 z$. $\tau(x)\ R\ y\ R_= \tau(z)$ implies $\tau(x)\ R\ \tau(z)$, and, therefore, by (3), $x\ R\ z$.

(5) $x \succ^{-1} y \succ^1 z$ implies $x \succ^0 z$. $x\ R_= \tau(y)\ R\ z$ implies $x\ R\ z$.

(6) $x \succ^m y \succ^n z$, $m\cdot n<0$, $|m|>1$ or $|n|>1$, implies $x \succ^{m+n} z$. Note, that relation $\succ^k$, $k\neq 0$, is equivalent to a sequence of $|k|$ relations $\succ^{sign(k)}$. Therefore, using 4), 5), and 1), 2), we eliminate terms of both sequences until we get a sequence with all binary relations of the same type. For example, $x \succ^{-2} y \succ^4 z$ implies $x \succ^{-1} x_1 \succ^{-1} y \succ^1 y_1 \succ^1 y_2 \succ^2 z$, implies, by 5), $x \succ^{-1} x_1 \succ^0 y_1 \succ^1 y_2 \succ^2 z$, implies, by 1), $x \succ^{-1} x_1 \succ^1 y_2 \succ^2 z$, implies $x \succ^0 y_2 \succ^2 z$, and, hence, by 1), $x \succ^2 z$.

Let $(\mathbf{x},\rho)$ be a chain such that $\mathbf{x}_0=a$, $\mathbf{x}_n=b$ and $\chi((\mathbf{x},\rho))\geq 0$. By Lemma 1, if $x \succ_+ y$, then $\tau(x)\ R\ y$, and, therefore, $x \succ^1 y$. Similarly, if $x \succ y$ then $x \succ^0 y$, and if $x \vdash y$ then $x \succ^{-1} y$. Hence, for a pair of successive alternatives $\mathbf{x}_{i-1}\ \rho_i\ \mathbf{x}_i$ of the chain $\mathbf{x}$, $\mathbf{x}_{i-1} \succ^{\chi((\mathbf{x}_{i-1},\mathbf{x}_i),(\rho_i))} \mathbf{x}_i$. Moreover, $\chi(\mathbf{x},\rho)=\sum_{i=1}^n \chi((\mathbf{x}_{i-1},\mathbf{x}_i),(\rho_i))$. Therefore, $\mathbf{x}_0 \succ^{\chi((\mathbf{x}_0,\mathbf{x}_1),(\rho_1))} \mathbf{x}_1 \ldots \succ^{\chi((\mathbf{x}_{n-1},\mathbf{x}_n),(\rho_n))} \mathbf{x}_n$ implies $\mathbf{x}_0 \succ^{\chi((\mathbf{x},\rho))} \mathbf{x}_n$. If $\chi((\mathbf{x},\rho))\geq 0$ then either $\chi((\mathbf{x},\rho))=0$, and, by definition of $\succ^0$, $\mathbf{x}_0\ R\ \mathbf{x}_n$, or $\chi((\mathbf{x},\rho))>0$, and, by definition of $\succ^k$ and (3), $\mathbf{x}_0\ R_= \tau(\mathbf{x}_0)\ R\,\mathbf{x}_1\ R_= \tau(\mathbf{x}_1)\ R \ldots \mathbf{x}_{n-1}\ R_= \tau(\mathbf{x}_{n-1})\ R\,\mathbf{x}_n$ implying $\mathbf{x}_0\ R\ \mathbf{x}_n$. In either case, $a\ R\ b$ holds.

Step 2. We now show that $a \succ_+^i b$ implies $a \succ_+^{\text{rev}} b$. There exists a chain $(\mathbf{x},\rho)$ such that $\mathbf{x}_0=a$, $\mathbf{x}_n=b$, $\chi(\mathbf{x})>0$. Let $\mathbf{x}^{(k)}$, $k=\overline{1,n}$, be the chain $(\mathbf{x}^{(k)},\rho^{(k)})=((\mathbf{x}_0,\ldots,\mathbf{x}_k),(\rho_1,\ldots,\rho_k))$, and, for convenience, let $(\mathbf{x}^{(0)},\rho^{(0)})=(\mathbf{x}_0,\emptyset)$, $\chi(\mathbf{x}^{(0)},\rho^{(0)})=0$. Then, for every $k=\overline{1,n}$, $\left|\chi(\mathbf{x}^{(k)},\rho^{(k)})-\chi(\mathbf{x}^{(k-1)},\rho^{(k-1)})\right|\leq 1$. Since $\chi(\mathbf{x}^{(0)},\rho^{(0)})=0$, $\chi(\mathbf{x}^{(n)},\rho^{(n)})>0$, there exists $K\in\overline{1,n}$ such that $\chi(\mathbf{x}^{(K)},\rho^{(K)})=1$, $\chi(\mathbf{x}^{(K-1)},\rho^{(K-1)})=0$. Let $u=\mathbf{x}_{K-1}$ and $v=\mathbf{x}_K$. It follows that $u \succ_+ v$, and, by Lemma 1, $\tau(u)\ R\ v$. If $K=1$, then $u=a$. If $K>1$, then $(\mathbf{x}^{(K-1)},\rho^{(K-1)})$ is a chain with zero characteristic connecting a and u, that is, $a \succ^i u$, and, by Step 1, $a\ R\ u$. In either case $a\ R_= u$ holds. Similarly, if $K=n$, then $v=b$, and if $K<n$, then $((\mathbf{x}_K,\ldots,\mathbf{x}_n),(\rho^{(K+1)},\ldots,\rho^{(n)}))$ is a chain with nonnegative characteristic, that is, $v \succ^i b$, and, by Step 1, $v\ R\ b$. In either case $v\ R_= b$. Now, by (3) and transitivity of R, $a\ R_= u$, $\tau(u)\ R\ v$, and $v\ R_= b$ imply $\tau(a)\ R\ b$. Since (R,τ) was chosen arbitrary, $a \succ_+^{\text{rev}} b$.

In a similar way, we can prove that $a \vdash^i b$ implies $a \vdash^{\text{rev}} b$. Indeed, if $a=b$, then, by (3), $a\ R_= \tau(a)$ holds for every $(R,\tau)\in\mathcal{P}_{C,A}$; otherwise, there exists a chain $(\mathbf{x},\rho)$ such that $\mathbf{x}_0=a$, $\mathbf{x}_n=b$, and $\chi(\mathbf{x},\rho)\geq -1$. If $\chi(\mathbf{x},\rho)\geq 0$, then $a \succ^i b$, and, by Step 1, $a\ R\ b$. Therefore, by (3), $a\ R\ \tau(b)$. If $\chi(\mathbf{x},\rho)=-1$, then we define $(\mathbf{x}^{(k)},\rho^{(k)})$, $k=\overline{0,n}$, as before, and let $K\in\overline{1,n}$ be such that $\chi(\mathbf{x}^{(K)},\rho^{(K)})=-1$, $\chi(\mathbf{x}^{(K-1)},\rho^{(K-1)})=0$. For u and v defined as before, $a\ R_= u$, $u\vdash v$, and $v\ R_= b$, by Lemma 1 and (3), imply $a\ R_= \tau(b)$. □

8.5 Proof of Theorem 2

Definition 12 Let (C, A) be a choice and aspiration function, and $\succ^*$ be a binary relation on X such that $\succ^*$ extends $\succ^d$. An extended chain ($\succ^*$-chain) (x, ρ) is an ordered sequence of alternatives $x = (\mathbf{x}_0, \ldots, \mathbf{x}_n)$ and binary relations $\rho = (\rho_1, \ldots, \rho_n)$ with $n \geq 1$ such that $\rho_i \in \{\succ_+, \succ^*, \vdash\}$ and $x_{i-1}\ \rho_i\ x_i$ holds for every $i = 1 \ldots n$. A characteristic of the extended chain $\chi\ (\mathbf{x}, \rho)$ is the number of times $\rho_i = \succ_+,\ i = 1 \ldots n$, minus the number of times $\rho_j = \vdash,\ j = 1 \ldots n$.[7]

Definition 13 Let (C, A) be a choice and aspiration function, and $\succ^*$ be a binary relation on X such that $\succ^*$ extends $\succ^d$. We define binary relation $R^{\succ^*}$ as follows: $x\ R^{\succ^*}\ y$ if and only if there exists an extended chain $(\mathbf{x}, \rho)$ such that $x_0 = x,\ x_n = y$, and $\chi\ (\mathbf{x}, \rho) \geq 0$.

Lemma 3 *Let (C, A) be a choice and aspiration function, and $\succ^*$ be a binary relation on X such that $\succ^*$ extends $\succ^d$. If $R^{\succ^*}$ is irreflexive[8], then (C, A) is a warm glow function, and there exists a warm glow pair (R, τ) underlying (C, A) such that for every pair of alternatives x and y, $x\ R^{\succ^*}\ y$ implies $x\ R\ y$.*

The proof is as follows. We first construct an order R that extends the binary relation $R^{\succ^*}$. We then show that R can be paired with some tolerance function τ such that (R, τ) is a warm glow pair underlying (C, A).

We construct R by induction. Let $k = 0$ and let $\succ_0 = \succ^*$. By assumption, $R^{\succ_0}$ is irreflexive. If $R^{\succ_0}$ is complete, then let $R = R^{\succ_0}$. Otherwise, we construct a series of binary relations $\{\succ_k\}$ such that for every $k > 0$, $\succ_k$ extends $\succ_{k-1}$ and $R^{\succ_k}$ is irreflexive.

So, assume that for a given k, $R^{\succ_k}$ is irreflexive and not complete. There exist two alternatives a and b such that $a \neq b$, and neither $a\ R^{\succ_k}\ b$ nor $b\ R^{\succ_k}\ a$ holds. Let $\succ_{k+1}$ be a binary relation such that $x \succ_{k+1} y$ if and only if $x \succ_k y$ or $x = b,\ y = a$. Note that $\succ_{k+1}$ extends $\succ_k$, and, by definition, $R^{\succ_{k+1}}$ extends $R^{\succ_k}$ (to see this, note that every $\succ_k$-chain is a $\succ_{k+1}$-chain).

Suppose now that $R^{\succ_{k+1}}$ is not irreflexive: There exists an alternative x such that $x\ R^{\succ_{k+1}}\ x$. Therefore, there exists $\succ_{k+1}$-chain $(\mathbf{x}, \rho)$ such that $\mathbf{x}_0 = \mathbf{x}_n = x$ and $\chi\ (\mathbf{x}, \rho) \geq 0$. If $(\mathbf{x}, \rho)$ does not contain new relation $b \succ_{k+1} a$ then $(\mathbf{x}, \rho)$ is $\succ_k$-chain and $x\ R^{\succ_k}\ x$. This contradicts the assumption that $R^{\succ_k}$ is irreflexive. Let $M = \{i \in 1 \ldots n \text{ s.t. } \mathbf{x}_{i-1} = b,\ \mathbf{x}_i = a,\ \rho_i = \succ_{k+1}\}$. M is not empty, and suppose M consists of the following indexes: $M = \{i_1, \ldots, i_m\}$, $i_1 < i_2 < \cdots < i_m$. Now,

[7] We may use $\chi\ (\mathbf{x})$ instead of $\chi\ (\mathbf{x}, \rho)$ whenever it is clear what relations between $\mathbf{x}$'s are assumed.

[8] Note, that, by definition, every chain is an extended chain. Therefore, $R^{\succ^*}$ extends $\succ^i$. Moreover, if $\succ^* = \succ$ then $R^{\succ^*} = R^{\succ} = \succ^i$.

$$\begin{aligned}\chi(\mathbf{x}) &= \chi\left(\mathbf{x}_0 = x, \ldots, \mathbf{x}_{i_1-1} = b\right) + \chi\left(\mathbf{x}_{i_1-1}, \mathbf{x}_{i_1}\right) + \chi\left(\mathbf{x}_{i_1} = a, \ldots, \mathbf{x}_{i_2-1} = b\right)\\ &\quad + \chi\left(\mathbf{x}_{i_2-1}, \mathbf{x}_{i_2}\right) + \chi\left(\mathbf{x}_{i_2} = a, \ldots, \mathbf{x}_{i_3-1} = b\right) + \cdots + \chi\left(\mathbf{x}_{i_m} = a, \ldots, \mathbf{x}_n = x\right)\\ &= \chi\left(\mathbf{x}_{i_m} = a, \ldots, \mathbf{x}_n = \mathbf{x}_0, \ldots, \mathbf{x}_{i_1-1} = b\right) + \chi\left(\mathbf{x}_{i_1-1}, \mathbf{x}_{i_1}\right) + \chi\left(\mathbf{x}_{i_1} = a, \ldots, \mathbf{x}_{i_2-1} = b\right)\\ &\quad + \chi\left(\mathbf{x}_{i_2-1}, \mathbf{x}_{i_2}\right) + \chi\left(\mathbf{x}_{i_2} = a, \ldots, \mathbf{x}_{i_3-1} = b\right) + \cdots + \chi\left(\mathbf{x}_{i_m-1}, \mathbf{x}_{i_m}\right)\\ &= \chi\left(\mathbf{x}_{i_m} = a, \ldots, \mathbf{x}_n = \mathbf{x}_0, \ldots, \mathbf{x}_{i_1-1} = b\right) + 0 + \chi\left(\mathbf{x}_{i_1} = a, \ldots, \mathbf{x}_{i_2-1} = b\right)\\ &\quad + 0 + \chi\left(\mathbf{x}_{i_2} = a, \ldots, \mathbf{x}_{i_3-1} = b\right) + \cdots + 0 \geq 0\end{aligned}$$

There exists at least one nonnegative term in the sum; hence, there exists a $\succ_k$-chain with nonnegative characteristic such that $\mathbf{x}_{i_{t-1}} = a$ and $\mathbf{x}_{i_t} = b$ ($i_0 = i_m$ if $t = 1$). This implies that $a\ R^{\succ_k}\ b$, and this contradicts the assumption that $a\ R^{\succ_k}\ b$ does not hold.

We increase k by one and continue extending relations $\{\succ_k\}$ until $R^{\succ_k}$ is complete. Since X is finite and $R^{\succ_{k+1}}$ strictly extends $R^{\succ_k}$, this process is finite. When $R^{\succ_k}$ is complete we define binary relation R to be equal to $R^{\succ_k}$. Note that R is irreflexive, and $\succ_k$ extends $\succ^*$, implying that R extends $R^{\succ^*}$.

(a) We show that R is an order. Constructed R is transitive. Indeed, if $x\ R\ y$ and $y\ R\ z$ then $x\ R^{\succ_k}\ y$ and $y\ R^{\succ_k}\ z$, and there exist two extended $\succ_k$-chains with nonnegative characteristics connecting x to y and y to z. The combined chain has nonnegative characteristic and connects x to z. Therefore, $x\ R^{\succ_k}\ z$ implying $x\ R\ z$. Now, R is irreflexive, complete and transitive. Therefore, R is a (complete) order.

(b) We define τ as follows. For every alternative x, let

$$L(x) = \{y \text{ s.t. there exist alternatives } u \text{ and } v \text{ s.t. } x\ R_=\ u,\ v\ R_=\ y, \text{ and } u \succ_+ v\}.$$

$L(x)$ is a set of all alternatives that must be intolerable by x given R and directly observable intolerances. By definition, if $y \in L(x)$ and $y\ R_=\ z$ for some alternative z, then $z \in L(x)$. Therefore, if $z \notin L(x)$ and $y\ R_=\ z$ then $y \notin L(x)$. Also, note that $x \notin L(x)$, and hence $X \backslash L(x)$ is not empty. Therefore, we define $\tau(x)$ as R-minimal element of $X \backslash L(x)$. If $z \notin L(x)$ then $z\ R_=\ \tau(x)$, and if $y \in L(x)$ then $\tau(x)\ R\ y$ (otherwise $y\ R_=\ \tau(x)$ and $\tau(x) \notin L(x)$ hold implying $y \notin L(x)$).

(c) We show that (3) holds for R and τ, and, therefore, it is a warm glow pair. For every alternative x, $x \notin L(x)$, and, therefore, $x\ R_=\ \tau(x)$. Suppose $x\ R\ y$. If $\tau(x) \in L(y)$, then there exist two alternatives u and v such that $x\ R\ y\ R_=\ u \succ_+ v\ R_=\ \tau(x)$. By transitivity of R, this implies $\tau(x) \in L(x)$ contradicting the definition of $\tau(x)$. Therefore, $\tau(x) \notin L(y)$ and $\tau(x)\ R_=\ \tau(y)$.

(d) (R, τ) underlies (C, A). Let B be an issue, $B \in \mathcal{B}$, and $C(B) = x$. We want to show that x must be selected by (R, τ). If $x \neq A(B) = y$, then $x \succ_+ y$, and for every $z \in B \backslash \{x\} : x \succ^d z$. $x \succ^d z$ is a chain with zero characteristic, therefore, $x\ R\ z$, and, since this holds for every $z \in B \backslash \{x\}$, $x = R(B)$. Now, by definition, $y = A(B) \in L(x)$ ($u = x,\ v = y$). Hence, $\tau(x)\ R\ y$ and (R, τ) selects x. Suppose now that $x = A(B)$. If $R(B) = x$, then x is selected by (R, τ). If $w = R(B) \neq x = C(B)$, then, by definition, $x \vdash w$. Suppose $x \in L(w)$. There exist two alternatives u and v such that $w\ R_=\ u$, $v\ R_=\ x$ and $u \succ_+ v$. If

$w\ R\ u$ then let $(\mathbf{x},\rho)$ be a $\succ_k$-chain such that $\mathbf{x}=(w,\dots,u)$ and $\chi(\mathbf{x},\rho)\geq 0$. If $w=u$ then let $(\mathbf{x},\rho)$ be $(w,\emptyset)$ and let $\chi(\mathbf{x},\rho)=0$.[9] If $v\ R\ x$ then let $(\mathbf{y},\rho')$ be a $\succ_k$-chain such that $\mathbf{y}=(v,\dots,x)$ and $\chi(\mathbf{y},\rho')\geq 0$. If $v=x$ then let $(\mathbf{y},\rho')=(x,\emptyset)$ and let $\chi(\mathbf{y},\rho')=0$. The characteristic of the combined $\succ_k$ chain $\chi(\mathbf{x},(u\succ_+ v),\mathbf{y},(x\vdash w))=\chi(\mathbf{x})+\chi(u\succ_+ v)+\chi(\zeta)+\chi(x\vdash w)\geq \chi(u\succ_+ v)+\chi(x\vdash w)=1-1=0$. Thus, $w\ R^{\succ_k}\ w$. This contradicts the irreflexivity of $R^{\succ_k}$. Therefore, $x\notin L(w)$, $x\ R_=\ \tau(w)$, and (R,τ) selects x.

We have shown that if $R^{\succ^*}$ is irreflexive, then there exists a warm glow pair (R,τ) underlying (C,A) such that R extends $R^{\succ^*}$. Therefore, (C,A) is a warm glow function and for every pair of alternatives x and y, $x\ R^{\succ^*}\ y$ implies $x\ R\ y$. □

Lemma 4 *Let (C,A) be a choice and aspiration function and suppose $\succ^i$ is irreflexive. For two alternatives x_0 and y_0, let Z be a set of alternatives such that $x_0,y_0\notin Z$, and for every $y\in Z$, $x_0\succ^i y$ does not hold (Z can be empty), and let W be a set of alternatives such that $x_0,y_0\notin W$, and for every $x\in W$, $x\succ^i y_0$ does not hold (W can be empty).*

(a) If $x_0\succ^i y_0$ does not hold, and for every $x\in W$, $y\in Z$, $x\succ^i_+ y$ does not hold, then there exists a warm glow pair (R,τ) underlying (C,A) such that $y_0\ R\ x_0$, $y_0\ R\ x$ for every $x\in W$, and $y\ R\ x_0$ for every $y\in Z$.[10]

(b) If $x_0\succ^i y_0$ holds, but $x_0\succ^i_+ y_0$ does not hold, and for every $x\in W$, $y\in Z$, $x\succ^i y$ does not hold, then there exists a warm glow pair (R,τ) underlying (C,A) such that $y\ R\ x_0\ R\ y_0\ R\ x$ for every $x\in W$, $y\in Z$.

We define binary relation $\succ^*$ as follows: $x\succ^* y$ if and only if $x\succ^d y$ or $x\in Z$, $y=x_0$ or $x=y_0$, $y\in W$ or, for a) only, $x=y_0$ and $y=x_0$, and show that $R^{\succ^*}$ is irreflexive.

Suppose it is not. Then, there exists $\succ^*$-chain $(\mathbf{x},\rho)$ such that $\mathbf{x}_0=\mathbf{x}_s=z$ and $\chi(\mathbf{x},\rho)\geq 0$. Let $\succ^{new}$ be a binary relation such that $x\succ^{new} y$ if and only if $x\succ^* y$ holds, but $x\succ^d y$ does not hold. Suppose that for every pair of successive alternatives $\mathbf{x}_{k-1}$ and $\mathbf{x}_k$, $\mathbf{x}_{k-1}\succ^{new}\mathbf{x}_k$ does not hold. In this case, if $\mathbf{x}_{k-1}\succ^*\mathbf{x}_k$, then $\mathbf{x}_{k-1}\succ^d\mathbf{x}_k$, and if not $\mathbf{x}_{k-1}\succ^*\mathbf{x}_k$, then $\mathbf{x}_{k-1}\ \rho_k\ \mathbf{x}_k$, where $\rho_k\in\{\succ^d,\succ_+,\vdash\}$. Thus, $(\mathbf{x},\rho)$ is a chain with nonnegative characteristic, and $z\succ^i z$, which contradicts the irreflexivity of $\succ^i$. Therefore, set of indexes $K=\{k=1,s \text{ s.t. } \mathbf{x}_{k-1}\succ^{new}\mathbf{x}_k\}$ is not empty. Let $K=\{k_1,\dots,k_t\}$, $k_1<\dots<k_t$, $\mathbf{x}'=(\mathbf{x}_{k_t},\dots z,\dots,\mathbf{x}_{k_1-1},\mathbf{x}_{k_1},\dots,\mathbf{x}_{k_2-1},\mathbf{x}_{k_2},\dots,\mathbf{x}_{k_t-1},\mathbf{x}_{k_t})$, $\rho'=(\rho_{k_t+1},\dots\rho_s,\rho_1,\dots,\rho_{k_t})$ $(k_0=k_t)$. $(\mathbf{x}',\rho')$ is $\succ^*$-chain, and $\chi\left(\mathbf{x}',\rho'\right)=\chi(\mathbf{x},\rho)\geq 0$. Now, for every $l\in 1,t$, $\chi\left((\mathbf{x}_{k_l-1},\mathbf{x}_{k_l}),(\rho_{k_l})\right)=0$, and $\mathbf{x}_{k_{l-1}}\ \rho_{k_{l-1}+1}\ \dots\ \rho_{k_l-1}\ \mathbf{x}_{k_l-1}$ is a chain connecting x_0 or some element in W and y_0 or some element in Z.

$$\begin{aligned}\chi\left(\mathbf{x}'\right)&=\chi(\mathbf{x}_{k_t},\dots,\mathbf{x}_{k_1-1})+\chi(\mathbf{x}_{k_1-1},\mathbf{x}_{k_1})+\dots+\chi(\mathbf{x}_{k_{t-1}},\dots,\mathbf{x}_{k_t-1})+\chi(\mathbf{x}_{k_t-1},\mathbf{x}_{k_t})\\&=\chi(\mathbf{x}_{k_t},\dots,\mathbf{x}_{k_1-1})+0+\dots+\chi(\mathbf{x}_{k_{t-1}},\dots,\mathbf{x}_{k_t-1})+0\geq 0,\end{aligned}$$

[9] Strictly speaking, $(\mathbf{x},\rho)=(w,\emptyset)$ is not an extended chain, but we define it this way for convenience.

[10] In a special case, when W and Z are empty, if $\succ^i$ is irreflexive, and $x\succ^i y$ does not hold for some pair for alternatives $x\neq y$, then (C,A) is a warm glow function and there exists $(R,\tau)\in\mathcal{P}_{C,A}$ such that $y\ R\ x$.

(a) Let $L = \{l \in 1, t \text{ s.t. } \mathbf{x}_{k_{l-1}} \in W \text{ and } \mathbf{x}_{k_l - 1} \in Z\}$. If $l \in L$, then $\chi\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right) \leq 0$, and $\mathbf{x}_{k_{l-1}-1} = y_0$, $\mathbf{x}_{k_l} = x_0$. Therefore, there exists at least one $l \notin L$, and the sum of characteristics of chains $\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right)$ for all $l \notin L$ is nonnegative. Therefore, there exists chain $\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right)$ such that $l \notin L$, and $\chi\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right) \geq 0$. If $l \notin L$, then $\mathbf{x}_{k_{l-1}} = x_0$, or $\mathbf{x}_{k_l - 1} = y_0$. Therefore, $x_0 \succ^i y_0$, or $x_0 \succ^i y$ for some $y \in Z$, or $x \succ^i y_0$ for some $x \in W$. This contradicts the assumption. Therefore, $R^{\succ^*}$ is irreflexive.

(b) Let $L = \{l \in 1, t \text{ s.t. } \mathbf{x}_{k_{l-1}} = x_0 \text{ and } \mathbf{x}_{k_l - 1} = y_0\}$. If $l \in L$, then $\chi\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right) \leq 0$, and $\mathbf{x}_{k_{l-1}-1} \in Z$, $\mathbf{x}_{k_l} \in W$. Therefore, there exists at least one $l \notin L$, and the sum of characteristics of chains $\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right)$ for all $l \notin L$ is nonnegative. Therefore, there exists chain $\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right)$ such that $l \notin L$, and $\chi\left(\mathbf{x}_{k_{l-1}}, \ldots, \mathbf{x}_{k_l - 1}\right) \geq 0$. If $l \notin L$, then $\mathbf{x}_{k_{l-1}} \in W$, or $\mathbf{x}_{k_l - 1} \in Z$. Therefore, for some $x \in W$ and for some $y \in Z$, $x \succ^i y_0$, or $x_0 \succ^i y$, or $x \succ^i y$. This contradicts the assumption. Therefore, $R^{\succ^*}$ is irreflexive.

If $a \succ^* b$ or $a \succ^i b$ then $a\ R^{\succ^*} b$. By lemma 3, (C, A) is a warm glow function and there exists $(R, \tau) \in \mathcal{P}_{C,A}$ such that if $a\ R^{\succ^*} b$ then $a\ R\ b$. □

Proof of Theorem 2 If for a given choice and aspiration function (C, A), $\succ^i$ is irreflexive then, by Lemma 3, (C, A) is a warm glow function.

Now suppose (C, A) is a warm glow choice and aspiration function. We need to show:

(a) $\succ^i$ is irreflexive, and $x \succ^{\text{rev}} y \Leftrightarrow x \succ^i y$;
(b) $x \succ^{\text{rev}}_{+} y \Leftrightarrow x \succ^i_{+} y$, $x \vdash^{\text{rev}} y \Leftrightarrow x \vdash^i y$.

Proof of (a). By Lemma 2, $x \succ^i y$ implies $x \succ^{\text{rev}} y$, and, therefore, $\succ^i$ must be irreflexive. Now, if $x \succ^{\text{rev}} y$, then $x\ R\ y$ holds for every warm glow pair (R, τ) underlying (C, A), and, by Lemma 4, this is possible only if $x \succ^i y$.

Proof of (b). By Lemma 2, $x \succ^i_{+} y$ implies $x \succ^{\text{rev}}_{+} y$, and $x \vdash^i y$ implies $x \vdash^{\text{rev}} y$. What left to be proved is that $x \succ^{\text{rev}}_{+} y$ implies $x \succ^i_{+} y$, and $x \vdash^{\text{rev}} y$ implies $x \vdash^i y$. We start with the tolerance relation $\vdash$ first. Assume that for every warm glow pair (R, τ) underlying (C, A), $x\ R_{=}\ \tau(y)$ holds. We need to show that if $x \neq y$, then there exists a chain $(\mathbf{x}, \rho) = ((x, \ldots, y), \rho)$ such that $\chi(\mathbf{x}, \rho) \geq -1$. If $x \succ^i y$, then there exists a chain $(\mathbf{x}, \rho) = ((x, \ldots, y), \rho)$ such that $\chi(\mathbf{x}, \rho) \geq 0$. Suppose now that $x \succ^i y$ does not hold (and $x \neq y$). Let $Z = \{z \in X \setminus \{x, y\} \text{ s.t. } x \succ^i z \text{ does not hold}\}$. Let $W = \{z \in X \setminus \{x, y\}$ s.t. $z \succ^i y$ does not hold, and for every $u \in Z$, $z \succ^i_{+} u$ does not hold$\}$. By Lemma 4, there exists $(R, \tau) \in \mathcal{P}_{C,A}$ such that $z\ R\ x$ for all $z \in Z$, and $y\ R\ z$ for all $z \in W$. Also, by assumption, $x\ R_{=}\ \tau(y)$.

Consider another pair (R, τ') such that $\tau'(y)$ is R-lowest alternative such that $\tau'(y)\ R\ x$, and if $z\ R\ y$ then $\tau'(z) = R - \max\left\{\tau(z), \tau'(y)\right\}$, if $y\ R\ z$, then $\tau'(z) = \tau(z)$. Note, that $\tau'(z) \in \left\{\tau(z), \tau'(y)\right\}$, and $\tau'(z)\ R_{=}\ \tau(z)$ for every $z \in X$. We show now that (3) holds for (R, τ'). Since $y\ R\ x$, by definition of $\tau'(y)$, $y\ R_{=}\ \tau'(y)$. If $z\ R\ y$, then $z\ R\ y\ R_{=}\ \tau'(y)$, and, by (3), $z\ R_{=}\ \tau(z)$, but $\tau'(z) \in \left\{\tau(z), \tau'(y)\right\}$, hence, $z\ R_{=} \tau'(z)$. If $y\ R\ z$, then $\tau'(z) = \tau(z)$, and, by (3), $z\ R_{=}\ \tau'(z)$. Now suppose $u\ R\ v$. By (3), $\tau(u)\ R_{=} \tau(v)$. If $u\ R_{=} y$, then $\tau'(u)\ R_{=} \tau(u)\ R_{=} \tau(v)$ and $\tau'(u)\ R_{=} \tau'(y)$, but $\tau'(v) \in \left\{\tau(v), \tau'(y)\right\}$, hence, $\tau'(u)\ R_{=}\ \tau'(v)$. If $y\ R\ u$, then

$y\ R\ v$, and $\tau'(u) = \tau(u)\ R_{=}\tau(v) = \tau'(v)$. Therefore, (R, τ') is a warm glow pair.

Now, since $\tau'(y)\ R\ x$, by assumption, (R, τ') does not underlie (C, A). There exists issue B such that (R, τ') does not select $C(B)$, while (R, τ) does. Since $\tau'(R(B))\ R_{=}\tau(R(B))$, it must be the case that $\tau'(R(B))\ R\,A(B)\ R_{=}\tau(R(B))$, and $C(B) = A(B)$. Now, if $y\ R\ R(B)$, then $\tau'(R(B)) = \tau(R(B))$. Hence, $R(B)\ R_{=}\ y$. Also, $\tau'(R(B)) \in \{\tau(R(B)), \tau'(y)\}$, $\tau'(R(B))\ R\,\tau(R(B))$ imply $\tau'(R(B)) = \tau'(y)\ R\,\tau(R(B))$, and, therefore, $x\ R_{=}\tau(R(B))$. To sum up, we have the following relations: $R(B)\ R_{=}\ y\ R_{=}\ \tau'(R(B)) = \tau'(y)\ R\,x\ R_{=}A(B)\ R_{=}\tau(R(B))\ R_{=}\tau(y)$.

Since $x\ R_{=}A(B)$, $A(B) \notin Z$, and $A(B) \neq y$. Therefore, $x \succ^{i}_{=} A(B)$. Since $R(B) \in B$, and $A(B) = C(B)$, by definition, $A(B) \vdash R(B)$. Now, $R(B)\ R_{=}\ y$ implies $R(B) \notin W$. Suppose for some $u \in Z$, $R(B) \succ^{i}_{+} u$. Then, by Lemma 2, $\tau(R(B))\ R\,u$. But since $u \in Z$, $u\ R\ x$ must hold, therefore, $\tau(R(B))\ R\,x$. Contradiction. Therefore, $R(B) \notin W$, but for every $u \in Z$, $R(B) \succ^{i}_{+} u$ does not hold. It follows that either $R(B) \in \{x, y\}$ or $R(B) \succ^{i} y$. Since $R(B)\ R\,x$, $R(B) \neq x$, and, therefore, $R(B) \succ^{i}_{=} y$. The relations $x \succ^{i}_{=} A(B) \vdash R(B) \succ^{i}_{=} y$ imply that there exists a chain $(\mathbf{x}, \rho) = ((x, \ldots, y), \rho)$ such that $\chi(\mathbf{x}, \rho) \geq -1$.

Now intolerance relation $\succ_{+}$. We show that if for every warm glow pair (R, τ) underlying (C, A), $\tau(x)\ R\ y$, then there exists a chain $(\mathbf{x}, \rho) = ((x, \ldots, y), \rho)$ such that $\chi(\mathbf{x}, \rho) \geq 1$. By (3), for every $(R, \tau) \in \mathcal{P}_{C,A}$, $x\ R_{=}\tau(x)$, and, hence, $x\ R\,y$. Therefore, $x \succ^{\text{rev}} y$ and $x \succ^{i} y$. Suppose $x \succ^{i}_{+} y$ does not hold. Let $Z = \{z \in X \setminus \{x, y\}$ s.t. $x \succ^{i} z$ does not hold$\}$. Let $W = \{z \in X \setminus \{x, y\}$ s.t. $z \succ^{i} y$ does not hold, and for every $u \in Z$, $z \succ^{i} u$ does not hold$\}$. By Lemma 4, there exists $(R, \tau) \in \mathcal{P}_{C,A}$ such that for all $z_x \in W$, $z_y \in Z$, $z_y\ R\ x\ R\ y\ R\ z_x$. Also, by assumption, $\tau(x)\ R\ y$.

Consider another pair (R, τ') such that if $x\ R_{=}\ z$, then $\tau'(z) = R{-}\min(\{\tau(z), y\})$, if $z\ R\,x$, then $\tau'(z) = \tau(z)$. Note, that $\tau'(x) = y$, $\tau'(z) \in \{\tau(z), y\}$, $\tau(z)\ R_{=}\tau'(z)$ for every $z \in X$, and if $y\ R_{=}\ z$, then, by (3), $y\ R_{=}\tau(y)\ R_{=}\tau(z)$, and $\tau'(z) = \tau(z)$. We show now that (3) holds for (R, τ'). Since $\tau(z)\ R_{=}\tau'(z)$, and, by (3), $z\ R_{=}\ \tau(z)$, $z\ R_{=}\ \tau'(z)$ for every $z \in X$. Now suppose $u\ R\ v$. By (3), $\tau(u)\ R_{=}\tau(v)$. If $u\ R\,x$ or $y\ R_{=}\ u$, then $\tau'(u) = \tau(u)\ R_{=}\tau(v)\ R_{=}\tau'(v)$, and, hence, $\tau'(u)\ R_{=}\tau'(v)$. If $x\ R_{=}\ u\ R\ y$, then $x\ R\ v$, and $\tau'(u) = R{-}\min(\{\tau(u), y\})\ R_{=}\ R{-}\min(\{\tau(v), y\}) = \tau'(v)$. Therefore, (R, τ') is a warm glow pair.

Now, since $\tau'(x) = y$, by assumption, (R, τ') does not underlie (C, A). There exists issue B such that (R, τ') does not select $C(B)$, while (R, τ) does. Since $\tau(R(B))\ R_{=}\tau'(R(B))$, it must be the case that $\tau(R(B))\ R\,A(B)\ R_{=}\tau'(R(B))$, and $C(B) = R(B)$. Moreover, since $\tau(R(B))\ R\,\tau'(R(B))$, $\tau(R(B)) \neq \tau'(R(B))$, and, therefore, $x\ R_{=}R(B)$, $\tau(R(B))\ R\,y$, and $\tau'(R(B)) = y$. Also, by (3), $R(B)\ R_{=}\tau(R(B))$. We have the following relations: $x\ R_{=}R(B)\ R_{=}\tau(R(B))\ R\,A(B)\ R_{=}\tau'(R(B)) = y$.

Since $x\ R_{=}R(B)$, $R(B) \notin Z$. Therefore, $R(B) \in \{x, y\}$ or $x \succ^{i} R(B)$. Since $R(B)\ R\ y$, $x \succ^{i}_{=} R(B)$. Now, $R(B)\ R\,A(B)$ implies $C(B) = R(B) \neq A(B)$, and, hence, by definition, $R(B) \succ_{+} A(B)$. Also, $A(B)\ R_{=}\ y$ implies $A(B) \notin W$. Suppose for some $u \in Z$, $A(B) \succ^{i} u$. Then, by Lemma 2, $A(B)\ R\,u$. But since $u \in Z$, $u\ R\ x$ must hold, and $A(B)\ R\,x$. Contradiction. Therefore, $A(B) \notin W$, but for every $u \in Z$, $A(B) \succ^{i} u$ does not hold. It follows that either $A(B) \in \{x, y\}$ or

$A(B) \succ^i y$. In either case, $A(B) \succ^i_= y$. The relations $x \succ^i_= R(B) \succ_+ A(B) \succ^i_= y$ imply that there exists a chain $(\mathbf{x}, \rho) = ((x, \ldots, y), \rho)$ such that $\chi(\mathbf{x}, \rho) \geq 1$. □

8.6 Proof of Theorem 1

Definition 14 Assume A is ordered. Let R^a be an order such that for any $B \in \mathcal{B}$

$$A(B) = R^a(B).$$

Let $\succ'$ be a binary relation such that $x \succ' y$ if and only if there exists an alternative z such that

$$x\ R^a\ z,\ y\ R^a\ z,\ C(x, z) = x,\ C(y, z) = z, \tag{15}$$

and $\succ^r$ be a binary relation such that

$$x \succ^r y \text{ if and only if } x \succ^d y \text{ or } x \succ' y. \tag{16}$$

Lemma 5 *If A is ordered, and the LWARP and WG axioms hold, then the following statements are true:*

Step 1. If $B \in \mathcal{B}^s$, $B \subseteq \tilde{B}$ and $A\left(\tilde{B}\right) \in B$, then $\tilde{B} \in \mathcal{B}^s$. Also, if $B_1, B_2 \in \mathcal{B}^s$, then $B_1 \cup B_2 \in \mathcal{B}^s$.
Step 2. $\succ^d$ is asymmetric and transitive, $\succ_+$ is transitive.
Step 3. $\succ'$ is asymmetric.
Step 4. If $x \succ' k$, $k \succ' j$ and $j\ R^a\ k$ then $x \succ' j$.
Step 5. $\succ'$ is acyclic.
Step 6. If $x \succ' k$, $k \succ^d j$ and $j\ R^a\ k$ then $x \succ' j$.
Step 7. $\succ^r$ is asymmetric.
Step 8. If $x \succ^d k$, $k \succ' j$ and $x\ R^a\ k$ then $x \succ' j$.
Step 9. $\succ^r$ is acyclic.

Step 1. Since $B \subseteq \tilde{B}$, $A\left(\tilde{B}\right) \in B$ and A is ordered, $A\left(\tilde{B}\right) = A(B)$ and $C(B) \in \tilde{B}$. Therefore, by the WG axiom, $B \in \mathcal{B}^s$ implies $\tilde{B} \in \mathcal{B}^s$. Now, let $B = B_1 \cup B_2$. Then $A(B) \in B_1$ or $A(B) \in B_2$. Thus, $B_1 \cup B_2 \in \mathcal{B}^s$.
Step 2. Assume, by contradiction, that $x \succ^d y$ and $y \succ^d x$. Then, there exist $B_1 \in \mathcal{B}^s$ and $B_2 \in \mathcal{B}^s$ such that $\{x, y\} \subseteq B_1$, $\{x, y\} \subseteq B_2$, $x = C(B_1)$ and $y = C(B_2)$. By Step 1, $B = B_1 \cup B_2 \in \mathcal{B}^s$. If $C(B) \in B_1$, then, by LWARP, $C(B) = x$. So, $C(B) \in \{x, y\} \subseteq B_2$. Hence, by LWARP, $C(B) = C(B_2) = y$. A contradiction. The proof for the case $C(B) \in B_2$ is analogous.
Now assume that $x \succ^d y$ and $y \succ^d z$. There exist $B_1 \in \mathcal{B}^s$ and $B_2 \in \mathcal{B}^s$ such that $\{x, y\} \subseteq B_1$, $\{y, z\} \subseteq B_2$, $x = C(B_1)$ and $y = C(B_2)$. By Step 1, $B = B_1 \cup B_2 \in \mathcal{B}^s$. So, either $C(B) \in B_1$ or $C(B) \in B_2$. If $C(B) \in B_2$, then, by LWARP, $C(B) = C(B_2) = y$. So, $y \succ^d x$ (because $x \in B$). This contradicts $x \succ^d y$ and the proved asymmetry of $\succ^d$. Hence, $C(B) \in B_1$. By LWARP, $C(B) = x$. So, $x \succ^d z$ (because $z \in B$).

Step 3. Assume, by contradiction, that $x \succ' j$ and $j \succ' x$. Then there are alternatives y_1 and y_2 such that, for $i = 1, 2$, $x\ R^a\ y_i$, $j\ R^a\ y_i$, $C(x, y_1) = x$, $C(j, y_1) = y_1$, $C(x, y_2) = y_2$, $C(j, y_2) = j$. In particular, $A(y_1, y_2, j) = j \in \{y_1, j\}$ and $\{y_1, j\} \in \mathcal{B}^s$. By step 1, $\{y_1, y_2, j\} \in \mathcal{B}^s$. If $C(y_1, y_2, j) = y_2$ then, by WG axiom, $\{y_2, j\} \in \mathcal{B}^s$. This contradicts $A(y_2, j) = C(y_2, j) = j$. So, $C(y_1, y_2, j) = y_1$, and $y_1 \succ^d y_2$. Analogously, $A(y_1, y_2, x) = x \in \{y_2, x\}$ and $\{y_2, x\} \in \mathcal{B}^s$ imply $\{y_1, y_2, x\} \in \mathcal{B}^s$, and $C(y_1, y_2, x) = y_1$ implies $\{y_1, x\} \in \mathcal{B}^s$. This contradicts $A(y_1, x) = C(y_1, x) = x$. So, $C(y_1, y_2, x) = y_2$, and $y_2 \succ^d y_1$. Contradiction to Step 2.

Step 4. By definition, there are alternatives y_1 and y_2 such that $k\ R^a\ y_i$, $x\ R^a\ y_1$, $j\ R^a\ y_2$, $C(y_1, x) = x$, $C(y_1, k) = y_1$, $C(y_2, k) = k$, and $C(y_2, j) = y_2$. So, $A(y_1, y_2, k) = k \in \{y_1, k\}$ and $\{y_1, k\} \in \mathcal{B}^s$. By step 1, $\{y_1, y_2, k\} \in \mathcal{B}^s$. If $C(y_1, y_2, k) = y_2$ then, by WG axiom, $\{y_2, k\} \in \mathcal{B}^s$. This contradicts $A(y_2, k) = C(y_2, k) = k$. So, $C(y_1, y_2, k) = y_1$. Thus, $y_1 \succ^d y_2$. Given that $j\ R^a\ k$, it follows that $j\ R^a\ y_i$, and $R^a(y_1, y_2, j) = j \in \{y_2, j\}$ and $\{y_2, j\} \in \mathcal{B}^s$. By step 1, $\{y_1, y_2, j\} \in \mathcal{B}^s$. If $C(y_1, y_2, j) = y_2$ then $y_2 \succ^d y_1$ contradicting $y_1 \succ^d y_2$. So, $C(y_1, y_2, j) = y_1$. Thus, by WG axiom, $(y_1, j) \in \mathcal{B}^s$. In addition, $A(y_1, j) = j$. So, $C(y_1, j) = y_1$. Now, $A(y_1, j) = j$, $A(y_1, x) = x$, $C(y_1, j) = y_1$, $C(y_1, x) = x$ imply $x \succ' j$.

Step 5. Step 3 shows that there are no cycles with 2 alternatives. Assume, by induction, that there are no cycles with $n - 1$ (or less) alternatives. Also assume, by contradiction, that $\{x_1, \ldots, x_n\}$ is a n-cycle, $n \geq 3$. So, $x_i \succ' x_{i+1}$, $i = 1, \ldots, n-1$, and $x_n \succ' x_1$. If $x_2\ R^a\ x_1$ then, by Step 4, $x_n \succ' x_2$. If $x_1\ R^a\ x_n$ then, by Step 4, $x_{n-1} \succ' x_1$. If $x_{i+1}\ R^a\ x_i$ $i = 2, \ldots, n-1$ then, by step 4, $x_{i-1} \succ' x_{i+1}$. Any of these cases produces a cycle with at most $n-1$ alternatives. This violates the induction hypothesis. Hence, $x_n\ R^a\ x_1$, and $x_i\ R^a\ x_{i+1}$, $i = 1, \ldots, n-1$. Therefore, R^a is cyclic. A contradiction.

Step 6. By definition there is an alternative y_1 such that $k\ R^a\ y_1$, $x\ R^a\ y_1$, $C(y_1, x) = x$, $C(y_1, k) = y_1$. So, $\{y_1, k\} \in \mathcal{B}^s$, $y_1 \succ^d k$ and $k\ R^a\ y_1$. Now, given that $j\ R^a\ k$ then $j\ R^a\ y_1$. By WG axiom and $A(y_1, k, j) = j$, $\{y_1, k\} \in \mathcal{B}^s$, $k \succ^d j$, it follows that $\{y_1, k, j\} \in \mathcal{B}^s$. If $C(y_1, k, j) = k$ then $k \succ y_1$ contradicting $y_1 \succ^d k$. So, $C(y_1, k, j) = y_1$. In addition, $A(y_1, j) = A(y_1, k, j) = j$. Thus, by WG axiom, $(y_1, j) \in \mathcal{B}^s$, and $C(y_1, j) = y_1$. Thus, $A(y_1, j) = j$, $C(y_1, j) = y_1$, $A(y_1, x) = x$, $C(y_1, x) = x$. So, $x \succ' j$.

Step 7. Assume, by contradiction, that $k \succ^r j$ and $j \succ^r k$. Then, there are four cases to consider. But if $k \succ^d j$ and $j \succ^d k$ or if $k \succ' j$ and $j \succ' k$ then a contradiction is immediately obtained given that $\succ^d$ and $\succ'$are asymmetric. So, assume, by contradiction, that $k \succ' j$ and $j \succ^d k$. Then, there exists an alternative y such that $A(y, j) = j$, $A(y, k) = k$, $C(y, j) = y$, $C(y, k) = k$. So, $\{y, j\} \in \mathcal{B}^s$. Hence, $\{y, k\} \in \mathcal{B}^s$ because $C(y, j) = y \in \{y, k\}$ and $A(y, j) = j \succ^d k = A(y, k)$. This is a contradiction because $A(y, k) = C(y, k) = k$.

Step 8. From $k \succ' j$ it follows that there exists y such that $A(y, j) = j$, $C(y, j) = y$, $A(y, k) = k$, $C(y, k) = k$. From $x\ R^a\ k$ and $k\ R^a\ y$ it follows that $A(y, x) = x$. Now assume that $C(y, x) = x$. Then, by definition, $x \succ' j$. Now assume that $C(y, x) = y$. Then, $\{y, x\} \in \mathcal{B}^s$. It follows that

$\{y, k\} \in \mathcal{B}^s$ (because $C(y, x) \in \{y, k\}$ and $A(y, x) = x \succ^d A(y, k) = k$). This is a contradiction because $A(y, k) = C(y, k) = k$.

Step 9. Step 7 shows that there are no cycles with 2 alternatives. Assume, by induction, that there are no cycles with $n-1$ (or less) alternatives. Also assume, by contradiction, that $\{x_1, \ldots, x_n\}$ is a $n-$cycle, $n \geq 3$. So, $x_i \succ^r x_{i+1}$, $i = 1, \ldots, n-1$, and $x_n \succ^r x_1$. By Step 2, there cannot be $i = 1, \ldots, n-1$ such that $x_i \succ^d x_{i+1} \succ^d x_{i+2}$. This would produce a cycle with $n-1$ alternatives which would violate the induction hypothesis. By Step 5, there must be some i^* such that $x_{i^*} \succ^d x_{i^*+1}$. Then, $x_{i^*-1} \succ' x_{i^*}$ and $x_{i^*+1} \succ' x_{i^*+2}$. By Step 6, $x_{i^*+1}\ R^a\ x_{i^*}$ implies $x_{i^*-1} \succ' x_{i^*+1}$. Now, by step 8, $x_{i^*}\ R^a\ x_{i^*+1}$ implies $x_{i^*} \succ' x_{i^*+2}$. Either way, a cycle with $n-1$ alternatives is produced. A contradiction. □

Proof of Theorem 1 The proof that the LWARP and WG axioms are satisfied under the warm glow theory with ordered aspirations is as follows: By Theorem 2 if (C, A) is a warm glow function then there is no chain with nonnegative characteristic connecting any alternative to itself. Suppose that LWARP does not hold, that is, $B, B' \in \mathcal{B}^s$, $B \subseteq B'$, $C(B') \in B$, and $C(B) \neq C(B')$. It follows that there is a chain $C(B) \succ^d C(B') \succ^d C(B)$ with zero characteristic. Next, suppose that the WG axiom does not hold. Then there exist two issues $B \in \mathcal{B}^s$ and B' such that $C(B) \in B'$, $A(B) \succ_= A(B')$ but $B' \notin \mathcal{B}^s$, i.e. $C(B') = A(B')$. Then the chain $C(B) \succ_+ A(B) \succ_= C(B') \vdash C(B)$ has zero characteristic. So, necessity is demonstrated by contradiction.

The proof that LWARP and WG are sufficient conditions is as follows: Assume now that a choice and aspiration function (C, A) is such that A is ordered and the LWARP and WG axioms are satisfied. By Lemma 5, $\succ^r$ is acyclical. An acyclical binary relation may be extended (not necessarily uniquely) to an order. Let R be any extension of $\succ^r$. So, R is a preference order such that

$$\text{if } w \succ^r y \text{ then } w\ R\ y. \tag{17}$$

Given $x \in X$, let $\mathcal{D}^x$ be the set of all alternatives z such that for some issue $B \in \mathcal{B}$, $x = C(B)$ and $z = A(B)$. Note that $z \in \mathcal{D}^x$ implies $x \succ^d z$. Let $d(x) \in \mathcal{D}^x$ be the element $d \in \mathcal{D}^x$ such that $d\ R_=\ z$ for any $z \in \mathcal{D}^x$. If $\mathcal{D}^x$ is empty then $d(x)$ is not defined. Let $\mathcal{L}^x$ be the set of all alternatives $d(y)$ where y is any alternative such that $x\ R_=\ y$ and $d(y)$ is defined. Let $\tau(x)$ be the alternative such that $\tau(x)\ R\ z$ for any $z \in \mathcal{L}^x$, and if $w \neq \tau(x)$ is such that $w\ R\ z$ for any $z \in \mathcal{L}^x$, then $w\ R\ \tau(x)$. If $\mathcal{L}^x$ is empty then $\tau(x)$ is such that $a\ R_=\ \tau(x)$ for every $a \in X$.

We now show that (R, τ) is a warm glow model that produces choice functions C.

(a) If $B \in \mathcal{B}^s$, then $C(B) = R(B)$. If $B \notin \mathcal{B}^s$, then $C(B) = A(B)$. So, $C(B) \in \{R(B), A(B)\}$. This follows immediately from (17).

(b) If $z \in \mathcal{D}^x$, then $\{x, z\} \in \mathcal{B}^s$, $A(x, z) = z$, $C(x, z) = x$ and $x \succ^d z$. By definition, there exists some issue $B \in \mathcal{B}^s$ such that $x = C(B)$ and $z = A(B) \neq x$. So, by (17), $x \succ^d z$ implies $x\ R\ z$. Let $\tilde{B} = \{x, z\}$. $C(B) = x \in \tilde{B}$ and $A(B) = A(\tilde{B}) = z$. By WG axiom, $\tilde{B} \in \mathcal{B}^s$, and $C(\tilde{B}) = x$.

(c) For any $x \in X$, and y such that $x\ R_=\ y$: if $\mathcal{D}^y \neq \varnothing$, then $x\ R\ d(y)$. If $y = x$, then $d(x) \in \mathcal{D}^x$ and, by (17), $x \succ^d d(x)$ implies $x\ R\ d(x)$. If $x\ R\ y$, then $x\ R\ y\ R\ d(y)$ implies, by transitivity of R, $x\ R\ d(y)$.

(d) If $x\ R\ y,\ z \in \mathcal{D}^y$ and $z\ R^a\ x$, then $z \in \mathcal{D}^x$. By b), $\{y, z\} \in \mathcal{B}^s$ and $A\ (y, z) = z$. By WG axiom, $A\ (x, y, z) = A\ (y, z)$ implies $\{x, y, z\} \in \mathcal{B}^s$. Thus, $C(x, y, z) \neq z$. If $C(x, y, z) = y$, then, $y \succ^d x$ contradicting $x\ R\ y$ and (17). So, $C(x, y, z) = x$ and $z \in \mathcal{D}^x$.

(e) For any $B \in \mathcal{B}$, if $\mathcal{D}^{R(B)}$ is empty or $A(B)\ R\ d(R(B))$, then $C(B) = A(B)$. Assume, by contradiction, that $A(B) \neq C(B)$. Then, $B \in \mathcal{B}^s$ and, by a), $C(B) = R(B)$. Let $x = C(B)$ and $z = A(B)$. Note that $z \in \mathcal{D}^x$ and, hence, $\mathcal{D}^x$ is not empty and $d(x)\ R_=\ z$. This contradicts the assumption.

(f) For any $B \in \mathcal{B}$, if $d(R(B))\ R_=\ A(B)$ then $C(B) = R(B)$. Let $x = R(B)$, $z = A(B)$ and $y = d(x)$. If $x = z$ then, by a), $C\ (B) = R\ (B)$. So, assume $x \neq z$. From $x \in B$ it follows that $A\ (x, z) = z$. Now either $C(x, z) = x$ or $C(x, z) = z$. Assume that $C(x, z) = z$. By definition, $y \in \mathcal{D}^x$. By b), $A\ (x, y) = y,\ C(x, y) = x$. So, $A\ (x, y) = y,\ C(x, y) = x,\ A\ (x, z) = z,\ C(x, z) = z$ implies $z \succ'\ y$. This contradicts $y\ R_=\ z$ and (17). So, $C(x, z) = x$. Then, $\{x, z\} \in B^s$. Moreover, $A(x, z) = A(B) = z$ and $C(x, z) \in B$. By WG axiom, $B \in B^s$. By a), $C(B) = R(B)$.

(g) τ satisfies (3). Consider an element $x \in X$. If $\mathcal{L}^x$ is empty, then $x\ R_=\ \tau\ (x)$. If $z \in \mathcal{L}^x$ then $z = d(y)$ for some alternative y such that $x\ R_=\ y$ and $\mathcal{D}^y \neq \varnothing$. So, by c), if $z \in \mathcal{L}^x$ then $x\ R\ z$. So, if $x \neq \tau(x)$ then, by definition, $x\ R\ \tau(x)$. Now assume that $a'\ R\ a$. Then, $\mathcal{L}^a \subseteq \mathcal{L}^{a'}$. This follows because if $z \in \mathcal{L}^a$ then $z = d(y)$ for some y such that $a\ R_=\ y$. By the transitivity of R, $a'\ R\ y$. So, $z \in \mathcal{L}^{a'}$. If $\mathcal{L}^{a'}$ is empty, then $\mathcal{L}^a$ is empty, and $\tau\ (a) = \tau\ \left(a'\right)$. If $\mathcal{L}^{a'}$ is not empty then, by definition, $\tau(a')\ R\ z$ for any $z \in \mathcal{L}^{a'}$. So, $\tau(a')\ R\ z$ for any $z \in \mathcal{L}^a$. Thus, if $\mathcal{L}^a$ is not empty and $\tau(a') \neq \tau(a)$ then, by definition, $\tau(a')\ R\ \tau(a)$. If $\mathcal{L}^a$ is empty, then $\tau(a')\ R_=\ \tau(a)$, by definition.

(h) If for some issue $B,\ A(B)\ R_=\ \tau(R(B))$ then $C(B) = A(B)$. If $\mathcal{D}^{R(B)}$ is not empty then, by definition, $\tau(R(B))\ R\ d(R(B))$. The conclusion now follows from the transitivity of R and step (e).

(i) If for some issue $B,\ \tau(R(B))\ R\ A(B)$ then $C(B) = R(B)$. Let $z = A(B)$ and $x = R(B)$. We can assume $x \neq z$ and that $\mathcal{L}^x$ is not empty. Otherwise, $\tau(x)\ R\ z$ cannot hold. It follows from $\tau(x)\ R\ z$ and $\mathcal{L}^x \neq \varnothing$ that there exists an alternative y such that $x\ R_=\ y$ and $d(y)\ R_=\ z$. If $d(x)\ R_=\ z$ then the conclusion follows from (f). So, we can assume, without loss of generality, that there exists an alternative y such that $x\ R\ y$ and $d(y)\ R_=\ z$. If $d(y) = z$, then $z \in \mathcal{D}^y,\ x\ R\ y$ and $z\ R^a\ x$, by step (d), imply $z \in \mathcal{D}^x$. The conclusion now follows from definition of $d(x)$ and (f). So, we can assume, without loss of generality, that exists an alternative y such that $x\ R\ y$ and $d(y)\ R\ z$. Now, either $x\ R^a\ d(y)$ or $d(y)\ R^a\ x$. Let's first consider the case $x\ R^a\ d(y)$. It follows that $z\ R^a\ d(y)$ (because $z\ R^a\ x$). So, $z\ R^a\ y$ (because $d(y)\ R^a\ y$). Now, it follows that $C(y, z) = y$. To see this assume, by contradiction, that $C(y, z) = z$. But, $A(z, y) = z$ and, by (b), $d(y) \in \mathcal{D}^y$ implies $C(y, d(y)) = y,\ A(y, d(y)) = d(y)$. So, by (15), $z \succ'\ d(y)$. This contradicts $d(y)\ R\ z$. Now from $C(y, z) = y$ and $A(y, z) = z$ it follows that $z \in \mathcal{D}^y$. Then, as above, $z \in \mathcal{D}^y,\ x\ R\ y$ and $z\ R^a\ x$, by step (d), imply $z \in \mathcal{D}^x$, and the conclusion follows from definition of $d(x)$ and (f). Now consider the remaining case in which $d(y)\ R^a\ x$. Let $B = \{y, d(y)\}$ and $\hat{B} = \{x, y, d(y)\}$. By (b), $B \in \mathcal{B}^s$ and

$A(B) = d(y) = A(\hat{B})$. This, combined with $C(B) \in \hat{B}$, implies, by WG axiom, that $\hat{B} \in \mathcal{B}^s$. So, $C(\hat{B}) \neq d(y)$. If $C(\hat{B}) = y$, then $y \succ x$. This contradicts $x\ R\ y$ and (17). So, $C(\hat{B}) = x$. Hence, $d(y) \in \mathcal{D}^x$. It now follows that $\mathcal{D}^x$ is not empty and $d(x)\ R_=\ d(y)$. But $d(y)\ R\ z$. So, $d(x)\ R\ z$. The conclusion now follows from (f).

By steps (g), (h) and (i), (R, τ) is a warm glow pair underlying (C, A). □

8.7 Proof of Proposition 1

If (C, A^*) is a warm glow choice and aspiration function, then, by definition, $\left(C, \tilde{A}\right)$ is a warm glow function. Now, suppose $\left(C, \tilde{A}\right)$ is a warm glow choice and partial aspiration function. In this case, $\mathcal{A}_{C,\tilde{A}}$ is not empty, and there exists aspiration function $A \in \mathcal{A}_{C,\tilde{A}}$ such that A extends $\tilde{A}$, and (C, A) is a warm glow function. Suppose, by contradiction, that (C, A^*) is not a warm glow function. There exists chain $\mathbf{x} = (\mathbf{x}_0, \ldots, \mathbf{x}_n)$ such that $\mathbf{x}_0 = \mathbf{x}_n$ and $\chi\ (\mathbf{x}) \geq 0$. For every issue B_i in the chain, if $\tilde{A}$ is defined on B_i, then $A^*\ (B_i) = \tilde{A}\ (B_i) = A\ (B_i)$, and if not, then $A^*\ (B_i) = C\ (B_i)$. In either case, the same cyclical chain with nonnegative characteristic can be constructed for (C, A), which contradicts the assumption that (C, A) is a warm glow function.

To show that revealed preference and tolerance relations are the same, note that every chain constructed for (C, A^*) is also a chain with the same characteristic for any other warm glow function (C, A) such that A extends $\tilde{A}$. Therefore, the intersection of all sets of revealed relations for all such models is equal to the set of revealed relations for (C, A^*). Moreover, since A^* extends $\tilde{A}$, all revealed relations for $\left(C, \tilde{A}\right)$ must be revealed for (C, A^*). □

References

Allouch, N.: A competitive equilibrium for a warm-glow economy. Econ. Theory (2012). doi:10.1007/s00199-012-0689-z

Ambrus, A., Rozen, K.: Revealed conflicting preferences. Working paper Harvard University (2008)

Andreoni, J.: Giving with impure altruism: applications to charity and ricardian equivalence. J. Political Econ. **97**, 1447–1458 (1989)

Andreoni, J.: An experimental test of the public-goods crowding-out hypothesis. Am. Econ. Rev. **83**, 1317–1327 (1993)

Andreoni, J.: Cooperation in public goods experiments: kindness or confusion. Am. Econ. Rev. **85**, 891–904 (1995)

Andreoni, J.: Philanthropy. In: Kolm, S.-C., Mercier Ythier, J. (eds.) Handbook of Giving, Reciprocity and Altruism, pp. 1201–1269. Amsterdam, North Holland (2006)

Andreoni, J., Miller, J.: Giving according to GARP: an experimental test of the consistency of preferences for altruism. Econometrica **70**, 737–753 (2002)

Berger, P., Smith, G.: The effect of diirect mail framing strategies and segmentation variables on university fundraising performance. J. Direct Mark. **2**, 30–43 (1997)

Coate, S., Conlin, M.: A group rule-utilitarian approach to voter turnout: theory and evidence. Am. Econ. Rev. **94**, 1476–1504 (2004)

Cherepanov, V., Feddersen, T., Sandroni, A.: Rationalization. Theor. Econ. http://econtheory.org/ojs/index.php/te/article/viewForthcomingFile/970/7435/1 (2012)

Dillenberger, D., Sadowski, P.: Ashamed to be selfish. Theor. Econ. **7**, 99–124 (2012)

Feddersen, T., Sandroni, A.: A theory of participation in elections. Am. Econ. Rev. **96**, 1271–1282 (2006)
Feddersen, T., Gailmard, S., Sandroni, A.: Moral bias in large elections: theory and experimental evidence. Am. Political Sci. Rev. **103**, 175–192 (2009)
Green, J., Hojman, D.: Choice, Rationality and Welfare Measurement. Harvard University, Working paper (2007)
Gul, F., Pesendorfer, W.: Temptation and self-control. Econometrica **6**, 1403–1435 (2001)
Kalai, G., Rubinstein, A., Spiegler, R.: Rationalizing choice functions by multiple rationales. Econometrica **70**, 2481–2488 (2002)
Levine, D., Palfrey, T.: The paradox of voter participation? A laboratory study. Am. Political Sci. Rev. **101**, 143–158 (2007)
Manzini, P., Mariotti, M.: Sequentially rationalizable choice. Am. Econ. Rev. **97**, 1824–1839 (2007)
Manzini, P., Mariotti, M.: Categorize then choose: boundedly rational choice and welfare. J. Eur. Econ. Assoc. **10**, 1141–1165 (2012)
Masatlioglu, Y., Ok, E.: Rational choice with status quo bias. J. Econ. Theory **121**, 1–29 (2005)
Noor, J., Takeoka, N.: Uphill self-control. Theor. Econ. **5**, 127–158 (2010)
Palfrey, T., Prisbrey, J.: Altruism, reputation, and noise in linear public goods experiments. J. Public Econ. **61**, 409–427 (1996)
Palfrey, T., Prisbrey, J.: Anomalous behavior in linear public goods experiments: how much and why? Am. Econ. Rev. **87**, 829–846 (1997)
Riker, W., Ordeshook, P.: A theory of the calculus of voting. Am. Political Sci. Rev. **62**, 25–43 (1968)
Sagi, J.: Anchored preference relations. J. Econ. Theory **130**, 283–295 (2006)
Salant, Y., Rubinstein, A.: Choices with frames. Rev. Econ. Stud. **75**, 1287–1296 (2008)
Shayo, M., Harel, A.: Non-consequentialist voting. J. Econ. Behav. Organ. **81**, 299–313 (2012)
Sigelman, L.: The nonvoting voter in voting research. Am. J. Political Sci. **26**, 47–56 (1982)
Segal, U., Sobel, J.: Tit for tat: foundations of preferences for reciprocity in strategic settings. J. Econ. Theory **136**, 197–216 (2007)