Econ Theory (2013) 54:623–674
DOI 10.1007/s00199-012-0728-9

RESEARCH ARTICLE

# Computing minimal state space recursive equilibrium in OLG models with stochastic production

**Jaime McGovern · Olivier F. Morand · Kevin Reffett**

Received: 24 April 2011 / Accepted: 19 October 2012 / Published online: 10 November 2012
© Springer-Verlag Berlin Heidelberg 2012

**Abstract** Using order-theoretic methods, we derive sufficient conditions for the existence, characterization, and computation of minimal state space recursive equilibrium (RE), as well as Stationary Markov equilibrium (SME) for various classes of stochastic overlapping generations models. In contrast to previous work, our methods focus on constructive methods. Our existence results are obtained for models that include public policy (e.g., social security policies, transfers, taxes, etc), production nonconvexities, elastic labor supply, non-monotone income processes, and long-lived agents. We distinguish conditions under which there exist various subclasses of minimal state space RE, including bounded, monotone, non-monotone, semicontinuous, Lipschitz continuous RE. Finally, we provide monotone equilibrium comparative statics results on the space of economies for some RE.

The authors would like to thank Elena Antoniadou, Lukasz Balbus, Bob Becker, Alex Citanna, Manjira Datta, Seppo Heikkila, Cuong LeVan, Len Mirman, Ed Prescott, Manuel Santos, Jayaram Sethuraman, Paulo Siconolfi, John Stachurski, Yiannis Vailakis, Lukasz Wozny, Itzhak Zilcha, seminar partipants at the 2008 SED meetings, the Centre d'Economie de la Sorbonne (CES), Arizona, two referees and Nicholas Yannelis for detailed remarks. Reffett also thanks Centre D'Economie de la Sorbonne, Universite de Paris 1, for its support of this research during visits to Paris in 2008, 2009, and 2012 as well the Dean's Award in Excellence Program at ASU in 2010 and 2011 for financial support.

J. McGovern · K. Reffett (✉)
Department of Economics, Arizona State University, PO Box 879801,
Tempe, AZ 85287-9801, USA
e-mail: kevin.reffett@asu.edu

O. F. Morand
University of Connecticut, 341 Mansfield Road, Unit 1063, Storrs,
CT 06269-1063, USA

**Keywords** Recursive equilibrium · Stationary Markov equilibrium · Order-theoretic fixed point theory · Equilibrium computation · Monotone Comparative statics

**JEL Classification** D5 · D62 · E20 · E6

## 1 Introduction

Since its introduction in the seminal work of Samuelson (1958) and Diamond (1965), the overlapping generations (OLG) model has been a workhorse in many areas of applied dynamic general equilibrium theory, including various models in macroeconomics, public finance with intergenerational risk sharing and social security, human capital formation and public education, labor economics, optimal taxation, economic growth, infrastructure and/or environmental degradation, and monetary economics. With a few exceptions, much of this applied work has focused on either numerical characterizations of minimal state space equilibrium using "direct" methods that construct approximate solutions in equilibrium versions of household Euler equations (e.g., projection methods as in Judd 1992) or more "indirect" methods for constructing Generalized Markov equilibrium (GME) using correspondence-based methods such "Euler equation APS methods" ala Feng et al. (2012), where approximate Markovian equilibria are computed on enlarged state spaces as selections from an equilibrium correspondence.[1]

In their current form, an important limitation of all these approaches is they provide little characterization of the structural properties of any recursive equilibrium (RE). Such structural characterizations prove useful if one wants to characterize rigorously the properties of both actual and particular approximate RE solutions. They are, of course, also of independent theoretical interest for questions such as equilibrium comparative statics, stochastic stability, as well as for understanding other important properties of RE. Perhaps most importantly, these methods say little about the state spaces where RE exist. For example, do *minimal state space* RE exist (i.e., defined on state spaces consisting only of current period endogenous and exogenous states), and, if so, can they be computed by successive approximation?[2] Further, how do RE

[1] These correspondence-based methods for computing Generalized Markov equilibrium are generalizations of the seminal work of Kydland and Prescott (1980) and Abreu et al. (1990) adapted to dynamic competitive economies. Typically, in macroeconomic applications with state variables and Euler equations, they are "dual" methods, constructing Markovian equilibrium on enlarged state spaces directly from the sequential equilibrium. These enlarged state variables often include Karash–Kuhn–Tucker (KKT) multipliers and/or envelope theorems. Such Generalized Markov equilibria are very different than "Prescott-Mehra" recursive equilibrium as they generally cannot be associated with dynamic programming representations of household decision problems in equilibrium, for example.

[2] In this paper, all references to "RE" will be to minimal state space Markovian equilibrium in the sense of Lucas–Prescott–Mehra (e.g., see Lucas and Prescott 1971; Prescott and Mehra 1980). This notation of "RE" is very different than that used in the literature on Generalized Markov Equilibrium, where one first constructs a sequential equilibrium and then asks whether there exists a set of state variables such that the sequential equilibrium admits a recursive representation. See Citanna and Siconolfi (2007, 2008, 2010) for a very nice discussion of memory and minimal state space RE and in particular the importance of short memory in the study of RE.

vary with the deep parameters of the economy? This latter question is actually of great interest for numerical methods, and not just of theoretical interest.

In this paper, we present an new collection of order-theoretic methods operating in function spaces for constructing Recursive Equilibria (RE) for interesting classes of stochastic OLG models that often appear in applied work. As our focus is on short-memory or minimal state space RE, our approach complements many of the existing methods found in the literature (e.g., especially, the correspondence-based methods of Feng et al. 2012). Further, our methods are constructive, and we provide iterative procedures for constructing least and greatest minimal state space RE in all the cases we study. Additionally, we show how equilibrium comparative statics results on the space of economies can be easily be obtained using our monotone methods and computed. We also provide constructive arguments for characterizing the set of equilibrium limiting distributions [or Stationary Markov equilibrium (SME)], and we provide situation where we can conduct equilibrium comparative statics for then set of SME.

Finally, in the last section of the paper, we extend our monotone map methods to economies where RE are *not* monotone.[3] That is, a common misconception in the literature when discussing so-called "monotone map methods" is that they do not work for models with (i) many state variables, and (ii) non-monotone RE (e.g, see Feng et al. 2012 for such remarks relative to the work of Coleman 1991; Datta et al. 2002; Mirman et al. 2008). To answer this criticism, we present three key extensions of the monotone map approach to models with (i) two-period lived agents, and general Lipschitzian income processes (i.e., not monotone); (ii) two-period lived agents and elastic labor supply, and (iii) long-lived agents, and with general local Lipschitzian income processes. Aside from addressing conditions for existing of minimal state space RE in these economies, in case (i), we also show how delicate uniqueness results are in even simple versions of these models. To obtain these extensions, in this last section of the paper, we propose a new monotone decomposition method, which we discuss in detail in the last section of the paper.

Our approach is complementary to the powerful new collection of direct methods that have been proposed in the important series of recent papers Citanna and Siconolfi (2007, 2008, 2010), where the authors develop an elegant approach to verifying the existence of minimal state space RE equilibrium in a very general class of stochastic OLG models based upon a generalized transversality theory. A key limitation of the Citanna–Siconolfi method concerns the weak nature of the characterization of global structural properties of any RE that one is able to obtain, as well as the difficulty one faces relating their results to approximation methods. In the end, aside from verifying the existence minimal state space RE, their method is unable to characterize such properties such as the continuity, monotonicity, etc. Of course, if existence issues (which is their focus) is one's sole concern, such issues are not critical, but if one is seeking to relate theoretical constructions to rigorous methods for constructing approximation solutions of particular RE (as is needed, for example, in applied work), this limitation can be a serious issue. We address all of these issues here.

[3] Our monotone methods are often referred to as "monotone map" methods in the literature. The term was coined in the seminal paper of Coleman (1991), where these sorts of methods were first discussed.

We provide the first results in this paper in the literature (of which we are aware) for computing OLG models with elastic labor supply, which is important as in applied work using stochastic OLG models, researchers often allow for endogenous labor supply decisions. Given the recent plethora of negative results associated with stability of Markovian equilibrium dynamics for OLG models reported in the work by Lloyd-Braga et al. (2007), our positive results provided here should be of interest to researchers attempting to compute RE in lifecycle models. We should note we only have results for the two-period lived agent case, so this is a limitation of our results.

Finally, in the last section of the paper, we consider existence of RE for the case of long-lived agents with borrowing constraints (i.e., an OLG version of a finite type Bewley model) and provide a constructive existence result using monotone operators for such economies. These results are also somewhat limited as we rule out the case of many assets. In all case, an important aspect of our approach is we work exclusively with operators defined in function spaces. This allows us to unify issues of topology and order when characterizing convergence structures for monotone iterative procedures. This means when our methods apply, we are able to improve a great deal upon the characterizations of RE obtained via correspondence-based methods recently proposed in the literature (e.g., Feng et al. 2012).[4] Also, we provide conditions for structural properties on RE (e.g., conditions for continuous or locally Lipschitz continuous RE), which are very important in approximation issues (e.g., for discretization procedures).

Our methodology to construct RE and SME is an direct extension of the approach outlined in the seminal papers of Lucas and Prescott (1971) and Prescott and Mehra (1980). That is, we partition state spaces for household decision problems into a "little k, big K" form, which allows us to restrict the parameterization of the continuation structure for the aggregate economy implied by collection of candidate RE functions. This formalization proves to be very important, as it avoids most (if not all) of the important technical problems that arise with multiplicities and dynamic indeterminacies that make studying the set of "self-fulfilling expectations equilibria" intractible in our enviroments. These latter issues are very elegantly discussed in, for example, Wang (1994). That is, unfortunately in situations where RE and/or sequential equilibria are not unique, self-fulfilling expectations equilibria (e.g., GME) are known to be very complicated to characterize. Using our methods, even with multiplicities of RE, this is *not* the case. That is, in the presence of multiplicities of our RE are present, the Lucas-Prescott-Mehra RE construction works as a equilibrium selection device, which amounts to a particular parametrization of an equilibrium selection in the expectations equilibrium correspondence (or Markovian equilibrium correspondence in the literature on GME using Euler equation APS type methods). This allows us to associate *minimal state* RE with *particular* SME without ever appealing to arguments in Duffie et al. (1994) (which focus only on SME, not the structure of the RE that actually generates them). Further, we can allow SME to only be an *invariant* measure, as opposed

[4] Of course, the correspondence-based methods of Feng et al. (2012) apply in more general settings than ours. This comes at a cost of not guaranteeing existence of minimal state space RE. Still, that cost might be acceptable to relax the conditions we need (especially, our restrictions on the set of assets in our models). So its best to think of our work as a selection device in Feng et al. (2012) that focuses on minimal state RE.

to Duffie et al. (1994) which must generate an ergodic measure as a SME.[5] Finally, unlike these correspondence-based methods for GME, we obtain sharp characterizations of *particular* minimal state space RE and its associated SME (as opposed to weak characterizations of some and/or all SME equilibria on enlarged state spaces).

More recently, Morand and Reffett (2007) extended the work of Wang (1994) to studying RE in models non-classical production and Markov shocks using monotone methods, providing successive approximation algorithms for computing extremal Markovian equilibria. Although we follow a similar approach, this paper differs from that work in several important ways. First, as Morand and Reffett (2007) study the Markov shock case, so they require very strong conditions on primitives (e.g., time separable utility and a limiting condition on capital income) to prove even existence of isotone RE (let alone to construct SME). Further, their results of existence of SME are very weak, providing, for example, no results on stochastic stability. In this paper, many of these conditions are relaxed, and given strong assumptions on the shocks (iid shocks), stronger results on SME are possible. Second, for the Markov shock case, they only show least and greatest RE that are measurable, whereas in this paper, we are able to construct a new space of measurable functions that actually forms a complete lattice (so our existence result here is much stronger, namely a complete lattice of measurable RE in a number of different subclasses of functions). Third, we give a context for the uniqueness results in Wang (1993) and Morand and Reffett (2007) (namely, we prove uniqueness in a space of continuous functions under capital income monotonicity is robust to a space of bounded increasing functions). Finally, in the last section of the paper, we provide a extensions of monotone methods to OLG models with non-monotone RE via monotone decompositions, as well as discuss the limitations of our methods, none of which are discussed in this previous work.

Finally, OLG models have found extensive application in the recent literature. For example, per a few recent applications, Constantinides et al. (2007) examine the role of overlapping generations in the study of bequest. In their model, the finite horizon for household decisions plays a key role in their results. In a related paper, Pestieau and Thibault (2012) examine the role of estate taxes in lifecycle model. De la Croix and Michel (2007) study the role of education and debt constraints in an simple OLG's model. Also, Prieur (2009) examines environmental policy in the context of a OLG model and is able to study the structure of the environmental Kuznet's curve in such a model. A final interesting recent paper on OLG models and bequest is the paper by (Barnett et al., 2012, Barnett, Bhattacharya, and Bunzel (2012)).

The paper has a very simple structure. In the next section, we discuss the economic environment. Section 3 addresses existence questions associated with the computation of RE. Section 4 studies existence of SME. Section 5 extends the results to economies continuous, but not monotone RE. The last section contains most of the proofs.

[5] In the construction of a SME in the sense of Duffie et al. (1994), given their existence argument, one must generate equilibrium ergodic measures and then select SME consistent with this equilibrium. Our argument for constructing SME works the other direction, i.e., we first compute explicitly a minimal state space RE and then show how to compute a corresponding non-trivial extremal invariant measure which is implied by this RE.

## 2 The economic environment

The baseline model has a large number of identical agents born each period who live for two periods. In their first period of life, they are endowed with a unit of time which they supply inelastically to the firm at the prevailing wage, and they consume and/or save. In their second period of life, they simply consume their savings which are subjected to a stochastic return. Preferences are represented by a non-time separable utility function. Utility is assumed to satisfy a standard intertemporal complementarity condition between consumption when young (denoted $c_1$) and consumption when old ($c_2$):

**Assumption 1** The utility function $u : X \times X \to \mathbb{R}$, for $X \subset \mathbb{R}$ is:

- I twice continuously differentiable;
- II strictly increasing in each of its arguments and jointly concave;
- III satisfies $\forall c_2 > 0$, $\lim_{c_1 \to 0^+} u_1(c_1, c_2) = +\infty$ and $\forall c_1 > 0$, $\lim_{c_2 \to 0^+} u_2(c_1, c_2) = +\infty$;
- IV is supermodular in $(c_1, c_2)$ (i.e., in this context, $u_{12} \geq 0$).

As in Wang (1993) and Hausenchild (2002), we assume iid production shocks with compact support.

**Assumption 2** The random variable $z_t$ follows an iid process characterized by the probability measure denoted $\gamma$. The support of $\gamma$ is the compact set $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ with $z_{\max} > z_{\min} > 0$.

Following recent work on the existence of RE in economies with public policy and non-classical production (e.g., Greenwood and Huffman 1995; Mirman et al. 2008), we consider equilibrium distortions that can be represented as a reduced-form production function with a non-classical specification. We denote this technology by $F(k, n, K, N, z)$, where we assume $F$ is constant returns to scale in private inputs $(k, n)$ for each level of aggregate inputs $(K, N)$. The following assumptions on $F$, adapted from the literature on nonoptimal stochastic growth, are completely standard. Anticipating $n = 1 = N$ in any equilibrium with inelastic labor supply, we state our assumptions as follows:

**Assumption 3** The production function $F(k, n, K, N, z) : X \times [0, 1] \times X \times [0, 1] \times Z \to \mathbb{R}_+$ is:

- I twice continuously differentiable in its first two arguments, and continuous in all arguments;
- II isotone in all its arguments, strictly increasing and strictly concave in its first two arguments;
- IIIa such that $r(k, z) = F_1(k, 1, k, 1, z)$ is decreasing and continuous in $k$, and $\lim_{k \to 0} r(k, z) = +\infty$;
- IIIb such that $w(k, z) = F_2(k, 1, k, 1, z)$ is increasing and continuous in $k$, and $\lim_{k \to 0^+} w(k, z) = 0$;
- IV such that there exists a maximal sustainable capital stock $k_{\max}$ (i.e., $\forall k \geq k_{\max}$ and $\forall z \in Z$, $F(k, 1, k, 1, z) \leq k_{\max}$), and with $F(0, 1, 0, 1, z) = 0$.

It is well known that Assumption 3 IV implies that the set of feasible capital stocks can be restricted to be in the compact interval $X = [0, k_{\max}]$ as long as we place the initial capital stocks in $X$. This condition, along with (IIIa and IIIb), also place restrictions on the amount of nonconvexity we can allow. The following two additional assumptions will help establish sharper properties of the RE, the latter being sufficient to exclude economies in which 0 may be the only RE (and will lead to the construction of minimal RE by successive approximations).

**Assumption 3′** Both $r(k, z)$ and $w(k, z)$ are continuous and isotone in $z$ for all $k$.

**Assumption 4** $\lim_{k \to 0^+} r(k, z_{\max})k = 0$.

## 3 Computing minimal state space RE

This section addresses the issues of existence, characterization and construction of extremal minimal state space RE. Our proofs rely on the Euler equation methods (see, for instance, Coleman 1991; Datta et al. 2002; Mirman et al. 2008).[6] As a direct consequence of Tarski's fixed point theorem, the set of fixed points of this operator will be a non-empty complete lattice, and by construction all fixed points but the trivial 0 are RE. As is well known, Tarski's theorem is not constructive; therefore, we shall then show we can construct lower bounds in some cases appealing to order continuity conditions. We will also remove the problem of trivial RE by finding least elements of our function spaces that map up under our operator.

### 3.1 Some useful complete lattices

We begin by defining the classes of complete lattices where we shall prove existence of RE.[7] First, given any bounded function $w : S \to \mathbb{R}^+$, define the set $W = \{h : S \to \mathbb{R}^+, 0 \leq h \leq w\}$ (the set of "bounded functions") endowed with the pointwise partial order $\leq$ is a complete lattice under the pointwise partial order. If $w$ is isotone (i.e., non-decreasing in its arguments), the set $H = \{h \in W, h \text{ isotone}\}$ is subcomplete in $W$. If in addition, $w$ is continuous in $k$ for each $z \in Z$ (in the usual topology on $\mathbb{R}$), define the set $H^u = \{h \in H, h \text{ upper semicontinuous in } k \in X \text{ for each } z \in Z\}$ (resp., $H^l = \{h \in H, h \text{ lower semicontinuous in } k \in X \text{ for each } z \in Z\}$), which are each subcomplete in $H$, as established in the following Proposition.

**Proposition 1** *The poset* $(H^u, \leq)$ *and* $(H^l, \leq)$ *are complete lattices. In addition, any* $h \in H^u$ *and* $h \in H^l$ *is measurable.*

*Proof* Given any $B \subset H^u$, denote $g(s) = \inf_{h \in B} h(s)$. Clearly $0 \leq g \leq w$, $g$ is isotone, and $g(., z)$ is usc for any given $z$. Thus $g$ is an upper bound of $B$, and it is easy to see that it is the least upper bound. Since $w$ is the top element of $H^u$, it is a complete lattice (e.g., Davey and Priestley 2002, Theorem 2.31). Next, since $X$ is a

6 The nonlinear operator in this paper differs from the one in the infinitely lived agent models cited above.

7 The order-theoretic terminology we use in the paper is not standard in the literature. Useful references for such terminology include Veinott (1992), Topkis (1998), and Davey and Priestley (2002).

compact interval of $\mathbb{R}$, denote by $\{x_0, x_1, \ldots .\}$ a countable dense subset of $X$. Given any $\alpha \in \mathbb{R}$, we claim that:

$$\{s \in S, h(s) \leq \alpha\} = \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}.$$

This property implies that $h$ is measurable (in the sense of jointly measurable): Indeed, since $h$ is isotone in $z$ for each $k$, it is $\mathcal{B}(Z)$-measurable for each $k$ which implies that $\{z \in Z, h(x_m, z) < \alpha + 1/n\} \in \mathcal{B}(Z)$, and that $\{s \in S, h(s) \leq \alpha\} \in \mathcal{B}(S)$. We prove now the stated claim. First, consider $(k, z)$ such that $h(k, z) \leq \alpha$. Such $h$ being usc and isotone in $k$ (for each $z$), it is necessarily right continuous at $k$, and we have that:

$$\forall n \in \mathbb{N}, \exists m \text{ such that } x_m - 1/n < k < x_m \text{ and } h(x_m, z) < \alpha + 1/n.$$

Thus:

$$\forall n \in \mathbb{N}, \exists m \text{ such that } (k, z) \in (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\},$$

which implies that:

$$\forall n \in \mathbb{N}, (k, z) \in \bigcup_{m=0}^{\infty} (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\},$$

and therefore that:

$$(k, z) \in \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} ]x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}.$$

Reciprocally, suppose that for all $n \in \mathbb{N}$, $(k, z)$ belongs to $\bigcup_{m=0}^{\infty} (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}$. This implies that for all $n$, there exists $m(n)$ such that $k \in (x_{m(n)} - 1/n, x_{m(n)}]$ and $h(x_{m(n)}, z) < \alpha + 1/n$. By construction the sequence $\{x_{m(1)}, x_{m(2)}, \ldots\}$ converges to $k$ and $x_{m(n)} \geq k$, so by continuity from the right at $k$ of $h(., z)$, $h(x_{m(n)}, z)$ converges to $h(k, z)$ and necessarily $h(k, z) \leq \alpha$. We note a similar result holds for the subset of $H^l$ of lsc functions, as:

$$\{(k, z) \in S, h(k, z) \geq \alpha\} = \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} [x_m, x_m + 1/n) \times \{z \in Z, h(x_m, z) > \alpha - 1/n\}.$$

□

This space of functions is very important (as it is a complete lattice of measurable functions). It will allow us to extend the results on existence of Morand and Reffett (2007) a great deal.

### 3.2 An Euler equation method for computing RE

Earning the competitive wage $w$ in the labor to the market, in a candidate RE $h \in W$, a typical young agent of any generation must decide what amount $y$ to save for next period consumption. To make this decision, given $h \in W$, the agent computes the expected continuation returns on her capital investment, as well as future competitive wages and returns on capital use the firms profit maximization problem with $w(k, z) = F_2(k, 1, k, 1, z)$ and $r(k, z) = F_1(k, 1, k, 1, z)$. Thus, given $s \in S^*$ and $h \in W$, a young agent seeks to solve:

$$\max_{y \in [0, w(s)]} \int_Z u(w(s) - y, r(h(s), z')y)\gamma(\mathrm{d}z'),$$

Let $y^*(s; h(s))$ be the optimal solution to this household problem. A RE therefore can be defined as follows:

**Definition 1** A Recursive Equilibrium (RE) is a bounded function $h^*(s) \in W$ and a policy function $y^*(s; h^*(s))$ such that (i) for all $s \in S^*$, $h^*(s) > 0$, we have $y^* = y^*(s; h^*(s)) = h^*(s)$, and $h^*(s) = 0$, else, and (ii)

$$\int_Z u_1(w(s) - y^*, r(h^*(s), z')h^*(s))\gamma(\mathrm{d}z')$$
$$= \int_Z u_2(w(s) - y^*, r(h^*(s), z')y^*(s))r(h^*(s), z')\gamma(\mathrm{d}z'). \tag{E}$$

Notice, in our definition, we restrict our attention to the case of RE that have memory only consisting of the current states of the economy.

To construct such RE, we introduce the nonlinear operator $A$ defined implicitly in the HH equilibrium Euler equation follows:

**Definition 2** Given any $h \in W$, define the operator $A$ as follows: If $h(s) > 0$, then $Ah(s)$ is the unique solution for $y$ to:

$$\int_Z u_1(w(s) - y, r(h(s), z')y)\gamma(\mathrm{d}z')$$
$$= \int_Z u_2(w(s) - y, r(h(s), z')y)r(y, z')\gamma(\mathrm{d}z'), \tag{E'}$$

and $Ah(s) = 0$ whenever $h(s) = 0$.[8]

A function $h \in W$ is a RE if and only if it is a nonzero fixed point of the operator $A$, and the issues of existence, characterization, and construction of extremal RE simply follow from the study of the set of non-trivial fixed points of $Ah$.

8 It is easy to verify the existence of a unique solution under Assumption 1.

We now prove our main existence result of this section. To do this, we first mention three lemmata. Understanding the importance of the first two lemmas well allows us to make our application of Tarski's theorem constructive via order continuity conditions (in the interval topology) for the operator $Ah$ (which allows use to make our methods constructive. e.g., see Dugundji and Granas 2003, Theorem 4.2, p. 15).

**Lemma 1** *Under Assumptions 1, 2, 3, A is an isotone self map on* $(W, \leq)$. *Under Assumptions 1, 2, 3, 3', A is an isotone self map on* $(H, \leq)$ *and on* $(H^u, \leq)$.

*Proof* By construction $A$ maps $W$ into itself, and it is easy to verify that $Ah \geq Ah'$ whenever $h \geq h'$. Clearly $Ah$ is isotone in $k$ whenever $h$ is, and Assumption 3' is sufficient for preservation of isotonicity in $z$, thus making $A$ an isotone map on $(H, \leq)$. Consider $h \in H^u$ and therefore right continuous at every $k \in [0, k_{\max}[$ given any $z$. Since the unique solution to (E') can be expressed as a continuous function of $h$, $w$, and $r$, Assumption 1, 2, 3, 3' imply that $Ah$ is right continuous in $k$ as well and therefore also usc in $k$ since isotone in $k$. Thus $A$ is an isotone self map on $H^u$.[9] □

We now define order continuity.

**Definition 3** A function $F : (P, \leq) \to (P, \leq)$ is order continuous if for any countable chain $C \subset P$ such that $\vee C$ and $\wedge C$ both exist,

$$\vee\{F(C)\} = F(\vee C) \text{ and } \wedge\{F(C)\} = F(\wedge C).$$

It is important to note that the hypothesis of order continuity in our computational fixed point results can be weakened to that isotonicity of $F$ and order continuity along monotone recursive generated $F$-sequences, that is, sequences of the form $\{x, F(x), \ldots, F^n(x), \ldots\}$ where either $x \leq F(x)$ or $x \geq F(x)$.[10] In that case, the partially ordered set need only be chain complete for the existence of a non-empty set of fixed points with minimal and maximal elements

We now show under pointwise partial orders, our operator $Ah$ is order continuous along recursively generated countable chains, so many cases, extremal RE can be computed.

**Lemma 2** *(i) Under Assumptions 1, 2, 3 the set of fixed points of A in* $(W, \leq)$ *is a non-empty complete lattice and A is order continuous along any monotone sequence in* $(W, \leq)$. *(ii) Under Assumptions 1, 2, 3, 3' the set of fixed points of A in* $(H, \leq)$ *(resp.* $(H^u, \leq)$, $(H^l, \leq)$*) is a non-empty complete lattice and A is order continuous along any monotone (resp. decreasing, increasing) sequence in* $(H, \leq)$ *(resp.* $(H^u, \leq)$, $(H^l, \leq)$*).*

*Proof* The complete lattice structure of these sets of fixed points follows from Tarski's fixed point theorem. Next, we prove order continuity along increasing sequences by

[9] Note that the same argument applies for semicontinuity in $z$ given any $k$. As a result, under Assumptions 1, 2, 3, 3' $A$ is an increasing self map on the set of isotone and (jointly) continuous functions. Unfortunately, as noted before, that set (with the pointwise partial order) is not a complete lattice.

[10] Order continuity along monotone $F$-sequences does not imply that $F$ is isotone though. Consider for instance $F : [0, 1] \to [0, 1]$ such that $F(x) = 1 - x$. $F$ is clearly continuous along the (only) monotone recursive $F$-sequence $\{1/2, 1/2, 1/2, \ldots\}$, but $F$ is not isotone.

showing that for an increasing sequence $\{g_n\}$ in $(W, \leq)$ or in $(H, \leq)$:

$$\sup(\{Ag_n(s)\}) = A(\sup\{g_n(s)\}).$$

For such a sequence and for all $s \in S$, the sequence of real numbers $\{g_n(s)\}$ is increasing and bounded above (by $w(s)$), thus $\lim_{n\to\infty} g_n(s) = \sup\{g_n(s)\}$. For the same reason $\lim_{n\to\infty} Ag_n(s) = \sup\{Ag_n(s)\}$. By definition, for all $n \in \mathbb{N}$, and all $s \in S^*$:

$$\int_Z u_1(w(s) - Ag_n(s), r(g_n(s), z')Ag_n(s))\gamma(\mathrm{d}z') = \int_Z u_2(w(s) - Ag_n(s), r(g_n(s), z')Ag_n(s))r(Ag_n(s), z')\gamma(\mathrm{d}z')$$

The functions $u_1$ and $u_2$ are continuous (Assumption 1), $r$ is continuous in its first argument (Assumption 3), hence taking limits when $n$ goes to infinity, we have:

$$\int_Z u_1(w(s) - \sup\{Ag_n(s)\}, r(\sup\{g_n(s)\}, z')\sup\{Ag_n(s)\})\gamma(\mathrm{d}z') = \int_Z u_2(w(s) - \sup\{Ag_n(s)\}, r(\sup\{g_n(s)\}, z')\sup\{Ag_n(s)\}) \times r(\sup\{Ag_n(s)\}, z')\gamma(\mathrm{d}z'),$$

which implies that $A(\sup\{g_n(s)\}) = \sup\{Ag_n(s)\}$. A symmetric argument can easily be made for any decreasing sequence $\{g_n\}$ in $(W, \leq)$ or in $(H, \leq)$. This establishes (i) and (ii). Finally, note that the proof also holds for a decreasing sequence in $(H^u, \leq)$ (since, as we have noted before, $\wedge_H$ and $\wedge_{H^u}$ coincide).[11] □

Our final lemma is particularly important for verifying the existence of non-trivial minimal RE. As is clear from the definition of $Ah$, in all cases of subsets of $W$, $h^* = 0$ is a trivial fixed point. Therefore, the next lemma find a minimal element of $H^l$ that maps up. Note, we construct this lower bound $h_0$ to be lsc so that the iterations $\{A^n h_0\}$ will be an increasing sequence of lsc functions, which therefore converge in order a lsc function $\vee\{A^n h_0\}$.

**Proposition 2** *Under assumptions 1, 2, 3, 4, there exists a function* $h_0 \in (H^l, \leq)$ *that is lower semicontinuous in* $k$ *and continuous in* $z$ *such that (i)* $\forall s \in S^*$, $Ah_0(s) > h_0(s) > 0$, *and (ii)* $\forall h \in (0, h_0]$, $Ah > h$ *on* $S^*$.

*Proof* See "Appendix A". □

We are now prepared to prove our first theorem on the existence of RE in the class of bounded functions $W$, as well as characterize the structure of the set of RE. In the Theorem, $h_0$ is the function constructed in Proposition 2.

[11] Similarly, $A$ is order continuous along any increasing sequence in the set of bounded isotone lsc functions.

**Theorem 1** *Under Assumptions 1, 2, 3, 3′,4: (i) there exist a non-empty complete lattice of non-trivial RE in* $W \cap [h_0, w]$*, (ii) The minimal RE in* $(W \cap [h_0, w], \leq)$ *(in* $(H \cap [h_0, w], \leq)$*) is an isotone lsc and measurable function* $h_{\min}$*; and the maximal RE in* $(W, \leq)$ *(in* $(H, \leq)$*) is an isotone usc and measurable function* $h_{\max}$*. Further, both extremal RE can be constructed by successive approximations, (iii) there exists a countable set* $\{h^n\}_{n\in\mathbb{N}}$ *of bounded measurable functions such that any bounded RE* $h \in W$ *satisfies* $h(s) \in cl\{h^1(s), h^2(s), \ldots\}$ *for each* $s$.

*Proof* (i) From Proposition 2, $Ah$ transforms the subcomplete set of bounded functions $[h_0, w] \subset W$. By Lemma 1, $Ah$ is isotone. The result then follows from Tarski's theorem (e.g.,Tarski 1955, Theorem 1). (ii) When restricted to the subcomplete order interval $H^l \cap [h_0, w]$, we have $0 < h_{\min} = \vee\{A^n h_0\}$ when $k > 0$ with

$$h_{\min}(s) = \vee\{A^n h_0\}(s) = \lim_{n\to\infty} A^n h_0(s) = \sup\{A^n h_0(s)\}.$$

Notice $h_{\min}$ is lsc as it is the upper envelope of a family of elements of lsc functions. It is therefore the minimal bounded isotone and lsc RE in $W \cap [h_0, w]$. It is also the minimal RE in $H$ with the addition of Assumption 3'. Similarly, the maximal RE in $(W, \leq)$ is obtained as the inf (pointwise limit) of a decreasing sequence beginning at $w$. That is, is:

$$h_{\max}(s) = \wedge\{A^n w\}(s) = \lim_{n\to\infty} A^n w(s) = \inf\{A^n w(s)\},$$

which implies that $h_{\max} \in H^u$ since it is the lower envelope of a family of elements of $(H^u, \leq)$. (ii) Note that the function $p : S^* \times X \to R$ defined as:

$$p(s, y) = -\left| \int_Z [u_1(w(s) - y, r(y, z')y) - u_2(w(s) - y, r(y, z')y)r(y, z')]\gamma(\mathrm{d}z') \right|$$

is continuous, and the correspondence $\Psi : S \to \mathbb{R}$ defined as $\Psi(s) = [0, w(s)]$ is non-empty, compact valued and measurable (the RE are constructed has the nonzero maximizers of $p$). As a consequence of the measurable maximum theorem (see for instance, Aliprantis and Border 1999, corollary 17.8), the correspondence $\Phi$ defined as $\Phi(s) = \arg\max_{y\in\Psi(s)} p(s, y)$ is measurable, non-empty and compact valued. By Castaing's theorem (see Aliprantis and Border 1999, Corollary 18.14), this implies that there exists a countable sequence $\{h^n\}_{n\in\mathbb{N}}$ of measurable selectors from $\Phi$ satisfying:

$$\forall s \in S, \Phi(s) = cl\{h^1(s), h^2(s), \ldots\}$$

□

We now prove a second existence theorem concerning the existence and computation of *non-trivial* least and greatest RE within the subclass of function $H^u$ and $H^l$ :

**Theorem 2** *Under Assumptions 1, 2, 3, 3', 4 the set of RE in* $(H^u, \leq)$ *is a non-empty complete lattice with minimal* $g_{\min}$ *and maximal elements* $h_{\max}$*, and both can be constructed by successive approximations. All the RE in* $(H^u, \leq)$ *are measurable. Further, when restricted to* $H^u \cap [h_0, w]$*, the set of RE is a non-empty complete lattice, with least and greatest fixed points constructed by successive approximations.*

*Proof* (i) Following the same argument as in Theorem 1, it is only a matter of correcting $h_{\min}$ at most at a countable number of points to obtain the minimal bounded isotone and usc RE. Specifically, the minimal RE in $(H^u, \leq)$ is the function $g_{\min} : S \to X$ defined as:

$$
\begin{aligned}
g_{\min}(s) &= \inf_{k'>k} \{\sup\{A^n h_0(k', z)\}\} \\
&= \inf_{k'>k} \{\vee\{A^n h_0\}(k', z)\} \forall s = (k, z) \in [0, k_{\max}) \times Z
\end{aligned}
$$

and $g_{\min}(k_{\max}, z) = \vee\{A^n h_0\}(k_{\max}, z)$. Indeed, by construction $g_{\min} \in H^u$, $g_{\min}(., z)$ and $q(., z) = \vee\{A^n h_0\}(., z)$ differ at most at the discontinuity points of $\vee\{A^n h_0\}(., z)$, and $g_{\min}(., z)$ is the smallest usc function greater than $\vee\{A^n h_0\}(., z)$. In addition, since $\vee\{A^n h_0\}$ is lsc, for any $s \in S$, $g_{\min}(s) = \lim_{k' \to k^+} \vee\{A^n h_0\}(k', z)$. For any $s = (k, z) \in [0, k_{\max}) \times Z$, and for all $k' > k$, by definition of $q(., z)$:

$$
\begin{aligned}
&\int_Z u_1(w(k', z) - q(k', z), r(q(k', z), z')q(k', z))\gamma(\mathrm{d}z') \\
&\quad = \int_Z u_2(w(k', z) - q(k', z), r(q(k', z), z')q(k', z))r(q(k', z), z')\gamma(\mathrm{d}z').
\end{aligned}
$$

Both functions $u_1$ and $u_2$ are continuous and $r$ is continuous in its first argument, taking limits when $k' \to k^+$ on both sides of the previous equality implies:

$$
\begin{aligned}
&\int_Z u_1(w(s) - g_{\min}(s), r(g_{\min}(s), z')g_{\min}(s))\gamma(\mathrm{d}z') \\
&\quad = \int_Z u_2(w(s) - g_{\min}(s), r(g_{\min}(s), z')g_{\min}(s))r(g_{\min}(s), z')\gamma(\mathrm{d}z'),
\end{aligned}
$$

which proves that $Ag_{\min}(s) = g_{\min}(s)$. The set of RE in $(H^u, \leq)$ is then the set of fixed point of $A$ that is bounded, isotone, and usc. For (ii), there are a non-empty complete lattice of RE follows from Tarski's theorem, noting the fact that $H^l \cap [h_0, w]$ is a complete lattice, and $Ah$ is isotone. The successive approximation result follows from a similar construction to part (ii) noting that as a consequence of the Theorem 8(ii), (a) $A$ must have a fixed point greater $h_0$, and (b) $Ah$ is order continuous when restricted to $H^l \cap [h_0, w]$, we must have $0 < h_{\min} = \vee\{A^n h_0\}$ when $k > 0$. □

Finally, note that it is easy to modify the usc function $h_{\max}$ at most at a countable number of points to construct the maximal bounded isotone and lsc RE.

### 3.3 Uniqueness of RE under capital income monotonicity

Under the additional assumption of capital income monotonicity (the only case discussed in Wang 1993), we prove the existence of a single Lipschitzian $h^*$ that is unique relative to a very large set of functions (namely, the set of bounded increasing functions $(H, \leq)$. The argument is direct: as under capital income monotonicity, any RE for investment that is semicontinuous must both be usc and lsc (and, therefore continuous). This turns out to imply the RE equilibrium consumption decision policy is also isotone (i.e., we have both $w - h^*$ and $h^*$ are jointly isotone). As under our assumptions, $w$ is also Lipschitz continuous in its arguments, both consumption and investment must Lipschitz continuous)

**Theorem 3** *Under Assumption 1, 2, 3, 3′, and 4, if $r(y, z)y$ is isotone in $y$ for all $z \in Z$ (an hypothesis we call "capital income monotonicity" (i) there exists a unique bounded isotone RE $h^*$ in $H$. Further, the corresponding (Markovian) equilibrium consumption policy, $w - h^*$ is also isotone, which implies that both $h^*$ and $w - h^*$ are Lipschitz continuous. Finally, (ii) the uniqueness result is robust relative to the space $(H \cap [h_0, w], \leq)$.*

*Proof* (i) Under capital income monotonicity, for all $s \in S^*$ the following equation in $y$:

$$\int_Z u_1(w(s) - y, r(y, z')y)\gamma(\mathrm{d}z') = \int_Z u_2(w(s) - y, r(y, z')y)r(y, z')\gamma(\mathrm{d}z').$$

has a unique solution, denoted $h^*(s)$. The function $h^*$ is thus the maximal and minimal RE and therefore usc and lsc in $k$, i.e., continuous in $k$. By definition, for all $s \in S^*$:

$$\begin{aligned}&\int_Z u_1(w(s) - h^*(s), r(h^*(s), z')h^*(s))\gamma(\mathrm{d}z')\\ &\quad = \int_Z u_2(w(s) - h^*(s), r(h^*(s), z')h^*(s))r(h^*(s), z')\gamma(\mathrm{d}z').\end{aligned} \tag{E''}$$

Suppose there exists $s = (k, z) \in X^* \times Z$ such that $w(k, z) - h^*(k, z)$ decreases with an increase in $k$. Then, for all $z' \in Z$, the expression:

$$u_1(w(k, z) - h^*(k, z), r(h^*(k, z), z')h^*(k, z))$$

increases with $k$ under the assumption of capital income monotonicity, and given that $h^*(k, z)$ is isotone in $k$, $u_{12} \geq 0$ and $u_{11} \leq 0$. However, for all $z' \in Z$, the expression:

$$u_2(w(k, z) - h^*(k, z), r(h^*(k, z), z')h^*(k, z))r(h^*(k, z), z')$$

necessarily decreases with an increase in $k$. Thus, right-hand side and the left-hand side in equation (E'') above move in opposite direction when $k$ increases, which is impossible. As a result, $w(k, z) - h^*(k, z)$ must be increasing in $k$. The same argument works to show that $w(k, z) - h^*(k, z)$ must be isotone in $z$. Finally, under the assumption that $w$ is continuous, if both the equilibrium investment and the equilibrium consumption policies are isotone, they both necessarily must be continuous. (ii) To see the uniqueness result in (i) is robust to the space $(H \cap [h_0, w], \leq)$, notice in the above argument, if $h^*(k, z) \in (H, \leq)$ is a fixed point, $Ah^*(k, z)$ has $w(k, z) - h^*(k, z)$ increasing when $k > 0$. As the set of fixed points in $(H \cap [h_0, w], \leq)$ is a complete lattice, wlog say we have two ordered fixed points, $h_0 \leq h_1^* \leq h_2^*$. When $k > 0$, by the definition of $Ah$ at $h_2^*$, we have

$$Z(h_2^*, k, z, h_2^*) = -\int_Z u_1(w - h_2^*, r(h_2^*, z')h_2^*)\gamma(\mathrm{d}z') + \int_Z u_2(w - h_2^*, r(h_2^*, z')h_2^*)r(h_2^*, z')\gamma(\mathrm{d}z') = 0$$

As by hypothesis, $h_1^* \neq h_2^*$ for some $\hat{k} > 0$, $h_1^*(\hat{k}, z) < h_2^*(\hat{k}, z)$, by capital income isotonicity, this implies $Z(h_1^*, \hat{k}, z, h_1^*) > 0$, which is a contradicts $h_1^*$ being a fixed point at $\hat{k}$. □

Finally, we stress three important facts relative to the claims made in the existing literature. First, our uniqueness under capital income isotonicity works on relative to the space $(H \cap [h_0, w], \leq)$. In particular, we cannot rule out other RE in the interval $[0, h_0)$, where $h_0$ is the lower positive solution (strictly positive when $k > 0$, all $z$) constructed Proposition 2. It remains an open question if any such uniqueness holds relative to larger sets of functions (even bounded increasing functions, let alone bounded functions) in the interval $[0, h_0)$. Second, our uniqueness result per RE is not implied by the uniqueness argument in Wang (1993) (Lemma 3.1) for "self-fulfilling equilibria" (even under capital income monotonicity). This, therefore, implies that our uniqueness result does *not* imply the "self-fulfilling expectations correspondence" for our models is a *function* even under capital income isotonicity (as our uniqueness result is relative only to the space $(H \cap [h_0, w], \leq)$ as discussed above).[12] Third, capital income isotonicity is not necessary for uniqueness of RE within the class of bounded increasing functions (as shown by the following example shows).

*Example 1* Consider the utility function:

$$\ln(c_t) + \ln(c_{t+1}),$$

in which case the maximization problem of an agent is:

$$\max_{y \in [0, w(s)]} \left\{ \ln(w(s) - y) + \int_Z \ln(r(h(s), z')y)\gamma(\mathrm{d}z') \right\},$$

[12] This correspondence is defined in Wang (1994), Lemma 3.1.

and the associated first-order condition is:

$$(w(s) - y) = y,$$

so that the unique RE is the function $h = .5w$.

## 4 Computing Stationary Markov equilibrium

We define a SME as a "non-trivial" (i.e., not all mass is concentrated at 0) invariant distribution, in line with the work of Hopenhayn and Prescott (1992) and Futia (1982), and in contrast to Wang (1993) and Wang (1994) who follows the path of Duffie et al. (1994) and focuses on ergodic distributions. Our main contribution in this section is to provide explicit iterative algorithms that converge in order (and topology) to extremal invariant probability measures corresponding to any isotone and measurable RE $h$. When the RE is also continuous (which is the case under capital income monotonicity), the stochastic operator has the Feller property and is thus an order continuous operator mapping the complete lattice $\Lambda(X, B(X))$ into itself. Applying Theorem 6 of Sect. 2, we prove that the set of SME is a non-empty complete lattice. When the isotone RE is only semicontinuous, the stochastic operator is at least order continuous along some recursive sequences, and this is sufficient for establishing the existence of minimal and maximal SME by Corollary 7 of Sect. 2. We treat the case of $h$ continuous first before proceeding with the general case.

### 4.1 SME associated with a continuous isotone RE

Any measurable bounded RE $h$ induces a Markov process for the capital stock represented by the transition function $P_h$ defined as:

$$\forall A \in \mathcal{B}(X),\, P_h(k, A) = \Pr\{h(k, z) \in A\} = \gamma(\{z \in Z, h(k, z) \in A\})$$
$$= \int\limits_Z \chi_A(h(k, z))\gamma(\mathrm{d}z).$$

and an associated operator $T_h^* : (\Lambda(X, \mathcal{B}(X)), \geq_s) \rightarrow (\Lambda(X, \mathcal{B}(X)), \geq_s)$ defined as:

$$\forall B \in \mathcal{B}(X),\, \mu_{t+1}(B) = T_h^* \mu_t(B) = \int P_h(k, B)\mu_t(\mathrm{d}k). \qquad \text{(M1)}$$

That is, $\mu_{t+1}(B)$ is the probability that $k_{t+1}$ lies in the set $B$ if $k_t$ is drawn according to the probability measure $\mu_t$. We define a Stationary Markov equilibria (SME) as a non-trivial fixed point of $T_h^*$.

**Definition 4** Given a measurable RE $h$, a SME is a probability measure $\mu \in \Lambda(X, \mathcal{B}(X))$ distinct from $\delta_0$ such that:

$$\forall B \in \mathcal{B}(X), \mu(B) = T_h^* \mu(B) = \int P_h(k, B)\mu(\mathrm{d}k).$$

It is easy to verify that if $h$ is isotone and continuous, the Markov operator $T_h^*$ is an isotone self map on $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ and $P_h$ has the Feller property or, equivalently, that $T_h^*$ is a weakly continuous and isotone operator (see, for instance, Exercises 8.10 and 12.7 in Stokey et al. 1989). These two properties imply that $T_h^*$ is order continuous along any monotone sequence. Indeed, if the sequence $\{\mu_n\}$ is increasing, then $\mu_n \Rightarrow \mu = \vee\{\mu_n\}$, so that $T_h^*(\mu_n) \Rightarrow T_h^*(\vee\{\mu_n\})$ by weak continuity. Since $T_h^*$ is an isotone operator, the sequence $\{T_h^*(\mu_n)\}$ is also increasing and therefore $T_h^*(\mu_n) \Rightarrow \vee\{T_h^*(\mu_n)\}$. By uniqueness of the limit, $T_h^*(\vee\{\mu_n\}) = \vee\{T_h^*(\mu_n)\}$, which proves continuity along any increasing sequence. The existence and computational results below follow directly from Theorem 6 of Sect. 2.

**Theorem 4** *For any continuous and isotone RE $h$, the set of fixed points of $T_h^*$ is a non-empty complete lattice with maximal and minimal elements, respectively $\wedge\{T_h^{*n}\delta_{k_{\max}}\}$ and $\vee\{T_h^*\delta_0\}$.*

Since our definition of SME excludes $\delta_0$, the previous result does not necessarily imply the existence of a SME. Indeed, suppose for instance that:

$$\forall (k, z) \in S^*, 0 < h^*(k, z) < k.$$

It is then easy to see that given any initial distribution of capital stock, in the long run, the capital stock will be 0. The only fixed point of $T_{h^*}^*$ is $\delta_0$, and the set of SME is therefore empty, a case taking place for instance when $w(k, z) < k$ for all $(k, z)$ in $S^*$. Thus, one needs sufficient conditions under which the set of SME is non-empty, and it is most useful to express any such condition in terms of restrictions on the primitives of the problem (unlike in Wang 1993).

While condition (I) in Assumption 5 below is necessary, condition (II) is sufficient for the existence of a specific element $h_0$ of $H$ to be mapped up strictly by $A$. It implies that the isotone operator $A$ maps the order interval $[h_0, w] \subset H$ (a complete lattice when endowed with the pointwise order) into itself, so that $A$ must have a fixed point in this interval. Since under the assumption of capital income monotonicity, the fixed point $h^*$ of $A$ in $H$ is unique, it must be that:

$$\forall k \in [0, k_0] \quad \text{and } \forall z \in Z, h^*(k, z) > h_0(k, z)(> k).$$

Given this property of $h^*$, we show that there exists a fixed point of $T_{h^*}^*$ that is distinct from $\delta_0$. The argument is the following: Consider any measure $\mu_0$ with support in $[0, k_0]$ and distinct from $\delta_0$ (we write $\mu_0 >_s \delta_0$). Since $h^*$ maps up strictly every point in $[0, k_0]$, $\mu_0$ is mapped up strictly by the operator $T_{h^*}^*$. By isotonicity of $T_{h^*}^*$ the sequence $\{T_{h^*}^{*n}\mu_0\}$ is increasing, and by order continuity along monotone sequences of $T_{h^*}^*$, it weakly converges to a fixed point of $T_{h^*}^*$. Clearly, by construction, this fixed point is strictly greater than $\delta_0$. We spend the rest of this subsection of the paper to formalize this argument.

**Assumption 5** Assume that:

(I) There exists a right neighborhood $\Delta$ of 0 such that for all $k \in \Delta$ and all $z \in Z$, $w(k, z) \geq k$.
(II) The following inequality holds:

$$\lim_{k \to 0^+} u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) < \lim_{k \to 0^+} u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}).$$

Note that for log separable utility, condition (II) in Assumption 5 is equivalent to:

$$\lim_{k \to 0^+} (w(k, z_{\min})/k) > 2,$$

and under a Cobb-Douglas production function with multiplicative shocks, it is trivially satisfied (and so is condition (I)). For a polynomial utility of the form $u(c_1, c_2) = (c_1)^{\eta_1}(c_2)^{\eta_2}$, the condition is equivalent to:

$$\lim_{k \to 0^+} (w(k, z_{\min})/k) > \left[1 + \frac{\eta_1 r(k, z_{\max})}{\eta_2 r(k, z_{\min})}\right],$$

also trivially satisfied with Cobb-Douglas production and multiplicative shocks.

We can prove a key proposition that extends the uniqueness result in Datta et al. (2002) and Mirman et al. (2008) obtained for infinite horizon economies to the present class of OLG models under Assumption 5. In particular, we show existence of minimal and maximal SME.

**Theorem 5** *Under Assumption 5, the set of SME associated with an isotone continuous RE h is a non-empty complete lattice. The maximal SME is* $\wedge\{T_{h^*}^{*n}\delta_{k_{\max}}\}$, *and there exists* $k_0 \in X$ *such that the minimal SME is* $\vee\{T_{h^*}^{*n}\delta_{k'}\}$ *for any* $0 < k' \leq k_0$.

*Proof* The proof is in two parts. Part 1 establishes the existence of $h_0$ that is mapped up strictly by the operator $A$, and Part 2 shows the existence of a probability measure $\mu_0$ that is mapped up $T_{h^*}^*$, where $h^*$ is the unique RE. Part 1. By continuity of all functions in $k$, the inequality in Assumption 5 must be satisfied in a right neighborhood of 0. That is, there exists $\Theta = (0, k_0] \subset \Delta$ such that, $\forall k \in \Theta$ :

$$u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) < u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}).$$

Consequently, $\forall k \in \Theta = (0, k_0]$:

$$\begin{aligned}
&\int_Z u_1(w(k,z)-k, r(k,z')k)G(\mathrm{d}z') \\
&\leq u_1(w(k,z_{\min})-k, r(k,z_{\max})k) \\
&< u_2(w(k,z_{\min})-k, r(k,z_{\max})k)r(k,z_{\min}) \\
&\leq \int_Z u_2(w(k,z)-k, r(k,z')k)r(k,z')G(\mathrm{d}z').
\end{aligned}$$

Next, consider the function $h_0 : X \times Z \to X$ defined as:

$$h_0(k,z) = \begin{array}{lll} 0 & \text{if} & k=0, z \in Z \\ k & \text{if} & 0 < k \leq k_0, z \in Z \\ k_0 & \text{if} & k \geq k_0, z \in Z \end{array}.$$

We prove now that $Ah_0 > h_0$. First, consider $0 < k \leq k_0$, $z \in Z$, and suppose that $Ah_0(k,z) \leq h_0(k,z) = k$. Then:

$$\begin{aligned}
&\int_Z u_1(w(k,z)-k, r(k,z')k)G(\mathrm{d}z') \\
&< \int_Z u_2(w(k,z)-k, r(k,z')k)r(k,z')G(\mathrm{d}z') \\
&\leq \int_Z u_2(w(k,z)-Ah_0(k,z), r(k,z')Ah_0(k,z))r(Ah_0(k,z),z')G(\mathrm{d}z'),
\end{aligned}$$

where the first inequality stems from the result just above, and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and $r$ decreasing in its first argument. By definition of $Ah_0$, this last expression is equal to:

$$\int_Z u_1(w(k,z)-Ah_0(k,z), r(k,z')Ah_0(k,z))G(\mathrm{d}z').$$

Thus, we have $Ah_0(k,z) \leq k$ and:

$$\begin{aligned}
&\int_Z u_1(w(k,z)-k, r(k,z')k)G(\mathrm{d}z') \\
&< \int_Z u_1(w(k,z)-Ah_0(k,z), r(k,z')Ah_0(k,z))G(\mathrm{d}z').
\end{aligned}$$

which contradicts the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. It must therefore be that for all $k \in (0, k_0]$ and all $z \in Z$, $Ah_0(k,z) > h_0(k,z) = k$, i.e., $A$ maps $h_0$ strictly up at least in the interval $]0, k_0]$. Finally, for $k > k_0$, since $Ah_0$ is isotone in its first argument:

$$Ah_0(k, z) \geq Ah_0(k_0, z) > h_0(k_0, z) = k_0 = h_0(k, z).$$

We have thus established that $A$ maps $h_0$ up (strictly). Since the order interval $[h_0, w]$ in $(H, \leq)$ is a complete lattice when endowed with the pointwise order, then by isotonicity of $A$ there must exist a fixed point of $A$ in that interval. Under capital income isotonicity, $h^* \in [h_0, w]$. Part 2. Consider any probability measure in $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ with support in the compact interval $[0, k_0]$ and distinct from $\delta_0$. We show that $T^*_{h^*}\mu_0 \geq_s \mu_0$. Consider any $f : X \to \mathbb{R}_+$ measurable, isotone and bounded, we have:

$$\int \left[ \int f(k') P_{h^*}(k, \mathrm{d}k') \right] \mu_0(\mathrm{d}k) = \int \left[ \int f(h^*(k, z))\lambda(\mathrm{d}z) \right] \mu_0(\mathrm{d}k)$$
$$= \int_{[0,k_0]} \left[ \int_Z f(h^*(k, z))\lambda(\mathrm{d}z) \right] \mu_0(\mathrm{d}k) + \int_{[k_0,k_{\max}]} \left[ \int f(h^*(k, z))\lambda(\mathrm{d}z) \right] \mu_0(\mathrm{d}k)$$
$$\geq \int_{[0,k_0]} f(k)\mu_0(\mathrm{d}k)$$

since $h^*(k, z) > k$ on $[0, k_0]$. Note that if $f$ is strictly positive on $[0, k_0]$ then the last inequality is strict. We have just demonstrated that $T^*_{h^*}\mu_0 \geq_s \mu_0$ and that $T^*_{h^*}\mu_0$ is distinct from $\mu_0$, so we write $T^*_{h^*}\mu_0 >_s \mu_0 (>_s \delta_0)$. By order continuity along any monotone sequence of $T^*_{h^*}$, necessarily the increasing sequence $\{T^{*n}_{h^*}\mu_0\}$ converges weakly to a fixed point of $T^*_{h^*}$ strictly greater than $\delta_0$. In addition, it is easy to see that there cannot be any fixed point of $T^*_{h^*}$ with support in $[0, k_0]$ other than $\delta_0$ so that the minimal non-trivial (i.e., distinct from $\delta_0$) fixed point of $T^*_{h^*}$, which is by definition the minimal SME, can be constructed as the limit of the sequence $\{T^{*n}_{h^*}\mu_0\}$, where $\mu_0 = \delta_{k'}$ for any $0 < k' \leq k_0$. This completes the proof that the set of SME is the non-empty complete lattice of fixed points of $T^*_{h^*}$ minus $\delta_0$ and that the maximal SME and minimal SME can be obtained as claimed. □

## 4.2 Constructing extremal SME for semicontinuous RE

Continuity of $h$, however, is not necessary for $T^*_h$ to be order continuous along recursive monotone $T^*_h$-sequences. Indeed, consider iid shocks as a special case of Markov shocks for which the transition function is defined as $Q(z, B) = G(B)$, and recall that a Markov transition function $Q$ satisfies Doeblin's condition is there exists $\delta \in \Lambda(Z, B(Z))$ and $\theta < 1$ and $\eta > 0$ such that:

$$\forall B \in \mathcal{B}(Z), \delta(B) \geq \theta \quad \text{implies that } \forall z \in Z, Q(z, B) \geq \eta.$$

In particular, since $Q(z, B) = \gamma(B)$, then any $\theta = \eta < 1$ and $\delta = \gamma$ show that iid shocks trivially satisfy Doeblin's condition. Clearly, if $Q$ satisfies Doeblin's condition, then the transition function $P_h$ corresponds to any measurable RE $h$ and is defined by:

$$\forall A \times B \in \mathcal{B}(S),\ P_h(x, z; A, B) = \begin{cases} Q(z, B) & \text{if} \quad h(x, z) \in A \\ 0 & \text{otherwise.} \end{cases}$$

also satisfies Doeblin's condition. Consequently, by Theorem 11.9 in Stokey et al. (1989), the n-average of any recursive $T_h^*$-sequence converges in the total variation norm and therefore weakly converges to a fixed point of $T_h^*$ (which is isotone). This implies that any monotone recursive $T_h^*$-sequence weakly converges and that the limit is a fixed point of $T_h^*$. This precisely proves that $T_h^*$ is order continuous along recursive monotone $T_h^*$-sequences. Notice, this is basically the argument followed in Morand and Reffett (2007) to prove that in an OLG model with Markov shocks associated with the transition function $Q$. That is, if $Q$ is increasing and satisfies Doeblin's condition, then the measurability of any isotone RE $h$ is sufficient for $T_h^*$ to be order continuous along recursive monotone $T_h^*$-sequences. By a standard argument (i.e., the Tarski-Kantorovich theorem), this type of order continuity permits the construction of extremal SME by successive approximations, as stated in the following result:

**Theorem 6** *Under Assumptions 1, 2, 3, 3′, 4, and 5, for any measurable RE $h$ in $H$, there exists a non-empty set of SME with maximal and minimal elements respectively given by* $\gamma_{\max}(h) = \wedge\{T_h^{*n}\delta_{(k_{\max}, z_{\max})}\}$ *and* $\gamma_{\min}(h) = \vee\{T_h^{*n}\mu_0\}$*, where* $\mu_0 = \delta_{(k', z_{\min})}$ *for any* $0 < k' \le k_0$*,* $k_0$ *constructed from Assumption 5.*

Since all elements of $H^u$ are measurable (see "Appendix B"), the following result also holds:

**Corollary 1** *Under Assumptions 1, 2, 3, 3″, 4, and 5, to any RE in $H^u$ corresponds a non-empty set of SME with maximal and minimal elements.*

Finally, for economies satisfying Assumption 4, by our results in the previous section of the paper, there exist minimal and maximal RE $h_{\min}$ and $h_{\max}$ in $H$, and both are measurable. Necessarily, any other RE $h$ in $H$ satisfies $h_{\min} \le h \le h_{\max}$, and therefore:

$$T_{h_{\min}}^* \mu_0 \le T_h^* \mu_0,$$

and recursively,

$$\gamma_{\min}(h_{\min}) = \vee\{T_{h_{\min}}^{*n}\mu_0\}_{n \in N} \le \vee\{T_h^{*n}\mu_0\}_{n \in N} = \gamma_{\min}(h).$$

By a similar argument:

$$\gamma_{\max}(h_{\max}) = \wedge\{T_{h_{\max}}^{*n}\delta_{(k_{\max}, z_{\max})}\}_{n \in \mathbb{N}} \ge \wedge\{T_h^{*n}\delta_{(k_{\max}, z_{\max})}\}_{n \in \mathbb{N}} = \gamma_{\max}(h),$$

and this proves that $\gamma_{\max}(h_{\max})$ and $\gamma_{\min}(h_{\min})$ are the greatest and least SME, respectively. We state this very general result in the last proposition of the paper.

**Theorem 7** *Under Assumptions 1, 2, 3, 3′, 4 and 5, the set of SME is non-empty and there exist maximal and minimal SME, respectively* $\gamma_{\max}(h_{\max}) = \wedge\{T^{*n}_{h_{\max}}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ *and* $\gamma_{\min}(h_{\min}) = \vee\{T^{*n}_{h_{\min}}\mu_0\}_{n\in N}$ *where* $\mu_0 = \delta_{(k',z_{\min})}$ *for any* $0 < k' \leq k_0$, $k_0$ *constructed from Assumption 5.*

## 4.3 Uniqueness of SME under capital income monotonicity

Significant progress has been made in proving uniqueness of SME in stochastic optimal growth economies, with distinct (but not unrelated) methods of proof showing some success. One line of proof uses a Liapunov function constructed from the Euler equation (see for instance, Nishimura and Stachurski 2005, among others). Another approach is presented in Mirman (1972, 1973), Brock and Mirman (1972) and Zhang (2007). This method rests on the stability properties of the "reverse Markov process" associated with the inverse of the optimal policy. Finally, a third alternative to stochastic stability is to prove directly the existence of a monotone mixing condition under a RE policy function (e.g., using the equilibrium Euler equation), as in Hopenhayn and Prescott (1992).It turns out all of these results are straightforward to apply to models with infinitely lived agents, where there is a set of stationary equilibrium Euler equations describing the stochastic dynamics in the model. This is not the case for stochastic OLG models, unfortunately.

One problem that immediately arises, for example, when applying the stochastic stability approach of Nishimura and Stachurski (2005) is finding the appropriate Liapunov function (or "norm-like function) for the problem. In a model with infinitely lived agents, one can use the equilibrium Euler equation (which is stationary in policies). Unfortunately, such a construction is far from obvious in stochastic OLG models (as consumption and investment, for example, are not stationary and depend on the age of cohorts (e.g., in a two-period model, consumption in first and second period are distinct functions, and knowing investment in the first period is not sufficient to build the L-function). This same exact issue also makes it difficult to pursue the monotone mixing condition approach in Hopenhayn and Prescott (1992).

It turn out, though, the approach of Mirman (1972, 1973), Brock and Mirman (1972) and Zhang (2007) does apply to stochastic OLG models under the assumption of capital income monotonicity. That is, the model just works on the properties of the policy directly (perhaps along with the equilibrium Euler equation where they are defined) and studies the stochastic structure of inverse Markov processes. We, therefore, just apply the results in this literature to verify conditions for stochastic stability. One nice feature with these techniques is one does not need to assume the shock process admits a density.

For this application, we must an assumption on multiplicative shocks.

**Assumption 6** The production function $F(k, n, K, N, z) : X \times [0, 1] \times X \times [0, 1] \times Z \to \mathbb{R}_+$ has positive multiplicative shocks (that is, $F(k, n, K, N, z) = F(k, n, K, N) \cdot z$

We now have the following Proposition.

**Proposition 3** *Under assumptions 1,2,3,3',4,5, 6, if $r(y,z)y$ is isotone in $y$ for all $z \in Z$ (capital income monotonicity), there is a unique SME, precisely equal to $\wedge\{T_{h^*}^{*n}\delta_{k_{\max}}\}$.*

*Proof* The proof follows as an application of Zhang (2007) for the special case of bounded shocks (e.g., see also Brock and Mirman 1972). We sketch the outline of the proof. By Theorem 3, the optimal consumption and optimal investment are monotone under capital income monotonicity under Assumptions 1,2,3,3',4 and capital income monotonicity. By Theorems 6 and Corollary 1, we have the existence of non-trivial probability measures mapped up and down and the existence of a non-trivial fixed point in Theorem 7 under Assumption 5 (as, for example, in Zhang 2007, Lemmas 5, 7 and Proposition 1). Assumptions 4 and 5 of Zhang (2007) are satisfied in the case of bounded strictly positive multiplicative shocks (our Assumptions 2 and 6). The uniqueness of a non-trivial fixed point, then, relies on the properties of the reverse Markov process introduced by Brock and Mirman (1972) and Zhang (2007), which are easily verified for the unique minimal state RE in Theorem 3. Then, the stability result in the Proposition follows exactly as in the proof of Zhang (2007) (Proposition 3 of Sect. 5.3).[13] □

One remark on Proposition 3. This uniqueness result of SME relies on the stochastic stability of any continuous RE (as in Theorem 3), the monotonicity of RE, as well as the uniqueness of minimal state space RE. As we shall show in the next section, if we "perturb" these space of economies in Sect. 2 to include production functions that imply income processes for households that are *not* increasing in states, we will lose existence of (i) monotone RE and (ii) uniqueness of continuous RE. In this case, Proposition 3 will fail. See Sect. 5.1.

## 5 Non-monotone minimal state space RE via isotone decompositions

We now extend our methods to economies where RE are not monotone. We study three cases: (i) models with two-period lived agents, but more general income processes, (ii) models with elastic labor supply, and (iii) models where agents of each generation live $N+1$ periods for $\infty > N > 1$. To study RE in these economies, we embed the actual system of RE functional equations (which, in general, neither defines an obvious monotone operator nor transforms a space of functions that are monotone) into a new system of functional equations defined on a enlarged state space. This new set of functional equations has very sharp monotone structure (namely it defines a monotone operator that transforms a space of functions defined on an enlarged set of aggregate state variables). Exploiting this monotone structure, we compute the fixed points of this new operator and then recover the actual set of minimal state space RE for our OLG economy as a restriction of these solutions along a particular subspace of this enlarged system of functional equation. We refer to this procedure as a *isotone decomposition method.*

[13] See also Brock and Mirman (1972), Sect. 4.

We begin with some definitions. Let $(X, \leq)$ be a poset, $f(x, y)$ a function. We say $f : X \times X \to X$ is *mixed monotone* if (i) the partial map $f_y(x)$ is isotone in $x$, each $y \in X$; (ii) $f_x(y)$ is antitone (i.e., monotone decreasing) in $y$, each $x \in X$. We denote by $(X^d, \leq^d)$ the space of $(X, \leq)$ endowed with its dual partial order $\leq^d$. Then, a mixed monotone function $f(x, y)$ is actually isotone (in each argument, not jointly) in the product space $X \times X^d$. We say a function $g(x)$ admits an *isotone decomposition* if $g(x)$ can be embedded into partially ordered space $X \times Y$ (with the product order where both $X$ and $Y$ are posets) as a diagonal of the function $f(x, y)$ such that $f(x, y)$ has (i) $f_y(x)$ isotone $x$, each $y \in X$, and (ii) $f_x(y)$ isotone in $y$, each $X$, with (iii) $g(x) = f(x, x)$, where $Y$ could either be $X$ or $X^d$. Notice, for an isotone decomposition, $f(x, y)$ is *not* jointly isotone. Similarly, a function $g(x)$ admits a *mixed-monotone decomposition* if we change condition (ii) for an isotone decomposition to the following condition: (ii)$'$ $f_x(y)$ antitone in $y$, each $x \in X$. So a function that admits a mixed-monotone decomposition admits an isotone decomposition on $X \times X^d$.

In the next section, we first provide a simple class of economies closely related to those in Sects. 2–4 but where (i) the minimal state space RE exist, (ii) capital income monotonicity holds, but (iii) the uniqueness result in Theorem 3 fails.

### 5.1 Failure of uniqueness of RE in simple models

We now consider a simple modification of economy in Sect. 2, but with primitive data for production that implies non-monotone lifecycle income processes. To make our point as simply as possible, we modify Assumption 3, so lifecycle income processes admit a mixed-monotone representation on an enlarged state space.

That is, recalling $X \subset \mathbb{R}$ is the space for the endogenous aggregate capital (endowed as before with the standard pointwise partial order), let $X^d$ be the space $X$ endowed with its *dual* partial order, and define $X^e = X \times X^d$, with typical element $k_e = (k, k^d) \in X^e$. So the expanded aggregate state variable will be $s_e = (k^e, z) \in S_e = X^e \times Z$ . Assume the (reduced-form) production function $F$ is consistent with wages $w$ and rental prices of capital $r$ from profit maximization both admit a mixed-monotone decomposition $w^e : S_e \to \mathbb{R}_+$ and $r^e : S_e \to \mathbb{R}_+$, respectively. The new version of Assumption A3 is therefore:

**Assumption 3$''$** The function $F(k, n, K, n, K^d, n, z) : X \times [0, 1] \times X \times [0, 1] \times X^d \times [0, 1] \times Z \to \mathbb{R}_+$ is:

- I twice continuously differentiable in its first two arguments;
- II isotone in all its arguments, strictly increasing and strictly concave in $k$ and $n$;
- IIIa $r^e(k, k^d, z) = F_1(k, 1, k, 1, k^d, 1, z)$ isotone and continuous in $k$, antitone and continuous in $k^d$, isotone and continuous in $z$, with $\lim_{k\to 0} r(k, k, z) = +\infty$;
- IIIb $w^e(k, k^d, z) = F_2(k, 1, k, 1, k^d, 1, z)$ isotone and continuous in $k$, antitone and continuous in $k^d$, isotone and continuous in $z$, with $\lim_{k\to 0^+} w(k, k, z) = 0$.

Examples of models that satisfy Assumption 3$''$ are easily produced. For example, consider OLG models with production nonconvexities (e.g., the non-classical model of growth described in Romer 1986). Other natural examples include models with taxes on current and future income are regressive with lump-sum transfers (e.g., Santos 2002).[14] Under Assumption 3″, first period income can be written as $m_1^e(s_e) = w^e(s_e)$, which is mixed-monotone decomposition of $w$, while second period income is $m_2^e(s_e) = r^e(s_e^{'})k'$, where $r^e$ is a mixed-monotone decomposition of $r$ that depends on tomorrow's expanded state variable $s_e^{'}$.

Define the sets $H^e = \{h^e | h^e : \mathbf{S}_e \to \mathbb{R}_+,\ 0 \leq h^e \leq w^e, h^e$ Borel measurable s.t. $h^e$ is isotone on $X^e\}$, and $H^C = \{h^e \in H^e | w^e - h^e$ isotone on $X^e\}$. Give each set the uniform topology and the pointwise partial order (noting the dual order on $X^d$). Then, the set $H^e$ is a countably chain complete when endowed the product order on $X^e \times Z$, while $H^C$ is countably chain subcomplete in $H^e$.[15] For the economies, under Assumptions 1, 2, 3″, 4, if households use the law of motion $h^e \in H^e$ to calculate the continuation of the aggregate economy in their first period of life in a candidate RE, and solve a standard optimization problem, letting initial states all be positive (i.e, for $S_e^* = X^* \times X^{d*} \times Z$, have initial state $s_e \in S_e^*$), then the young agents at $h^e \in H^e$ solve:

$$\max_{y \in [0, m_1^e]} \int_Z u(m_1^e(s_e) - y, r(h^e(s_e), h^e(s_e)), z')y)\gamma(\mathrm{d}z'),$$

Notice, this decision problem coincides with the household's decision problem in a minimal state space RE along the restriction to the subpace of the enlarged state space where $k = k^d$, each $z$ (i.e., the actual household's problem is embedded in a decision problem with a larger set of state variables).

To compute RE, we modify our previous Euler equation method to accommodate this more general framework as follows: for $k^e >> 0$, any $h^e \in H^e$, define $Ah^e$ as follows: if $h^e \in W^e$ (resp, $h^e \in H^e$), $h^e >> 0$, define $Ah^e(s_e)$ as the unique solution for $y$ in:

---

14 For a very simple and important example, consider the case of nonconvexities in production ala Romer (1986). Assume $f(k, K) = k^\alpha K^\beta$, with $\infty > \alpha + \beta > 1$ ($\alpha, \beta$ both positive). Here, we could take

$$r^e(k^d, k) \cdot z = \alpha(k^d)^{\alpha-1}(k)^\beta \cdot z$$

so in equilibrium we have increasing returns socially.
An example where this issue arises because of taxes is the following. Say the production function $F$ is concave, CRS, and supermodular in all arguments (e.g., Cobb-Douglas). If we also had a regressive tax on capital income as in Santos (2002), the return on capital would admit a mixed-monotone decomposition as follows:

$$r^e(k^e) \cdot z = r(k^d) \cdot z \cdot (1 - \tau_k(k, z)).$$

where $r = f'(k^d)$, and the tax $\tau_k$ is decreasing in $k$. Alternatively, if the tax was progressive for labor income (i.e, increasing in $k$), we would have

$$w^e(k^e) = w(k)(1 - \tau(k^d))$$

Notice, of these would be isotone on $X^e = X \times X^d$.

15 Neither are complete lattices because of the measurability conditions in the definitions of the spaces.

$$Z(y, s_e, h^e) = \int_Z [u_1(m_1^e(s_e) - y, r(y, h^e(s^e), z')y)$$
$$-u_2(m_1^e(s_e) - y, r(y, h^e(s^e), z')y) r(y, h^e(s^e), z')] \gamma(\mathrm{d}z') = 0$$

and $Ah^e(s_e) = 0$ whenever $h^e(s_e) = 0$ elsewhere.

Noting the dual order on $X^e$, we have $Z$: (i) strictly increasing in $y$ for each $(s_e, h^e)$; (ii) antitone in $(s_e, h^e)$ for each $y \in \mathbb{R}_+$. As a result:

**Lemma 3** *Under Assumptions 1, 2, 3″, the operator A is an isotone self map on* $(H^e, \leq)$.

*Proof* For fixed $(s_e, h^e)$, $h^e > 0$, $h^e \in H^e$, $Z$ is well defined. Further, as $Z$ is continuous and strictly increasing in $y$, $Ah^e(s_e)$ is also well defined (i.e., non-empty and single valued) and bounded. Further, by comparative statics in (i) and (ii) above imply when $h^e \in H^e$, $h^e > 0$, we have $Ah^e \in H^e$. Finally, if $h^e \in H^e, h^e > 0$, $Z$ is Caratheodory function (continuous in $y$ and measurable in $s_e$, each $h^e$). By Fillipov's measurable selection theorem (e.g., Aliprantis and Border 1999, Theorem 18.17), $Ah^e(s_e)$ admits a measurable selection. As $Ah^e(s_e)$ is single valued, it is the measurable selection. Noting the definition of $Ah^e$ elsewhere, if $h^e \in H^e$ (respectively, $H^e$), $Ah^e$ is measurable. All these facts together imply $Ah^e \in H^e$. Also, that for fixed $s^e$, $Ah^e$ is an isotone operator on $H^e$ that follows from the comparative statics in (i) and (ii), noting the definition of $Ah^e$ when $h^e \ngtr 0$. □

Of course, the actual state variable for the economy is $s \in S = X \times Z$ (not, $s_e \in S_e$); but, by construction, $S$ is embedded in $S_e$ as the product of (a) the diagonal of $X \times X^d$, and (b) the shocks $Z$. Let $H_s^e$ be the of functions $h \in H^e$ restricted to $s \in S$, and $H_s^C$ be the space of functions $h \in H^C$ restricted to $S$, and note $w^e(k, k, z) = w \in H_s^C$ by construction. Before we proceed to our main result, consider the following version of Assumption A4, which we shall refer to as *capital income mixed-monotonicity:*[16]

**Assumption A4′** Assume $F$ is such $\lim_{k \to 0^+} r(k, k^d, z_{\max})k \to 0$, with $r(k, k^d, z)k$ increasing in $k$ and $r(k, k^d, z)$ falling in $k^d$.

We now prove the main theorem in this section:

**Theorem 8** *(i) Under Assumptions 1, 2, 3″, the set of fixed points of A in* $(H^e, \leq)$ *is a non-empty countable chain complete poset, such that* $\inf_n A^n(w^e) \to h_*^e \in H^e$. *Further, a minimal state space RE for this economy is* $h^* = h_*^e(k, k, z) \in H_s^e$, *with* $h^*(k, z) > 0$ *when* $k > 0$. *Finally, under generalized capital income monotonicity in Assumption* 4′, *the RE* $h^*(k, z)$ *is continuous in k, and measurable in z.*

*Proof* $(H^e, \leq)$ is a countably chain complete poset.[17] That the operator $Ah^e$ is order continuous in $W^e$ (resp, $H^e$) follows from an argument similar to Lemma 2, noting

[16] For a simple example, again take the production function in Romer (1986), with

$$F(k, K, z) = k^\alpha K^\beta z$$

but now for $\alpha, \beta \geq 0$, $0 < \alpha + \beta < 1$. Then, we could take $r^e(k, k^d, z) = \alpha (k^d)^{\alpha - 1} K^\beta z$ , which is both falling when $k = k^d$, and has $r(k, k, z)k$ increasing in $k$, with the limiting condition also holding.

[17] I.e., complete only with respect to arbitrary sequences.

the pointwise convergence of $\Psi$ in $y$ and $h^e$. Then, the set of fixed points of $Ah^e$ is countably chain complete from a theorem in Balbus, Reffett, and Wozny (?, Theorem 2.1). Noting further that $w^e \in H^e$, we have $\inf_n A^n(w^e) \to h^e_* \in H^e$ the greatest fixed point follows from the Tarski-Kantorovich theorem. In particular, as $w(k, k, z) \in H^C$, we have $h^*(k, z) = h^e_*(k, k^e, z) \in H^C_s$ the greatest fixed point in $H^C_s$. The fact that we have $h^*(k, z) > 0$ when $k > 0$ follows from the Inada condition in A1, and hence $\inf_n A^n(w^e) \to h^e_*(k, k, z) = h^*(k, z) > 0$ when $k > 0$. This all implies $h^*(k, z)$ is a (non-trivial) RE for this economy. Finally, that $h^*(k, z) \in H^*$ is continuous follows the continuity part of the argument in Theorem 3 (i.e., as at any fixed point $h^*_e$, when $k^e_1 \geq k^e_2$, under the generalized capital income monotonicity Assumption in A4, the second term of $Z$ is falling in $k^e$ at $h^e_*$. Hence, we must have $Ah^e_*(k^e, z)$ such that $(m^e_1 - Ah^e_*)(k^e, z)$ must be rising (so that the first term of $Z$ is falling). □

One key remark needs to be made. As $h^e_*(k, k, z) = h^*(k, z)$ is *not* monotone necessarily under capital income mixed-monotonicity, the uniqueness part of the argument for RE in Theorem 3 *fails*. That is, one can perturb the vector $(k, k^d)$ in a manner such that the monotone comparative statics needed for the uniqueness part of the proof of Theorem 3 fails (as comparative statics of two terms in $Z$ under that perturbation at $h^e_*(k, k^e, z)$ are ambiguous). That is true, in particular, when $k = k^d$ (as $h^*(k, k, z)$ is not required to be monotone in $k$). So, many zeros of this equation are now possible in any RE at $(k, k, z)$, and this is true even under a version of the capital income mixed-monotonicity condition (e.g., OLG models with Romer technologies).

### 5.2 RE in models with elastic labor

We new use our isotone decomposition method to compute RE in two-period stochastic OLG models with elastic labor supply. For this section, we adopt the following variation of our original assumptions in Sect. 2:

**Assumption 1′** The utility function is $U(c, l) = u(c) + v(l)$, where $u\colon\mathbf{K} \to \mathbf{R}_+$ or $u(c) = \ln c$, $v : [0, 1] \to \mathbf{R}$, where $U(c, l)$,

- I twice continuously differentiable;
- II strictly increasing in each of its arguments and jointly concave;
- III $\lim_{l\to 0^+} v'(l) = +\infty$; $\lim_{c\to 0} u'(c) = +\infty$
- IV $u'(rx)r$ is increasing in $r$, each $x > 0$

Also, for the next two subsections of this paper, we shall assume:

**Assumption 2′** The random variable $z$ is iid with probability measure denoted $\gamma$ with support a countable subset set $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ with $z_{\max} > z_{\min} > 0$.

Finally, we need a slightly modified version of Assumption 3 to accommodate elastic labor in the production decisions (i.e., we need a notion of "labor income monotonicity"):

**Assumption 3‴** The production function $F(k, n, K, N, z) : X \times [0, 1] \times X \times [0, 1] \times Z \to \mathbb{R}_+$ in constant returns to scale in its first 4 arguments and has $f_2(k.n, k, n, z) \cdot n$ increasing in $n$, each $k$.

Assumption $1'$ is the only assumption we need to discuss. It is standard in applied work using lifecycle models, as the condition is satisfied, for example, for $u(c)$ and $v(l)$ power utility (while the separability condition in assumption A$1'$(i) is typical in applied work lifecycle models). We also remark, assumption $2'$, for this section, is just a simplifying assumption and is used to remove non-essential technical issues in this section associated with measurability.[18] It is worth mentioning that although A$2'$ is restrictive, it is also typical in the theoretical literature on existence of RE in stochastic OLG models (e.g., Citanna and Siconolfi 2010). Assumption $3'''$ places restrictions on the class of production functions and equilibrium distortions that we allow. It is satisfied for OLG models with nonconvexities in production (e.g., models with production externalities ala Romer 1986), as well as the class of tax structures considered in the infinite horizon case with elastic labor (e.g., Datta et al. 2002).

Let $N(k, z)$ be a continuous feasible aggregate labor supply decision (where feasibility requires $0 \leq N \leq 1$ for all $(k, z)$). Then, when $N > 0$, $s \in S^*$, for an aggregate law of capital $h \in W$ (where now the function space $W$ is defined using the upper bound $w_N(s) = w(k, N(k, z), z)$ for the elastic labor case), a young agent solves the following problem:

$$\max_{y \in [0, w_N(s)], n \in [0,1]} u(w_N(s)n - y) + v(1-n) + \int u\left(r\left(\frac{h(s)}{N(h(s), z')}\right), z') \cdot y\right)\gamma(\mathrm{d}z') \tag{1}$$

where, under Assumption $2'$, the integral is just a sum. When $c > 0$, imposing $n^* = N^* = N$, using the first-order condition on labor supply, we can define an equilibrium labor supply function to be the $n^*(c, k, z)$ the solves:

$$\frac{v'(1 - n^*(c, k, z))}{u'(c)} = w(k, n^*(c, k, z))$$

If we additionally let $n^*(0, k, z) = 1$, under Assumptions A$1'$ and A$3'''$, $n^*(c, k, z)$ is increasing in $k$, decreasing in $c$, and continuous in all its arguments. Also, define $n_f(k, z)$ as the solution to

$$\frac{v'(1 - n_f(k, z))}{u'(w(k, n_f(k, z), z))} = w(k, n_f(k, z), z)$$

where $n_f$ is the lower bound for labor supply and is positive when $k > 0$ by the Inada conditions.[19] Then, the household's income process when young at $n^*$ can be written

[18] We shall remark when discussing the existence theorem how this assumption can be relaxed to continuous shock spaces. Basically, we lose the complete lattice structure of the set of RE, but the least/greatest fixed points still exist, and the set of RE is a countably chain complete poset.

[19] By a global version of the implicit function theorem, one can show both $n^*(c, k, z)$ and $n_f(k, z)$ are $C^1$, and hence locally Lipschitz. We use this fact in our arguments below.

in equilibrium as

$$m = m(k, n^*(c, k, z), z)) = w(k, n^*(c, k, z), z)) \cdot n^*(c, k, z))$$

which under Assumption $3'''$ is increasing in $k$, and decreasing in $c$. Therefore, in a RE equilibrium, the next period's capital stock for any given current level of consumption $c$ for the young must be given by

$$\begin{aligned} k' &= m(k, n^*(c, k, z), z) - c \\ &= m_c(k, z) \end{aligned}$$

which is also increasing in $k$, and decreasing in $c$.

As in the previous section, enlarge the aggregate state variable to be $p^e = (k, k^d) \in P^e = X \times X^d$, where $X^d$ is again just the original state space $X$, but endowed with its dual partial order. Define the new state variable as $s_e = (k, k^d, z) \in \mathbf{S}_e = P^e \times Z$. Consider the following collection of functions:

$$\begin{aligned} H^c(\mathbf{S}_e) = \{c | 0 \leq c(s_e) \leq m(s_e) \leq m_f(s_e) \text{ all } s_e, c(s_e) \text{ increasing in } p^e \\ = (k, k^d), \text{ each } z \\ \text{such that } (m - c)(s_e) \text{ is increasing } p^e, \text{ each } z\} \end{aligned}$$

with $m_f = w(k, n_f(k, z), z)) \cdot n_f(k, z))$. By the Arzela-Ascoli Theorem, $H^C$ is compact in each argument (as its closed, pointwise bounded, and equicontinuous). For $c \in H^c(\mathbf{S}_e)$, the law of motion for capital in equilibrium is

$$\begin{aligned} k' &= m(k, n^*(c, k, z), z) - c \\ &= m_c(k, z) \end{aligned}$$

which increasing in $(k, z)$.[20] Also, it is convenient to define

$$m_y(k, z) = m(k, n^*(y, k, z), z) - y$$

which is increasing in $(k, z)$, and decreasing in $y$.

We now are ready to compute RE in two steps. For a pair of functions $(c, \hat{c}) \in H^C \times H^C$, $k > 0$, $\hat{c} < m$ for all states, noting the CRS assumption on Assumption $3'''$, define the marginal return on investment tomorrow (the second term of the Euler equation) to be

$$\Psi_2(y, s_e; c, \hat{c}) = \int u'(R^*(y, s_e, z'; c, \hat{c}) \cdot m_{\hat{c}}(k^d, z)) \cdot R^*(y, s_e, z'; c, \hat{c})\gamma(\mathrm{d}z')$$

[20] Notice, we shall use this law of motion for both $k'$ and $k^{d'}$.

where tomorrow's price of capital is

$$R^*(y, s_e, z'; c, \hat{c}) = r\left(\frac{k'}{n'}, z\right)$$
$$= r\left(\frac{m_y(k, z)}{n^*(c(m_y(k, z)), m_{\hat{c}}(k^d, z), z'), m_{\hat{c}}(k^d, z), z')}, z'\right) \quad (2)$$

The following lemma describes the comparative statics of $\Psi_2(y, s_e; c, \hat{c})$:

**Lemma 4** *Under assumptions A1", A2", A3‴, we have (i) for fixed* $(c, \hat{c}) \in H^C \times H^C$, $k > 0, \hat{c} < m_f$ *and* $z \in Z$, $\Psi_2(y, s_e; c, \hat{c})$ *is increasing in y, and decreasing in* $p^e = (k, k^d)$*; (ii) for fixed* $(y, s_e)$*, when* $\hat{c} < m_f$, $\Psi_2(y, s_e; c, \hat{c})$ *decreasing in* $(c, \hat{c})$.

*Proof* See "Appendix B". □

For fixed $\hat{c} \in H^C$ with $\hat{c} < m$, $k > 0$, the Euler equation associated with the household's problem in equilibrium can be rewritten as the following:

$$\Psi(y, s_e, c; \hat{c}) = \Psi_2(y, s_e, z, c; \hat{c}) - u'(y)$$

Consider an operator $A(c; \hat{c})(s_e)$ defined implicitly using $\Psi$ for $\hat{c} < m$,

$$A(c; \hat{c})(s_e) = x^* \text{ s.t. } \Psi(y^*, s_e, c; \hat{c}) = 0 \text{ when } k > 0, 0 < c \le m,$$
$$= 0 \text{ else}$$

$A_{\hat{c}}(c)(s_e)$ is our "first step" operator (i.e., $A(c; \hat{c})(s_e)$ parameterized at fixed $\hat{c} \in H^C$) when $\hat{c} < m$. Let $\Phi_A(\hat{c})$ be the set of fixed points of $A$ at $\hat{c} \in H^C$, $\hat{c} < m$. We have the following Lemma about the fixed points of operator $A_{\hat{c}}(c)(s_e)$ at such $\hat{c}$:[21]

**Lemma 5** *Under Assumptions A1′, A2′, A3‴ for* $0 \le \hat{c} < m_f$, *(i)* $\Phi_A(\hat{c})(s_e)$ *is a non-empty complete lattice,(ii) the successive approximations* $\inf_n A^n_{\hat{c}}(m_f)(s_e) \to \vee\Phi_A(\hat{c})(s_e)$ *converges in order (and uniformly* $(k, k^d)$*) to* $\vee\, \Phi_A(\hat{c})(s_e)$*, with* $\vee\, \Phi_A(\hat{c})(s_e) > 0$ *when* $k > 0$*; and (iii)* $\vee\Phi_A(\hat{c})(s_e)$ *is increasing in* $\hat{c}$.

*Proof* See "Appendix B". □

Using the greatest fixed point of the "first step" operator, consider a "second step" operator $B(\hat{c})$ defined as follows

$$B(\hat{c}) = \vee\Phi_A(\hat{c}) \text{ for } \hat{c} < m$$
$$= m \text{ else}$$

Let $\Phi_B$ be the set of fixed points of the operator $B(\hat{c})$. We now prove the main theorem of this section.

[21] It is here where Assumption A2′ is used in the argument, namely to get a non-empty complete lattice of fixed points for $A(c; \hat{c})$ (and, similarly for the second stage operator in the next theorem below). If we relax A2′ to A2, we have a countably chain complete set of fixed points at each stage. Aside from this detail, nothing else changes in the arguments (as least and greatest fixed points remain isotone selections). In particular, the definition of $B(\hat{c})$ does not change.

**Theorem 9** *Under A1′, A2′, A3‴, there exists a RE. Further, this RE can be computed as* $\sup_n B^n(0) = \lim_{n\to\infty} B^n(0) \to \wedge\Phi_B$.

*Proof* Let $\Phi_B$ be the fixed points of $B(\hat{c})$. As $H^c$ is a complete lattice, and by Lemma 5, we have $B(\hat{c})$ is isotone in $\hat{c}$, then by Tarski's theorem, $\Phi_B \subset H^C$ is a non-empty complete lattice (with trivial maximal fixed point $y$). Consider $\hat{c} = 0$. Then, for all $s_e$ when $k = k^d$, $k > 0$, the iterations $\{B^n(0)(s_e)\}_n$ when $k = k^d > 0$ must satisfy the functional equation

$$\Psi(B^{n+1}(0), s; B^{n+1}, B^n(0)) = \Psi_2(B^{n+1}(0), s; B^{n+1}, B^n(0)) - u'(B^n(0)) = 0$$

Therefore, for all such $s_e$, by the Inada condition in A1″, the iterations $\lim_n B^n(0) < m(s_e)$ for all such $s_e$. By the monotonicity of $B(\hat{c})$ for each fixed $s_e$, the sequence $\{B^n(0)\}$ is an increasing sequence pointwise, and therefore has $\lim_n B^n(0)(s_e) \to c^*(s_e) < m(s_e)$ for each $s_e$. Further, as $\{B^n(0)(s_e)\}$ is a countable chain in $H^c$, we have $\sup_n B^n(0)(s_e) = B^n(0)(s_e)$ (where the sup here is with respect to the pointwise order on $H^c$). By equicontinuity in each argument, we have $\sup_{n\to\infty} B^n(0)(s_e) = \lim_n B^n(0)(s_e) \to c^*(s_e) < m(s_e)$. Further, as each element of $\{B^n(0)(s_e)\} \in H^c$, the convergence for $\lim_n B^n(0) \to c^*(k, k^d, z)$ is uniform in $(k, k^d)$, when $k = k^d$, each $z \in Z$. Additionally, as $c^*(s_e)$ is bounded in $z$, each $(k, k^d)$, $k = k^d > 0$, we have $\sup_{n\to\infty} B^n(0)(s_e) = \lim_n B^n(0) \to c^* \in \Phi_B \subset H^c$, with $c^*(s_e) < m(s_e)$ when $k = k^d$, $k > 0$. Finally, as the iterations $B^n(0)$ have initial element $0 = \wedge H^c$, by monotonicity of $B(\hat{c})$, there does not exist another fixed point, say $c^{*\prime} \in \Phi_B$, such that $c^{*\prime} \leq c^*$. Therefore, $c^* = \wedge\Phi_B$ for such $s^e$. Noting the definition of $B(\hat{c})(s_e)$ when $k = k^d = 0$, each $z$, $c^*$ is then the least fixed point of $B(\hat{c})$ in $\Phi_B$. As $0 < c^*(k, k, z) < m_f(k, k, z)$ for all $s_e$, such that $k > 0$, $c^*(k, k, z) = c^*(k, z)$ is the least RE in the set $\Phi_B$. □

### 5.3 RE in models with long-lived agents

We finally consider models where each generation born is long lived (i.e., they live $N + 1$ periods, for $\infty > N > 2$). We assume no borrowing (i.e., an OLG version of a "Bewley" model with aggregate risk).[22] The assumption of long-lived agents greatly complicates matters, but our monotone decomposition method still can be used to compute a minimal state RE. For this section, assume now there are a continuum of infinitely lived household/firm agents born each period, but each living $N + 1$ periods. Their lifetimes can be divided into three stages: initial period ($n = 1$), midlife ($n \in \{2, \ldots, N\}$) and retirement ($t = N + 1$). In all but the terminal period, agents each period will be given a unit of time which they supply inelastically, consume and save, and in the terminal period they retire (hence, do not work). When born, they possess no capital, and there is no bequest. We shall also limit our attention to RE where agents of the same generation are treated identically in a RE. Let the

[22] As the household live a finite number of periods, and we do not allow default, the borrowing constraint case is actually more difficult than the case with no borrowing constraints. We can easily adapt of methods to the case of borrowing.

subscript $j \in \mathbf{J} = \{1, 2, \ldots, N+1\}$ denote the period of the agent's lifecycle, and for convenience (and without loss of generality), we normalize the mass of agents to be the unit interval, with $\eta_j = \frac{1}{N+1} > 0$, of each type $j$, and there is no population growth.

For this section, we assume time separable utility with constant discounting. For households of age $j$, they have period preferences represented by a utility index $u_j(c)$ and discount future utilities at rate $\beta \in (0, 1)$. Therefore, household lifetime utility is

$$\int \sum_{j=1}^{N+1} \beta^{t-1} u_j(c_j)$$

where the integral is again a sum under Assumptions A2′.

The assumptions on the period utility function are as follows:[23]

**Assumption 1″** $u_j : \mathbf{R}_+ \mapsto \mathbf{R}_+$ or $u_j(c) = \ln c$[24], for $j \in \{1, 2, \ldots, N+1\}$ where

(i) $u_j(c)$ is twice continuously differentiable, strictly increasing, strictly concave;
(ii) $u_j^{'}(c)$ satisfies Inada conditions, i.e.,

$$\lim_{c \to 0} u_j^{'}(c) = \infty \text{ and } \lim_{c \to \infty} u_j^{'}(c) = 0.$$

We remark that in this section, the assumption of time separability is needed (as without it, we cannot transform the space of functions under our current operator). Also, unlike the last section, we also need assumption A2′ (i.e., countable shocks) to avoid a technical problems when defining our operator associated with measurability. We make a more specific remarks per the need for these two assumptions after our existence argument.

For production technology, given a continuous function $F(k, n, K_m, N, z)$, where $F$ is constant returns to scale in $(k, n, k, n) \in \mathbf{X} \times [\mathbf{0}, \mathbf{1}] \times \mathbf{X} \times [\mathbf{0}, \mathbf{1}]$, where $\mathbf{X} \subset \mathbf{R}_+$ is a compact. The mean capital stock will be denoted by $K_m = \sum_{j=2}^{N+1} K_j$, and $N = 1$ is the average per capital stock of labor, which is unity by assumption in equilibrium. Denote the cross-sectional distribution (by age) of individual capital stocks by $k = (k_2, k_3, \ldots k_{N+1}) \in \mathbf{X}^N$, and their aggregate per capita counterparts by $K \in \mathbf{X}^N$. We make the following assumptions on the production function $F$:

**Assumption 3‴′** $F : \mathbf{X} \times [0, 1] \times \mathbf{X} \times [0, 1] \times Z \to \mathbf{R}_+$ is CRS in $(k, n, K_m, N)$ such that:

(i) $F(k, n, K_m, N, z) > 0$ for all $k > 0, z \in Z$ whenever $k = K_m > 0, n = N = 1$, and $F(0, 1, K_m, N, z) = F(k, 0, K_m, N, z) = 0$;
(ii) $F(k, n, K_m, N, z)$ is continuous, strictly increasing, twice continuously differentiable, and strictly concave in $(k, n)$ for each $(K_m, N, z)$;

[23] A careful examination of our arguments for the existence of RE will show that we can also allow for different discount rates in each period of life also, but we abstract from this to keep notation at a minimum.

[24] For the logarithmic case, obviously take the domain to be $\mathbf{R}_{++}$.

(iii) there exists $\widehat{k} > 0$ such that $F(\widehat{k}, n, \hat{k}, N, z) = \widehat{k}$ and $F(k, n, k, N, z) < k$ for all $k > \hat{k}(z)$ for all $z \in Z$, $k = K$ and $n = N = 1$.

Define $\mathbf{X}^N = \times_{j=2}^{N+1}[0, \bar{k}_j]$ with $\sup_z \hat{k}(z) \leq \bar{k}_j < \infty$ and choose the initial capital stocks as elements of $\mathbf{X}_*^N = \mathbf{X}^N \backslash 0$ where 0 is zero vector on $\mathbf{R}^n$ (i.e., assume $(k_0, K_0) \in \mathbf{X}^N \times \mathbf{X}_*^N$). Denote the aggregate state space for the economy as $S = [K, z] = [K_{2,\ldots,} K_{N+1}, z] \in \mathbf{S}_* = \mathbf{X}_*^n \times Z$ , with $\mathbf{S} = \mathbf{X}^N \times Z$. Finally, for household of age $j > 1$ entering the period with capital stock $k_j$, the state of an individual household is $s_j = [k_j, S]$. Given the CRS assumptions on technologies, the prices of capital and labor are now evaluated in equilibrium using $k_m = K_m$

$$r = F_1(k_m, z) = F_1(k_m, 1, k_m, 1, z)$$
$$w = F_2(k_m, z) = F_2(k_m, 1, k_m, 1, z)$$

along equilibrium paths.

We now describe agent decision problems in a RE. At age $j \in J$, the household enters the period in state $(k_j, K, z)$, faces feasibility constraints given by a well-defined correspondence,

$$\Upsilon_j(s_j) = \{(c_j, k_j') : c_j + k_j' = m^j(k_j, K, z), c_j, k_j' \geq 0\}$$

where the income process at each age is given by:

$$\begin{aligned} m^1(K, z) &= w(K, z) \\ m^i(k_i, K, z) &= r(K, z)k_i + w(K, z), i = \{2, 3, \ldots, N\} \\ m^{N+1}(k_{N+1}, K, z) &= r(K, z)k_{N+1} \end{aligned}$$

where the index $i$ indicates the "midlife" stages of life $i = 2, \ldots, N$. Under Assumption 3'''', when $K \neq 0$, $\Upsilon_j(s_j)$ is a non-empty, compact, convex-valued, and (locally Lipschitz) continuous correspondence in $(k_j, S)$.

To define a recursive representation of the households decision problem, let the aggregate laws of motion on the distribution of capital be described by a vector of functions for $k = K$

$$K' = h(k, z) \in \mathbf{H}^f = \{h | 0 \leq h_j(k_j, k, z) \leq m^j(k_j, k, z)\}$$

where are the set of feasible aggregate laws of motion. The terminal value function for any generation is

$$v_{N+1}(k_{N+1}, K, z) = u_{N+1}(r(K, z)k_{N+1})$$

so we have recursively for $h \in \mathbf{H}^f$, when at least one $h_j > 0$, $K \neq 0$, the following:

$$v_j(k_j, K, z) = \max_{x_j \in [0, y_j(k_j, K, z)]} u(m^j - x_j) + \beta \int v_{j+1}(x_j, h, z)\gamma(\mathrm{d}z')$$

Conjugating this sequence of primal problems with the obvious Lagrangian dual formulation (noting we have strong duality under our assumptions for the resulting sequence of decision problems with strict concavity in $x_j$), we arrive at the following system of necessary and sufficient of RE functional equations via the dual[25]

$$u^{'}(m^1(s_1)-x_1^*(s_1))-\beta\int u'((m^2-h_2)(x_1^*(s_1),h,z'))\cdot r(h,z')\gamma(\mathrm{d}z')+\phi_1^*=0$$

$$u^{'}(m^i(s_1)-x_i^*(s_1))-\beta\int u'((m^{i+1}-h_{i+1})(x_i^*(s_i),h,z'))\cdot r(h,z')\gamma(\mathrm{d}z')+\phi_i^*=0;\quad i=2,\ldots,N$$

$$u'(m_N(s_N)-x_N^*(s_N))-\beta\int u'_{N+1}(r(h,z')x_N^*(s_N))\cdot r(h,z')\gamma(\mathrm{d}z')+\phi_N^*=0$$

plus the standard complementarity slackness conditions determining the vector of Karash–Kuhn–Tucker (KKT) multipliers $\phi=(\phi_1,\phi_2,\ldots,\phi_n)$.[26] As the household's problem has a constraint system for its sequential problem that trivially satisfies a linear independence constraint qualification, and the feasible correspondence is locally Lipschitzian (hence, uniformly compact), by Kyparisis' Theorem (Kyparisis 1985, Theorem 1), the set of KKT points are compact. As the problem is also strictly concave, this set of KKT points are bounded and unique and given by $\phi^*=(\phi_1^*,\ldots,\phi_n^*)$ at each $s$. Let the range of $\phi(s)\in\Phi(s)\subset\mathbf{R}_+^n$.

To study the existence of RE, we proceed as before. That is, let $k^d\in\mathbf{X}^d$ be the space $\mathbf{X}^N$ given its dual componentwise Euclidean order, $p^e=(k,k^d)$, with $s_j^e=(k_j,p^e,z)\in\mathbf{S}_j^e=\mathbf{X}\times\mathbf{X}^N\times\mathbf{X}^d\times Z$.[27] Under our assumptions on production, the income process for a household of age $j$ can be rewritten on $\mathbf{S}_j^e$ as

$$\hat{m}^j(s_j^e)=r(k^d,z)k_j+w(k,z)$$

where for $j=1$, we just delete the first term of $\hat{m}^1(s_j^e)$, and for $j=N+1$, we just delete the second term of $\hat{m}^{N+1}(s_j^e)$. Notice, viewing each these income processes from the vantage point of the standard Euclidean partial order on $\mathbf{S}_j^e$, $\hat{m}^j$ is increasing in $(k_j,k,z)$ for each $k^d$, and decreasing in $k^d$ for each $(k_j,k,z)$ (i.e, "mixed monotone" in $(k_jk;k^d)$ on $\mathbf{S}_j^e=\mathbf{X}\times\mathbf{X}^N\times\mathbf{X}^d\times Z$).

Let $\mathbf{R}_+^*=[0,\infty]$, $\mathbf{\Omega}=\mathbf{R}_+^{n*}$ with product order, and $\mathbf{S}_e=\mathbf{X}^N\times\mathbf{X}^N\times\mathbf{X}^d\times\mathbf{Z}$ and define the exponential space $\mathbf{\Omega}^{\mathbf{S}_e}$ of all nonnegative mappings $h:\mathbf{S}_e\to\mathbf{\Omega}$. Give $\mathbf{\Omega}^{\mathbf{S}_e}$ its pointwise partial order (hence, $\mathbf{\Omega}^{\mathbf{S}_e}$ is a complete lattice). Let $S^{e*}=\{s^e\in\mathbf{S}_e|s^e$ has $k^d=0\}$, and define by $\mathbf{C}^+(\mathbf{S}_e)\subset\mathbf{\Omega}^{\mathbf{S}_e}$ the space of functions $h(s^e)$ that are (i) continuous and (ii) bounded on $\mathbf{S}_e\backslash S_e^*$, with typical element $h(s^e)=(h_1,\ldots,h_N)(s^e)$. Equip $\mathbf{C}^+(\mathbf{S}_e)$ with the topology of uniform convergence on compacta (i.e., the compact-open topology), and its pointwise partial order. Under this partial order, $\mathbf{C}^+(\mathbf{S}_e)$ is

[25] In this problem, we cannot guarantee interiority of savings decision for every age of life. Our envelope theorems are simple applications of the results in Rincon-Zapatero and Santos (2009).

[26] e.g., $\phi_i^*x_i^*=0$, $\phi_i^*\geq 0$.

[27] In this section, we use $s^e$ for the expanded state variable (so we can subscript on age cohort $i=1,2,\ldots n$).

lattice. Finally, consider a subset $\mathbf{H}^C \subset \mathbf{C}^+(\mathbf{S}_e)$ of functions to be:[28]

$$\mathbf{H}^C = \{h \in \mathbf{C}^+ | h_i(k_i, p^e, z) \text{ is increasing in } s_i^e = (k_i, p^e), \text{ each } z, \\ \text{such that } \hat{m}_i - h_i \text{ is increasing in } (k_i, p^e), \text{ each } z, i = 1, 2, \ldots, n\}$$

$\mathbf{H}^C$ is an equicontinuous subcollection in $\mathbf{C}^+(\mathbf{S}_e)$ that is also pointwise subcomplete for all $s^e$; hence, $\mathbf{H}^c$ is a subcomplete lattice in $\mathbf{C}^+(\mathbf{S}^e)$ in the pointwise partial order. So it is compact in its interval topology and hence a complete lattice. Further, by a version of Arzela-Ascoli's theorem (e.g., Kelley 1955, Theorem 18, p. 234), the subclass $\mathbf{H}^C \subset \mathbf{H}^C$ is compact in $\mathbf{C}^+$ on any compact subset $S_1^e \subset \mathbf{S}_e \backslash S^{e*}$.[29] We summarize these observations in Lemma 6.

**Lemma 6** *Under Assumption A3'''', $\mathbf{H}$ is a subcomplete order interval in $\mathbf{C}^+(\mathbf{S}^e)$. Further, $\mathbf{H}^C(S_1^e)$ is also a compact in $\mathbf{C}^+(S_1^e)$ on any compact subset $S_1^e \subset \mathbf{S}^e \backslash S^{e*}$ such that each $h \in \mathbf{H}^C$, $h_i$ is locally Lipschitz in $(k_i, p^e)$ on this compact covering $S_1^e$.*

Define an operator in the system of household Euler inequalities as follows: Let $\hat{m}_h^j(\cdot) = \hat{m}^j(\cdot) - h_i(\cdot)$, and define the extended real-valued mapping $\hat{\Psi} : \mathbf{X}^N \times \mathbf{\Phi} \times \mathbf{S}_e \backslash S_e^* \times \mathbf{H}^C \times \mathbf{H}^C \to \mathbf{R}^{N*} = \mathbf{R}^N + \{-\infty, \infty\}^N$,[30] where each component of $\hat{\Psi}$ is given as:

$$\begin{aligned}
\hat{\Psi}_1(y, \phi, s^e, h, \hat{h}) &= u'(\hat{m}^1 - y_1) + \phi_1 \\
&\quad - \beta \int_{\Theta} u'(\hat{m}_{h_2}^2(y_1, y, \hat{h}, z') \cdot r(y, z'))\gamma(\mathrm{d}z') \\
\hat{\Psi}_i(y, \phi, s^e, h, \hat{h}) &= u'(\hat{m}^i(s^e) - x_2) + \phi_2 \\
&\quad - \beta \int_{\Theta} u'(\hat{m}_{h_2}^{i+1}(y_i, y, \hat{h}, z') \cdot r(y, z'))\gamma(\mathrm{d}z') \\
\hat{\Psi}_N(y, \phi, s^e; \hat{h}) &= u'(\hat{m}^N(s^e) - y_N) + \phi_N \\
&\quad - \beta \int_{\Theta} u'(r(\hat{h}, z')y_N) \cdot r(y, z'))\chi(\theta, d\theta')
\end{aligned}$$

or more compactly,

$$\hat{\Psi}(y, \phi, s^e; h, \hat{h}) = \Psi_1(y, s^e) + \phi - \Psi_2(y, s^e; h, \hat{h})$$

[28] For a discussion of the compact-open topology, see Kelley (1955), Chapter 6.

[29] That is, we have order convergence in the interval topology coinciding with uniform convergence on any *compact* subset of the space $\mathbf{S}^e \backslash S^{e*}$. For a discussion of local uniform convergence, see Amann and Escher (2005), Corollary 2.9, p. 374.

[30] Given that $\hat{\Psi}$ is extended real valued, notice we are careful to avoid any comparison of $\infty$ and $-\infty$ in the definition of our operators. Also, the partial order on $\mathbf{R}^{N*}$ is the product order, and $\mathbf{R}^{N*}$ is a complete lattice under this order.

where $\Psi_1$ (resp, $\Psi_2$) denotes the first term (resp., second term) in the HH's system of Euler inequalities in equilibrium, but only embedded into our larger system of functional equations $\hat{\Psi}$. In addition, we denote the complementarity slackness conditions on $\phi$ as:

$$\hat{\Psi}^e_\phi(y,\phi)(s^e;h,\hat{h}):\ \phi_i y_i \geq 0, \phi_i \geq 0$$

with $\hat{\Psi}^e = (\hat{\Psi}, \hat{\Psi}^e_\phi)$ the systems of Euler inequalities that we shall solve, where $\Psi_1$ (resp, $\Psi_2$) denotes the first term (resp., second term) in the system $\hat{\Psi}$.

Let the vector of income processes be $\hat{m} = (\hat{m}^1, \ldots, \hat{m}^{N+1})$. For $(h, \hat{h}) \in \mathbf{H}^C \times \mathbf{H}^C$, $h < \hat{m}$, $\hat{h} < \hat{m}$, $k^d > 0$, when $\hat{h} < \hat{m}$, define the correspondence $Y^*(s^e, h, \hat{h})$ by

$$Y^*(s^e, h, \hat{h}) = \{y^*, \phi^* | \hat{\Psi}(y^*, \phi^*, s^e; h, \hat{h}) = 0\}$$

Then, for fixed $\hat{h} < \hat{m}$, define our first step operator $A(h; \hat{h})(s^e)$ as follows:

$$\begin{aligned} A(h;\hat{h})(s^e) = y(s^e, h, \hat{h}) \in Y^*(s^e, h, \hat{h}) : (h, \hat{h}) \in \mathbf{H}^C \times \mathbf{H}^C, h < \hat{m}, , k^d > 0; \\ y(p^e, z; h, \hat{h}) \text{ isotone in } p^e, = \hat{m}^j \text{ for } h_j = m^j, \text{any state } s^e \\ = 0, \text{else} \end{aligned}$$

We first verify the required order continuity properties of the operator $A(h; \hat{h})$ in $h$ when $\hat{h} < \hat{m}$:

**Proposition 4** *Under Assumption A1$''$, A2$'$, and A3$''''$, we have for fixed $\hat{h} < \hat{m}$, $\hat{h} \in \mathbf{H}^C$, (i) $A(h;\hat{h}) : \mathbf{H}^C \times \mathbf{H}^C \to \mathbf{H}^C$ is well defined, with $A(h;\hat{h}) \subset \mathbf{H}^C$, and isotone on $\mathbf{H}^C$; and (ii) $A(h;\hat{h})$ is order continuous on $\mathbf{H}^C$.*

*Proof* See “Appendix C”.

With Proposition 4 in place, we can now prove our final existence theorem for the paper.

**Theorem 10** *Under Assumptions A1$''$, A2$'$, and A3$''''$, there exists a RE. Further, it can be computed by successive approximations from the minimal element $\wedge\mathbf{H}^C$.*

*Proof* Fix $\hat{h} \in \mathbf{H}^c$, $\hat{h} < \hat{m}$, and let $\Phi_A(\hat{h})(s^e)$ denote the set of fixed points of $A(h;\hat{h})(s^e)$ at any such $\hat{h}$. First, by Lemma 6, $\mathbf{H}^C$ is a complete lattice. Further, by Proposition 4, $A(h;\hat{h})(s^e)$ is isotone on $\mathbf{H}^C$ for each such $\hat{h}$. Therefore, by Tarski's theorem, the fixed point correspondence $\Phi_A(\hat{h})(s^e)$ is a non-empty complete lattice for each $\hat{h} < \hat{m}$. Define $B(\hat{h})(s^e)$ as follows

$$\begin{aligned} B(\hat{h})(s^e) &= \wedge\Phi_A(\hat{h})(s^e) \text{ for } \hat{h} < \hat{m} \\ &= \hat{m}_j \text{ when } \hat{h}_j = \hat{m}_j \text{ for any } s^e \end{aligned}$$

By Veinott's Fixed point comparative statics theorem (e.g., Veinott 1992, Chapter 4, Theorem 14 or Topkis 1998, Theorem 2.5.2), as the operator $A(h;\hat{h})(s^e)$ is jointly increasing in $(h, \hat{h}(p^e), p^e) \in \mathbf{H}^C \times \mathbf{H}^C \times \mathbf{X}^N \times \mathbf{X}^d$ with respect to the product order when $\hat{h} < \hat{m}$,[31] the least selection $\wedge\Phi_A(\hat{h})(s^e)$ is an increasing selection jointly in $(\hat{h}(p^e), p^e)$ when $\hat{h} < \hat{m}$. Noting the definition of $B(\hat{h})(s^e)$ elsewhere, $B(\hat{h})(s^e)$ is isotone in $(\hat{h}(p^e))$ on $\mathbf{H}^C$. Also, $\mathbf{H}^c$ is a non-empty complete lattice. Therefore, denoting by $\Phi_B$ the set of fixed point of $B(\hat{h})(s^e)$, again by Tarski's theorem, we conclude $\Phi_B$ is a non-empty complete lattice. Finally, consider the iterations $\{B^n(0)(s^e)\} = \{(B_i^n)(s^e))_i\}$, with associated dual variables $\{(\phi_i^n(s^e))_i\}$, with $\phi_i^n(s^e) \geq 0$ when $B_i^n(0)(s^e) = 0$. For each $s^e \in \mathbf{S}^e$, the iterations $\{B^n(0)(s^e)\}$ form an increasing chain that satisfies the system of Euler inequalities

$$\hat{\Psi}^e = (\hat{\Psi}, \hat{\Psi}^e_\phi)(B^n(0)(s^e), \phi^n(s^e), s^e, B^{n-1}(0), B^{n-1}(0)) = 0$$

with

$$\begin{aligned}&\hat{\Psi}(B^n(0), \phi^n, s^e, B^{n-1}(0), B^{n-1}(0))\\&= \Psi_1(B^n(0), s^e) + \phi^n(s^e) - \Psi_2(B^n(0), s^e; B^{n-1}(0), B^{n-1}(0))\\&= 0\end{aligned}$$

Under the differentiability assumptions in A1$''$ and A3$''''$, iterations converge in order (and pointwise) with $(B^n(0), (\phi_i^n)(s^e) \to (h^*, \phi^*)(s^e)$ for each $s^e \in \mathbf{S}^e$, with $h^*(s^e) < \hat{m}(s^e)$ by the Inada condition in assumption A1$''$. This verifies that $B(\hat{h})$ is order continuous along the chain $\{B^n(0)(s^e)\}$, and we have $h^* \in \Phi_B$. Further, given the equicontinuity of the collection $\mathbf{H}^C$, we have $\{B^n(0)(s^e)\} \to h^*(s^e)$ uniformly on every compact subset $S_1^e \subset \mathbf{S}^e \backslash S^{e*}$. As $0 = \wedge\mathbf{H}^C$, by the monotonicity of $B(\hat{h})(s^e)$, there does not exist any other fixed point $\hat{h}^*$ such that $\hat{h}^* \leq h^*$. As $\Phi_B$ is a non-empty complete lattice, we must have $h^* = \wedge\Phi_B$. Finally, let $h^*(k, k, z) = h^*(k, z)$ be the fixed point $h^*(k, k^d, z)$ defined on the diagonal $k = k^d$. By the Inada conditions, we must have $0 \leq h^*(k, z) < \hat{m}$, such that when $k > 0$, the vector of consumptions $c^*(s) = (m - h^*)(k, k, z) > 0$ when $k > 0$. Therefore, $h^*(k, z)$ is the (non-trivial) minimal state space RE. □

A few remarks on this theorem. First, although in principle, this result is "constructive", our proof of existence uses an operator defined as any monotone selection in the correspondence $Y^*(s^e, h, \hat{h})$. That monotone selection exists by Smithson's theorem (e.g., see Smithson 1971), but Smithson's theorem is actually proven using Zorn's lemma. As Zorn's Lemma is well known to be equivalent to the Axiom of choice, this step of the argument is actually not constructive in the sense of the rest of the paper.

Second, our existence result in Theorem 10 is different than one that can be obtained using the Euler equation APS methods as in Feng et al. (2012). In particular, we verify the existence of minimal state space RE (as opposed to Generalized Markovian equilibrium). As mentioned before, correspondence-based APS methods could be used to compute RE in versions of our economies with many assets (albeit at

[31] We make explicit that the monotonicity in is the joint variables $(\hat{h}(p^e), p^e)$ on $\mathbf{H}^C \times \mathbf{X}^N \times \mathbf{X}^d$ in the notation to make clear that we can raise either $\hat{h}$ for fixed $p^e$, or $p^e$ for fixed $\hat{h}$.

the expense of the minimality of the equilibrium state space). Also notice, a careful reading of the proof of this claim reveals also that our RE are continuous in each argument (but, necessarily jointly continuous). In principle, a similar issue could arise in the results for elastic labor in Sect. 5.2. What is different is that in the elastic labor case, as the operator is the unique interior root when $k > 0$ of an *single* Euler equation, joint continuity can be established it turns out by application of Clarke's Implicit function theorem (e.g., Clarke 1983, Corollary, p. 256). Unfortunately, in this section, no similar nonsmooth implicit function theorem can be applied because of the presence of the KKT system and non interior investment decision rules in a RE for some generations (hence, it is not clear how to improve upon this result).

Third, unlike the previous section, we *cannot* relax Assumption A2$'$ in this argument. The problem is although if $(h, h) \in \mathbf{H}^C \times \mathbf{H}^C$ are additionally measurable, $Y^*(s^e, h, \hat{h})$ can be shown to be a measurable correspondence on $\mathbf{S}^e$ (and, hence, measurable selections exist), the monotone selection that exists in $Y^*(s^e, h, \hat{h})$ by Lemmas 10 and 12 not be the measurable selection. So, unlike the situation in the previous section for elastic labor supply, we actually need Assumption A2$'$ to eliminate the measurability issues associated with this selection.

### 5.4 Equilibrium comparative statics for RE

We conclude the paper with a simple example of equilibrium comparative statics (many others exist by a similar argument). For this example, we take a version of the model of Hausenchild (2002) incorporating a social security system in the overlapping generation model of Wang (1993) and prove an equilibrium comparative statics result for the set of both minimal state space RE and SME. Recall that in Hausenchild (2002), a RE equilibrium investment policy is a function $h$ satisfying:

$$\begin{aligned}\int_Z & u_1((1-\tau)w(k,z) - h(k,z), r(h(k,z),z')h(k,z) + \tau w(h(k,z),z')))\gamma(\mathrm{d}z') \\ &= \int_Z u_2((1-\tau)w(k,z) - h(k,z), r(h(k,z),z')h(k,z) + \tau w(h(k,z),z'))) \\ &\quad \times r(h(k,z),z')\gamma(\mathrm{d}z'). \end{aligned} \tag{B1}$$

Consider the following equation in $y$ :

$$\begin{aligned}\int_Z & u_1((1-\tau)w(k,z) - y, r(h(k,z),z')y + \tau w(y,z')))\gamma(\mathrm{d}z') \\ &= \int_Z u_2((1-\tau)w(k,z) - y, r(h(k,z),z')y + \tau w(y,z')))r(y,z')\gamma(\mathrm{d}z'). \end{aligned}$$

For any $(k, z) \in X \times Z$ and $h \in E$, denote $Ah(k, z)$ the unique solution to this equation. It is easy to see that, in addition to being an order continuous isotone operator mapping $E$ into itself, $A$ is also isotone in $-\tau$. Consequently, an increase in $\tau$ generates a decrease (in the pointwise order) of the extremal RE equilibrium investment policies $h_{\tau,\max}$ and $h_{\tau,\min}$.[32]

Next, any equilibrium investment policy $h$ induces a Markov process for the capital stock defined by the following transition function $P_h$:

$$\text{For all } A \in \mathcal{B}(X),\ P_h(k, A) = \Pr\{h(k, z) \in A\} = \lambda(\{z \in Z, h(k, z) \in A\}).$$

Consider two RE equilibrium policies $h' \geq h$ and their respective transition functions $P_{h'}$ and $P_h$. For any $k \in X$ and any function $f : X \to \mathbb{R}_+$ bounded, measurable and isotone:

$$\int f(k')P_{h'}(k, \mathrm{d}k') = \int f(h'(k, z))\lambda(\mathrm{d}z) \geq \int f(h(k, z))\lambda(\mathrm{d}z) = \int f(k')P_h(k, \mathrm{d}k').$$

Thus, for any $\mu \in \Lambda(X, \mathcal{B}(X))$ :

$$\begin{aligned}\int f(k')T^*_{h'}\mu(\mathrm{d}k') &= \int \left[\int f(k')P_{h'}(k, \mathrm{d}k')\right]\mu(\mathrm{d}k)\\ &\geq \int \left[\int f(k')P_h(k, \mathrm{d}k')\right]\mu(\mathrm{d}k) = \int f(k')T^*_h\mu(\mathrm{d}k'),\end{aligned}$$

which establishes that $T^*_{h'}\mu \geq T^*_h\mu$. Thus, the natural ordering on the set of taxes $\tau$ induces an ordering by stochastic dominance of the corresponding extremal Stationary Markov equilibria in the following way:

$$\tau' \geq \tau \text{ implies } h_{\tau,\max} \geq h_{\tau',\max} \text{ implies } \lim_{n\to\infty} T^{*n}_{\tau}\delta_{k\max} \geq_s \lim_{n\to\infty} T^{*n}_{\tau'}\delta_{k\max}.$$

In particular, we obtain Proposition 2 in Hausenchild (2002) by taking $\tau = 0$ (a “pure economy”) and $\tau' > 0$.

## Appendix

### Appendix A: Proof of Proposition 2

We prove Proposition 2 in three steps.

**Lemma 7** *Under Assumption 4, for all $k \in X^*$, there exists a right neighborhood $\Omega = (0, \bar{k}]$ with $0 < \bar{k} \leq w(k, z_{\min})$ and $M > 0$ such that, for all $x \in \Omega$,*

$$u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) > M.$$

[32] Notice, we need Theorems 8 and 9 in this example to obtain equilibrium comparative statics in Veinott’s strong set order (i.e., to avoid the least RE in $(H^u, \leq)$ and $(H^l, \leq)$ from being trivial.

*Proof* If $\lim_{x\to 0^+} r(x, z_{\max})x = 0$ then for all $k \in X^*$:

$$\lim_{x\to 0^+} u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) = u_2(w(k, z_{\min}), \lim_{x\to 0^+} r(x, z_{\max})x) = \infty.$$

The expression $u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)$ can therefore be made arbitrarily large in a right neighborhood of 0, and the existence of $\Omega$ thus follows. □

**Lemma 8** *For all $s = (k, z) \in S^*$, there exists $h_0(s) \in (0, w(s))$ such that:*

$$\begin{aligned} &\int_Z u_1(w(s) - h_0(s), r(h_0(s), z')h_0(s))G(dz') \\ &< \int_Z u_2(w(s) - h_0(s), r(h_0(s), z')h_0(s))r(h_0(s), z')G(dz'). \end{aligned} \tag{E0}$$

*In addition, $h_0$ can be chosen isotone in $k$ for each $z$, constant in $z$ (and therefore continuous and isotone in $z$) for each $k$.*

*Proof* Fix $k \in X^*$. For all $z \in Z$:

$$\begin{aligned} &\lim_{x\to 0^+} \int_Z u_1(w(k, z) - x, r(x, z')x)G(\mathrm{d}z') \\ &= \int_Z u_1(w(k, z), 0)G(\mathrm{d}z') \\ &\leq u_1(w(k, z_{\min}), 0). \end{aligned}$$

Thus, there exists a right neighborhood of 0, denoted $\Psi = (0, \overline{x}]$, such that, for all $x \in \Psi$:

$$\begin{aligned} &\int_Z u_1(w(k, z) - x, r(x, z')x)G(\mathrm{d}z') \\ &< .5u_1(w(k, z_{\min}), 0). \end{aligned}$$

Next, for $x \in \Omega$ :

$$\begin{aligned} &\int_Z u_2(w(k, z) - x, r(x, z')x)r(x, z')G(\mathrm{d}z') \\ &\geq \int_Z u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)r(x, z')G(\mathrm{d}z') \\ &\geq \int_Z Mr(x, z')G(\mathrm{d}z'), \end{aligned}$$

where the first inequality stems from $u_{12} \geq 0$ and $u_2$ antitone, and the second from the Lemma above. This last expression can be made arbitrarily large, independently of $z$, by choosing $x$ in $\Omega$ sufficiently close to 0. That is, it is always possible to choose $x^*$ sufficiently small in $\Omega \cap \Psi$ so that:

$$\int_Z Mr(x^*, z')F(\mathrm{d}z') \geq .5u_1(w(k, z_{\min}), 0). \tag{E1}$$

Pick such an $x^*$ and set $\delta_0(k, z) = x^*$ for all $z \in Z$. By construction, any $x \in ]0, \delta_0(s)]$ satisfies:

$$\begin{aligned}
&\int_Z u_1(w(s) - x, r(x, z')x)G(\mathrm{d}z') \\
&< .5u_1(w(k, z_{\min}), 0) \\
&\leq \int_Z Mr(x, z')G(\mathrm{d}z') \\
&\leq \int_Z u_2(w(s) - x, r(x, z')x)r(x, z')G(\mathrm{d}z').
\end{aligned}$$

That is, for all $x \in (0, \delta_0(s)]$:

$$\begin{aligned}
&\int_Z u_1(w(s) - x, r(x, z')x)G(\mathrm{d}z') \\
&< \int_Z u_2(w(s) - x, r(x, z')x)r(x, z')G(\mathrm{d}z').
\end{aligned} \tag{E2}$$

We repeat the same operation for each $k$ in $X^*$, thus constructing a function $\delta_0 : S \to X$, setting $\delta_0(0, z) = 0$. By construction, for each $k \in X$, $\delta_0(k, z)$ is constant in $z$, and therefore isotone in $z$. In addition, any function smaller (pointwise) than $\delta_0$ also satisfies (E2). In particular, the function $p_0 : X \times Z \to X$ defined as:

$$p_0(k, z) = \min_{k' \geq k}\{\delta_0(k', z)\}.$$

satisfies (E2), which is isotone in $k$ for all $z$ and constant in $z$ for all $k$ (and thus continuous in $z$ for all $k$). Finally, the function $h_0$ defined as follows:

$$h_0(k, z) = \left\{ \begin{array}{ll} \sup_{0<k'<k} p_0(k', z) & \text{for } (k, z) \in X^* \times Z \\ 0 & \text{for } k = 0, z \in Z \end{array} \right\}$$

is smaller than $p_0$ (and, therefore, than $\delta_0$, hence it satisfies (E2)), isotone in $k$ for all $z$, constant in $z$ for all $k$, and lower semicontinuous in $k$ for all $z$. □

**Proposition 5** $\forall s \in S^*$, $h_0(s) \geq h(s) > 0$ *implies that* $Ah(s) > h(s) > 0$.

*Proof* Suppose that there exists $s \in S^*$ such that $Ah(s) \leq h(s)$. Then:

$$\begin{aligned}
&\int_Z u_1(w(s) - h(s), r(h(s), z')h(s))G(\mathrm{d}z') \\
&< \int_Z u_2(w(s) - h(s), r(h(s), z')h(s))r(h(s), z')G(\mathrm{d}z') \\
&\leq \int_Z u_2(w(s) - Ah(s), r(h(s), z')Ah(s))r(Ah(s), z')G(\mathrm{d}z'),
\end{aligned}$$

where the first inequality stems from ($E2$) (since $0 < x = h(s) \leq h_0(s) \leq \delta_0(s)$) and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and $r$ antitone in its first argument. By definition of $Ah$, this last expression is equal to:

$$\int_Z u_1(w(s) - Ah(s), r(h(s), z')Ah(s))G(\mathrm{d}z').$$

Summarizing, we have:

$$\begin{aligned}
&\int_Z u_1(w(s) - h(s), r(h(s), z')h(s))G(\mathrm{d}z') \\
&< \int_Z u_1(w(s) - Ah(s), r(h(s), z')Ah(s))G(\mathrm{d}z').
\end{aligned}$$

which is contradicted by the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. Thus, necessarily, $Ah(s) > h(s)$. In particular, $A$ maps $h_0$ strictly up. □

Appendix B: Proofs for elastic labor section

*Proof of Lemma 4*

*Proof* To prove the lemma, given assumption *A1′*(IV),it suffices to show for each $(c, \hat{c}) \in H^C \times H^C, k > 0$, $\hat{c} < m_f$, $z \in Z$, that $R^*$ is (i.a) increasing $y$ and (i.b) decreasing in $p^e$ for fixed $(c, \hat{c}) \in H^C \times H^C$, $\hat{c} < m_f$, and $z \in Z$, and (i.c) decreasing in $(c, \hat{c}) \in H^C \times H^C$, when $\hat{c} < m_f$, for each $(y, s_e)$. To see (i.a), fix $(c, \hat{c}) \in H^C \times H^C$, $k > 0$, $\hat{c} < m_f$. As $m_y(k, z)$ is falling in $y$, the numerator in the first argument of $R^*$ is falling in $y$. Further, as for $c \in H^C$, $c$ is increasing in $k$, and $m_y$ is falling in $y$, $c(m_y, \cdot, \cdot)$ is falling in $y$, so $n^*(c(m_y, \cdot.\cdot), \cdot.\cdot)$ is rising in $y$. So the first argument of $R^*$ is falling in $y$. Therefore, under A3$'''$, $R^*$ is increasing in $y$. To see (i.b), as $m_y(k, \cdot)$ is rising in $k$, the numerator in the first argument of $r$ is rising in $k$. As for the denominator, as $c(k, \cdot, \cdot) \in H^c$ is rising in $k$, $c(m_y(k, \cdot), \cdot, \cdot)$

is rising in $k$. Further, as $n^*$ is falling in $c$, we have $n^*(c(m_y(k,\cdot),\cdot,\cdot),\cdot,\cdot)$ falling in $k$. Also, as $c(\cdot,k^d,\cdot)$ is rising in $k^d$, noting the dual order on $K^d$, $m_c(k^d,\cdot)$ is falling in $k^d$, so $c(\cdot,m_c(k^d,\cdot),\cdot)$ is rising in $k^d$; therefore, as $n^*$ is again falling in $c$, $n^*(c(\cdot,m_{\hat{c}}(k^d,\cdot),\cdot),\cdot,\cdot)$ falling in $k^d$. Finally, as $n^*$ is rising in $k$, again noting the dual order on $K^d$, as $m_{\hat{c}}(k^d,\cdot)$ is falling in $k^d$, $n^*(c(\cdot,\cdot,\cdot),m_c(k^d,\cdot),\cdot)$ is falling in $k^d$. So, we conclude the denominator is falling in $p^e=(k,k^d)$. Therefore, we have the ratio

$$\frac{m_y(k,z)}{n^*(c(m_y(k,z)),m_{\hat{c}}(k^d,z),z'),m_{\hat{c}}(k^d,z),z')}$$

rising in $p^e$, so $R^*$ is decreasing in $p^e$. (i.c) Fix $(y,s_e)$. As $n$ is falling in $c$, $n^*(c,\cdot,\cdot)$ is falling in $c$. So the ratio is rising in $c$, and $R^*$ is falling in $c$. Further, as $m_{\hat{c}}(\cdot,\cdot)$ is falling in $\hat{c}$, $c$ increasing in $k^d$ in the dual order on $K$ (hence, falling in the natural order), we have $n^*(c(\cdot,m_{\hat{c}},\cdot),\cdot,\cdot)$ falling in $\hat{c}$ through this term. Further, as $n^*$ is rising in $k$, we have $n^*(c(\cdot,\cdot,\cdot),m_{\hat{c}},\cdot)$ also falling in $\hat{c}$. So in both case, the denominate is falling $(c,\hat{c})$ failing, so first argument is of $R^*$ is rising, and therefore, $R^*$ is falling in $(c,\hat{c})$. Then, as (i.a)–(i.c) are true, noting that $m_{\hat{c}}(\cdot,\cdot)$ is also falling in $\hat{c}$, $R^*$ is falling in $(c,\hat{c})$ is preserved to $\Psi_2(y,s_e,z;c,\hat{c})$ by under Assumption A1′(IV). □

*Proof of Lemma 5*

*Proof* (i) We first show $A_{\hat{c}}(c)(s_e)\in H^c$. We first show for each $\hat{c}<m_f$, $\hat{c}\in H^c$, $\hat{c}<m_f$, when $c\in H^c$, $k>0$, $A_{\hat{c}}(c)(s_e)\in H^C$. Noting the comparative statics proven in Lemma 4 for $\Psi_2(y,s_e;c,\hat{c})$, it is clear that $\Psi(y,s_e;c,\hat{c})$ has the same comparative statics. For $c\in H^c$,as $c$ is continuous in its first argument, and $\Psi$ is increasing in $y$, $A_{\hat{c}}(c)(s_e)$ is well defined (i.e., exists and is single valued), and increasing in $p^e$, each $(c,\hat{c})$. Further, as for $c\in H^c$, and $\hat{c}(s_e)$ fixed (so $m_{\hat{c}}(s_e)$ is fixed),[33] letting $x^*_{A_{\hat{c}}(c)(s_e)}=m_{A_{\hat{c}}(c)(p)}(s_e)=(m-A_{\hat{c}}(c))(s_e)$,when $(k_1,k_1^d)\geq(k_2,k_2^d)$, we have to have

$$\begin{aligned}&\Psi_2(x^*_{A_{\hat{c}}(c)(s_e)},s_e,z;c,\hat{c})-u'(A_{\hat{c}}(c)(k_1,k_2^d,z)\\&\leq\Psi_2(x^*_{A_{\hat{c}}(c)(s_e)},s_e;c,\hat{c})-u'(A_{\hat{c}}(c)(k_2,k_2^d,z))\end{aligned}$$

by the monotonicity of $A_{\hat{c}}(c)(s_e)$ in $p^e$,where

$$\begin{aligned}\Psi_2(x^*_{A_{\hat{c}}(c)(p)},s_e;c,\hat{c})=\int &u'(R^*(x^*_{A_{\hat{c}}(c)(s_e)},s_e,z';c,\hat{c})\cdot m_{\hat{c}}(k^d,z))\\&\cdot R^*(x^*_{A_{\hat{c}}(c)(p)},s_e,z';c,\hat{c})\gamma(\mathrm{d}z')\end{aligned}$$

33 Notice, in this argument, $m_{\hat{c}}(p)$ fixed means both the argument $p$ and $\hat{c}$.

where tomorrow's price of capital is

$$R^*(x^*_{A_{\hat{c}}(c)(s_e)}, s_e, z'; c, \hat{c}) = r\left(\frac{k'}{n'}, z\right) \\ = r\left(\frac{x^*_{A_{\hat{c}}(c)(p)}}{n^*(x^*_{A_{\hat{c}}(c)(s_e)}, m_{\hat{c}}(k^d, z), z'), m_{\hat{c}}(k^d, z), z')}, z'\right) \tag{3}$$

where by a similar argument to Lemma 4, as $R^*$ is falling in $x^*_{A_{\hat{c}}(c)(p)}$, the term $\Psi_2(x^*_{A_{\hat{c}}(c)(p)}, s_e; c, \hat{c})$ is falling in $x^*_{A_{\hat{c}}(c)(s_e)}$ by Assumption A3. Therefore, the operator $A_{\hat{c}}(c)(s_e)$ must be such that that $\Psi_2$ fails when $(k, k^d)$ rises, i.e., $x^*_{A_{\hat{c}}(c)(s_e)} = m_{A_{\hat{c}}(c)(s_e)}(k, z) = (m - A_{\hat{c}}(c))(s_e)$ increasing in $(k, k^d)$, each $z \in Z$. Therefore, $A_{\hat{c}}(c)(s_e) \in H^C$ at all such points. Noting the definition $A_{\hat{c}}(c)(s_e)$ elsewhere for $c$ and $s_e$, we conclude $A_{\hat{c}}(c)(s_e) \in H^c$ for each $0 \le \hat{c} < m_f$. Further, noting the comparative statics in Lemma 4 (again, noting the definition of $A_{\hat{c}}(c)(s_e)$ when $c \not< m_f$ or $k = 0$, $A_{\hat{c}}(c)(s_e)$ is isotone in $c$ for each $0 \le \hat{c} < m_f$. Therefore, by Tarski's theorem, the set of first step fixed points is $\Phi_A(\hat{c})(s_e)$, which is a non-empty complete lattice for each $0 \le \hat{c} < m_f$. (ii) Under A1$'$ and A3$'''$, $u'(c)$ and $r(k)$ are both continuous in their arguments, and $n^*(c, k, z)$ is continuous in $c$, we have $\Psi_2(y, s_e; c_n, \hat{c}) \to \Psi_2(y, s_e; c, \hat{c})$ pointwise (hence, $\Psi(y, s_e; c_n, \hat{c}) \to \Psi(y, s_e; c, \hat{c})$ pointwise). Therefore, noting the definition of $A_{\hat{c}}(c)(s_e)$ elsewhere, by an argument similar to that in Lemma 2, $A_{\hat{c}}(c)(s_e)$ is order continuous in $c$ when $\hat{c} < m_f$. Then, by the Tarski-Kantorovich theorem, we have

$$\inf_n A_{\hat{c}}(c)(s_e) \to \vee\Phi_A(\hat{c})(s_e)$$

Noting the equicontinuity of $H^c$ in $p^e = (k, k^d)$ for each $z \in Z$, this convergence is uniform in $p^e$ in each argument. Finally, the fact that the greatest fixed point $\vee \Phi_A(\hat{c})(s_e) > 0$ when $k > 0$ follows now from a standard argument for infinite horizon problems adapted to our operator (e.g., Greenwood and Huffman 1995, Main theorem), noting the Inada condition on assumption A1$'$. (iii). Noting the comparative statics result in Lemma 4, under the assumptions of the lemma, $A_{\hat{c}}(c)(s_e)$ is isotone in $\hat{c}$ for each $(s_e; c(s_e))$ when $c \in H^C$. Then that $\vee\Phi_A(\hat{c})(s_e)$ is increasing in $\hat{c}$ follows from Veinott's fixed point monotone comparative statics theorem (e.g., Veinott 1992, Chapter 4, Theorem 14 or Topkis 1998, Theorem 2.5.2). □

## Appendix C: Proofs for long-lived agent case

We first prove Proposition 4. To do this, we need to prove number of lemmas.

**Lemma 9** *Under assumptions A1$''$, A2$'$, and A3$''''$, for $(h, \hat{h}) \in \mathbf{H}^C \times \mathbf{H}^C$, $h < \hat{m}$, $\hat{h} < \hat{m}$, $p^e \neq 0$, the mapping $\hat{\Psi}(y, \phi, s^e; h, \hat{h})$ is (i) decreasing in $(h, \hat{h})$, each $(y, \phi, s^e)$, (ii) strictly decreasing in $p^e$, each $(z, h, \hat{h}(s^e))$; and (iii) increasing and continuous in $y$, strictly increasing in $y_i$ for each $i$ when $y_i > 0$, each $(s^e, h, \hat{h})$.*

*Proof* (i) Fix $(y, \phi, s^e)$, $p^e \neq 0$. and let $(h_1, \hat{h}_1) \geq (h_2, \hat{h}_2), (h_i, \hat{h}_i) < (\hat{m}, \hat{m})$. Under A1$''$, $\Psi_2$ is increasing in $h$ (as the vector $u(c)$ is concave). Further, as $h \in \mathbf{H}^c$, $m - h$ is increasing in $k^d$ (with the dual order for $\mathbf{X}^d$), hence $\hat{m}^j_{hj}(\cdot, \cdot, h)$ is falling in $h$, so as $u'(m^j_{hj}(\cdot, \cdot, h))$ rising in $\hat{h}$, $\Psi_2$ is rising in $\hat{h}$. That is, $\hat{\Psi}(y, \phi, s^e; h_1, \hat{h}_1) \geq \hat{\Psi}(y, \phi, s^e; h_2, \hat{h}_2)$. (ii) Fix $(y, z, h, \hat{h}(s^e)), h < \hat{m}, \hat{h} < \hat{m}$, and let $p^e \neq 0$, and take $p^e_1 \geq p^e_2$. Noting the dual order for $k^d$, for fixed $\hat{h} = \hat{h}(s^e)$, the second term $\Psi_2$ is independent of $p^e$. As $m^j(p^e, z)$ is increasing in $p^e$, and $u'(c)$ is falling, $\Psi_1$ is falling in $p^e$. (iii) Fix $(z, h, \hat{h})$, with $(h, \hat{h}) \in \mathbf{H}^C \times \mathbf{H}^C, h < \hat{m}, \hat{h} < \hat{m}, p^e \neq 0$, and consider $y_1 \geq y_2$. As $h \in \mathbf{H}^c$, by assumptions A1$''$ and A3$''''$, $\Psi_2$ is falling. Further, under A1$''$, $\Psi_1$ increases in $y$; therefore $\hat{\Psi}(y_1, \phi, s^e; h, \hat{h}) \geq \hat{\Psi}(y_2, \phi, s^e; h, \hat{h})$. Continuity of $\hat{\Psi}(y, \phi, s^e; h, \hat{h})$ in $y$ follows from the fact that (a) under Assumption A1$''$, $u'(c)$ is continuous, (b) A3$''''$ implies $r(\cdot, z)$ is continuous, and (c) $h_j \in \mathbf{H}^c$, under A3$''''$, $u'(\hat{m}^j_{h_j}(k_j, k, \cdot))$ is (locally Lipschitz) continuous in $(k_i, k)$. Finally, the fact that $\hat{\Psi}(y, \phi, s^e; h, \hat{h})$ is strictly increasing in $y_i$ when $y_i > 0$ follows from Assumption A1$''$. □

The operator $A(h, \hat{h})(s^e)$ is defined to be an isotone selection in the correspondence $Y^*(s^e, h, \hat{h})$ when $h < \hat{m}$, $\hat{h} < \hat{m}$, and $k^d > 0$. We need to show this operator is well defined. First, a few definitions. Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be a partially ordered sets. We say a correspondence or multifunction is *ascending* in the set relation $\geq_S$ to $P(Y) \subset 2^Y \backslash \varnothing$ if $Y(x') \geq_S Y(x)$, when $x' \geq_X x$. If a set relation $\geq_S$ induces a partial order on $P(Y)$, we refer $Y(x)$ as an *isotone correspondence*. An *antitone* correspondence is isotone in the dual order in its domain $X$.

To study the properties of ascending and/or isotone correspondences, Smithson (1971) and Veinott (1992) have developed set relations for correspondences that guarantee the existence of isotone selections. In this paper, we focus on two such set relations. Then the first set relation we define is (i) the *Weak Induced Set relation* $\geq_{wi}$on $2^Y \backslash \varnothing : B_1 \geq_{wi} B_2$, $B_1, B_2 \in 2^Y \backslash \varnothing$, if (C1) (ascending upward) $\forall\, b \in B_2$, there exists an $a \in B_1$ such that $a \geq b$; and (C2) (ascending downward) $\forall\, a \in B_1$, there exists a $b \in B_2$ such that $a \geq b$. Now, let $Y$ also be a lattice. Then define (ii) the *Veinott-Strong Set Order* $\geq_v$on $L(Y) = \{A | A \subset Y,\ A$ a non-empty sublattice$\}$ : for $A_1$ and $A_2$ in $L(Y)$, $A_1 \geq_v A_2$, if $\forall\, a \in A_1$, $\forall b \in A_2$, we have $a \wedge b \in A_2$ and $a \vee b \in A_1$.[34]

We now provide a few key lemmas.

**Lemma 10** *Let* $(\mathbf{X}, \geq_X)$ *and* $(\mathbf{Y}, \geq_Y)$ *be chain complete partially ordered sets,* $Y^*(x) : \mathbf{X} \to 2^{\mathbf{Y}} \backslash \varnothing$ *be a non-empty and chain subcomplete valued for each* $x \in \mathbf{X}$*. Say* $Y^*(x)$ *is ascending in the weak induced set order. Then,* $Y^*(x)$ *admits an isotone selection.*

*Proof* Follows from Smithson (1971, Theorem 1.7), noting $Y^*(x)$ is weak induced order ascending (i.e., both (C1) and (C2) hold), and $Y^*(x)$ is also chain subcomplete valued. □

[34] We will only use Smithson's relation in the next Proposition. Veinott's strong set order is used all other places in the paper.

For the sake of simplifying notation, let $t = (p^e, h, \hat{h}) \in \mathbf{T} = \{t | t \in \mathbf{K}^N \times \mathbf{K}^d \times \mathbf{H}^c \times \mathbf{H}^c, h < \hat{m}, \hat{h} < \hat{m}, k^d > 0\}$. The operator $A(h, \hat{h})(s^e)$ is defined at all $t$, each $z$, as an isotone selection in $t$, each $z$. In the next two lemmata, we prove (i) $Y^*(t, z)$ is non-empty for each $(t, z)$, and (ii) $Y^*(t, z)$ is ascending in the weak induced set order in $t$, each $z$ (and hence by Smithson's theorem in Lemma 10, the operator is well defined for such $t \in \mathbf{T}$)

**Lemma 11** *Under Assumptions A1″, A2′, and A3″″, the correspondence $Y^*(t, z)$ is non-empty for each $(t, z) \in T \times Z$.*

*Proof* For any $(t, z) \in \mathbf{T} \times Z$, it suffices to check for the existence of solutions $\hat{\Psi}(y, 0, s^e, h, \hat{h})$ (noting by the Inada conditions in A1″, when $y_j^* = 0$ for any equation $j$, we have $\infty > \phi_j^*(s^e, h, h) = \Psi_{1j}(y_{-j}^*, 0, p^e, z) - \Psi_2(y_{-j}^*, 0, p^e, z; h, \hat{h}) \geq 0$ by the complementary slackness conditions). By Lemma 9, we have $\hat{\Psi}$ is increasing and continuous $y$, with $\hat{\Psi}_j = 0$ with $\phi_j \geq 0$ when $y_j = \hat{m}_j$. Further, for each $(t, z) \in \mathbf{T} \times Z$, by the Inada conditions in A1″ and A3″″ on $u'$ and $r$, there exists a pair of vectors $(y_\wedge(t, \theta), y^\vee(t, \theta)) \in [0, \hat{m}(t, z)] \times [0, \hat{m}(t, z)]$ such that (a) $\hat{\Psi}(y_\wedge(t, z), t, z) \leq 0$; (b) $\hat{\Psi}(y^\vee(t, z), t, z) \geq 0$, and (c) $0 \leq y_\wedge(t, z) \leq y^\vee(t, z) < \hat{m}$ in the standard component product Euclidean order on $\mathbf{R}^N$, with $y_\wedge(t, z) \neq 0$. As $\hat{\Psi}$ is strictly increasing and continuous jointly in $y$ on $[y_\wedge(t, z), \hat{m}(K, \theta))$, the mapping $\hat{\Psi}(y, t, z)$ satisfies the semi-continuity conditions in the generalized intermediate value theorem on arbitrary product spaces of Guillerme (1995, Theorem 3) on the connected interval $[y_\wedge(t, z), y^\vee(t, z)]$, and we conclude the correspondence $Y^*(t, z)$ is non-empty for each $(t, z) \in \mathbf{T} \times Z$. □

**Lemma 12** *Under Assumptions A1″, A2′, and A3″″, the correspondence $Y^*(t, z)$ is ascending in $t$ in the weak set induced set order $\geq_{wi}$, for each $z \in Z$, and chain subcomplete valued for each $(t, z)$.*

*Proof* As $\hat{\Psi}$ increasing in $y$ on $[0, \hat{m}(t, z)]$, and strictly increasing when $y_i > 0$, we have the set of roots of $Y^*(t, z) = \{y | \Psi^e(y^*(t, z), 0, t, z) = 0\}$ *antichain*-valued for each $(t, z) \in \mathbf{T} \times Z$ relative to the componentwise order on $\mathbf{R}^N$ (Dacic 1979, Proposition 1.1).[35] To see $Y^*(t, z)$ is ascending in the weak induced set order in $t$, each $z$, fix $z \in Z$, and consider $y(t, z) \in Y^*(t, z)$ for any $t \in \mathbf{T}$. By the definition $Y^*$, for any element $y(t, z) \in Y^*(t, z)$, we have $\hat{\Psi}^e(y^*(t, z), \phi^*(t, z), t, z) = 0$ with $\phi^*(t, z) \geq 0$. Consider $t' \geq t, t \neq t'$, $t$ and $t'$ in $\mathbf{T}$. By Lemma 9, as $\hat{\Psi}$ is antitone in $t$, we have $-\infty < \hat{z} = \hat{\Psi}(y(t, z), \phi^*(t, z), t', z) \leq 0$. Define the directed up-set (e.g., filter) $G(y^*(t, z)) = \{y | y \geq y^*(t, z), \hat{\Psi}(y, \phi^*(y), t, z) = z \geq 0 \geq \hat{z}, z \text{ finite}\}$, where $\phi_i(y) \geq 0$ if $y_i = 0$ by the complementary slackness conditions. Notice, $G(y^*(t, z))$ is order-closed downward (i.e., chain subcomplete for all decreasing chains) as $\hat{\Psi}$ is continuous in $y$. Compute the point $y^T = \sup G(x(t, z)) \in [0, \hat{m}(t, z)]$, where the existence of $y^T$ follows from $[y(t, z), \hat{m}(t, z)]$ as complete sublattice of $\mathbf{R}_+^N$, and the strict bound $y^T = \sup G(x(t, z)) < \hat{m}$ follows from the Inada condition on the vector of period utilities $u(c)$ in A1″ for each agent $i$. As $\hat{\Psi}$ is continuous and increasing in $y$

[35] Let $X$ be a partially ordered set. We say a subset $X_1 \subset X$ is *antichained* if no two elements of $X_1$ are ordered.

on the connected order interval $[y(t,z), y^T]$, we conclude by Guillerme's intermediate value theorem applied to $\hat{\Psi}$, $\exists$ a $y^*(t', z)$ such that $y(t,z) \leq y^*(t', z) \in Y^*(t', z)$ in the order interval $[y(t,z), y^T]$ for all $t \leq t'$, each $z$. Therefore, as $y(t,z) \in Y^*(t,z)$ was arbitrary, we conclude $Y^*(t,z)$ is weak induced set order ascending upward (C1). A similar argument proves $Y^*(t,z)$ is weak induced set order ascending downward (C2). Therefore, $Y^*(t,z)$ is weak induced set order ascending. □

*Proof of Proposition 4*

*Proof* For this proof, fix $\hat{h} \in \mathbf{H}^c$, $\hat{h} < \hat{m}$, and recall $t \in \mathbf{T} = \{t | t \in \mathbf{S}^e \times \mathbf{H}^c \times \mathbf{H}^c, h < \hat{m}, \hat{h} < \hat{m}, k^d > 0\}$, each $z \in Z$. (i) By Lemma 11, $Y^*(t,z)$ is non-empty for all $(t,z)$. Further, by Lemma 12, $Y^*(t,z)$ is weak set order ascending in $t$, each $z$. Therefore, by Lemma 10, $Y^*(t,z)$ admits an increasing selection on $\mathbf{T} \times Z$ in $t$, each $z$. Noting the definition of $A(h;\hat{h})(s^e)$ elsewhere, $A(h;\hat{h})(s^e)$ is well defined and isotone in $(p^e, h, h)$. We need to now verify $A(h;\hat{h})(p^e, z) \in \mathbf{H}^c$ when $\hat{h} < \hat{m}$. Fix $h \in \mathbf{H}^c$, and let $p_1^e = (k_1, k_1^d) \geq (k_2, k_2^d) = p_2^e$. Then as $A(h;\hat{h})(p^e, z)$ is isotone in $p^e$, we have $A(h;\hat{h})(p_1^e, z) \geq A(h;\hat{h})(p_2^e, z)$. Using the definition of $A(h;\hat{h})(p^e, z)$ at these any two such $p^e$, we have when $\hat{h} = \hat{h}(p)$ is fixed [36]

$$\Psi_2(A(h;\hat{h})(p_1^e, z), s^e; h, \hat{h}) \geq \Psi_2(A(h;\hat{h})(p_2^e, z), s^e; h, \hat{h})$$

which implies by the definition of the operator,$A(h;\hat{h})(p^e, z)$ must be such each component of the vector

$$u'(\hat{m}(p^e, z) - A(h;\hat{h})(p^e, z), p^e, z)) + \phi^*(p^e, z; h, \hat{h})$$

is falling in $p^e$. When $A_i(h;\hat{h})(p_1^e, z) = A_i(h;\hat{h})(p_2^e, z) = 0$, as $\hat{m}_i \in \mathbf{H}^c$, $A_i(h;\hat{h})(p_1^e, z) \in \mathbf{H}^c$, and $\phi_i^*(p^e, z; h, \hat{h}) > 0$ for both $p_1^e$ and $p_2^e$ . So, the interesting case occurs when $\hat{m}(p_1^e, z) - A(h;\hat{h})(p_1^e, z), p_1^e, z) \geq 0$, such that for cohort $i$, $\hat{m}_i(p_2^e, z) - A_i(h;\hat{h})(p_2^e, z) > 0$. For this case, we must have

$$u'(\hat{m}(p_1^e, z) - A(h;\hat{h})(p_1^e, z), p_1^e, z)) \geq \hat{m}(p_2^e, z) - A(h;\hat{h})(p_2^e, z), p_2^e, z)$$

Therefore, $A(h;\hat{h})(p^e, z), p^e, z)$ must be such that $\hat{m}_i(p^e, z) - A_i(h;\hat{h})(p^e, z)$ is increasing in $p^e$. Noting the definition of $A(h;\hat{h})(p^e, z)$ elsewhere, we have $A(h;\hat{h})(p^e, z) \in \mathbf{H}^C$. (ii) That $A(h;h)(s^e)$ is isotone in $h$ on $\mathbf{H}^C$ follows from (i) (as $h$ is a component of $t$). Consider an increasing countable chain $H = \{h_n\} \subset \mathbf{H}^C$, $h^u = \sup(H)$. The point $h^u$ is well defined as $\mathbf{H}$ is a complete lattice. By the equicontinuity of the collection $\{h_n\}$ at any $s^e \in S_1^e$ for any $S_1^e \subset \mathbf{S}^e \backslash S^{e*}$, $S_1^e$ compact, $h_n \to h = h^u$, we have $A(h_{i\in I}) \to A(h^u)$. Further, as $\mathbf{H}$ is equicontinuous on any such $S_1^e$, we have $\vee H = h = h^u$ (see for example, Heikkila and Reffett 2006, Lemma 4.1). Since $A(h;\hat{h})(s^e)$ isotone in $h$, and $A(h_n)$ is a

[36] Notice, the claim is also true when we allow the arguments of $\hat{h}$ vary with $p^e$. As in the last section, we compute the problem in two steps, so for this step of the argument, we fixed $\hat{h} = \hat{h}(p)$ for this step.

chain, we have $\lim_n A(h_n) = \vee A(h_i) = A(h^u) = A(\vee H) = A(\lim h_n)$. Further, given the definition of $A(h; \hat{h})(s^e)$, $A(h; \hat{h})(s^e)$ is order continuous on $\mathbf{H}^C$. □

## Appendix D: Mathematical tools and results

This is a brief exposition of some of the mathematical tools and results used in the paper. Throughout the paper, we consider the compact intervals $X = [0, k_{\max}] \subset \mathbb{R}$, $Z = [z_{\min}, z_{\max}]$ with $0 < z_{\min} \leq z_{\max}$ (although many results can be generalized to compact subsets of $\mathbb{R}^n_+$). Denote by $X^* = X \setminus \{0\}$, $S = X \times Z$, $S^* = X^* \times Z$, and let $\mathcal{B}(S)$ be the Borel algebra associated with $S$. All compact subsets are endowed with the pointwise partial order $\leq$ and the usual topology on $\mathbb{R}^n$.

Given a partially ordered set, or poset, $(Y, \leq)$, a *chain* $C$ is a subset of $X$ that can be linearly ordered. The poset $(Y, \leq)$ is a *lattice* if $\forall (x, x') \in Y^2$, both the infimum $\wedge(x, x')$ and supremum $\vee(x, x')$ exist and belong to $Y$. If $B \subset Y$ and $(B, \leq)$ is a lattice, then we say that $B$ is a *sublattice* of $Y$. A lattice $(Y, \leq)$ is *complete* if $\vee B$ and $\wedge B$ exist for any $B \subset Y$. If $X_1 \subset Y$ is a sublattice, and $X_1$ is complete in its relative partial order, then $X_1$ is *subcomplete*. If every chain $C$ in $Y$ is complete, then $Y$ is *chain complete partially ordered set (or a CPO)*. If every chain $C_c$ in $Y$ is countable and complete, then $Y$ is *countable chain complete*. Finally, a countable chain $\{x_n\}$ with $x_i \leq x_j$ *order converges* to $x$ iff $\vee\{x_n\} = x$. A function (mapping) $f : (X, \leq_X) \to (Y, \leq_Y)$ is *isotone* if it is "order-preserving", that is if $x' \geq_X x$ implies $f(x') \geq_Y f(x)$, (this definition generalizes that of a "non-decreasing function"). Similarly, a mapping $f$ is *antitone* (i.e., order-reversing) if $x' \geq_X x$ implies $f(x) \geq_Y f(x')$. A mapping that is either isotone or antitone is *monotone*. The mapping $f$ is a *self-mapping* or *transformation* of $X$ if $Y = X$.

**Proposition 6** *The poset $(H^u, \leq)$ is a complete lattice. In addition any $h \in H^u$ is measurable.*

*Proof* Given any $B \subset H^u$, denote $g(s) = \inf_{h \in B} h(s)$. Clearly $0 \leq g \leq w$, $g$ is isotone, and $g(., z)$ is usc for any given $z$. Thus, $g$ is a lower bound of $B$, and it is easy to see that $g = \wedge B$. Since $w$ is the top element of $H^u$, the first result follows the theorem above. Next, since $X$ is a compact interval of $\mathbb{R}$, denote by $\{x_0, x_1, \ldots.\}$ a countable dense subset of $X$. Given any $\alpha \in \mathbb{R}$, we claim that:

$$\{s \in S, h(s) \leq \alpha\} = \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}.$$

This property implies that $h$ is measurable (in the sense of jointly measurable): Indeed, since $h$ is isotone in $z$ for each $k$, it is $\mathcal{B}(Z)$-measurable for each $k$ which implies that $\{z \in Z, h(x_m, z) < \alpha + 1/n\} \in \mathcal{B}(Z)$, and that $\{s \in S, h(s) \leq \alpha\} \in \mathcal{B}(S)$. We prove now the stated claim. First, consider $(k, z)$ such that $h(k, z) \leq \alpha$. Such $h$ being usc and isotone in $k$ (for each $z$), it is necessarily right continuous at $k$, and we have that:

$$\forall n \in \mathbb{N}, \exists m \text{ such that } x_m - 1/n < k < x_m \text{ and } h(x_m, z) < \alpha + 1/n.$$

Thus:

$$\forall n \in \mathbb{N}, \exists m \text{ such that } (k, z) \in (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\},$$

which implies that:

$$\forall n \in \mathbb{N}, (k, z) \in \bigcup_{m=0}^{\infty} (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\},$$

and therefore that:

$$(k, z) \in \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} (x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}.$$

Reciprocally, suppose that for all $n \in \mathbb{N}$, $(k, z)$ belongs to $\bigcup_{m=0}^{\infty}(x_m - 1/n, x_m] \times \{z \in Z,\ h(x_m, z) < \alpha + 1/n\}$. This implies that for all $n$, there exists $m(n)$ such that $k \in (x_{m(n)} - 1/n, x_{m(n)}]$ and $h(x_{m(n)}, z) < \alpha + 1/n$. By construction the sequence $\{x_{m(1)}, x_{m(2)}, \ldots\}$ converges to $k$ and $x_{m(n)} \geq k$, so by continuity from the right at $k$ of $h(., z)$, $h(x_{m(n)}, z)$ converges to $h(k, z)$ and necessarily $h(k, z) \leq \alpha$. Finally, we note that a similar result holds for the subset of $H$ of lsc functions, since it can be shown that:

$$\{(k, z) \in S, h(k, z) \geq \alpha\} = \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} [x_m, x_m + 1/n) \times \{z \in Z, h(x_m, z) > \alpha - 1/n\}.$$

□

*Remark* Note that for any $P \subset (H^u, \leq)$, $\wedge_{H^u} P$ coincides with $\wedge_H P$ (the pointwise inf) but that $\vee_{H^u} P$ and $\vee_H P$ (the pointwise sup) may differ.

A second class of lattices of interest for this paper are the spaces of probability measures, which we use to study the existence of Stationary Markov equilibria. Denote by $\Lambda(X, \mathcal{B}(X))$ the set of probability measures defined on the measurable space $(X, \mathcal{B}(X))$, and endow $\Lambda(X, \mathcal{B}(X))$ with the partial order $\geq_s$ of stochastic dominance:

$$\mu \geq_s \mu' \text{ if } \int_X f(k)\mu(\mathrm{d}k) \geq \int_X f(k)\mu'(\mathrm{d}k),$$

for every isotone, and bounded function $f : X \to \mathbb{R}_+$.

**Proposition 7** *The poset $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ is a complete lattice with minimal and maximal elements $\delta_0$ and $\delta_{k_{\max}}$.*

*Proof* It is easy to show that the set $\mathbb{D}(X)$ of functions $F : X \to [0, 1]$, that are isotone, upper semicontinuous, and satisfy $F(b) = 1$, is a complete lattice when endowed with the pointwise order. $\mathbb{D}(X)$ has maximal and minimal elements (respectively, the function $F(k) = 1$ for all $k \in X$, and the function $G(k) = 1$ if $k = b$ otherwise $G(k) = 0$). $\mathbb{D}(X)$ is in fact the set of probability distributions over the compact set $X$. It is well known that to any probability measure $\mu \in \Lambda(X, \mathcal{B}(X))$ corresponds a unique distribution function $F_\mu \in \mathbb{D}(X)$ and vice versa, and $\mu \geq_s \mu'$ is equivalent to $F_\mu \leq F_{\mu'}$ (see, for instance, Stokey et al. 1989).[37] $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ is thus isomorphic to $(\mathbb{D}(X), \leq)$, and is therefore a complete lattice with minimal element the singular probability measure $\delta_0$, and maximal element the singular probability measure $\delta_{k_{\max}}$. □

When $\Lambda(X, \mathcal{B}(X))$ is endowed with the weak topology, a sequence of probability measures $\{\mu_n\}$ in $\Lambda(X, \mathcal{B}(X))$ is said to weakly converge to $\mu \in \Lambda(X, \mathcal{B}(X))$ if for all continuous functions $f : X \to \mathbb{R}$:

$$\lim_{n\to\infty} \int_X f(k)\mu_n(\mathrm{d}k) = \int_X f(k)\mu'(\mathrm{d}k), \tag{CV}$$

In this case, we write $\mu_n \Longrightarrow \mu$, and we refer to $\mu$ as the weak limit of the sequence $\{\mu_n\}$. It is easy to see that when $X \subset \mathbb{R}$, increasing sequences $\{\mu_n\}$ in $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ necessarily converges to their supremum, that is:

$$\mu_n \Rightarrow \mu = \vee\{\mu_n\}.$$

Similarly, for a decreasing sequence to their infimum. □

Endowed with the stochastic dominance order, the set $\Lambda(S, \mathcal{B}(S))$ of probability measures defined on the measurable space $(S, \mathcal{B}(S))$ fails to be a complete lattice, although it is chain complete (see Hopenhayn and Prescott 1992).

**Proposition 8** *$(\Lambda(S, \mathcal{B}(S)), \geq_s)$ is a chain complete lattice with minimal and maximal elements $\delta_{(0,z_{\min})}$ and $\delta_{(k_{\max},z_{\max})}$.*

Existence proofs in this paper are based on an extension of Tarski's fixed point theorem for order continuous operators related to Theorem 4.2 in Dugundji and Granas (2003). Order continuity is defined as follows:

**Definition 5** A function $F : (P, \leq) \to (P, \leq)$ is order continuous if for any countable chain $C \subset P$ such that $\vee C$ and $\wedge C$ both exist,

$$\vee\{F(C)\} = F(\vee C) \text{ and } \wedge\{F(C)\} = F(\wedge C).$$

It is important to note that, in view of the above proof, the hypothesis of order continuity in (b) and (c) can be weakened to that of isotonicity of $F$ and order continuity along monotone recursive $F$-sequences, that is, sequences of the form $\{x, F(x), \ldots, F^n(x), \ldots\}$ where either $x \leq F(x)$ or $x \geq F(x)$.[38] In that case, $(P, \leq)$

[37] This is not true if $X \subset \mathbb{R}^l$ with $l \geq 2$, and this is one fundamental reason why this argument does not trivially generalize to economies with Markov shocks.

[38] Order continuity along monotone $F$-sequences does not imply that $F$ is isotone. Consider for instance $F : [0, 1] \to [0, 1]$ such that $F(x) = 1 - x$. $F$ is clearly continuous along the (only) monotone recursive $F$-sequence $\{1/2, 1/2, 1/2, \ldots\}$, but $F$ is not isotone.

needs only be chain complete for the existence of a non-empty set of fixed points with minimal and maximal elements. We shall exploit this property, which we state in the following corollary:

**Corollary 2** *With $F : (P, \geq) \to (P, \geq)$ isotone and order continuous along monotone $F$-sequences and $(P, \leq)$ chain complete with maximal element $p_{\max}$ and minimal element $p_{\min}$, the set of fixed points of $F$ is non-empty, and (b) and (c) in the previous Theorem hold true.*

## References


Abreu, D., Pearce, D., Stacchetti, E.: Toward a theory of repeated games with discounting. Econometrica **58**, 1041–1063 (1990)

Aliprantis, C., Border, K.: Infinite Dimensional Analysis: A Hitchhiker's Guide. Springer, Berlin (1999)

Amann, H., Escher, J.: Analysis 1. Springer, Berlin (2006)

Barnett, R., Bhattacharya, J., Bunzel, H.: Deviant generations, Ricardian equivalence, and growth cycles. Econ Theory. doi:10.1007/s00199-011-0645-3

Birkhoff, G.: Lattice Theory. AMS Press, New York (1967)

Blume, L.: New techniques for the study of stochastic equilibrium processes. J. Math. Econ. **9**, 61–70 (1982)

Brock, W., Mirman, L.: Optimal growth and uncertainty: the discounted case. J. Econ Theory **4**, 479–513 (1972)

Citanna, A., Siconolfi, P.: Short-memory equilibria in stochastic overlapping generations models. J. Econ. Theory **134**, 448–469 (2007)

Citanna, A., Siconolfi, P.: On the nonexistence of recursive equilibrium in stochastic OLG models. Econ Theory **37**(3), 417–437 (2008)

Citanna, A., Siconolfi, P.: Recursive equilibria in stochastic OLG economies. Econometrica **78**(1), 309–347 (2010)

Clarke, F.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)

Coleman, W.J.: Equilibrium in a production economy with an income tax. Econometrica **59**, 1091–1104 (1991)

Constantinides, G., Donaldson, J., Mehra, R.: Junior is rich: bequests a consumptions. Econ. Theory **32**, 125–155 (2007)

Dacic, R.: Fixed points of antitone mappings in conditionally complete Posets. Pub de L'Institut Mathematique. **40**, 83–90 (1979)

Datta, M., Mirman, L., Reffett, K.: Existence and uniqueness of equilibrium in distorted dynamic economies with capital and labor. J. Econ. Theory **103**, 377–410 (2002)

Davey, B., Priestley, H.: Introduction to Lattices and Order, 2nd edn. Cambridge University Press, Cambridge (2002)

de la Croix, D., Michel, P.: Education and growth with endogenous debt constraints. Econ. Theory **33**, 509–530 (2007)

Diamond, P.: National debt in a neoclassical growth model. Am. Econ. Rev. **55**, 1126–1150 (1965)

Dugundji, J., Granas, A.: Fixed Point Theory. Springer, New York (2003)

Duffie, D., Geanakoplos, J.D., Mas-Colell, A., McLennann, A.: Stationary Markov equilibria. Econometrica **62**, 745–782 (1994)

Feng, Z., Miao, J., Peralva-Alta, A., Santos, M.: Numerical simulation of nonoptimal dynamic equilibrium models. Working Paper 2009-18C, St Louis Federal Reserve Bank (2012)

Forni, L.: Social security as Markov equilibrium in OLG models. Rev. Econ. Dyn. **8**, 178–194 (2005)

Futia, Carl A.: Invariant distributions and the limiting behavior of Markovian economic models. Econometrica **50**(2), 377–408 (1982)

Greenwood, J., Huffman, G.: On the existence of nonoptimal equilibria in dynamic stochastic economies. J. Econ. Theory **65**, 611–623 (1995)

Guillerme, J.: Intermediate value theorems and fixed point theorems for semi-continuous functions in product spaces. Proc. Am. Math. Soc. **123**(7), 2119–2122 (1995)

Heikkila, S., Reffett, K.: Fixed point theorems and their applications to Nash equilibria. Nonlinear Anal. **64**, 1415–1436 (2006)

Hausenchild, N.: Capital accumulation in a stochastic overlapping generation model with social security. J. Econ. Theory **106**, 201–216 (2002)
Hopenhayn, H., Prescott, E.: Stochastic monotonicity and stationary distributions for dynamic economies. Econometrica **60**(6), 1387–1406 (1992)
Judd, K.: Projection methods for solving aggregate growth models. J. Econ. Theory **58**(2), 410–452 (1992)
Judd, K., Yeltekin, S., Conklin, J.: Computing supergame equilibria. Econometrica **71**(4), 1239–1254 (2003)
Kelley, J.: General Topology. Von Nostrand Press, New York (1955)
Kubler, F., Polemarchakis, H.: Stationary Markov equilibria for overlapping generations. Econ. Theory **24**, 1–22 (2004)
Kydland, F., Prescott, E.: Dynamic optimal taxation, rational expections, and optimal control. J. Econ. Dyn. Control **2**, 79–91 (1980)
Kyparisis, J.: On uniqueness of Kuhn-Tucker multipliers in nonlinear programing. Math. Program. **32**, 242–246 (1985)
Lloyd-Braga, T., Nourry, C., Venditti, A.: Indeterminacy in dynamic models: when Diamond meets Ramsey. J. Econ. Theory **134**, 513–536 (2007)
Lucas R.E. Jr., Prescott, E.: Investment under uncertainty. Econometrica **39**, 659–681 (1971)
Mirman, L.: On the existence of steady measures in one-sector growth models with uncertain technology. Int. Econ. Rev. **13**(2), 271–286 (1972)
Mirman, L.: The steady state behavior of a class of one sector growth models with uncertain technology. J. Econ. Theory **6**(3), 219–242 (1973)
Mirman, L., Morand, O., Reffett, K.: A Qualitative approach to Markovian equilibrium in infinite horizon economies with capital. J. Econ. Theory **139**, 75–98 (2004)
Morand, O., Reffett, K.: Stationary Markov equilibria in overlapping generations models with Stochastic Production. J. Math. Econ. **43**(3–4), 501–522 (2007)
Nishimura, K., Stachurski, J.: Stability of stochastic optimal growth models: a new approach. J. Econ. Theory **122**, 118–130 (2005)
Pestieau, P., Thibault, E.: Love they children, reflections on debt neutrality and estate taxes. Econ Theory doi:10.1007/s00199-010-0578-2
Prescott, E., Mehra, R.: Recursive equilibrium: the case of homogeneous agents. Econometrica **48**, 1365–1379 (1980)
Prieur, F.: The environmental Kuznet curve in a world of irreversibility. Econ. Theory **40**, 57–90 (2009)
Romer, P.: Increasing returns and long-run growth. J. Polit. Econ. **94**, 1002–1037 (1986)
Rincon-Zapatero, J., Santos, M.: Differentiability of the value function without interiority assumptions. J. Econ. Theory **144**, 1948–1964 (2009)
Samuelson, P.A.: An exact consumption-loans model of interest with or without the social contrivance of money. J. Polit. Econ. **66**, 467–482 (1958)
Santos, M.: On non existence of Markov equilibria in competitive-market economies. J. Econ. Theory **105**, 73–98 (2002)
Santos, M., Peralta-Alva, A.: Accuracy of simulations for stochastic dynamic models. Econometrica **73**(6), 1939–1976 (2005)
Smithson, R.: Fixed points of order preserving multifunctions. Proc. Am. Math. Soc. **28**(1), 304–310 (1971)
Stokey, N., Lucas, R.E., Prescott, E.: Recursive methods in economics dynamics. Harvard Press, Cambridge (1989)
Tarski, A.: A lattice-theoretical fixpoint theorem and its applications. Pac. J. Math. **5**, 285–309 (1955)
Topkis, D.: Supermodularity and Complementarity. Princeton Press, Princeton (1998)
Veinott, A.: Lattice Programming: Qualitative Optimization and Equilibria. MS. Stanford University, Stanford (1992)
Wang, Y.: Stationary equilibria in an overlapping generations economy with stochastic production. J. Econ. Theory. **61**, 423–435 (1993)
Wang, Y.: Stationary Markov equilibrium in an OLG model with correlated shocks. Int. Econ. Rev. **35**(3), 731–744 (1994)
Zhang, Y.: Stochastic optimal growth with non-compact state space. J. Math. Econ. **43**, 115–129 (2007)