Econ Theory (2013) 54:27–44
DOI 10.1007/s00199-012-0726-y

RESEARCH ARTICLE

Essential equilibria of discontinuous games

Vincenzo Scalzo

Received: 15 October 2011 / Accepted: 6 October 2012 / Published online: 17 October 2012

Abstract We introduce spaces of discontinuous games in which games having essential Nash equilibria are the generic case. In order to prove the existence of essential Nash equilibria in such spaces, we provide new results on the Ky Fan minimax inequality. In the setting of potential games, we show that games with essential Nash equilibria are the generic case when their potentials satisfy a condition called weak upper pseudocontinuity that is weaker than upper semicontinuity.

Keywords Discontinuous strategic form games · Potential games · Essential Nash equilibria · Ky Fan minimax inequality

JEL Classification C72

1 Introduction

Let G be a strategic form game. In this paper, we are interested in the existence of Nash equilibria of G which satisfy the additional property that any game close to G has Nash equilibria close to them. Such equilibria were introduced and called *essential Nash equilibria* by Wu and Jiang (1962).

We consider games where the payoffs are not necessarily continuous functions (henceforth *discontinuous games*). Remarkable examples of such games are the

The author is grateful to G. Carmona, P. Prokopovych and the anonymous referees for their precious comments and suggestions.

V. Scalzo (✉)
Dipartimento di Matematica e Statistica, Università di Napoli Federico II,
via Cinthia (Monte S. Angelo), 80126 Naples, Italy
e-mail: vincenzo.scalzo@unina.it

Bertrand oligopoly (see Bertrand 1883) and the Hotelling linear city model (see Hotelling 1929; D'Aspemont et al. 1979), which have been a source of inspiration for a wide body of the literature on equilibrium existence in games with discontinuous payoffs. In an early paper, Dasgupta and Maskin (1986) studied the existence of Nash equilibria in pure strategies under a condition more general than the lower semicontinuity of the payoff functions with respect to the other players' strategies. Afterward, both upper and lower semicontinuity of payoffs have been relaxed in several ways. Baye et al. (1993) showed that a compact game has a Nash equilibrium if its aggregator function is *diagonally transfer continuous* and *diagonally transfer quasi-concave*; see also Tian and Zhou (1995).[1] Reny (1999) introduced the *better-reply secure* games and proved that every compact, quasi-concave, and better-reply secure game has a Nash equilibrium. Some related results and generalizations can be found in Carmona (2009), Bich (2009), Carmona (2011a,b), De Castro (2011), Prokopovych (2011), Reny (2011). Morgan and Scalzo (2007) introduced the class of *pseudocontinuous* functions and showed that a compact and quasi-concave game has a Nash equilibrium whether the payoffs are pseudocontinuous functions. Moreover, see Lebrun (1995) for the existence of equilibria in discontinuous games with imperfect information.

In the present paper, we study *essential* Nash equilibria of discontinuous games; in particular, we are interested in spaces $\mathfrak{g}$ of discontinuous games which have the following property: there exists a dense subset $\mathfrak{q}$ of $\mathfrak{g}$ such that every game which belongs to $\mathfrak{q}$ has essential Nash equilibria. If a space $\mathfrak{g}$ satisfies the property, we say that games with essential Nash equilibria are the generic case in $\mathfrak{g}$. Some spaces of discontinuous games with this property were described by Yu (1999) and by Carbonell-Nicolau (2010); their results rely on a theorem of Fort (1949) on continuity points of set-valued functions.

In addition to Fort's theorem, we also rely on the Ky Fan minimax inequality (Fan 1972): *find* $z \in X$ *such that* $\Phi(x, z) \leq 0$ *for any* $x \in X$. In particular, we introduce the class of *generalized positively quasi-transfer continuous* functions and prove that the space, denoted by $\mathfrak{f}_1$, of generalized positively quasi-transfer continuous functions Φ such that the Ky Fan inequality has solutions is a non-empty and complete metric space (equipped with the sup-norm metric). When X is a convex and compact subset of a metrizable and locally convex topological vector space, a non-empty and complete subset of $\mathfrak{f}_1$, denoted by $\mathfrak{f}_2$, is given by the generalized positively quasi-transfer continuous functions Φ which are 0-diagonally quasi-concave.

Since the set of Nash equilibria of a game coincides with the solution set of the Ky Fan minimax inequality corresponding to a suitable function Φ, using the properties of $\mathfrak{f}_1$ and $\mathfrak{f}_2$, we give new spaces of discontinuous games where games having essential Nash equilibria are the generic case. In the setting of *potential games* (Monderer and Shapley 1996), we use a condition called *weak upper pseudocontinuity* (Morgan and Scalzo 2006; Scalzo 2009) that is weaker than upper semicontinuity.

The outline of the paper is as follows. In Sect. 2, we recall some definitions and introduce the class of generalized positively quasi-transfer continuous functions. In Sect. 3, we give new results on the Ky Fan minimax inequality. Sufficient conditions

[1] See Scalzo (2010) for an application of transfer continuity to the existence of *efficient* Nash equilibria.

for the existence of essential Nash equilibria are provided in Sect. 4. Section 5 deals with potential games. Concluding remarks are given in Sect. 6.

2 Setting and preliminaries

We recall that a *strategic form game* (in short: *game*) is a set of data $G = \langle X_i, u_i \rangle_{i \in I}$ where, for any player $i \in I$, X_i is a non-empty set of strategies and the payoff u_i is a real-valued function defined on $X = \prod_{j \in I} X_j$. We assume that the set of players I is finite. The elements of X are called *strategy profiles*; if x is a strategy profile, we use the notation $x = (x_i, x_{-i})$, where $x_i \in X_i$ and $x_{-i} \in X_{-i} = \prod_{i \neq j \in I} X_j$. Assume that any player wishes to maximize his/her own payoff. We recall that a strategy profile x^* is a *Nash equilibrium*—Nash (1950)—if $u_i(x^*) \geq u_i(x_i, x^*_{-i})$ for any $x_i \in X_i$ and any $i \in I$.

Throughout the paper, we assume that X is a non-empty and compact subset of a metrizable topological space (the metric is denoted by d) and we consider games such that X is the set of their strategy profiles. Moreover, we assume that the payoffs of any game are bounded functions on X. Let $\mathfrak{g}^b$ be the space of such games and let ρ be the metric defined on $\mathfrak{g}^b$ as below (see Wu and Jiang 1962; Yu 1999):

$$\rho(G, G') = \sum_{i \in I} \sup_{x \in X} |u_i(x) - u'_i(x)| \; . \tag{1}$$

It is easy to see that $\mathfrak{g}^b$ is a complete metric space.

Definition 1 Let $\mathfrak{g} \subseteq \mathfrak{g}^b$ and $G \in \mathfrak{g}$. A strategy profile x is said to be an *essential Nash equilibrium of G relative to* $\mathfrak{g}$ if it is a Nash equilibrium of G and for any $\varepsilon > 0$, there exists $\delta > 0$ such that any game $G' \in \mathfrak{g}$ with $\rho(G, G') < \delta$ has at least one Nash equilibrium x' such that $d(x, x') < \varepsilon$.

Our aim is to introduce classes of discontinuous games where games having essential Nash equilibria are the generic case, that is, if $\mathfrak{g}$ is a class of games, there exists a dense subset $\mathfrak{q}$ of $\mathfrak{g}$ such that every game belonging to $\mathfrak{q}$ has essential Nash equilibria relative to $\mathfrak{g}$. We use the *Ky Fan minimax inequality* (Fan 1972):

$$\text{find } z \in X \text{ such that } \Phi(x, z) \leq 0 \;\text{ for any } x \in X \; , \tag{2}$$

where Φ is a real-valued function defined on $X \times X$. It is easy to see that x^* is a Nash equilibrium of a game $G = \langle X_i, u_i \rangle_{i \in I}$ if and only if $\Phi(x, x^*) \leq 0$ for any $x \in X$, where Φ is defined on $X \times X$ as below:

$$\Phi(x, z) = \sum_{i \in I} \left[u_i(x_i, z_{-i}) - u_i(z) \right] \; . \tag{3}$$

We consider games G where the functions Φ defined by (3) are generalized positively quasi-transfer continuous (see the definition below). First, let us recall that a correspondence (set-valued function) $\mathfrak{T} : Y \rightrightarrows Z$, where Y and Z are topological spaces, is said to be *upper semicontinuous* if, for any $y \in Y$ and for any open set O which includes $\mathfrak{T}(y)$, there exists an open neighborhood U of y such that $\mathfrak{T}(y') \subseteq O$ for any $y' \in U$ (see, for example, Aliprantis and Border 1999). A correspondence is said to be *well-behaved* if it is upper semicontinuous and non-empty, convex and compact valued.

Definition 2 We say that a function $\Phi : X \times X \longrightarrow \mathbb{R}$ is *generalized t-quasi-transfer continuous*[2] if, whenever $\Phi(x, z) > t$ for some $(x, z) \in X \times X$, there exists a neighborhood U_z of z and a well-behaved correspondence $\xi : U_z \rightrightarrows X$ such that $\Phi(\xi(z'), z') > t$ for any $z' \in U_z$. We say that Φ is *generalized positively quasi-transfer continuous* if it is generalized t-quasi-transfer continuous for any $t > 0$.[3]

Definition 3 We say that a function $\Phi : X \times X \longrightarrow \mathbb{R}$ is *positively quasi-transfer continuous* if, whenever $\Phi(x, z) > t$ for some $(x, z) \in X \times X$ and $t > 0$, there exists a neighborhood U_z of z and $x' \in X$ such that $\Phi(x', z') > t$ for any $z' \in U_z$.

Obviously, any positively quasi-transfer continuous function is generalized positively quasi-transfer continuous. Example 5 shows that the converse is not true. The next proposition shows that, if Φ is the function defined by (3) for a generalized payoff secure game where the sum of the payoff functions is upper semicontinuous, then Φ is generalized positively quasi-transfer continuous. We recall that a game $G = \langle X_i, u_i \rangle_{i \in I}$ is *generalized payoff secure* (see, for example, Carbonell-Nicolau 2010) if for any $x \in X$, any $\varepsilon > 0$ and any player i, there exists a neighborhood U^i of x and a well-behaved correspondence $\xi_i : U^i \rightrightarrows X_i$ such that $u_i(\xi_i(x'), x'_{-i}) > u_i(x) - \varepsilon$ for all $x' \in U^i$. When the correspondence ξ_i is constant and single-valued on U^i for any x, any $\varepsilon > 0$ and any i, then G is said to be *payoff secure* (see Reny 1999).

Proposition 1 *Let $G = \langle X_i, u_i \rangle_{i \in I}$ be a generalized payoff secure game where the sum of the payoff functions is upper semicontinuous. Then, the function Φ defined by (3) is generalized positively quasi-transfer continuous.*

Proof Assume that $\Phi(x, z) > t > 0$ and let $\varepsilon > 0$ such that $\Phi(x, z) > \varepsilon(1 + |I|) + t$, where $|I|$ denotes the number of players. So, we have :

$$\sum_{i \in I} u_i(x_i, z_{-i}) - |I|\varepsilon > \varepsilon + \sum_{i \in I} u_i(z) + t \ . \tag{4}$$

Since G is generalized payoff secure, for any player i, there exists a neighborhood $U^i = U^i_{x_i} \times \prod_{l \neq i} U^i_{z_l}$ of (x_i, z_{-i}) and a well-behaved correspondence $\xi_i : U^i \rightrightarrows X_i$ such that[4]

$$u_i\left(\xi_i(z'), z'_{-i}\right) > u_i(x_i, z_{-i}) - \varepsilon \qquad \forall\, z' \in U^i \ .$$

2 Another possible name, suggested by a referee, is *generalized t-transfer lower semicontinuous*.

3 $\Phi(\xi(z'), z') > t$ denotes that $\Phi(s, z') > t$ for each $s \in \xi(z')$.

4 $u_i(\xi_i(z'), z'_{-i}) > u_i(x_i, z_{-i}) - \varepsilon$ denotes that $u_i(s'_i, z'_{-i}) > u_i(x_i, z_{-i}) - \varepsilon$ for any $s'_i \in \xi_i(z')$.

For any $l \in I$, we set $U_{z_l} = \bigcap_{j \neq l} U^j_{z_l}$ and $U'_z = \prod_{l \in I} U_{z_l}$. So, one gets:

$$\sum_{i \in I} u_i \left(\xi_i(x_i, z'_{-i}), z'_{-i} \right) > \sum_{i \in I} u_i(x_i, z_{-i}) - |I|\varepsilon \qquad \forall\, z' \in U'_z \,. \tag{5}$$

On the other hand, since the sum of the payoff functions is upper semicontinuous, there exists a neighborhood U''_z of z such that:

$$\varepsilon + \sum_{i \in I} u_i(z) > \sum_{i \in I} u_i(z') \qquad \forall\, z' \in U''_z \,. \tag{6}$$

Let $U_z = U'_z \cap U''_z$ and $\xi : U_z \rightrightarrows X$ be the correspondence defined by $\xi(z') = \prod_{i \in I} \xi_i(x_i, z'_{-i})$ for any $z' \in U_z$—we recall that $\xi_i(x_i, z'_{-i}) \subseteq X_i$ for any $i \in I$. It is clear that ξ is well-behaved. From (4), (5), and (6), we obtain:

$$\Phi \left(\xi(z'), z' \right) = \sum_{i \in I} \left[u_i \left(\xi_i(x_i, z'_{-i}), z'_{-i} \right) - u_i(z') \right] > t \qquad \forall\, z' \in U_z \,,$$

which concludes the proof. □

Remark 1 In light of Proposition 1, the class of games $\mathfrak{g}_X$ considered in Theorem 2 by Carbonell-Nicolau (2010) is included in the class of games where the functions Φ defined according to (3) are generalized positively quasi-transfer continuous. We recall that $\mathfrak{g}_X$ is the set of generalized payoff secure games $G = \langle X_i, u_i \rangle_{i \in I}$ such that the sum of the payoff functions is upper semicontinuous and, for any $i \in I$, X_i is a convex and compact subset of a metric space, u_i is bounded and $u_i(\cdot, x_{-i})$ is quasi-concave for any x_{-i}. Let $\mathfrak{g}_1$ be the space of games where any game has Nash equilibria and the function Φ defined by (3) is generalized positively quasi-transfer continuous. Since any game belonging to $\mathfrak{g}_X$ has Nash equilibria, in light of Proposition 1, we have that $\mathfrak{g}_1$ is non-empty and includes $\mathfrak{g}_X$. The following Example 1 shows that this inclusion is strict. In Sect. 4, we prove that games having essential Nash equilibria are the generic case in $\mathfrak{g}_1$.

Example 1 Let $G = \langle X_1, X_2, u_1, u_2 \rangle$ be the game where $X_1 = X_2 = [0, 1]$ and the payoffs are defined as follows:

$$u_1(x_1, x_2) = \begin{cases} 2 - x_2 & \text{if } (x_1, x_2) \in [0, 1] \times [0, 1[\\ 0 & \text{if } (x_1, x_2) \in [0, 1] \times \{1\} \end{cases} ;$$

$u_2(x_1, x_2) = 0$ for each $(x_1, x_2) \in [0, 1] \times [0, 1]$. Note that such functions are bounded and $u_i(\cdot, x_{-i})$ is quasi-concave for any x_{-i} and any i. The sum $u_1 + u_2$ is not upper semicontinuous; for instance, $u_1(1, 1) = 0$ but $u_1(1, x^n_2)$ converges to 1 for any sequence $x^n_2 \longrightarrow 1$ where $x^n_2 < 1$ for n sufficiently large. So, $G \notin \mathfrak{g}_X$. On the other hand, it is easy to see that $\Phi(x, z) = 0$ for any x and z which belong to $X = X_1 \times X_2$. Hence, Φ is generalized positively quasi-transfer continuous. Moreover, the set of Nash equilibria of G coincides with X.

In the setting of discontinuous *potential games*, we use a generalization of upper semicontinuity that is called weak upper pseudocontinuity (see the definition below by Morgan and Scalzo 2006; Scalzo 2009). We recall that a game $G = \langle X_i, u_i \rangle_{i \in I}$ is said to be a *potential game* (Monderer and Shapley 1996) if there exists a function P such that, for each player i:

$$u_i(x_i, x_{-i}) - u_i(x'_i, x_{-i}) = P(x_i, x_{-i}) - P(x'_i, x_{-i})$$

for any x_i and x'_i belonging to X_i and for any $x_{-i} \in X_{-i}$. The function P is the *potential* of G. Examples of potential games are the congestion games (see Rosenthal 1973; Monderer and Shapley 1996) and the quasi-Cournot competition (see Monderer and Shapley 1996). Let us note that if G is a potential game, any maximizer of its potential is a Nash equilibrium of G, but the converse is not true (see Example 3).

Definition 4 A real-valued function F defined on a metric space X is said to be *weakly upper pseudocontinuous* if, for any x and x' belonging to X such that $F(x) < F(x')$, we have:

$$\limsup F(x_n) \leq F(x') \quad \text{for any sequence} \quad x_n \longrightarrow x \ .$$

Remark 2 If X is a compact metric space and F is weakly upper pseudocontinuous, then the set of maximum points of F is non-empty: see Scalzo (2009).

Any upper semicontinuous function is weakly upper pseudocontinuous, but the converse is not true. In fact, the Dirichlet's function F, defined on the set of real numbers by $F(x) = 0$ if x is a rational number and $F(x) = 1$ otherwise, is weakly upper pseudocontinuous, but it is not upper semicontinuous. We note that the utility functions representing *weakly lower continuous* preferences (see Campbell and Walker 1990) are not necessarily upper semicontinuous, but weakly upper pseudocontinuous. A potential game where the potential is weakly upper pseudocontinuous but not upper semicontinuous, as well as the sum of the payoff functions is not upper semicontinuous, is given by the example below.

Example 2 Let $G = \langle X_1, X_2, u_1, u_2 \rangle$ be the game where $X_1 = [0, 1]$, $X_2 = [0, 2]$ and

$$u_1(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in [0, 1[\times[0, 1] \\ 2 & \text{if } (x_1, x_2) \in [0, 1[\times]1, 2] \\ 1 & \text{if } (x_1, x_2) \in \{1\} \times [0, 2] \end{cases} ,$$

$$u_2(x_1, x_2) = \begin{cases} 1 - x_2 & \text{if } (x_1, x_2) \in [0, 1] \times [0, 1[\cup\{(1, 1)\} \\ 0 & \text{if } (x_1, x_2) \in [0, 1[\times[1, 2] \\ -1 & \text{if } (x_1, x_2) \in \{1\}\times]1, 2] \end{cases} .$$

The sum $u_1 + u_2$ is not upper semicontinuous: consider $(1, x_2)$ with $x_2 \in]1, 2]$ and $(x_1^n, x_2^n) \longrightarrow (1, x_2)$ where $x_1^n < 1$ for any n. So, we have $(u_1 + u_2)(1, x_2) = 0$ and $(u_1 + u_2)(x_1^n, x_2^n) = 2$ for any n. Moreover, the functions u_1 and u_2 are not upper semicontinuous, but weakly upper pseudocontinuous. In fact, for the function u_1, one has $u_1(1, 1) = 1$ but, for any $(x_1^n, x_2^n) \longrightarrow (1, 1)$ where $x_1^n < 1$ and $x_2^n > 1$, we have $u_1(x_1^n, x_2^n) = 2$ for each n. It is easy to prove that u_1 is weakly upper pseudocontinuous. The function u_2 is not upper semicontinuous at $(1, x_2)$ for any $x_2 \in]1, 2]$. In fact, if $x_2 \in]1, 2]$ and $(x_1^n, x_2^n) \longrightarrow (1, x_2)$ with $x_1^n < 1$ for each n, one gets $u_2(x_1^n, x_2^n) = 0$ for each n and $u_2(1, x_2) = -1$. Besides, if $u_2(1, x_2) < u_2(\bar{x}_1, \bar{x}_2)$ and $\{(x_1^n, x_2^n)\}_n$ is a sequence converging to $(1, x_2)$, we have $u_2(\bar{x}_1, \bar{x}_2) \geq 0$ and $u_2(x_1^n, x_2^n) \in \{-1, 0\}$ for each n. Hence, u_2 is weakly upper pseudocontinuous at $(1, x_2)$ for all $x_2 \in]1, 2]$. Similarly, one can prove that u_2 is weakly upper pseudocontinuous on $[0, 1[\times[0, 2] \cup \{1\} \times [0, 1]$. Finally, one can prove that G is a potential game with potential $P = u_2$.

The class of potential games having weakly upper pseudocontinuous potentials is not included in the class of games where the functions Φ defined by (3) are generalized positively quasi-transfer continuous. In fact, we have:

Example 3 Let $G = \langle X_1, X_2, v_1, v_2 \rangle$ be the game such that $X_1 = [0, 1]$, $X_2 = [0, 2]$ and $v_1 = v_2 = u_1$, where u_1 is the function considered in Example 2. Obviously, $P = u_1$ is a potential of G. Let $z = (1, z_2)$ with $z_2 \in]1, 2]$ and $x \in [0, 1[\times[0, 2]$. We have:

$$\Phi(x, z) = P(x_1, z_2) + P(z_1, x_2) - 2P(z) = 1 \ .$$

If U is a neighborhood of z, for any $z' \in U \cap [0, 1[\times]1, 2]$ and for any $x' \in [0, 1] \times [0, 2]$, we get $P(z') = 2$ and $\Phi(x', z') \leq 0$. Therefore, Φ is not generalized positively quasi-transfer continuous. Note that the set of Nash equilibria of G is $[0, 1[\times]1, 2] \cup \{(1, x_2) : x_2 \in [0, 1]\}$, while the set of maximum points of the potential is $[0, 1[\times]1, 2]$.

Remark 3 Let $P : X \longrightarrow \mathbb{R}$ and Υ be the function defined on $X \times X$ by $\Upsilon(x, z) = P(x) - P(z)$. The weak upper pseudocontinuity is equivalent to the following condition *a*): $\Upsilon(x, z) > 0$ implies that for any $\varepsilon > 0$, there exists a neighborhood U of z such that $\Upsilon(x, z') + \varepsilon \geq 0$ for any $z' \in U$. Assume by way of contradiction that P is weakly upper pseudocontinuous and condition *a*) does not hold. So, for at least one $(x, z) \in X \times X$ and $\varepsilon > 0$, we have $\Upsilon(x, z) > 0$ and, for any neighborhood U_n of z in a countable local base decreasing with respect to inclusion, there exists $z_n \in U_n$ such that $\Upsilon(x, z_n) + \varepsilon = P(x) - P(z_n) + \varepsilon < 0$. Since the sequence $(z_n)_n$ converges to z, we get the contradiction $P(x) + \varepsilon \leq \limsup P(z_n) \leq P(x)$. Finally, assume that Υ satisfies *a*). Let $P(z) < P(x)$ and $z_n \longrightarrow z$. So, $\Upsilon(x, z) > 0$ and, for any $\varepsilon > 0$, we have $P(z_n) \leq P(x) + \varepsilon$ for n sufficiently large, which gives $\limsup P(z_n) \leq P(x) + \varepsilon$ for any $\varepsilon > 0$. So $\limsup P(z_n) \leq P(x)$.[5]

In the next sections, we prove that games having essential Nash equilibria are the generic case in the spaces of discontinuous games $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_1^p$, that are defined below. First, let us recall that a function $\Phi : X \times X \longrightarrow \mathbb{R}$ is said to be 0-*diagonally quasi-concave in the first argument* if for any $F = \{x_1, \ldots, x_k\} \subset X$ and for any $z \in \mathrm{co}F$ (we denote by $\mathrm{co}A$ the convex hull of a set A) such that $z = \sum_{j=1}^{l} \lambda_{i_j} x_{i_j}$ and $\lambda_{i_j} > 0$ for $j = 1, \ldots, l$, we have $\min\{\Phi(x_{i_j}, z) : j = 1, \ldots, l\} \leq 0$ (see Zhou

5 The author thanks a referee that pointed out the equivalence between weakly upper pseudocontinuity and condition *a*).

and Chen 1988; Tian 1992). Assume that X is the compact set of strategy profiles of any game:

$\mathfrak{g}_1$ is the space of games where, for any game, the payoffs are bounded, the function Φ defined by (3) is generalized positively quasi-transfer continuous, and the solution set of inequality (2) is non-empty.

$\mathfrak{g}_2$ is the space of games such that: X is a convex subset of a metrizable and locally convex topological vector space; for any game, the payoffs are bounded and the function Φ defined by (3) is generalized positively quasi-transfer continuous and 0-diagonally quasi-concave in the first argument.

$\mathfrak{g}_1^p$ is the space of potential games where, for any game, the payoffs are bounded and the potential is weakly upper pseudocontinuous.

Let us remark that: $\mathfrak{g}_2 \subset \mathfrak{g}_1$ and the set of Nash equilibria of every game which belongs to $\mathfrak{g}_2$ is non-empty and compact (see the following Proposition 2); $\mathfrak{g}_X \subset \mathfrak{g}_1$ (see Proposition 1 and Remark 1), where $\mathfrak{g}_X$ is the space of games considered in Theorem 2 by Carbonell-Nicolau (2010); $\mathfrak{g}_1^p \not\subseteq \mathfrak{g}_1$ (see Example 3). Of course, neither $\mathfrak{g}_1$ nor $\mathfrak{g}_2$ is included in $\mathfrak{g}_1^p$. Moreover, $\mathfrak{g}_1^p$ is included neither in the space of potential games where, for any game, the sum of the payoff functions is upper semicontinuous nor in the space of potential games where, for any game, the potential is upper semicontinuous (see Example 2).

3 Generalized positive quasi-transfer continuity and Ky Fan minimax inequality

Let $\mathfrak{f}_0$ be the set of bounded and generalized positively quasi-transfer continuous functions defined on $X \times X$. We denote by $\mathfrak{f}_1$ the subset of functions $\Phi \in \mathfrak{f}_0$ such that the solution set of inequality (2) is non-empty. Note that if Φ is defined by (3) for a game $G \in \mathfrak{g}_X$, then $\Phi \in \mathfrak{f}_1$: see Remark 1. Moreover, in light of the following Proposition 2, another class of functions included in $\mathfrak{f}_1$ is given by the functions Φ that are generalized positively quasi-transfer continuous and 0-diagonally quasi-concave in the first argument.

Proposition 2 *Assume that X is a convex and compact subset of a metrizable and locally convex topological vector space. If Φ is generalized 0-quasi-transfer continuous and 0-diagonally quasi-concave in the first argument, then the solution set of inequality (2) is non-empty and compact.*[6]

Proof Let S be the solution set of inequality (2). First, we prove that $S \neq \emptyset$. By contradiction, assume that for any $z \in X$, there exists $x \in X$ such that $\Phi(x, z) > 0$. Since Φ is generalized 0-quasi-transfer continuous, for any $z \in X$, there exists a neighborhood U_z of z and a well-behaved correspondence $\xi_z : U_z \rightrightarrows X$ such that $\Phi\left(\xi_z(z'), z'\right) > 0$ for any $z' \in U_z$. By compactness of X, we have $X = \bigcup_{j=1}^k U_{z_j}$. Let $\{\beta_j\}_{j=1}^k$ be a partition of the unity subordinate to $\{U_{z_j}\}_{j=1}^k$ (see, for example,

[6] The author thanks P. Prokopovych for his hint about the existence of solutions to inequality (2) with generalized 0-quasi-transfer continuous functions.

Aliprantis and Border 1999) and let $\xi : X \rightrightarrows X$ be the correspondence defined by $\xi(x) = \sum_{\beta_j(x)>0} \beta_j(x)\xi_{z_j}(x)$. It is easy to see that ξ is upper semicontinuous with non-empty, convex and compact values. So, in light of the Kakutani–Fan–Glicksberg fixed point theorem (see, for example, Aliprantis and Border 1999), ξ has a fixed point x^*. Assume that $x^* = \sum_{\beta_j(x^*)>0} \beta_j(x^*)s_j^*$, where $s_j^* \in \xi_{z_j}(x^*)$. Since $x^* \in U_{z_j}$ for any $j \in \{1, \ldots, k\}$ such that $\beta_j(x^*) > 0$, we have $\min\{\Phi(s_j^*, x^*) : \beta_j(x^*) > 0\} > 0$. On the other hand, Φ is 0-diagonally quasi-concave in the first argument and we get the following contradiction:

$$0 \geq \min \left\{ \Phi(s_j^*, x^*) : \beta_j(x^*) > 0 \right\} > 0 \,.$$

So, S is non-empty. Finally, it is easy to prove that S is a closed subset of X. □

It is easy to see that a function Φ is 0-diagonally quasi-concave in the first argument whether, for any $z \in X$, $\Phi(\cdot, z)$ is quasi-concave and $\Phi(z, z) \leq 0$. Hence, Proposition 2 implies:

Corollary 1 *Assume that X is a convex and compact subset of a metrizable and locally convex topological vector space. If Φ is generalized 0-quasi-transfer continuous and, for any $z \in X$, $\Phi(\cdot, z)$ is quasi-concave and $\Phi(z, z) \leq 0$, then the solution set of inequality (2) is non-empty and compact.*

In the following, we assume that $\mathfrak{f}_0$ is endowed with the sup-norm metric ρ_1:

$$\rho_1(\Phi, \Phi') = \sup_{(x,z) \in X \times X} \left| \Phi(x, z) - \Phi'(x, z) \right| \,.$$

Proposition 3 $\mathfrak{f}_0$ *is a complete metric space.*

Proof Since the space of bounded functions is complete in the metric ρ_1 (see, for example, Aliprantis and Border 1999), it is sufficient to prove that if a sequence $(\Phi_n)_n \subseteq \mathfrak{f}_0$ is converging to a function Φ, then $\Phi \in \mathfrak{f}_0$. Assume that $\Phi(x, z) > t > 0$. Let t_1 such that $\Phi(x, z) > t_1 > t$. Since $\rho_1(\Phi_n, \Phi)$ converges to 0, we have $\rho_1(\Phi_n, \Phi) < t_1 - t$ for any n greater than some n_o. On the other hand, $\Phi(x, z) > t_1$ implies $\Phi_n(x, z) > t_1$ for any $n \geq n_1$, where $n_1 \geq n_o$. Let $n \geq n_1$. Because Φ_n is generalized positively quasi-transfer continuous, there exists a neighborhood U_z of z and a well-behaved correspondence $\xi : U_z \rightrightarrows X$ such that $\Phi_n(s, z') > t_1$ for any $s \in \xi(z')$ and for any $z' \in U_z$, which implies:

$$\Phi_n(s, z') - t > t_1 - t > \rho_1(\Phi_n, \Phi) \geq \Phi_n(s, z') - \Phi(s, z') \,.$$

So, $\Phi\left(\xi(z'), z'\right) > t$ for any $z' \in U_z$, that is: Φ is generalized positively quasi-transfer continuous.[7] □

[7] The author thanks G. Carmona and P. Prokopovych for streamlining the proof.

Remark 4 Baye et al. (1993) have obtained the existence of solutions to inequality (2) with functions Φ that are *diagonally transfer quasi-concave* and *diagonally transfer continuous*.[8] Note that any diagonally transfer continuous function is generalized 0-quasi-transfer continuous, but the converse is not true. However, neither the class of generalized 0-quasi-transfer continuous functions nor the class of diagonally transfer continuous functions are complete spaces in the sup-norm metric: see Example 4. A function which is generalized positively quasi-transfer continuous but not diagonally transfer continuous is given in Example 5. We remark that the existence of solutions to inequality (2) can be obtained in several ways and under quasi-concavity-like conditions weaker than that considered in Proposition 2: see Chang (2010) and references therein. Nevertheless, for the purposes of the present paper, we need to consider a property that is robust with respect to the convergence in the sup-norm metric (the space of 0-diagonally quasi-concave in the first argument functions is complete: see Lemma 1). Finally, see Yannelis (1991) for an in-depth discussion on the classical existence results of solutions to the Ky Fan minimax inequality.

Example 4 For any $n \in \mathbb{N}$, let Φ_n be the function defined on $[0,1] \times [0,1]$ by $\Phi_n(x,z) = u_n(x) - u_n(z)$, where u_n is as below:

$$u_n(x) = \begin{cases} x & \text{if } x \in \left[0, 1 - \frac{1}{n}\right] \\ 2\left(1 - \frac{1}{n}\right) - x & \text{if } x \in \left]1 - \frac{1}{n}, 1\right[\\ 0 & \text{if } x = 1 \end{cases} .$$

The function Φ_n is diagonally transfer continuous because u_n is transfer upper continuous[9]. Since $(u_n)_n$ converges to the function u defined by $u(x) = x$ for each $x \in [0, 1[$ and $u(1) = 0$, the sequence $(\Phi_n)_n$ converges to the function $\Phi(x,z) = u(x) - u(z)$ in the metric ρ_1, but Φ is not generalized 0-quasi-transfer continuous at any point $(x, 1)$ with $x > 0$.

Example 5 Let $f : [0,1] \longrightarrow [0,1]$ be a continuous and strictly increasing function such that $f(0) = 0$. We denote by X the set $[0,1] \times [0,1]$ and define the function $u : X \longrightarrow \mathbb{R}$ as below:

$$u(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 = f(x_2) \\ 0 & \text{otherwise} \end{cases} .$$

Let $\Phi(x,z) = u(x_1, z_2) - u(z)$ for all $(x,z) \in X \times X$. Assume that $\Phi(x,z) > t > 0$: this is possible only in the case where $x_1 = f(z_2) > 0$ and $z_1 \neq f(z_2)$, and we have $\Phi(x,z) = x_1 > t$. Since $z_1 \neq f(z_2)$ and $x_1 = f(z_2) > t$, in light of the continuity of f, there exists a neighborhood U_z of z such that $z'_1 \neq f(z'_2)$ and $f(z'_2) > t$ for any

[8] A function Φ is said to be: *diagonally transfer quasi-concave* if for any finite subset $F \subset X$ there exists a finite subset $F' \subset X$ such that to any $x_{i_j} \in F$ corresponds an element $x'_{i_j} \in F'$ and $\min\{\Phi(x_{i_j}, z) : j = 1, \ldots, l\} \leq 0$ for any $z \in \text{co}\{x'_{i_1}, \ldots, x'_{i_l}\}$; *diagonally transfer continuous* if $\Phi(x,z) > 0$ implies that there exists a neighborhood U of z and x' such that $\Phi(x', z') > 0$ for any $z' \in U$.

[9] We recall that u is said to be *transfer upper continuous* if $u(x) > u(z)$ implies that there exists a neighborhood U of z and x' such that $u(x') > u(z')$ for any $z' \in U$ (see Tian and Zhou 1995).

$z' \in U_z$. Let $\xi : U_z \longrightarrow X$ be the correspondence defined by $\xi(z') = \{(f(z'_2), x_2)\}$ for all $z' \in U_z$. It is clear that ξ is a well-behaved correspondence. So, $\Phi(\xi(z'), z') = f(z'_2) > t$ for all $z' \in U_z$, which proves that Φ is generalized positively quasi-transfer continuous. On the other hand, for any neighborhood U'_z of z and for any $x' \in X$, there are points $z' \in U'_z$ such that $x'_1 \neq f(z'_2)$, which implies $\Phi(x', z') \leq 0$. Hence, Φ is not diagonally transfer continuous.

For any function Φ, we denote by $\mathfrak{S}(\Phi)$ the set of solutions to inequality (2) and we set $\mathfrak{S} : \Phi \in \mathfrak{f}_0 \to \mathfrak{S}(\Phi) \subseteq X$. Note that $\mathfrak{S}(\Phi) \neq \emptyset$ for all $\Phi \in \mathfrak{f}_1$.

We recall that a set-valued function $\mathfrak{T} : Y \rightrightarrows X$, where Y and X are metric spaces, is said to be *closed* (see, for example, Aliprantis and Border 1999) if, for any sequence $(y_n)_n$ converging to y in Y and for any sequence $(x_n)_n$ converging to x in X with $x_n \in \mathfrak{T}(y_n)$ for n sufficiently large, one has $x \in \mathfrak{T}(y)$.

Remark 5 Let $\mathfrak{T} : Y \rightrightarrows X$, where Y and X are metric spaces and X is compact. Then, $\mathfrak{T}$ is upper semicontinuous with compact values if and only if $\mathfrak{T}$ is closed (see, for example, Aliprantis and Border 1999).

Proposition 4 *The set-valued function* $\mathfrak{S}$ *is closed.*

Proof By contradiction, assume that: $\rho_1(\Phi_n, \Phi) \longrightarrow 0$, $z_n \longrightarrow z$, $z_n \in \mathfrak{S}(\Phi_n)$ for n sufficiently large and $z \notin \mathfrak{S}(\Phi)$. Then, there exists $x \in X$ such that $\Phi(x, z) > t$ for some $t > 0$. Since Φ is generalized positively quasi-transfer continuous, there exists a neighborhood U_z of z and a well-behaved correspondence $\xi : U_z \rightrightarrows X$ such that $\Phi\left(\xi(z'), z'\right) > t$ for any $z' \in U_z$. So, for n sufficiently large, we have $\Phi(s_n, z_n) > t > \rho_1(\Phi_n, \Phi)$, where $s_n \in \xi(z_n)$, which implies:

$$\Phi(s_n, z_n) > \rho_1(\Phi_n, \Phi) \geq \Phi(s_n, z_n) - \Phi_n(s_n, z_n) .$$

Hence, $\Phi_n(s_n, z_n) > 0$ for n sufficiently large, which contradicts $z_n \in \mathfrak{S}(\Phi_n)$. □

Corollary 2 $\mathfrak{f}_1$ *is a complete metric space.*

Proof In light of Proposition 3, it is sufficient to prove that if $(\Phi_n)_n \subseteq \mathfrak{f}_1$ and $\rho_1(\Phi_n, \Phi) \longrightarrow 0$, then $\mathfrak{S}(\Phi) \neq \emptyset$. Let $z_n \in \mathfrak{S}(\Phi_n)$ for any n. Since X is compact, there exists a subsequence of $(z_n)_n$ which converges to a point $z \in X$: suppose that $z_n \longrightarrow z$. From Proposition 4, we have $z \in \mathfrak{S}(\Phi)$, that is, $\mathfrak{S}(\Phi)$ is non-empty. □

Now, assume that X is a convex and compact subset of a metrizable and locally convex topological vector space and let $\mathfrak{f}_2$ be the set of functions which are generalized positively quasi-transfer continuous and 0-diagonally quasi-concave in the first argument on $X \times X$. In light of Proposition 2, we have $\mathfrak{f}_2 \subset \mathfrak{f}_1$.

Lemma 1 *Assume that* $(\Phi_n)_n$ *is a sequence of* 0*-diagonally quasi-concave in the first argument functions that converge to a function* Φ *in the metric* ρ_1*. Then,* Φ *is* 0*-diagonally quasi-concave in the first argument.*

Proof By contradiction, suppose that there exists a subset $\{x_1, \ldots, x_k\}$ of X and $z = \sum_{j=1}^{k} \lambda_j x_j \in \mathrm{co}\{x_1, \ldots, x_k\}$, where $\lambda_j > 0$ for any j, such that $\min\{\Phi(x_j, z) : j = 1, \ldots, k\} > 0$. Since any function Φ_n is 0-diagonally quasi-concave in the first argument, we have that $\min\{\Phi_n(x_j, z) : j = 1, \ldots, k\} \leq 0$ for any n. So, there exists $\bar{x} \in \{x_1, \ldots, x_k\}$ and a subsequence (Φ_{n_l}) of (Φ_n) such that $\Phi_{n_l}(\bar{x}, z) = \min\{\Phi_{n_l}(x_j, z) : j = 1, \ldots, k\} \leq 0$ for l sufficiently large. Since $\rho_1(\Phi_{n_l}, \Phi) \longrightarrow 0$, we get a contradiction. □

Finally, from Propositions 2, 3 and 4 and Lemma 1, we have:

Corollary 3 $\mathfrak{f}_2$ *is a complete metric space and the set-valued function* $\mathfrak{S} : \Phi \in \mathfrak{f}_2 \to \mathfrak{S}(\Phi)$ *is closed and non-empty compact valued.*

4 Essential Nash equilibria of discontinuous games

In this section, we prove that games having essential Nash equilibria are the generic case for any $\mathfrak{g} \in \{\mathfrak{g}_1, \mathfrak{g}_2\}$, that is, there exists a dense subset $\mathfrak{q}$ of $\mathfrak{g}$ such that, for every game which belongs to $\mathfrak{q}$, every Nash equilibrium is an essential equilibrium relative to $\mathfrak{g}$. Let $\mathfrak{N} : \mathfrak{g} \rightrightarrows X$ be the set-valued function defined by $\mathfrak{N}(G) = \{\text{Nash equilibria of } G\}$ for any $G \in \mathfrak{g}$. Note that $\mathfrak{N}(G) = \mathfrak{S}(\Phi)$, where Φ is defined by (3). We recall that a set-valued function $\mathfrak{T} : Y \rightrightarrows X$ is *lower semi-continuous* (see, for example, Aliprantis and Border 1999) if, for any $y \in Y$ and for any open set O such that $\mathfrak{T}(y) \cap O \neq \emptyset$, there exists a neighborhood U of y such that $\mathfrak{T}(y') \cap O \neq \emptyset$ for any $y' \in U$. It is clear that every Nash equilibrium of a game $G \in \mathfrak{g}$ is an essential equilibrium relative to $\mathfrak{g}$ if and only if $\mathfrak{N}$ is lower semicontinuous at G. So, useful is the following theorem of Fort (1949):

Lemma 2 *Let* Y *and* X *be metric spaces and* Y *be complete. Assume that* $\mathfrak{T} : Y \rightrightarrows X$ *is an upper semicontinuous set-valued function with non-empty and compact values. Then, there exists a dense* $\mathcal{G}_\delta$ *subset* Q *of* Y *such that* $\mathfrak{T}$ *is lower semicontinuous at any point belonging to* Q.[10]

In order to apply Lemma 2, we need to prove that any $\mathfrak{g} \in \{\mathfrak{g}_1, \mathfrak{g}_2\}$ is a complete metric space and the set-valued function $\mathfrak{N}$ is upper semicontinuous with non-empty and compact values on $\mathfrak{g}$.

Proposition 5 $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are complete metric spaces.*

Proof Let $\mathfrak{g} \in \{\mathfrak{g}_1, \mathfrak{g}_2\}$ and $(G_n)_n \subseteq \mathfrak{g}$ be a Cauchy sequence. Since $\mathfrak{g}^b$ is a complete metric space, the sequence $(G_n)_n$ converges to a game $G = \langle X_i, u_i \rangle_{i \in I} \in \mathfrak{g}^b$ in the metric ρ. Now, let $(\Phi_n)_n$ be the sequence of functions such that, for any n, Φ_n is

[10] We recall that a subset of a topological space is a $\mathcal{G}_\delta$*-set* (in short $\mathcal{G}_\delta$) if it is a countable intersection of open subsets. We note that the theorem of Fort holds also in the more general case in which Y is a Baire topological space. A *Baire's space* is a topological space where any countable intersection of open dense subsets is a dense subset. Examples of Baire spaces are the complete metric spaces (see Aliprantis and Border 1999).

defined by (3) for the game G_n. The sequence $(\Phi_n)_n$ is included in $\mathfrak{f}_1$, if $\mathfrak{g} = \mathfrak{g}_1$, or in $\mathfrak{f}_2$, if $\mathfrak{g} = \mathfrak{g}_2$. One has:

$$\rho_1(\Phi_h, \Phi_k) \leq 2\rho(G_h, G_k) \quad \text{for any } h \text{ and } k \ ,$$

which implies that $(\Phi_n)_n$ is a Cauchy sequence. So, $(\Phi_n)_n$ converges to a function Φ which belongs to $\mathfrak{f}_1$, if $\mathfrak{g} = \mathfrak{g}_1$ (see Corollary 2), or to $\mathfrak{f}_2$, if $\mathfrak{g} = \mathfrak{g}_2$ (see Corollary 3). Finally, since $\Phi_n(x, z) \longrightarrow \Phi(x, z)$ and

$$u_i^n(x_i, z_{-i}) - u_i^n(z) \longrightarrow u_i(x_i, z_{-i}) - u_i(z)$$

for any $(x, z) \in X \times X$ and any $i \in I$, we have:

$$\Phi(x, z) = \sum_{i \in I} \left[u_i(x_i, z_{-i}) - u_i(z)\right] \quad \text{for any} \quad (x, z) \in X \times X \ ,$$

that is, Φ is the function defined by (3) for G. □

Proposition 6 *The set-valued function $\mathfrak{N}$ is upper semicontinuous with non-empty and compact values for each $\mathfrak{g} \in \{\mathfrak{g}_1, \mathfrak{g}_2\}$.*

Proof Let $(G_n)_n$ be a sequence converging to G in the space $\mathfrak{g} \in \{\mathfrak{g}_1, \mathfrak{g}_2\}$. Consider the sequence $(\Phi_n)_n$ and Φ, where, for any n, Φ_n and Φ are defined by (3) for the games G_n and G, respectively. So, $(\Phi_n)_n$ converges to Φ in the sup-norm metric; moreover, $\mathfrak{N}(G) = \mathfrak{S}(\Phi)$ and $\mathfrak{N}(G_n) = \mathfrak{S}(\Phi_n)$ for any n. From Proposition 4 and Remark 5, it follows that $\mathfrak{N}$ is upper semicontinuous on $\mathfrak{g}$. Finally, if $\mathfrak{g} = \mathfrak{g}_1$, we have that $\mathfrak{N}(G)$ is non-empty and therefore compact for any $G \in \mathfrak{g}_1$. If $\mathfrak{g} = \mathfrak{g}_2$ and $G \in \mathfrak{g}_2$, $\mathfrak{N}(G)$ is non-empty and compact in light of Proposition 2. □

From Lemma 2 and Propositions 5 and 6, we obtain:

Theorem 1 *For any $\mathfrak{g} \in \{\mathfrak{g}_1, \mathfrak{g}_2\}$, there exists a dense $\mathcal{G}_\delta$ subset $\mathfrak{q}$ of $\mathfrak{g}$ such that, for every game in $\mathfrak{q}$, every Nash equilibrium is an essential equilibrium relative to $\mathfrak{g}$.*

Remark 6 In the paper by Carbonell-Nicolau (2010)—see Theorem 2—it is proved that games having essential Nash equilibria are the generic case in the space $\mathfrak{g}_X$. In light of Proposition 1 and Example 1, $\mathfrak{g}_X$ is strictly included in $\mathfrak{g}_1$. So, Theorem 2 by Carbonell-Nicolau (2010) can be obtained as a corollary from Theorem 1.

5 Essential Nash equilibria of discontinuous potential games

In this section, we deal with potential games. We prove that there exists a dense subset $\mathfrak{q}$ of $\mathfrak{g}_1^p$ such that, if $G \in \mathfrak{q}$, every maximizer of the potential of G is an essential Nash equilibrium of G relative to $\mathfrak{g}_1^p$. For any potential game $G \in \mathfrak{g}_1^p$, we denote by $\mathfrak{M}(G)$ the set of maximizers of its potential. So, the maximizers of the potential of G are essential Nash equilibria relative to $\mathfrak{g}_1^p$ if and only if the set-valued function $\mathfrak{M} : G' \in \mathfrak{g}_1^p \longrightarrow \mathfrak{M}(G')$ is lower semicontinuous at G.

If $G = \langle X_i, u_i \rangle_{i \in I}$ is a potential game, the potential P of G is unique up to an additive constant (see Monderer and Shapley 1996, Lemma 2.7) and it is given by:

$$P(x) = \sum_{i \in I} \left[u_i(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_{|I|}) - u_i(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_{|I|}) \right] , \quad (7)$$

where $\bar{x}$ is a fixed strategy profile and $|I|$ denotes the number of players. So, the potential of a game with bounded payoff functions is bounded, and we have:

Proposition 7 $\mathfrak{g}_1^p$ *is a complete metric space.*

Proof Let $(G_n)_n \subset \mathfrak{g}_1^p$ be a Cauchy sequence. Since $\mathfrak{g}^b$ is complete, $(G_n)_n$ converges to a game G in the metric ρ. For any n, let P_n be the potential of G_n defined according to (7). We have:

$$\begin{aligned} P_h(x) - P_k(x) &= \sum_{i \in I} \left[u_i^h(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_{|I|}) - u_i^k(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_{|I|}) \right] \\ &\quad + \sum_{i \in I} \left[u_i^k(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_{|I|}) - u_i^h(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_{|I|}) \right] . \end{aligned}$$

So, $\rho_1(P_h, P_k) \le 2\rho(G_h, G_k)$, which implies that $(P_n)_n$ is a Cauchy sequence and, in light of Proposition 2 by Scalzo (2009), $(P_n)_n$ converges to a weakly upper pseudocontinuous function P. The function P is the potential of G: in fact, since the sequences $\{\rho(G_n, G)\}_n$ and $\{\rho_1(P_n, P)\}_n$ converge to 0 and

$$u_i^n(x_i, x_{-i}) - u_i^n(x_i', x_{-i}) = P_n(x_i, x_{-i}) - P_n(x_i', x_{-i})$$

for any x_i and x_i' belonging to X_i, for any $x_{-i} \in X_{-i}$, for any $i \in I$ and for any n, one has:

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = P(x_i, x_{-i}) - P(x_i', x_{-i}) ,$$

which concludes the proof. □

Let us note that, if $G \in \mathfrak{g}_1^p$, $\mathfrak{M}(G)$ is not necessarily a compact set: see Example 3. So, we consider the set-valued function $\mathfrak{C} : G \in \mathfrak{g}_1^p \to \text{cl}\mathfrak{M}(G)$ (we denote by $\text{cl}A$ the closure of a set A). The following property of weakly upper pseudocontinuous functions allows us to prove that $\mathfrak{C}$ is upper semicontinuous.

Proposition 8 *Let F be a weakly upper pseudocontinuous function on X and let $(x_n)_n$ be a sequence converging to x in X. Then:*

$$\left.\begin{array}{r} F(x) < F(x') \\ F(x_n) < F(x') \\ \text{for } n \text{ sufficiently large} \end{array}\right\} \Longrightarrow \limsup F(x_n) < F(x') .$$

Proof If there exists z such that $F(x) < F(z) < F(x')$, we have $\limsup F(x_n) \leq F(z) < F(x')$. Otherwise, assume that there are no $z \in X$ such that $F(x) < F(z) < F(x')$. In this case, since $F(x_n) < F(x')$, we have $F(x_n) \leq F(x)$ for n sufficiently large and we get $\limsup F(x_n) \leq F(x) < F(x')$. □

Proposition 9 *The set-valued function $\mathfrak{C}$ is upper semicontinuous with non-empty and compact values.*

Proof The values of $\mathfrak{C}$ are non-empty and compact in light of Remark 2. So, from Remark 5, it is sufficient to prove that $\mathfrak{C}$ is closed. By contradiction, assume that: $(G_n)_n$ converges to G in $\mathfrak{g}_1^p$, $(x_n)_n$ converges to $x \notin \mathfrak{C}(G)$ and $x_n \in \mathfrak{C}(G_n)$ for any n. Let P be the potential of G and, for any n, let P_n be the potential of G_n, where the potentials are given by (7). Because $(G_n)_n$ converges to G, we have $\rho_1(P_n, P) \longrightarrow 0$. Since $x \notin \mathfrak{C}(G)$, there exists a neighborhood U of x such that $U \cap \mathfrak{M}(G) = \emptyset$. Moreover, because x_n belongs to $\mathfrak{C}(G_n)$, there exists a sequence $(z_n)_n$ converging to x such that, for n sufficiently large, z_n belongs to $\mathfrak{M}(G_n)$ but not to $\mathfrak{M}(G)$. Let $\bar{x} \in \mathfrak{M}(G)$. We get:

$$P(x) < P(\bar{x}) \qquad P(z_n) < P(\bar{x}) \qquad z_n \longrightarrow x$$

for n sufficiently large and, in light of Proposition 8, one has:

$$\limsup P(z_n) < P(\bar{x}) \ . \tag{8}$$

On the other hand, we have:

$$P(z_n) \geq P_n(z_n) - \rho_1(P_n, P) \geq P_n(\bar{x}) - \rho_1(P_n, P)$$

for any n. Since $\rho_1(P_n, P)$ converges to 0, we get

$$\limsup P(z_n) \geq P(\bar{x}) \ ,$$

which contradicts (8). □

Finally, we obtain:

Theorem 2 *There exists a dense $\mathcal{G}_\delta$ subset $\mathfrak{q}$ of $\mathfrak{g}_1^p$ such that, for every game in $\mathfrak{q}$, every maximizer of the potential is an essential Nash equilibrium relative to $\mathfrak{g}_1^p$.*

Proof In light of Lemma 2 and Proposition 9, there exists a dense G_δ subset $\mathfrak{q}$ of $\mathfrak{g}_1^p$ such that the set-valued function $\mathfrak{C}$ is lower semicontinuous on $\mathfrak{q}$. Lemma 16.22 by Aliprantis and Border (1999) implies that the set-valued function $\mathfrak{M}$ is lower semicontinuous on $\mathfrak{q}$, and the thesis follows. □

Remark 7 In the framework of the present paper, a set-valued function is upper semicontinuous if and only if it is closed. Let us consider the set-valued function $\mathfrak{N}$ defined on $\mathfrak{g}_1^p$ by $\mathfrak{N}(G) = \{\text{Nash equilibria of } G\}$. The set-valued function $\mathfrak{N}$ is not closed in general; for instance, $\mathfrak{N}$ is not closed at the game considered in Example 3. For this reason, one cannot use the theorem of Fort in proving the existence of a dense subset $\mathfrak{q}$ of $\mathfrak{g}_1^p$ such that, for every game in $\mathfrak{q}$, every Nash equilibrium is essential.

We conclude the section with some remark on the results that can be obtained by using (generalized) positively quasi-transfer continuous functions in the setting of potential games.

For any potential game G, let Υ be the real-valued function defined on $X \times X$ by $\Upsilon(x, z) = P(x) - P(z)$, where P is the potential of G. So, the set of maximizers of P—that we denote by argmax$P$—coincides with the solution set of the Ky Fan inequality:

$$\text{find } z \in X \text{ such that } \Upsilon(x, z) \leq 0 \text{ for any } x \in X .$$

Let $\mathfrak{g}_2^p$ be the space of potential games where, for any game, X is the set of strategy profiles and the function Υ is positively quasi-transfer continuous. It is easy to see that a game G belongs to $\mathfrak{g}_2^p$ if and only if the potential P of G satisfies the following property: *if* $P(x) > P(z) + t$ *for some* $t > 0$, *then there exists a neighborhood* U_z *of* z *and* $x' \in X$ *such that* $P(x') > P(z') + t$ *for any* $z' \in U_z$. We call *positively transfer upper continuous* a function that satisfies the property. It is clear that any upper semicontinuous function is positively transfer upper continuous. Indeed, the positive transfer upper continuity is a property equivalent to the upper semicontinuity when X is compact. In fact, we have:

Lemma 3 *Let P be a real-valued function defined on a non-empty compact topological space X. So, P is upper semicontinuous if and only if it is positively transfer upper continuous.*

Proof Let P be positively transfer upper continuous. Assume that P is not upper semicontinuous. So, for at least one $z \in X$, we have:

$$P(z) < \limsup_{z' \longrightarrow z} P(z') .$$

Let $x \in$ argmax$P$, which is non-empty and compact in light of Theorem 2 by Tian and Zhou (1995). Then, there exists $t > 0$ for which

$$P(x) - \limsup_{z' \to z} P(z') < t < P(x) - P(z) .$$

It follows from the definition of lim sup that in every neighborhood U of z, there exists z_U such that $P(x) < P(z_U) + t$. Since P is positively transfer upper continuous, we get a contradiction. The converse is trivial. □

In light of Lemma 3, $\mathfrak{g}_2^p$ coincides with the space of potential games where the potential of each game is upper semicontinuous. So, $\mathfrak{g}_2^p \subset \mathfrak{g}_1^p$ and, since the space of upper semicontinuous function is complete in the sup-norm metric (see, for example, Aliprantis and Border 1999), Proposition 9 and Lemma 2 imply that games having essential Nash equilibria are the generic case in $\mathfrak{g}_2^p$.

Finally, let $\mathfrak{g}_3^p$ be the space of potential games where, for any game, the function Υ is generalized positively quasi-transfer continuous and 0-diagonally quasi-concave in the first argument. Following the arguments of Sect. 4, it can be proved that there exists a dense subset $\mathfrak{q}$ of $\mathfrak{g}_3^p$ such that any game which belongs to $\mathfrak{q}$ has essential Nash equilibria relative to $\mathfrak{g}_3^p$. However, since $\mathfrak{g}_3^p \subset \mathfrak{g}_2^p \subset \mathfrak{g}_1^p$, this property of $\mathfrak{g}_3^p$ can be obtained from the results presented in this section. In fact, let $G \in \mathfrak{g}_3^p$. In light of Proposition 2, the set of solutions to the Ky Fan inequality corresponding to Υ—which coincides with the set of maximizers of the potential P of G—is non-empty and compact. Assume that $P(x) > P(z) + t$, where $t > 0$. Since Υ is generalized positively quasi-transfer continuous, there exists a neighborhood U_z of z and a well-behaved correspondence $\xi_z : U_z \rightrightarrows X$ such that $P(s') > P(z') + t$ for any $s' \in \xi_z(z')$ and for any $z' \in U_z$. Let $x' \in \operatorname{argmax} P$. So, we have $P(x') > P(z') + t$ for any $z' \in U_z$ and, in light of Lemma 3, P is upper semicontinuous. On the other hand, if $G \in \mathfrak{g}_3^p$ and P is the potential of G, since Υ is 0-diagonally quasi-concave in the first argument, we have that P is quasi-concave. So, $\mathfrak{g}_3^p$ is strictly included in $\mathfrak{g}_2^p$.

6 Conclusions

In this paper, we have proved that games having essential Nash equilibria are the generic case in the spaces of discontinuous games $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_1^p$ (see Sect. 2). The first two spaces concern the general case of strategic form games, while the last one deals with potential games. We have introduced the class of generalized positively quasi-transfer continuous functions and, among other properties, we have proved that such a class is a complete metric space in the sup-norm metric. Using generalized positively quasi-transfer continuous functions and the Ky Fan minimax inequality, we have obtained that games having essential Nash equilibria are the generic case in the spaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$. In the setting of discontinuous potential games, we have used a weakening of upper semicontinuity called weak upper pseudocontinuity (Morgan and Scalzo 2006; Scalzo 2009). A theorem of Fort (1949) on the existence of dense subsets of points of continuity for set-valued functions has played a central role in the main results of the paper.

References

Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis. Springer, Berlin (1999)

Baye, M.R., Tian, G., Zhou, J.: Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. Rev. Econ. Stud. **60**, 935–948 (1993)

Bertrand, J.: Théorie Mathématique de la richesse sociale. J. Savants **48**, 499–508 (1883)

Bich, P.: Existence of pure Nash equilibria in discontinuous and non quasiconcave games. Int. J. Game Theory **38**(3), 395–410 (2009)

Campbell, D.E., Walker, M.: Optimization with weak continuity. J. Econ. Theory **50**, 459–464 (1990)

Carbonell-Nicolau, O.: Essential equilibria in nornal-form games. J. Econ. Theory **145**, 421–431 (2010)

Carmona, G.: An existence result for discontinuous games. J. Econ. Theory **144**, 1333–1340 (2009)

Carmona, G.: Understanding some recent existence results for discontinuous games. Econ. Theory **48**(1), 31–45 (2011a)

Carmona, G.: Symposium on: existence of Nash equilibria in discontinuous games. Econ. Theory **48**(1), 1–4 (2011b)

Chang, S.-Y.: Inequalities and Nash equilibria. Nonlinear Anal. **73**, 2933–2940 (2010)

D'Aspemont, C., Gabszewicz, J.J.: On Hotelling's stability in competition. Econometrica **47**(5), 1145–1150 (1979)

Dasgupta, P., Maskin, E.: The existence of equilibrium in discontinuous economic games, I: theory. Rev. Econ. Stud. **53**, 1–26 (1986)
De Castro, L.: Equilibrium existence and approximation of regular discontinuous games. Econ. Theory **48**(1), 67–85 (2011)
Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequalities, vol. 3. Academic Press, New York (1972)
Fort, M.K.: A unified theory of semi-continuity. Duke Math. J. **16**, 237–246 (1949)
Hotelling, H.: The stability of competition. Econ. J. **39**, 41–57 (1929)
Lebrun, B.: Existence of an equilibrium in first price auctions. Econ. Theory **7**(3), 421–443 (1995)
Monderer, D., Shapley, L.S.: Potential games. Games Econ. Behav. **14**, 124–143 (1996)
Morgan, J., Scalzo, V.: Discontinuous but well-posed optimization problems. SIAM J. Opt. **17**(3), 861–870 (2006)
Morgan, J., Scalzo, V.: Pseudocontinuous functions and existence of Nash equilibria. J. Math. Econ. **43**(2), 174–183 (2007)
Nash, J.: Equilibrium points in n-person games. Proc. Natl. Acad. Sci. USA **36**, 48–49 (1950)
Prokopovych, P.: On equilibrium existence in payoff secure games. Econ. Theory **48**(1), 5–16 (2011)
Reny, P.J.: On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica **67**(5), 1029–1056 (1999)
Reny, P.J.: Strategic approximations of discontinuous games. Econ. Theory **48**(1), 17–29 (2011)
Rosenthal, R.W.: A class of games possessing pure-strategy Nash equilibria. Int. J. Game Theory **2**, 65–67 (1973)
Scalzo, V.: On the unifor convergence in some classes of non-necessarily continuous functions. Int. J. Contemp. Math. Sci. **4**(13), 617–624 (2009)
Scalzo, V.: Pareto efficient Nash equilibria in discontinuous games. Econ. Lett. **107**, 364–365 (2010)
Tian, G.: Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity. J. Math. Anal. Appl. **170**, 457–471 (1992)
Tian, G., Zhou, J.: Transfer continuities, generalizations of the Weierstrass and maxi- mum theorems: a full characterization. J. Math. Econ. **24**, 281–303 (1995)
Wu, W.T., Jiang, J.H.: Essential equilibrium points of n-person noncooperative games. Sci. Sinica **11**, 1307–1322 (1962)
Yannelis, N.C.: The core of an economy without ordered preferences. In: Khan, M.A., Yannelis, N.C. (eds.) Equilibrium Theory in Infinite Dimensional Spaces. Springer, Berlin (1991)
Yu, J.: Essential equilibria of n-person noncooperative games. J. Math. Econ. **31**, 361–372 (1999)
Zhou, J., Chen, G.: Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities. J. Math. Anal. Appl. **132**, 213–225 (1988)