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RESEARCH ARTICLE

The single deviation property in games with discontinuous payoffs

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Abstract We study equilibrium existence in normal form games in which it is possible to associate with each nonequilibrium point an open neighborhood, a set of players, and a collection of deviation strategies, such that at any nonequilibrium point of the neighborhood, a player from the set can increase her payoff by switching to the deviation strategy designated for her. An equilibrium existence theorem for compact, quasiconcave games with two players is established as an application of a general equilibrium existence result for qualitative games. A new form of the better-reply security condition, called the strong single deviation property, is proposed.

Keywords Better-reply secure game · Discontinuous game · Single deviation property · Majorized correspondence · Qualitative game

JEL Classification C65 · C72

1 Introduction

A number of generalizations and strengthenings of Reny's equilibrium existence theorem for better-reply secure games (Reny 1999) have been proposed recently. Among them are the papers by Barelli and Soza (2009), Carmona (2011a,b), De Castro (2011),

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McLennan et al. (2011), Bich (2009), and Reny (2009). Within the framework of the endogenous sharing rule approach by Simon and Zame (1990), Balder (2011) presents an equilibrium closure result for a sequence of games and associated mixed strategy Nash equilibria. De Castro (2011) explores a link between Reny's better-reply security and Simon and Zame's approach.

In this paper, we look at the equilibrium existence problem in normal form and qualitative games through the prism of a property called, by Reny (2009), the single deviation property. Both better-reply secure games and diagonally transfer continuous games (Baye et al. 1993) possess this property. According to it, if a strategy profile is not a Nash equilibrium, then there exist an open neighborhood and a full profile of deviation strategies—one for each player—such that, at any point of the neighborhood, a player can increase her payoff by switching to her deviation strategy. If the single deviation property holds and the game has no Nash equilibria in pure strategies, we can associate with each strategy profile a neighborhood and a collection of constant-valued correspondences defined on the neighborhood. Barelli and Soza (2009, Theorem 2.2) glue the locally defined correspondences together into an upper hemicontinuous correspondence, defined on the Cartesian product of the players' strategy sets, to which Kakutani's fixed point theorem can be applied. A strengthening of Barelli and Soza's equilibrium existence result for qualitative games, Theorem 5, is established in Sect. 4 via using majorized correspondences.

As shown with the aid of a three-player example by Reny (2009, Section 3), replacing the better-reply security condition with the single deviation property does not result in a complete set of sufficient conditions for the existence of a pure strategy Nash equilibrium in compact, quasiconcave games.[1] In that example, even though it is possible to find, for every point, a neighborhood and a collection of constant-valued correspondences defined on the neighborhood, one cannot glue them together into a well-behaved correspondence having the Cartesian product of the players' strategy sets as its domain. Nessah and Tian (2010) show that if, instead of quasiconcavity, a property related to but stronger than diagonal transfer quasiconcavity is assumed, then the existence of an equilibrium obtains.

The weak single deviation property, introduced in this paper, is weaker than the single deviation property in two respects: (a) deviation strategies need not be defined for all players, and (b) neighborhoods of nonequilibrium points may contain equilibrium points, as in second-price sealed-bid auction games with bidders having different valuations.

Intuitively, if a single player can increase her payoff using the same deviation strategy at every point of an open neighborhood of a nonequilibrium point, there is no need in defining deviation strategies for the rest of the players. On the other hand, if deviation strategies are not necessarily defined for all players, then glueing locally defined correspondences together might become a rather unwieldy problem.

The main result of the game theoretical part of this paper, Theorem 3, states that every compact, quasiconcave, two-person game with the weak single deviation property has a pure strategy Nash equilibrium if the players' strategy sets lie on the real

[1] However, if the mixed extension of a compact Borel game has the single deviation property, then the game has a mixed strategy Nash equilibrium (see also Reny 2011).

line. To show Theorem 3, we proceed by contradiction, assuming that the game has no equilibrium. Then, we construct an open cover of the Cartesian product of the players' strategy sets that satisfies the hypotheses of Theorem 2, a particular case of Theorem 5 for normal form games.

In Sect. 3, we introduce a strengthening of the weak single deviation property, the strong single deviation property. This property is not a generalization of the better-reply security condition, but another, slightly improved form of it. Lemmas 1 and 2 show that the strong single deviation property is equivalent to the better-reply security condition in compact games with no Nash equilibria in pure strategies. As we demonstrate on the example of a timing game, the strong single deviation property makes it possible to apply Reny's equilibrium existence theorem to games with a noncompact set of pure strategy Nash equilibria. In Remark 1, we describe a possible way of modifying the notion of a better-reply secure game.

The equilibrium existence results for qualitative games presented in the second part of the paper serve as the cornerstone of its game theoretical part. Since the groundbreaking work of Borglin and Keiding (1976) and Yannelis and Prabhakar (1983), majorized correspondences have been used as a powerful tool for analyzing qualitative and generalized games. At the heart of the proof of the aforementioned Theorem 5 lies the notion of a domain L-majorized correspondence.

Implicitly, the idea of domain L-majorization has been present in the literature studying majorized correspondences and their applications for quite a while. For instance, Yuan (1999) considers a correspondence whose values majorize values of the correspondence under study and that has a multivalued selection with open lower sections. This very idea also stands behind L_{F_C}-majorized correspondences (Ding and Xia 2004). Domain L-majorization, introduced in Sect. 4, goes a little farther: We do not majorize the values of the correspondence under study, only its domain.

Lemma 5 provides a set of sufficient conditions for a correspondence to be domain L-majorized that are equivalent, in the context of qualitative games, to Barelli and Soza's equilibrium existence conditions (Corollary 3).

Intuitively, Theorem 5 deals with qualitative games having a generalized weak single deviation property. Some of its applications are provided in Sect. 4. Corollary 4 is a generalization of the Fan-Browder collective fixed point theorem, and Corollary 5 is an equilibrium existence theorem for qualitative games.

2 The weak single deviation property

We consider a compact game $G = (X_i, u_i)_{i\in N}$ where $N = \{1, \ldots, n\}$ denotes the set of players, each player i's pure strategy set X_i is a nonempty, compact subset of a Hausdorff topological vector space, and each payoff function u_i is a bounded function from the Cartesian product $X = \Pi_{i\in N} X_i$, equipped with the product topology, to $\mathbb{R}$. Under these conditions, $G = (X_i, u_i)_{i\in N}$ is called a compact game. A game $G = (X_i, u_i)_{i\in N}$ is quasiconcave if each X_i is convex and $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is quasiconcave for all $i \in N$ and all $x_{-i} \in X_{-i}$, where $X_{-i} = \Pi_{k\in N\setminus\{i\}} X_k$. Denote the set of all pure strategy Nash equilibria of G in X by E_G and the graph of G by $\mathrm{Gr}G = \{(x, u) \in X \times \mathbb{R}^n \mid u_i(x) = u_i \text{ for all } i \in N\}$. For a subset B of a topological vector space Y, denote the interior of B in Y by $\text{int}_Y B$, the boundary of B by ∂B, the closure of B by $\text{cl}B$, and the convex hull of B by $\text{co}B$.

Definition 1 Player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$ if there exists $d_i \in X_i$ such that $u_i(d_i, x'_{-i}) \geq \alpha$ for all x'_{-i} in some open neighborhood of x_{-i}.

Definition 2 A game $G = (X_i, u_i)_{i \in N}$ is better-reply secure if, whenever $(x^*, u^*) \in \text{clGr}G$ and $x^* \in X \backslash E_G$, some player i can secure a payoff strictly above u_i^* at x^*.

The following theorem is the main result of Reny (1999).

Theorem 1 *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.*

It is not difficult to verify that every better-reply secure game possesses the following property (see Reny 2009; Nessah and Tian 2010; or Lemma 2 below).

Definition 3 A game $G = (X_i, u_i)_{i \in N}$ has the single deviation property if, whenever $x \in X \backslash E_G$, there exist a profile of deviation strategies $d \in X$ and a neighborhood $U_X(x)$ of x in X such that, for every $x' \in U_X(x)$, there is a player i for whom $u_i(d_i, x'_{-i}) > u_i(x')$.

We modify Definition 3 in two aspects: First, the requirement that a deviation strategy, d_i, be defined for each player i can be detrimental in applications (see, e.g., Example 1), and second, we ought not to require that there be a player able to increase her payoff for those $x' \in U_X(x)$ which are Nash equilibria of G.

Definition 4 A game $G = (X_i, u_i)_{i \in N}$ has the weak single deviation property if, whenever $x \in X \backslash E_G$, there exist an open neighborhood $U_X(x)$ of x, a set of players $I(x) \subset N$, and a collection of deviation strategies $\{d_i(x) \in X_i : i \in I(x)\}$ such that, for every $x' \in U_X(x) \backslash E_G$, there is a player $i \in I(x)$ for whom $u_i(d_i(x), x'_{-i}) > u_i(x')$.

To simplify notation, we will write d_i instead of $d_i(x)$ if it is clear for which neighborhood $U_X(x)$ player i's deviation strategy d_i is used.

Theorem 2 is an equilibrium existence result for normal form games with the weak single deviation property. Its proof is omitted since the theorem is a particular case of Theorem 5 whose proof is given in Sect. 4. For a set A, let $\langle A \rangle$ denote the family of its nonempty finite subsets. In Theorem 2, it is assumed that $d_i(x) = \{\varnothing\}$ for all $i \in N \backslash I(x)$.

Theorem 2 *Let $G = (X_i, u_i)_{i \in N}$ be a compact game. Suppose that*

(i) *G has the weak single deviation property: that is, for each $x \in X \backslash E_G$, there exist an open neighborhood $U_X(x)$ of x, a set of players $I(x) \subset N$, and a collection of deviation strategies $\{d_i(x) \in X_i : i \in I(x)\}$ such that, for every $x' \in U_X(x) \backslash E_G$, there exists $i \in I(x)$ with $u_i(d_i(x), x'_{-i}) > u_i(x')$;*
(ii) *for each $A \in \langle X \backslash E_G \rangle$ and every $z \in \cap_{x \in A} U_X(x)$, there exists $i \in \cup_{x \in A} I(x)$ such that $z_i \notin \text{co}\{\cup_{x \in A} d_i(x)\}$.*
Then, G has a pure strategy Nash equilibrium.

The only demanding assumption of Theorem 2, (ii), is naturally satisfied in a number of games. Among them are both quasiconcave games, such as second-price sealed-bid auctions, and nonquasiconcave games, such as the duopoly game from Example 1 of Baye et al. (1993). Example 1 illustrates the importance of allowing $I(x)$ to be a proper subset of N.

Example 1 The following notion of a weakly transfer continuous game is introduced in Nessah (2011, Definition 3.1). A game $G = (X_i, u_i)_{i\in N}$ is weakly transfer continuous if, whenever $x \in X\backslash E_G$, there exist a player i, a deviation strategy $d_i \in X_i$, and a neighborhood $U_X(x)$ of x such that $\inf_{x'\in U_X(x)}\{u_i(d_i, x'_{-i}) - u_i(x')\} > 0$. To show that the infimum operation in the last inequality is excessive, we introduce the following definition: A game has the single player deviation property if, whenever $x \in X\backslash E_G$, there exist a player i, a deviation strategy $d_i \in X_i$, and a neighborhood $U_X(x)$ of x such that $u_i(d_i, x'_{-i}) - u_i(x') > 0$ for all $x' \in U_X(x)$. Obviously, every game that possesses the single player deviation property also has the weak single deviation property.

Assume, by way of contradiction, that there is a compact, quasiconcave game with the single player deviation property which has no Nash equilibrium in pure strategies (see also Nessah 2011, Corollary 3.1). Then, for every $x \in X$, there exist an open neighborhood $U_X(x)$ of x, a player $i(x)$, and a deviation strategy $d_{i(x)}(x) \in X_{i(x)}$ such that, for every $x' \in U_X(x)$, $u_{i(x)}(d_{i(x)}(x), x'_{-i(x)}) > u_{i(x)}(x')$. Since the game is quasiconcave, (ii) of Theorem 2 is satisfied. Therefore, the game has a pure strategy Nash equilibrium by Theorem 2, a contradiction.[2]

At the same time, adding the assumption that $I(x) = N$ for all $x \in X$ would make the application of Theorem 2 unfeasible.

A compact, quasiconcave game with the weak single deviation property need not have a pure strategy Nash equilibrium. The next example is borrowed from Reny (2009).

Example 2 Consider a three-player game $G = (X_i, u_i)_{i\in\{1,2,3\}}$ with $X_1 = X_2 = X_3 = [0, 1]$. The payoff functions are defined as follows. Let, for $r \in [0, 1]$,

$$u_0(r) = \begin{cases} 0 & \text{if} \quad r > 0, \\ 1 & \text{if} \quad r = 0, \end{cases}, \quad u_1(r) = \begin{cases} 0 & \text{if} \quad r < 1, \\ 1 & \text{if} \quad r = 1. \end{cases}$$

Then, for $x_3 \in [0, \frac{1}{2}]$,

	$x_2 \in [0, \frac{1}{3}]$	$x_2 \in (\frac{1}{3}, \frac{2}{3})$	$x_2 \in [\frac{2}{3}, 1]$
$x_1 \in [0, \frac{1}{2}]$	$u_0(x_1), u_1(x_2), u_0(x_3)$	$u_1(x_1), u_1(x_2), u_0(x_3)$	$u_1(x_1), u_1(x_2), u_1(x_3)$
$x_1 \in (\frac{1}{2}, 1]$	$u_0(x_1), u_1(x_2), u_0(x_3)$	$u_1(x_1), u_1(x_2), u_1(x_3)$	$u_1(x_1), u_1(x_2), u_1(x_3)$

[2] If we considered generalized weakly transfer continuous games (Nessah 2011, Definition 3.2), we would have to invoke the more general Theorem 5.

and, for $x_3 \in (\frac{1}{2}, 1]$,

	$x_2 \in [0, \frac{1}{3}]$	$x_2 \in (\frac{1}{3}, \frac{2}{3})$	$x_2 \in [\frac{2}{3}, 1]$
$x_1 \in [0, \frac{1}{2}]$	$u_0(x_1), u_0(x_2), u_0(x_3)$	$u_0(x_1), u_0(x_2), u_0(x_3)$	$u_1(x_1), u_0(x_2), u_1(x_3)$
$x_1 \in (\frac{1}{2}, 1]$	$u_0(x_1), u_0(x_2), u_0(x_3)$	$u_0(x_1), u_0(x_2), u_1(x_3)$	$u_1(x_1), u_0(x_2), u_1(x_3)$

where the first coordinate of each entry corresponds to player 1, the second coordinate to player 2, and the third to player 3. It is not difficult to see that the game is compact and quasiconcave and has the weak single deviation property. At the same time, it has no Nash equilibrium in pure strategies. Therefore, it is impossible to find, for each $x \in X \backslash E_G$, an open neighborhood $U_X(x)$, a set of players $I(x) \subset N$, and a collection of deviation strategies $\{d_i(x) \in X_i : i \in I(x)\}$ satisfying the hypotheses of Theorem 2. However, this conclusion does not hold in two-player games.

The following theorem covers a broad class of compact, quasiconcave games with two players, including second-price sealed-bid auctions and diagonally transfer continuous and better-reply secure games.[3]

Theorem 3 *If a two-player, compact, quasiconcave game $G = (X_i, u_i)_{i=1}^2$ has the weak single deviation property and each X_i is a subset of the real line $\mathbb{R}$, then G has a pure strategy Nash equilibrium.*

Intuitively, the first step of the proof of Theorem 3 is clear: Assume, by way of contradiction, that the game has no pure strategy Nash equilibrium. Then, since the game has the weak single deviation property, we consider a cover of X consisting of open neighborhoods $U_X(x)$, with some corresponding sets of players $I(x)$ and collections of deviation strategies $\{d_i(x) \in X_i : i \in I(x)\}$. The compactness of X implies that the cover has a finite subcover, and it is tempting to conclude that what is left is to apply Theorem 2. However, Example 3 demonstrates that it is not so.

Example 3 Consider a two-player game $G = (X_i, u_i)_{i\in\{1,2\}}$ with $X_1 = X_2 = [0, 1]$, and $u_1 : X \to \mathbb{R}$ defined by

$$u_1(x) = \begin{cases} 1 & \text{if} \quad x \in \{\frac{1}{2}\} \times [0, \frac{1}{2}) \text{ and } \quad x \in (0, \frac{1}{2}] \times [\frac{1}{2}, 1], \\ 2 & \text{if} \quad x \in \{0\} \times [\frac{1}{2}, 1], \\ 0 & \text{otherwise,} \end{cases}$$

and $u_2 : X \to \mathbb{R}$ defined by

$$u_2(x) = \begin{cases} 1 & \text{if} \quad x \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1) \text{ and } \quad x \in (\frac{1}{2}, 1] \times \{\frac{1}{2}\}, \\ 2 & \text{if} \quad x \in [0, \frac{1}{2}] \times \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

[3] First-price sealed-bid auctions with two bidders do not have the weak single deviation property and do not have pure strategy Nash equilibria if the bidders have different valuations. There are a number of ways to ensure equilibrium existence. For instance, the sharing rule can be determined endogenously, as in Simon and Zame (1990), or, when information is incomplete, communication can be allowed, as in Lebrun (1996) and Jackson et al. (2002).

This game is compact and quasiconcave and has the weak single deviation property. Denote by $B_X(x, r)$ the open ball in X of radius $r > 0$ centered at $x \in X$.

Consider $U_X(\frac{1}{2}, \frac{1}{2}) = B_X((\frac{1}{2}, \frac{1}{2}), \frac{1}{10})$. For this open neighborhood of $x = (\frac{1}{2}, \frac{1}{2})$, $I(x) = \{1, 2\}$, and, unfortunately, there are two possible ways of choosing the set $\{d_i \in X_i : i \in I(x)\}$, namely $(d_1^1, d_2^1) = (\frac{1}{2}, 1)$ and $(d_1^2, d_2^2) = (0, \frac{1}{2})$. Since $x_i \in \text{co}\{d_i^1, d_i^2\}$ for $i = 1, 2$, it is quite possible that (ii) of Theorem 2 does not hold for a finite subcover consisting of open balls, irrespective of how small its elements are. As a result, we have to modify the finite cover to make Theorem 2 applicable (see Example 3 continued after the proof of Theorem 3 in Appendix A).

3 The strong single deviation property

In this section, we introduce a strengthening of the weak single deviation property and show that it is another, slightly weakened form of the better-reply security condition.

Definition 5 A game $G = (X_i, u_i)_{i \in N}$ has the strong single deviation property if whenever $x \in X \backslash E_G$, there exist an open neighborhood $U_X(x)$ of x, a set of players $I(x) \subset N$, a family of open neighborhoods $\{U_{X_{-i}}(x_{-i}) : i \in I(x)\}$, a collection of deviation strategies $\{d_i(x) \in X_i : i \in I(x)\}$, and a number $\varepsilon(x) > 0$, such that, for every $x' \in U(x) \backslash E_G$, there exists $i \in I(x)$ with $u_i(d_i(x), z_{-i}) - \varepsilon(x) > u_i(x')$ for all $z_{-i} \in U_{X_{-i}}(x_{-i})$ such that $(d_i(x), z_{-i}) \in X \backslash E_G$.

A game with the strong single deviation property need not be better-reply secure in the sense of Definition 2.

Example 4 Consider a timing game between two players with $X_1 = X_2 = [0, 1]$. Player i's payoff function is given by

$$u_i(x_i, x_{-i}) = \begin{cases} 1 & \text{if } x_i < x_{-i}, \\ \varphi_i(x_i) & \text{if } x_i = x_{-i}, \\ -1 & \text{if } x_i > x_{-i}, \end{cases}$$

where $\varphi_i(x_i) = 1$ if $x_i = x_{-i}$ and $x_i < 0.5$, and $\varphi_i(x_i) = 0$ if $x_i = x_{-i}$ and $x_i \geq 0.5$. The set of pure strategy Nash equilibria of this game is $E_G = \{x \in [0, \frac{1}{2}) \times [0, \frac{1}{2}) : x_1 = x_2\}$. It is easy to see that the game is not better-reply secure at $(\frac{1}{2}, \frac{1}{2})$.

In order to show that the game has the strong single deviation property, we have to consider the following two cases.

If $x \in X$ is such that $x_{-i} < x_i$ for some $i \in \{1, 2\}$, then put $r = \frac{x_i - x_{-i}}{2}$, $I(x) = \{i\}$, $d_i(x) = 0$, $U_X(x) = B_X(x, r)$, $U_{X_{-i}}(x_{-i}) = B_{X_{-i}}(x_{-i}, r)$, and $\varepsilon(x) = 1$.

If $x \in X$ is such that $x_1 = x_2$ and $x_1 \geq \frac{1}{2}$, then put $r = \frac{x_1}{2}$, $I(x) = \{1, 2\}$, $(d_1(x), d_2(x)) = (0, 0)$, $U_X(x) = B_X(x, r)$, $U_{X_{-i}}(x_{-i}) = B_{X_{-i}}(x_{-i}, r)$, and $\varepsilon(x) = \frac{1}{2}$.

Verifying that the game possesses the strong single deviation property is a straightforward exercise in both cases.

The difference between the strong single deviation property and the better-reply security condition is in the way Nash equilibria are treated.

Lemma 1 *If a compact game $G = (X_i, u_i)_{i \in N}$ with no Nash equilibria in pure strategies has the strong single deviation property, then it is better-reply secure.*

Proof Consider $x^* \in X$ and $u^* \in \mathbb{R}^n$ such that $(x^*, u^*) \in \text{clGr}G$. Since the game has the strong single deviation property, there exist an open neighborhood $U_X(x^*)$ of x^*, a set of players $I(x^*) \subset N$, a family of open neighborhoods $\{U_{X_{-i}}(x^*_{-i}) : i \in I(x^*)\}$, a collection of deviation strategies $\{d_i \in X_i : i \in I(x^*)\}$, and a number $\varepsilon(x^*) > 0$, such that, for every $x' \in U_X(x^*)$, there exists $i \in I(x)$ with $u_i(d_i, z_{-i}) - \varepsilon(x^*) > u_i(x')$ for all $z_{-i} \in U_{X_{-i}}(x^*_{-i})$.

We shall show that some player i can secure a payoff strictly above u^*_i at x^*. Fix a net $\{x^\beta\}$ converging to x^* such that the corresponding net $\{u(x^\beta)\}$ tends to u^*. Then, there exists $\widehat{\beta}$ such that $x^\beta \in U_X(x^*)$ and $\left|u^*_i - u_i(x^\beta)\right| < \frac{\varepsilon(x^*)}{2}$ for all $\beta \succeq \widehat{\beta}$ and all $i \in I(x^*)$. In particular, by the strong single deviation property, the inclusion $x^{\widehat{\beta}} \in U_X(x^*)$ implies that there exists $i \in I(x^*)$ such that $u_i(d_i, z_{-i}) - \varepsilon(x^*) > u_i(x^{\widehat{\beta}})$ for all $z_{-i} \in U_X(x^*_{-i})$. Therefore, $u_i(d_i, z_{-i}) - \frac{\varepsilon(x^*)}{2} > u^*_i$ for all $z_{-i} \in U_X(x^*_{-i})$, which means that player i can secure $u^*_i + \frac{\varepsilon(x^*)}{2}$ at x^*. □

The following corollary follows from Theorem 1 and Lemma 1.

Corollary 1 *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave and has the strong single deviation property, then it possesses a pure strategy Nash equilibrium.*

Remark 1 It is not difficult to relax the better-reply security condition to cover, for example, the above timing game. Instead of considering the graph of G, we can introduce the "nonequilibrium" graph of G by $\text{Grn}G = \{(x, u) \in X \times \mathbb{R}^n \mid u_i(x) = u_i$ for all $i \in N$ and $x \in X \backslash E_G\}$ and replace the set $\text{clGr}G$ in Definition 2 with $\text{clGrn}G$, which will expand the scope of applications of Theorem 1.

The next lemma, along with Lemma 1, shows that the strong single deviation property is another, slightly weakened form of the better-reply security condition.

Lemma 2 *If a compact game $G = (X_i, u_i)_{i \in N}$ is better-reply secure, then it has the strong single deviation property.*

The proof of Lemma 2 is given in Appendix B.

The closest to the strong single deviation property is the lower single deviation property, introduced by Reny (2009). Its definition is as follows. For each $i \in N$, let $\underline{u}_i : X \to \mathbb{R}$ be defined by $\underline{u}_i(x_i, x_{-i}) = \liminf_{x'_{-i} \to x_{-i}} u_i(x_i, x'_{-i})$. A game $G = (X_i, u_i)_{i \in N}$ has the lower single deviation property if whenever $x \in X \backslash E(G)$, there exists $d \in X$ and a neighborhood U of x such that for every $x' \in U$, there is a player i for whom $\underline{u}_i(d_i, y_{-i}) > \underline{u}_i(x')$ for all $y \in U$.

Since $\varepsilon(x)$ in Definition 5 does not depend on x' and $u_i(x') \geq \underline{u}_i(x')$ for each $i \in N$ and every $x' \in X$, a game with no Nash equilibria in pure strategies that has the strong single deviation property also has the lower single deviation property. The latter property is a generalization of the better-reply security condition. In its turn, the strong single deviation property may be considered as a slightly improved version of the better-reply security condition. The lower single deviation property can also be improved upon in a similar manner.

4 Equilibrium existence in qualitative games

In this section, we first introduce domain L-majorized correspondences and then prove Theorem 5, a general equilibrium existence result for qualitative games which is used in the proof of Theorem 3.

4.1 Domain L-majorized correspondences

Let X be a nonempty subset of a topological space, Y be nonempty, convex subset of a vector space, and $\theta : X \to Y$ be a single-valued function. A correspondence $F : X \twoheadrightarrow Y$ has open lower sections in X if $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for every $y \in Y$; F is of class L_θ with respect to θ if it has open lower sections in X and $\theta(x) \notin \text{co}F(x)$ for all $x \in X$. In the special cases when $Y = X$ and θ is the identity map on X and when $X = \Pi_{i=1}^n X_i$, $Y = X_i$, and $\theta : X \to X_i$ is the projection of X onto X_i, we will write L in place of L_θ. The domain of F is defined by $\text{Dom}F = \{x \in X : F(x) \neq \varnothing\}$. If $\text{Dom}F = X$, then we say that F is strict.

Given $F : X \twoheadrightarrow Y$, $\theta : X \to Y$, and $x \in X$, a correspondence $F_x : X \twoheadrightarrow Y$ is an L_θ-majorant of F at x if F_x is of class L_θ and there exists an open neighborhood U_x of x in X such that $F(z) \subset F_x(z)$ for every $z \in U_x$.[4] The correspondence F is locally L_θ-majorized if, for each $x \in \text{Dom}F$, there exists an L_θ-majorant of F at x, and F is L_θ-majorized if there exists a correspondence $\overline{F} : X \twoheadrightarrow Y$ of class L_θ such that $F(x) \subset \overline{F}(x)$ for every $x \in X$.

The next maximal element existence result is equivalent to Browder's fixed point theorem (see Browder 1968, Theorem 1; Yannelis and Prabhakar 1983, Theorems 3.1 and 5.1).

Lemma 3 *Let X be a nonempty, compact, convex subset of a Hausdorff topological vector space, and let $F : X \twoheadrightarrow X$ be a correspondence of class L. Then, there exists $\widehat{x} \in X$ such that $F(\widehat{x}) = \varnothing$.*

The following lemma says that, from the standpoint of applications, there are no differences between L_θ-majorized and locally L_θ-majorized correspondences (see Yannelis and Prabhakar 1983, Corollary 5.1; or Bagh 1998, Lemma 1.5).

Lemma 4 *Let X be a nonempty, compact subset of a Hausdorff topological vector space, and Y be a nonempty, convex subset of a vector space. Let $\theta : X \to Y$ and $F : X \twoheadrightarrow Y$ be a strict correspondence. Then, F is locally L_θ-majorized if and only if it is L_θ-majorized.*

Among the assumptions of Lemma 4 is a nonstandard one, namely that F is a strict correspondence. It is not restrictive since every proof using majorized correspondences proceeds by contradiction.

The proof of Lemma 4 follows along the lines of the proof of Corollary 1 of Borglin and Keiding (1976) (for details, see Ding et al. 1994, Theorem 1; Ding and Tan 1993, Lemma 2).

[4] In this section, we will write U_x instead of $U_X(x)$ since there is no ambiguity regarding the space in which the neighborhood is considered.

Corollary 2 (Yannelis and Prabhakar 1983, Corollary 5.1) follows from Lemmas 3 and 4 by way of contradiction.

Corollary 2 *Let X be a nonempty, compact, convex subset of a Hausdorff topological vector space and $F : X \twoheadrightarrow X$ be a locally L-majorized correspondence. Then there exists $\widehat{x} \in X$ such that $F(\widehat{x}) = \varnothing$.*

Now, we introduce a generalization of the notion of an L-majorized correspondence.

Definition 6 Let X be a nonempty subset of a topological space and Y be a nonempty, convex subset of a vector space, and let $\theta : X \to Y$. A correspondence $F : X \twoheadrightarrow Y$ is domain L_θ-majorized if there exists a correspondence $\overline{F} : X \twoheadrightarrow Y$ of class L_θ such that $\operatorname{Dom} F \subset \operatorname{Dom} \overline{F}$.

Clearly, if F is L_θ-majorized, then it is domain L_θ-majorized. Obviously, the converse does not necessarily hold.

Theorem 4 is a maximal element existence theorem for domain L-majorized correspondences.

Theorem 4 *Let X be a nonempty, compact, convex subset of a Hausdorff topological vector space, and let $F : X \twoheadrightarrow X$ be a domain L-majorized correspondence. Then, there exists $\widehat{x} \in X$ such that $F(\widehat{x}) = \varnothing$.*

Proof Since $F : X \twoheadrightarrow X$ is domain L-majorized, there is a correspondence $\overline{F} : X \twoheadrightarrow X$ of class L such that $\operatorname{Dom} F \subset \operatorname{Dom} \overline{F}$. Then, by Lemma 3, $\overline{F}(\widehat{x}) = \varnothing$ for some $\widehat{x} \in X$, which implies that $F(\widehat{x}) = \varnothing$. □

The next lemma provides a set of sufficient conditions for a correspondence to be domain L_θ-majorized.

Lemma 5 *Let X be a compact Hausdorff topological space and Y be a nonempty, convex subset of a vector space. Let $\theta : X \to Y$ and $F : X \twoheadrightarrow Y$ be a strict correspondence such that*

(i) *for each $x \in X$, there exist an open neighborhood U_x of x in X and a correspondence $F_x : X \twoheadrightarrow Y$ with $\operatorname{Dom} F_x = U_x$ and open lower sections in X;*
(ii) *for each $A \in \langle X \rangle$ and every $z \in \cap_{x \in A} U_x$, $\theta(z) \notin \operatorname{co}\{\cup_{x \in A} F_x(z)\}$.*
Then F is domain L_θ-majorized.

The proof of Lemma 5 is given in Appendix B, where is shown that (i) and (ii) imply the existence of a strict correspondence $\overline{F} : X \twoheadrightarrow Y$ of class L_θ.

It is worth noticing that another set of sufficient conditions obtains if (ii) is replaced with the more conventional assumption that $\cap_{x \in A} U_x \subset \operatorname{Dom}(\cap_{x \in A} F_x)$ for each $A \in \langle X \rangle$ (see, e.g., Yuan 1999, Theorem 3.1). However, the latter assumption has a strong flavor of value majorization.

4.2 Qualitative games

As before, let $N = \{1, \ldots, n\}$ be a finite set of players. Each player i's strategy set X_i is a nonempty, compact, and convex subset of a Hausdorff topological vector space. Let $X = \Pi_{i \in N} X_i$, and let $P_i : X \twoheadrightarrow X_i$ denote player i's preference correspondence.

Consider a qualitative game $\Gamma = (X_i, P_i)_{i\in N}$. A strategy profile $x \in X$ is an equilibrium of Γ if $P_i(x) = \varnothing$ for all $i \in N$.

For a qualitative game $\Gamma = (X_i, P_i)_{i\in N}$, we call the set $\mathrm{Dom}\Gamma = \cup_{i\in N}\mathrm{Dom}P_i$ the domain of Γ.

Definition 7 A game $\Gamma = (X_i, P_i)_{i\in N}$ is domain L-majorized if there exists a correspondence $\overline{F} : X \twoheadrightarrow X$ of class L such that $\mathrm{Dom}\Gamma \subset \mathrm{Dom}\overline{F}$.

Obviously, if Γ is domain L-majorized, then it has an equilibrium $\widehat{x} \in X$; that is, $P_i(\widehat{x}) = \varnothing$ for each $i \in N$. Therefore, if we want to show the existence of an equilibrium in a qualitative game, a legitimate way of doing that is to show that the game is domain L-majorized. However, it is important to keep in mind that the correspondence $\overline{F}$ should not only have open lower sections but also satisfy the condition that $x \notin \mathrm{co}\overline{F}(x)$ for all $x \in X$.

Extending Lemma 5 to qualitative games produces an equilibrium existence result, which is analogous to Theorem 2.2 of Barelli and Soza (2009).

Corollary 3 *Let X_i be a nonempty, compact, convex subset of a Hausdorff topological vector space, and let $\Gamma = (X_i, P_i)_{i\in N}$ be a qualitative game. Suppose, for each $x \in \mathrm{Dom}\Gamma$, there exists an $(n+1)$-tuple $(D_x^1, \ldots, D_x^n; U_x)$, where $D_x^i : X \twoheadrightarrow X_i$ and U_x is an open neighborhood of x in X, such that*

(i) $\mathrm{Dom}D_x^i = U_x$ *and* D_x^i *has open lower sections in* X *for all* $i \in N$;
(ii) *for each* $A \in \langle\mathrm{Dom}\Gamma\rangle$ *and every* $z \in \cap_{x\in A}U_x$, *there exists* $i \in N$ *such that* $z_i \notin \mathrm{co}\{\cup_{x\in A}D_x^i(z)\}$.
Then Γ *has an equilibrium.*

Proof Assume, by contradiction, that Γ has no equilibrium. For each $x \in X$, consider $F_x : X \twoheadrightarrow X$ defined by $F_x(z) = (D_x^1(z), \ldots, D_x^n(z))$. Since assumptions (i) and (ii) of Lemma 5 are satisfied, Γ is domain L-majorized, a contradiction. □

From the standpoint of applications, assumption (i) of Corollary 3 is too strong. For example, if, for some i, $P_i(z) = \varnothing$ for all $z \in U_x$, then $\mathrm{Dom}D_x^i$ should be equal to the empty set as well. Moreover, assuming that (i) holds for all $i \in N$ makes it more difficult to satisfy (ii) (see Example 1).

Theorem 5 *Let X_i be a nonempty, compact, convex subset of a Hausdorff topological vector space, and let $\Gamma = (X_i, P_i)_{i\in N}$ be a qualitative game. Suppose, for each $x \in \mathrm{Dom}\Gamma$, there exist $I(x) \subset N$, and an $(n+1)$-tuple $(D_x^1, \ldots, D_x^n; U_x)$, where $D_x^i : X \twoheadrightarrow X_i$ and U_x is an open neighborhood of x in X, such that*

(i) $\mathrm{Dom}D_x^i = U_x$ *and* D_x^i *has open lower sections in* X *for all* $i \in I(x)$, *and* $\mathrm{Dom}D_x^i = \varnothing$ *for all* $i \in N\backslash\{I(x)\}$;
(ii) *for each* $A \in \langle\mathrm{Dom}\Gamma\rangle$ *and every* $z \in \cap_{x\in A}U_x$, *there exists* $i \in \cup_{x\in A}I(x)$ *such that* $z_i \notin \mathrm{co}\{\cup_{x\in A}D_x^i(z)\}$.
Then Γ *has an equilibrium.*

Proof Assume, by contradiction, that Γ has no equilibrium, that is, $\mathrm{Dom}\Gamma = X$. Since X is compact, the open cover $\{U_x : x \in X\}$ of X contains a finite subcover

$\{U_{x_j} : j \in J\}$, where J is a finite set. Let $\{V_{x_j} : j \in J\}$ be an open refinement of $\{U_{x_j} : j \in J\}$ such that $\mathrm{cl}V_{x_j} \subset U_{x_j}$ for every $j \in J$ (see Aliprantis and Border 2006, p. 169). For each $j \in J$ and each $i \in N$, define a correspondence $F_i^j : X \twoheadrightarrow X_i$ by

$$F_i^j(z) = \begin{cases} D^i_{x_j}(z) \cup_{\{s\in J\setminus\{j\}:z\in V_{x_s}\}} D^i_{x_j}(z) & \text{if} \quad z \in \mathrm{cl}V_{x_j} \quad \text{and} \quad i \in I(x_j), \\ X_i & \text{if} \quad z \notin \mathrm{cl}V_{x_j} \quad \text{or} \quad i \notin I(x_j). \end{cases}$$

It is not difficult to see that each F_i^j has open lower sections. Then, for each $i \in N$, the correspondence $F_i : X \twoheadrightarrow X_i$ defined by $F_i(z) = \cap_{j\in J} F_i^j(z)$ has open lower sections. Therefore, the correspondence $\overline{F} : X \twoheadrightarrow X$ defined by $\overline{F}(z) = \cap_{i\in N}\{\Pi_{k\in N\setminus\{i\}} X_k \times F_i(z)\}$ also has open lower sections.

Fix some $z \in X$. It lies in some V_{x_j}. We have to show that $z_{i'} \notin \mathrm{co}F^j_{i'}(z)$ for some $i' \in I(x_j)$. Denote $A = \{s \in J : z \in V_{x_s}\}$. By (ii), there exists $i' \in \cup_{x\in A} I(x)$ such that $z_{i'} \notin \mathrm{co}\{\cup_{j\in A} D^{i'}_{x_j}(z)\}$. Since, by definition, $F^j_{i'}(z) = \cup_{j\in A} D^{i'}_{x_j}(z)$, we conclude that $z_{i'} \notin \mathrm{co}F^j_{i'}(z)$. Therefore, $z_{i'} \notin \mathrm{co}F_{i'}(z)$, and consequently, $z \notin \mathrm{co}\overline{F}(z)$.

Since $\overline{F}$ is of class L and $\mathrm{Dom}\Gamma = \mathrm{Dom}\overline{F} = X$, Γ is domain L-majorized, a contradiction. □

Corollary 4 is a version of the Fan-Browder collective fixed point theorem. In Lassonde and Schenkel (1992, Theorem 5), it follows from a generalization of the KKM lemma, which is a reflection of the fact that the KKM lemma and Browder's fixed point theorem are two equivalent results (see Yannelis 1991, pp. 105–109, for an in-depth explanation).[5]

Corollary 4 *Let $X_1, \ldots, X_n$ be nonempty, compact, convex subsets of Hausdorff topological vector spaces, and $X = \Pi_{i\in N} X_i$. For each $i \in N$, let $D_i : X \twoheadrightarrow X_i$ have open lower sections. If, for each $x \in X$, there exists $i \in N$ such that $D_i(x) \neq \varnothing$, then there exists $\overline{x} \in X$ and $i \in N$ such that $\overline{x} \in \mathrm{co}D_i(\overline{x})$.*

Proof Assume, by way of contradiction, that each correspondence D_i is of class L. Consider the game $\Gamma = (X_i, D_i)_{i\in N}$. For each $x \in X$, put $I(x) = \{i \in N : D_i(x) \neq \varnothing\}$ and fix a neighborhood U_x of x such that $U_x \subset \mathrm{Dom}D_i$ for all $i \in I(x)$. Then define $D_i^x : X \twoheadrightarrow X_i$ as a restriction of D_i to U_x for $i \in I(x)$ and put $\mathrm{Dom}D_i^x = \varnothing$ for $i \in N\setminus I(x)$. By Theorem 5, Γ has an equilibrium, a contradiction.

Corollary 5 extends 4 to qualitative games.

Corollary 5 *Let each X_i be a nonempty, compact, convex subset of a Hausdorff topological vector space, and let $\Gamma = (X_i, P_i)_{i\in N}$ be a qualitative game. If, for each $i \in N$, the correspondence $P_i : X \twoheadrightarrow X_i$ is domain L-majorized, then Γ has an equilibrium in X.*

[5] In order to show Reny's equilibrium existence theorem for better-reply secure games, Prokopovych (2011) applies the Fan-Browder collective fixed point theorem to well-behaved selections of ε-best-reply correspondences.

Proof Since the players' preference correspondences are domain L-majorized, for each $i \in N$, there exists $F_i : X \twoheadrightarrow X_i$ of class L such that $\mathrm{Dom}P_i \subset \mathrm{Dom}F_i$.

By Corollary 4, there exists $x \in X$ such that $F_i(x) = \varnothing$ for all $i \in N$. Since $\mathrm{Dom}\Gamma = \cup_{i\in N}\mathrm{Dom}P_i \subset \cup_{i\in N}\mathrm{Dom}F_i$, Γ has an equilibrium. □

Appendix A

Proof of Theorem 3

Assume, by contradiction, that G has no Nash equilibrium in pure strategies. Since G has the weak single deviation property, for every $x \in X$, there exist an open ball $B_X(x, 3r(x))$ of x in X, a set of players $I(x) \subset \{1, 2\}$, and a collection of deviation strategies $K(x) = \{d_i \in X_i : i \in I(x)\}$, such that for every $x' \in B_X(x, 3r(x))$, $u_i(d_i, x'_{-i}) > u_i(x')$ for some $i \in I(x)$.

We will modify the initial open cover of X, $\{B_X(x, r(x)) : x \in X\}$, in a number of steps. First, we additionally assume that, for every $x \in X$, $r(x)$, $I(x)$, and $K(x)$ satisfy the following three conditions:

(a) if $d_i \neq x_i$ for some $i \in I(x)$, then $|d_i - x_i| > 5r(x)$;
(b) $I(x)$ is minimal in the following sense: If $I(x) = \{1, 2\}$, there are no $r > 0$ and $i \in \{1, 2\}$ such that $u_i(d_i, x'_{-i}) > u_i(x')$ for all $x' \in B_X(x, r)$;
(c) if $d_i = x_i$ for some $i \in I(x)$, then d_i cannot be replaced in $K(x)$ with $\overline{d}_i \in X_i\backslash\{d_i\}$ such that, for some $r > 0$ and every x' in $B_X(x, 3r)$, at least one of the following inequalities holds: $u_i(\overline{d}_i, x'_{-i}) > u_i(x')$ or $u_{-i}(d_{-i}, x'_i) > u_{-i}(x')$.

As is shown below, if the elements of the cover satisfy these three simple conditions, we do not have to further modify them in most cases. Condition (a) is not restrictive since, for every $x \in X$, the radius $r(x)$ can be chosen arbitrarily small. If $I(x) = \{1, 2\}$, then condition (b) states that, given $K(x)$, it is impossible to reduce the number of elements of $I(x)$ by choosing a smaller $r(x)$. Condition (c) is also not burdensome. If $d_i = x_i$ for some $i \in I(x)$ and there are $r > 0$ and $\overline{d}_i \in X_i\backslash\{d_i\}$ such that $u_i(\overline{d}_i, x'_{-i}) > u_i(x')$ for every $x' \in B_X(x, 3r)$ at which $u_i(d_i, x'_{-i}) > u_i(x')$ and $u_{-i}(d_{-i}, x'_i) \leq u_{-i}(x')$, then we replace $B_X(x, r(x))$ in the cover with $B_X(x, r)$ and d_i in $K(x) = \{d_1, d_2\}$ with $\overline{d}_i$. A useful fact to keep in mind is the following: If $d_i = x_i$ for some $i \in I(x)$, then d_{-i} does not coincide with x_{-i}.

The compactness of X implies that the open cover $\{B_X(x, r(x)) : x \in X\}$ of X contains a finite subcover $\{B_X(x^j, r(x^j)) : j \in J\}$, where $J = \{1, \ldots, k\}$. It is useful to notice that if $B_X(x^s, r(x^s))) \cap B_X(x^t, r(x^t)) \neq \varnothing$ and $r(x^s) > r(x^t)$ for some $s, t \in J$, then $B_X(x^t, r(x^t)) \subset B_X(x^s, 3r(x^s)))$. Hence, for every $x' \in B_X(x^t, r(x^t))$, there exist $i \in I(x^s)$ and $d^s_i \in K(x^s)$ such that $u_i(d^s_i, x'_{-i}) > u_i(x')$.

Without loss of generality, we assume that $r(x^s) > r(x^t)$ if $s, t \in J$ and $s < t$ and that each $B_X(x^j, r(x^j))$ contains some points of X that do not lie in any of the other elements of the subcover. The latter assumption will help us avoid dealing with empty sets in the course of modifying the cover.

Let us show that, for our purposes, it is enough to focus attention on the intersections of just two elements of the cover. In the reasoning below, the fact that the open sets are balls is not essential. So, we want to show that if for some $\{l_1, \ldots, l_m\} \subset \{1, \ldots, k\}$, there exists $z' \in \cap_{j=1}^m B_X(x^{l_j}, r(x^{l_j}))$ such that $z'_i \in \text{co}\{\cup_{j=1}^m d_i^{l_j}\}$ for $i = 1, 2$ (here we assume that $d_i^{l_j} = \{\varnothing\}$ for $i \in \{1, 2\}\backslash I(x^{l_j})$), then $z'_i \in \text{co}\{d_i^s, d_i^t\}$, $i = 1, 2$, for some $s, t \in \{l_1, \ldots, l_m\}$.

Without loss of generality, $\{l_1, \ldots, l_m\} = \{1, \ldots, m\}$, $u_1(d_1^1, z'_2) > u_1(z')$ for $d_1^1 \in K(x^1)$, and $d_1^1 < z'_1$. Since $z'_1 \in \text{co}\{d_1^1, d_1^s\}$ for some $s \in \{2, \ldots, m\}$ and u_1 is quasiconcave in x_1, we have that $z'_1 \leq d_1^s$ and $u_2(z'_1, d_2^s) > u_2(z')$. Then for some $t \in \{1, \ldots, m\}$ such that $z'_2 \in \text{co}\{d_2^s, d_2^t\}$, the inequality $u_1(d_1^t, z'_2) > u_1(z')$ holds. It follows from the quasiconcavity of u_1 in x_1 that $d_1^t < z'_1$. That is, $z'_i \in \text{co}\{d_i^s, d_i^t\}$ for $i = 1, 2$, as claimed.

Now consider the intersection of the first two elements of the cover. Assume that there is a point $z' \in B_X(x^1, r(x^1)) \cap B_X(x^2, r(x^2))$ such that $z'_i \in \text{co}\{d_i^1, d_i^2\}$, $i = 1, 2$. First, we will show that it might happen only in one case, Case 7, and then we will describe how to modify the cover to preclude Case 7 for every pair of intersecting elements of the cover.

As before, we assume that $r(x^1) > r(x^2)$ and that $u_1(d_1^1, z'_2) > u_1(z')$ with $d_1^1 < z'_1$ and $u_2(z'_1, d_2^2) > u_2(z')$ with $d_2^2 > z'_2$ (if it is not so, renumber the players and/or redirect one or both axes). Then the inclusions $z'_i \in \text{co}\{d_i^1, d_i^2\}$ for $i = 1, 2$ imply that $d_1^2 \geq z'_1$ and $d_2^1 \leq z'_2$.

We claim that if $z'_i \in \text{co}\{d_i^1, d_i^2\}$ for $i = 1, 2$, then the following six cases are impossible (see the proof of this claim below).

Case 1. $d_1^1 = x_1^1$ and $d_1^2 = x_1^2$.

Cases 2–3. $d_1^1 = x_1^1$ and $d_1^2 \neq x_1^2$.

Therefore, if Cases 1-3 are impossible, then it must be the case that $d_1^1 \neq x_1^1$.

Case 4. $d_1^1 \neq x_1^1$ and $d_1^2 \neq x_1^2$.

Case 5. $d_1^1 \neq x_1^1$, $d_1^2 = x_1^2$, and $d_2^1 \neq x_2^1$.

Cases 4 and 5 complement the picture. We conclude from Case 4 that if $z'_i \in \text{co}\{d_i^1, d_i^2\}$, $i = 1, 2$, then $d_1^2 = x_1^2$, and from Case 5 that, moreover, $d_2^1 = x_2^1$. The next case is also impossible.

Case 6. $d_2^1 = x_2^1$, $d_1^2 = x_1^2$, and $x_2^2 > d_2^1$.

It might happen that $z'_i \in \text{co}\{d_i^1, d_i^2\}$, $i = 1, 2$, in Case 7. As a result, we have to modify the cover so that to ensure that every two of its elements do not satisfy the conditions of Case 7.

Case 7. Let $d_1^2 = x_1^2, d_2^1 = x_2^1$, and $x_2^2 \leq d_2^1$. Denote $B_X(x^1, r(x^1))$ by $V_X^1(x^1, r(x^1))$ and replace $B_X(x^1, r(x^1))$ in the cover $\{B_X(x^j, r(x^j)) : j \in J\}$ with $V_X^2(x^1, r(x^1)) = B_X(x^1, r(x^1))\backslash \text{cl}B_X(x^2, r(x^2))$. We have to add to the cover a finite number of open balls covering the compact set $A = \partial B_X(x^2, r(x^2)) \cap \text{cl}B_X(x^1, r(x^1))$. For every $x \in A$ with $x_2 \neq d_2^1$, pick an open ball $B_X(x, r(x))$ such that $\left|x_2 - d_2^1\right| > 5r(x)$ and find a minimal $I(x) \subset I(x^1)$ (for $B_X(x, r(x))$) with $K(x) \subset K(x^1)$ such that for every $x' \in B_X(x, 3r(x))$, there exists $i \in I(x)$ with $u_i(d_i, x'_{-i}) > u_i(x')$.

Since $x_2^2 \leq d_2^1$ and $z'_2 \geq d_2^1$, we deduce that $(d_1^2, d_2^1) \notin A$. Then for every $x \in A$ with $x_2 = d_2^1$, it is possible to choose an open ball $B_X(x, r(x))$ such that $\left|x_1 - d_1^2\right| > 5r(x)$ and find a minimal $I(x) \subset I(x^2)$ with $K(x) \subset K(x^2)$ such that, for every $x' \in B_X(x, 3r(x))$, there exists $i \in I(x)$ with $u_i(d_i, x'_{-i}) > u_i(x')$.

Since A is a compact set, it has a finite subset $\{x_A^1, \ldots, x_A^T\}$ such that $A \subset \cup_{t=1}^T B_X(x_A^t, r(x_A^t))$. Without loss of generality, $r(x^k) > r(x_A^1)$ and $r(x_A^t) > r(x_A^s)$ if $t, s \in \{1, \ldots, T\}$ and $t < s$. Let $x^{k+t} = x_A^t$ for all $t \in \{1, \ldots, T\}$. Consider the finite cover of X consisting of $V_X^2(x^1, r(x^1))$, $B_X(x^2, r(x^2))$, …,$B_X(x^k, r(x^k))$, $B_X(x^{k+1}, r(x^{k+1}))$, …, $B_X(x^{k+T}, r(x^{k+T}))$.

For the open set $V_X^2(x^1, r(x^1))$, we use the same sets $I(x^1)$ and $K(x^1)$ as for $B_X(x^1, r(x^1))$. The balls that we have added to the initial cover satisfy conditions (a)–(c). Moreover, $d_i^{k+t} \neq x_i^{k+t}$ for all $i \in I(x^{k+t})$ and all $t \in \{1, \ldots, T\}$. Hence, if a ball added to the cover, denoted by $B_X(x^s, r(x^s))$, intersects another element of the cover, denoted by $V(x^j)$, then, for every $z' \in B_X(x^s, r(x^s)) \cap V(x^j)$, we have that $z'_i \notin \text{co}\{d_i^s, d_i^j\}$ for $i = 1, 2$. This is so because $V(x^j)$ is either a subset of or equal to an open ball satisfying conditions (a)-(c), and $d_i^s \neq x_i^s$ for all $i \in I(x^s)$, which precludes Case 7. It is worth mentioning that x^1 need not belong to $V_X^2(x^1, r(x^1))$.

Then consider the sets $V_X^2(x^1, r(x^1))$ and $B_X(x^3, r(x^3))$. If needed, we again modify the cover of X with the aid of the just described technique, denoting the difference of $V_X^2(x^1, r(x^1))$ and $\text{cl}B_X(x^3, r(x^3))$ by $V_X^3(x^1, r(x^1))$. Otherwise we put $V_X^3(x^1, r(x^1)) = V_X^2(x^1, r(x^1))$. After considering all the pairs $V_X^{j-1}(x^1, r(x^1))$ and $B_X(x^j, r(x^j))$, $j = 2, \ldots, k$, we denote $V(x^1) = V_X^k(x^1, r(x^1))$ and proceed to considering $B_X(x^2, r(x^2))$ and $B_X(x^3, r(x^3))$, and so on. If needed, the technique is applied again. The last ball that might need modifying is $B_X(x^{k-1}, r(x^{k-1}))$. Put $V(x^s) = B_X(x^s, r(x^s))$ for $s \geq k$. So, after a finite number of rounds, we will get a finite open cover of X, $\{V(x^j) : j = 1, \ldots, R\}$ with $I(x^j)$ and $\{d_i^j \in X_i : i \in I(x^j)\}$, such that

(a) for every $x' \in V(x^j)$, $u_i(d_i^j, x'_{-i}) > u_i(x')$ for some $i \in I(x^j)$;
(b) for every pair $s, t \in \{1, \ldots, R\}, s \neq t$, if $z' \in V(x^s) \cap V(x^t)$, then $z'_i \notin \text{co}\{d_i^s, d_i^t\}$ for some $i \in I(x^s) \cup I(x^t)$, where again we assume that $d_i^j = \{\varnothing\}$ if $i \in \{1, 2\} \backslash I(x^j)$.

Let x be some point of X. Since $\{V(x^1), \ldots, V(x^R)\}$ is a cover of X, there exists $V(x^j)$, $j \in \{1, \ldots, R\}$, such that $x \in V(x^j)$. Put $U_X(x) = V(x^j)$ and $I(x) = I(x^j)$. For each $i \in I(x)$, set $d_i(x) = d_i^j$. Then the hypotheses of Theorem 2 are satisfied, a contradiction.

Proofs for Cases 1–6 of Theorem 3

The following fact will be used below frequently: It follows from the quasiconcavity of $u_1(u_2)$ in $x_1(x_2)$ and the inclusion $z'_1 \in \text{co}\{d_1^1, d_1^2\}$ ($z'_2 \in \text{co}\{d_2^1, d_2^2\}$) that if $u_1(d_1^1, z'_2) > u_1(z')$ ($u_2(z'_1, d_2^2) > u_2(z')$), then $u_1(d_1^1, z'_2) > u_1(d_1^2, z'_2)$

$(u_2(z_1', d_2^2) > u_2(z_1', d_2^1))$. So it is important to keep in mind that our assumptions imply that $u_1(d_1^1, z_2') > u_1(d_1^2, z_2')$ and $u_2(z_1', d_2^2) > u_2(z_1', d_2^1)$.

Case 1. Assume, by contradiction, that $d_1^1 = x_1^1$ and $d_1^2 = x_1^2$. Then $u_2(d_1^2, d_2^2) > u_2(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$. Then the containment $B_X(x^2, r(x^2)) \subset B_X(x^1, 3r(x^1))$, the quasiconcavity of u_2 in x_2, and condition (a) imply that it must be the case that $u_1(d_1^1, z_2) > u_1(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$, which contradicts condition (c).

Case 2. We now assume that $d_1^1 = x_1^1$, $d_1^2 \neq x_1^2$, and $(d_1^1, z_2') \in B_X(x^2, 3r(x^2))$. Since $(d_1^1, z_2') \in B_X(x^2, 3r(x^2))$ and $u_1(d_1^1, z_2') > u_1(d_1^2, z_2')$, we have that $u_2(d_1^1, d_2^2) > u_2(d_1^1, z_2') \geq u_2(d_1^1, d_2^1)$. However, the inclusion $(d_1^1, z_2') \in B_X(x^1, r(x^1))$ implies that it must be the case that $u_2(d_1^1, d_2^1) > u_2(d_1^1, z_2')$, a contradiction.

Case 3. Let $d_1^1 = x_1^1$, $d_1^2 \neq x_1^2$, and $(d_1^1, z_2') \notin B_X(x^2, 3r(x^2))$. It is worth noticing that if $x_1^2 \leq d_1^1$, then (d_1^1, z_2') would be located closer to x^2 than z', which is impossible since $z' \in B_X(x^2, r(x^2))$. Therefore, $x_1^2 > d_1^1$. We have to consider the following two subcases: $d_2^2 = x_2^2$ and $d_2^2 \neq x_2^2$.

Case 3.1. If $d_2^2 = x_2^2$, then $u_1(d_1^2, d_2^2) > u_1(z_1', d_2^2) \geq u_1(d_1^1, d_2^2)$. Since $(z_1', d_2^2) \in B_X(x^1, 3r(x^1))$ and $u_1(z_1', d_2^2) \geq u_1(d_1^1, d_2^2)$, it must be the case that $u_2(z_1', d_2^1) > u_2(z_1', d_2^2)$, a contradiction.

Case 3.2. Let $d_2^2 \neq x_2^2$. First, we claim that $u_2(z_1, d_2^2) > u_2(z_1, z_2') \geq u_2(z_1, d_2^1)$ for all $(z_1, z_2') \in B_X(x^2, r(x^2))$. Assume, to the contrary, that $u_2(z_1, d_2^2) \leq u_2(z_1, z_2')$ for some $(z_1, z_2') \in B_X(x^2, r(x^2))$. Since $(d_1^1, z_2') \notin B_X(x^2, 3r(x^2))$ and $(z_1, z_2') \in B_X(x^2, r(x^2))$, we have that $d_1^1 < z_1$. Then $u_1(d_1^2, z_2') > u_1(z_1, z_2') \geq u_1(d_1^1, z_2')$, a contradiction.

Now we claim that, in contradiction to condition (b), $u_2(z_1, d_2^2) > u_2(z)$ for all $z \in B_X(x^2, r(x^2))$ with z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$.

Assume, by contradiction, that $u_2(z_1, d_2^2) \leq u_2(z)$ for some $z \in B_X(x^2, r(x^2))$ with $z_2 > z_2'$ and z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$. Then $u_1(d_1^2, z_2) > u_1(z)$, and, therefore, $u_1(z) \geq u_1(d_1^1, z_2)$, which, in its turn, implies that $u_2(z_1, d_2^1) > u_2(z) \geq u_2(z_1, d_2^2)$, a contradiction.

Case 4. Let $d_1^1 \neq x_1^1$ and $d_1^2 \neq x_1^2$. Then $u_1(d_1^1, z_2') > u_1(z_1, z_2')$ for all $(z_1, z_2') \in B_X(x^2, r(x^2))$. Let us show this for the sake of completeness. Assume, by contradiction, that $u_1(d_1^1, z_2') \leq u_1(z_1, z_2')$ for some $(z_1, z_2') \in B_X(x^2, r(x^2))$ with $z_1 < z_1'$. Then $u_2(z_1, d_2^1) > u_2(z_1, z_2')$, and, hence, $u_1(d_1^2, z_2') > u_1(z_1, z_2')$. This implies that $u_1(d_1^2, z_2') > u_1(d_1^1, z_2')$, a contradiction.

Therefore, $u_2(z_1, d_2^2) > u_2(z_1, z_2') \geq u_2(z_1, d_2^1)$ for all $(z_1, z_2') \in B_X(x^2, r(x^2))$. We have to consider the following two subcases: $d_2^2 = x_2^2$ and $d_2^2 \neq x_2^2$.

Case 4.1. Let $d_2^2 = x_2^2$. Since $u_1(d_1^2, d_2^2) > u_1(z_1', d_2^2)$, it must be the case that $u_2(z_1', d_2^1) > u_2(z_1', d_2^2)$, a contradiction.

Case 4.2. Let $d_2^2 \neq x_2^2$. Then, in contradiction to the minimality property of $I(x^2)$, $u_2(z_1, d_2^2) > u_2(z)$ for all $z \in B_X(x^2, r(x^2))$ with z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$. Let us show this.

Assume, by contradiction, that $u_2(z_1, d_2^2) \leq u_2(z)$ for some $z \in B_X(x^2, r(x^2))$ with $z_2 > z_2'$ and z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$. Then $u_1(d_1^2, z_2) > u_1(z)$, and, hence, $u_2(z_1, d_2^1) > u_2(z) \geq u_2(z_1, d_2^2)$, a contradiction.

Case 5. Let $d_1^1 \neq x_1^1, d_2^1 \neq x_2^1$, and $d_1^2 = x_1^2$. Since $u_2(d_1^2, d_2^2) > u_2(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$, we have that $u_1(d_1^1, z_2) > u_1(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$. In order to obtain a contradiction with condition (c), it is enough to show that $u_1(d_1^1, z_2) > u_1(z)$ for all $z \in B_X(x^2, r(x^2))$. Assume that $u_1(d_1^1, z_2) \leq u_1(z)$ for some $z \in B_X(x^2, r(x^2))$ with $z_1 < d_1^2$. Then it must be the case that $u_2(z_1, d_2^1) > u_2(z)$, and, therefore, $u_1(d_1^2, z_2) > u_1(z) \geq u_1(d_1^1, z_2)$, a contradiction.

Case 6. Let $d_2^1 = x_2^1, d_1^2 = x_1^2$, and $x_2^2 > d_2^1$. Since $u_2(d_1^2, d_2^2) > u_2(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$, we have that $u_1(d_1^1, z_2) > u_1(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$ with $z_2 \geq d_2^1$. Then one can show that $u_1(d_1^1, z_2) > u_1(z)$ for all $z \in B_X(x^2, r(x^2))$ with $z_2 \geq d_2^1$, which contradicts condition (c).

Example 3 (continued)

We now explain the intuition behind the modifying technique used in the proof of Theorem 3.

Let $U_X(x^1) = B_X(x^1, r(x^1))$ with $x^1 = (\frac{1}{2}, \frac{1}{2}), r(x^1) = \frac{1}{10}, I(x^1) = \{1, 2\}$, and $(d_1^1, d_2^1) = (0, \frac{1}{2})$, and $U_X(x^2) = B_X(x^2, r(x^2))$ with $x^2 = (\frac{1}{2}, \frac{5}{12}), r(x^2) = \frac{1}{11}, I(x^2) = \{1, 2\}$ and $(d_1^2, d_2^2) = (\frac{1}{2}, 1)$. It is not difficult to see that conditions (a)-(c) of the proof of Theorem 3 are satisfied for these two open balls. Denote $C = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$. Then $z_i' \in \text{co}\{d_i^1, d_i^2\}, i = 1, 2$, for every $z' \in U_X(x^1) \cap U_X(x^2) \cap C$.

We replace $B_X(x^1, r(x^1))$ with $V_X(x^1, r(x^1)) = B_X(x^1, r(x^1)) \backslash \text{cl} B_X(x^2, r(x^2))$ and keep $B_X(x^2, r(x^2))$ unchanged. Obviously, the open sets $V_X(x^1, r(x^1))$ and $B_X(x^2, r(x^2))$ do not intersect. However, in order to cover the compact set $A = \partial B_X(x^2, r(x^2)) \cap \text{cl} B_X(x^1, r(x^1))$, we have to add a finite number of new elements to the initial cover.

For every $x \in A$ with $x_2 \neq d_2^1$, we pick an open ball $B_X(x, r(x))$ such that $|x_2 - d_2^1| > 5r(x)$ and find a minimal $I(x) \subset I(x^1)$ (for $B_X(x, r(x))$) with $K(x) \subset K(x^1)$ such that for every $x' \in B_X(x, 3r(x))$, there exists $i \in I(x)$ with $u_i(d_i, x_{-i}') > u_i(x')$.

For every $x \in A$ with $x_2 = d_2^1$, we pick $B_X(x, r(x))$ such that $|x_1' - d_1^2| > 5r(x)$ and find a minimal $I(x) \subset I(x^2)$ with $K(x) \subset K(x^2)$ such that for every $x' \in B_X(x, 3r(x))$ there exists $i \in I(x)$ with $u_i(d_i, x_{-i}') > u_i(x')$.

At first glance, it looks like not much has changed. However, it is not so. For example, if, for some $x \in A \cap \text{int} C$, there is $z' \in B_X(x, r(x)) \cap B_X(x^2, r(x^2))$ such that $z_i' \in \text{co}\{d_i, d_i^2\}, i = 1, 2$, then, obviously, $I(x) = \{1, 2\}$. As we have shown in the proof of Theorem 3, it is possible only if $d_i = x_i$ for some $i \in \{1, 2\}$ (Case 7). However, this is not the case by construction. Therefore, the minimal $I(x)$ is a one-element set.

Appendix B

Proof of Lemma 2

Fix $x \in X \backslash E_G$. Let $A(x)$ be the set of $\alpha \in \mathbb{R}^n$ such that $(x, \alpha) \in \text{clGr}G$. For each $i \in N$, define $\underline{u}_i : X \to \mathbb{R}$ by $\underline{u}_i(x_i, x_{-i}) = \liminf_{x'_{-i} \to x_{-i}} u_i(x_i, x'_{-i})$. By construction, $\underline{u}_i$ is lower semicontinuous in x_{-i}. For each $i \in N$ and every $x_{-i} \in X_{-i}$, define $\delta_i : X_{-i} \to \mathbb{R}$ by $\delta_i(x_{-i}) = \sup_{y_i \in X_i} \underline{u}_i(y_i, x_{-i})$. It is clear that each δ_i, as the supremum of a collection of lower semicontinuous functions, is lower semicontinuous (see also Reny 1999, p. 1037).

Since G is better-reply secure, for each $\alpha = (\alpha_1, \ldots, \alpha_N) \in A(x)$ there is $i(\alpha) \in N$ such that $\delta_{i(\alpha)}(x_{-i(\alpha)}) > \alpha_{i(\alpha)}$. Pick $\varepsilon_\alpha > 0$ and $r_\alpha > 0$ such that $\delta_{i(\alpha)}(x_{-i(\alpha)}) > \alpha'_{i(\alpha)} + \varepsilon_\alpha$ for all $\alpha' \in B_{\mathbb{R}^n}(\alpha, r_\alpha)$. We can say that player $i(\alpha)$ secures the neighborhood $B_{\mathbb{R}^n}(\alpha, r_\alpha)$ at x.

Since $A(x)$ is compact, the cover $\{B_{\mathbb{R}^n}(\alpha, r_\alpha) : \alpha \in A(x)\}$ contains a finite subcover $\{B_{\mathbb{R}^n}(\alpha^j, r_{\alpha^j}) : j = 1, \ldots, k\}$. Let $\varepsilon(x) = \frac{1}{2} \min_{j \in \{1,\ldots,k\}} \varepsilon_{\alpha_j}$. Denote by $J_i(x)$ the collection of all $j \in \{1, \ldots, k\}$ such that player i secures $B_{\mathbb{R}^n}(\alpha^j, r_{\alpha^j})$ at x. Let $I(x) = \{i \in N : J_i(x) \neq \varnothing\}$ and $\overline{\alpha}_i = \max_{j \in J_i}(\alpha^j + r_{\alpha^j})$. Then, by the definition of the least upper bound, for each $i \in I(x)$ there exists $d_i \in X_i$ such that $\underline{u}_i(d_i, x_{-i}) > \overline{\alpha}_i + \varepsilon(x)$. From the lower semicontinuity of $\underline{u}_i$ in x_{-i}, we deduce that $\underline{u}_i(d_i, x'_{-i}) > \overline{\alpha}_i + \varepsilon(x)$ for all x'_{-i} in some open neighborhood $U_{X_{-i}}(x_{-i})$ of x_{-i}.

We claim that there exists an open neighborhood $U_X(x)$ of x such that, for every $x' \in U_X(x)$, there is some $i \in I(x)$ with $u_i(d_i, z_{-i}) - \varepsilon(x) > u_i(x')$ for all $z_{-i} \in U_{X_{-i}}(x_{-i})$. If it is not so, then one can construct a net $\{x^\beta\}$ converging to x such that, for each β and each $i \in I(x)$, $u_i(d_i, z^\beta_{-i}) - \varepsilon(x) \leq u_i(x^\beta)$ for some $z^\beta_{-i} \in U_{X_{-i}}(x_{-i})$. Since the payoff functions are bounded, there is no loss of generality in assuming that the net $\{u(x^\beta)\}$ converges to some $\alpha \in A(x)$. Then, for some $j \in \{1, \ldots k\}$, there exists $\widehat{\beta}$ such that $u(x^\beta) \in B_{\mathbb{R}^n}(\alpha^j, r_{\alpha^j})$ for all $\beta \succeq \widehat{\beta}$. Therefore, for some $i \in I(x)$, $u_i(d_i, x'_{-i}) \geq \underline{u}_i(d_i, x'_{-i}) > \overline{\alpha}_i + \varepsilon(x) > u_i(x^\beta) + \varepsilon(x)$ for all $x'_{-i} \in U_{X_{-i}}(x_{-i})$ and all $\beta \succeq \widehat{\beta}$, a contradiction.

Proof of Lemma 5

Since X is compact, the open cover $\{U_x : x \in X\}$ of X contains a finite subcover $\{U_{x_j} : j \in J\}$, where J is a finite set. Let $\{G_{x_j} : j \in J\}$ be a closed refinement of $\{U_{x_j} : j \in J\}$ such that $G_{x_j} \subset U_{x_j}$ for each $j \in J$. For each $j \in J$, define a correspondence $F_j : X \twoheadrightarrow Y$ by

$$F_j(z) = \begin{cases} \cup_{\{s \in J : z \in U_{x_s}\}} F_{x_s}(z) & \text{if} \quad z \in G_{x_j}, \\ Y & \text{if} \quad z \notin G_{x_j}. \end{cases}$$

By (ii), $\theta(z) \notin \text{co}F_j(z)$ for all $z \in G_{x_j}$. Each F_j has open lower sections in X since, for each $y \in Y$,

$$
\begin{aligned}
F_j^{-1}(y) &= \{z \in G_{x_j} : y \in (\cup_{\{s \in J : z \in U_{x_s}\}} F_{x_s}(z))\} \cup (X \backslash G_{x_j}) \\
&= (G_{x_j} \cap (\cup_{s \in J} (U_{x_s} \cap F_{x_s}^{-1}(y))) \cup (X \backslash G_{x_j}) \\
&= \cup_{s \in J} (U_{x_s} \cap F_{x_s}^{-1}(y)) \cup (X \backslash G_{x_j}).
\end{aligned}
$$

Therefore, $\overline{F} : X \twoheadrightarrow Y$ defined by $\overline{F}(z) = \cap_{j \in J} F_j(z)$ also has open lower sections. By construction, $\mathrm{Dom}\overline{F} = X$ and $\overline{F}$ is of class L_θ.

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