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RESEARCH ARTICLE

Large extensive form games

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Abstract This paper provides a self-contained definition and a characterization of the class of extensive form games that are adequate for applications, but still do not employ any finiteness assumptions. In spite of its simplicity, the resulting definition is more general than the classical ones. Moreover, we show that this class satisfies the basic desiderata that strategies induce outcomes and do so uniquely. Within the class of playable extensive forms, the characterization is by the existence of an immediate predecessor function on the set of moves.

Keywords Extensive form games · Sequential decision theory · Trees

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1 Introduction

Non-cooperative game theory is the theory of games with complete rules. The tool to verify that rules are complete is the representation of the game in *extensive form*. This

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representation also lends itself naturally to an intuitive understanding of the players' interaction. As a consequence, a number of commonly used solution concepts that relate to backwards induction, like subgame perfection or perfect Bayesian equilibrium, are only defined for extensive form games.

Much of the early theoretical work on extensive form games (Kuhn 1953; Selten 1975) tacitly adopted finiteness assumptions in order to avoid technicalities and focus on conceptual issues. Likewise, more recent definitions as those by Kreps and Wilson (1982) or Mas-Colell et al. (1995, p. 227) are restricted to finite settings. On the other hand, applications of game theory soon transcended these limitations: games with continuum action spaces and infinitely repeated games are examples of obvious interest to economists. For decades, this has led to a parallel development with applications of large games on the one hand and theoretical analysis of finite games on the other. Yet, as far as the extensive form representation of a game is concerned, this gap is easily closed.

In this paper, we show that all finiteness assumptions for extensive form representations of games can be dispensed with. This point is not new and we have already emphasized it in previous papers (briefly described below). But our previous work aimed at necessity on top of sufficiency, in order to precisely delimit the confines of traditional game theory. This has led to statements that are occasionally complicated by technical subtleties. In this paper, we focus on a framework that is *sufficient* for (most) applications and that is the one which has effectively been employed in practice. And it turns out that this framework is considerably simpler than the general one.

This is, of course, an informed endeavor which uses previous results on what is necessary for game theoretic analysis. And it has to be, because merely dropping finiteness assumptions without appropriate substitutes can produce pathologies. It is well known, for instance, that extensive form games in continuous time ("differential games") may allow strategies that produce no outcome at all or a continuum of outcomes (e.g., Simon and Stinchcombe 1989; Stinchcombe 1992; Alós-Ferrer and Ritzberger 2008). But this is not an artifact of continuous time. For, replace the reals for the time axis by all rational numbers of the form $t = 1/n$ for $n = 1, 2, \ldots,$ together with $t = 0$. Suppose that at each $t \in \{0, \ldots, 1/2, 1\}$, player t, say, can choose an action $a \in [0, 1]$ from the unit interval. For every $b \in [0, 1]$, define a strategy profile as follows: choose $a = b$ at "time" 0, choose $a = 0$ at any positive "time" if $a = b$ has been chosen by all players before the present one, and choose $a = 1$ otherwise. If $b = 1$, this strategy profile induces no outcome. For, constantly playing $a = 1$ is not compatible with this strategy, as players would switch to 0 at any positive "time." On the other hand, there cannot be any $n = 1, 2, \ldots,$ such that player $t = 1/n$ chooses $a = 0$ either. For, if there were, then all players $1/(n + \tau)$, for $\tau = 1, 2, \ldots$, before the present one must have chosen action 1; but in that case, player $1/(n + 1)$ must already have chosen action 0 according to the strategy profile, prompting player $t = 1/n$ to choose $a = 1$, a contradiction. Since the strategy profile chooses only between 0 and 1, it induces no outcome. On the other hand, if $b = 0$, the associated strategy profile induces multiple outcomes: constantly playing 0, and starting with $a = 0$ at $t = 0$ and playing 1 at all positive "times".

That a strategy combination induces no outcome or many is problematic, because neither the empty set nor (nonsingleton) sets of outcomes belong to the domain of players' preferences. Therefore, players cannot evaluate such strategies, as they are supposed to, for instance, in equilibrium analysis. In other words, dropping finiteness assumptions calls for a careful specification of the objects of extensive form analysis.

In an earlier paper (Alós-Ferrer and Ritzberger 2005b), we have shown that it involves no loss of generality to view the *tree* of an extensive form as a collection of subsets of an underlying set of outcomes (with a particular structure), as proposed by von Neumann and Morgenstern (1944, p. 65). This formalization makes the game tree an appropriate domain for decision theory, as nodes are sets of "states" like any other event. And this formalism covers all known examples of extensive form games, inclusive of infinitely repeated games, infinite bilateral bargaining (Rubinstein 1982), stochastic games (Shapley 1953), the "transfinite cheap-talk" by Aumann and Hart (2003), and "differential games."

Related work (Alós-Ferrer and Ritzberger 2008) has investigated when strategies, as derived from an extensive form, satisfy the following two desiderata: (A1) Every strategy combination induces an outcome/play; and (A2) if a strategy combination induces an outcome, this outcome is unique. The main result of that paper is that (A1) and (A2) hold true *if and only if* the tree satisfies two conditions: it is "regular" and "up-discrete" (precise definitions are below). The good news is that this characterization does—almost—not rule out any of the classical examples of extensive form games. All aforementioned cases are covered, except for differential games (which are not up-discrete).

In practice, most applications use more structure than what is necessary and sufficient to obtain (A1) and (A2). The key step from what is necessary to what is sufficient is a property which we call "discreteness," but which does not exclude, e.g., continuum action sets. More precisely, discreteness has two parts. Up-discreteness (toward the root of the tree) is necessary for extensive form analysis, as pointed out above. The other part, down-discreteness, is the added structure that dramatically simplifies the formalism and still maintains enough generality for applications. In particular, the combination of up- and down-discreteness still allows one to study games with large (continuum) action spaces and/or a long (infinite) time horizon.

In this paper, we characterize the class of *discrete* extensive forms within the abstract framework developed in previous papers. It is shown that discrete trees are characterized by the existence of an (onto) immediate predecessor function on the set of moves. Equivalently, this means that the history of every decision point in the tree can be uniquely reconstructed in a finite number of steps—yet, infinite-horizon games are not excluded. As a result, an elementary definition of extensive forms (on a discrete tree) is obtained that lends itself easily to applications, but is still more general than the classical ones. The four defining consistency conditions of the general definition collapse to two intuitive properties: One says that the decisions of the relevant players at any given move lead to a unique new node. The other says that players cannot deduce from the menu of available choices more information than what they have according to their information sets.

A key advantage of the present approach is that it is immaterial whether plays or nodes are taken as the primitives. In particular, for discrete trees, there is an alternative definition with the nodes as primitives—"simple extensive forms." This is due to the fact that certain ("infinite") terminal nodes need not be part of the tree. In the end, then, one obtains a simple framework for extensive form analysis that still covers almost all relevant cases and preserves the freedom to take plays or nodes as primitives.

The remainder is organized as follows. Section 2 presents the definition of a game tree, the classification of nodes, and extensive decision problems in the general framework developed by Alós-Ferrer and Ritzberger (2005b; 2008, henceforth AR1 and AR2) and discusses the characterization of playable games. Section 3 characterizes discrete trees and the associated notion of a discrete extensive form. In this section, we also show that (A1) and (A2) hold for all discrete extensive forms. Section 4 exploits a property of discrete trees to provide a definition and characterization of "simple" trees and associated extensive forms, where nodes (rather than plays) serve as the primitives. Section 5 concludes. Proofs are given in the text only for the main theorems; all other proofs are gathered in an Appendix.

2 Extensive form games: the general case

2.1 Game trees

Extensive form games are defined on (decision) trees. The concept of a tree, as needed in game theory, is studied in detail in AR1. The results in that paper show that there is no loss of generality in assuming a game tree, as in the following definition, when working with decision trees.[1]

Definition 1 A (rooted) **game tree** $T = (N, \supseteq)$ is a collection of nonempty subsets $x \in N$ (called nodes) of a given set W partially ordered by set inclusion such that $W \in N$ and

(GT.i) $h \subseteq N$ is a chain if and only if there is $w \in W$ such that $w \in x$ for all $x \in h$,[2]

(GT.ii) if $w, w' \in W$ satisfy $w \neq w'$, then there are $x, x' \in N$ such that $w \in x \setminus x'$ and $w' \in x' \setminus x$.

For a game tree $(N, \supseteq)$ and a node $x \in N$ define the *up-set* (or *order filter*) $\uparrow x$ and the *down-set* (or *order ideal*) $\downarrow x$ by

$$\uparrow x = \{y \in N \,|\, y \supseteq x\} \text{ and } \downarrow x = \{y \in N \,|\, x \supseteq y\}. \tag{1}$$

By the "if"-part of (GT.i) $\uparrow x$ is a chain for all $x \in N$. A *play* is a chain of nodes $h \subseteq N$ that is maximal in N, i.e. there is no $x \in N \setminus h$ such that $h \cup \{x\}$ is a chain. For every chain in N, there exists a play that contains it. This is the Hausdorff Maximality Principle—a version of the Axiom of Choice (see Hewitt and Stromberg 1965, chp. 1).

The main advantage of game trees is that the set of plays can be one-to-one identified with the underlying set W (Theorem 3(c) of AR1). A node can then be identified with

[1] Henceforth $\subseteq$ denotes weak set inclusion and $\subset$ denotes proper inclusion.

[2] A chain is a subset of a partially ordered set that is *completely* ordered.

the set of plays passing through it, and the underlying set W represents all plays. An element $w \in W$ can thus be seen either as a possible outcome (element of some node) or as a play (maximal chain of nodes). When a distinction is called for, we write w for the outcome and $\uparrow\{w\}$ for the play (chain of nodes), where $\uparrow\{w\} = \{x \in N \mid w \in x\}$ is the play consisting of all nodes that contain w. If h is a play, there exists a unique outcome $w \in W$ such that $\cap_{x \in h} x = \{w\}$, or, equivalently, $\uparrow\{w\} = h$.

Textbook formulations typically start from a given set of nodes and define trees as graphs on this set, that is, as a partially ordered set (poset) where the order relation $\geq$ is precedence (e.g., the "arborescences" in Kreps and Wilson 1982). Every game tree as defined above can be also viewed as a graph by taking set inclusion as the relevant partial order, i.e. a node x precedes another node y if and only if $x \supseteq y$. Conversely, if one starts from a tree as a graph, it is shown in AR1 (Theorem 1, p. 775) that it is always possible to construct an equivalent game tree as defined above. The construction is as follows. Let M be a finite set with a partial order $\geq$. If $y_1 \geq y_2$ say that y_1 is a predecessor of y_2 or that y_2 is a successor of y_1. Call this poset a tree if the set of predecessors of any given node is a chain and every non-terminal node has at least two different proper successors (a nontriviality requirement). For each $y \in M$, let $W(y)$ denote the set of maximal chains that y belongs to. The collection of all $W(y)$ ordered by set inclusion is a game tree in the sense of Definition 1. Every node gets represented by the set of plays passing through it.

For instance, consider the finite set $\{y_1, \ldots, y_5\}$ endowed with the partial order given by $y_1 \geq y_i$ for all $i = 1, \ldots, 5$, $y_3 \geq y_4$, and $y_3 \geq y_5$. There are three maximal chains in this graph, $w_1 = \{y_1, y_2\}$, $w_2 = \{y_1, y_3, y_4\}$, and $w_3 = \{y_1, y_3, y_5\}$. Now represent this graph as a game tree by $x_1 = \{w_1, w_2, w_3\}$, $x_2 = \{w_1\}$, $x_3 = \{w_2, w_3\}$, $x_4 = \{w_2\}$, and $x_5 = \{w_3\}$. This procedure works beyond finite examples, as shown in detail in AR1 (see also Sect. 4).

The following example (taken from Section 2.2.5 of AR1) shows that game trees even cover "differential games," that is, decision problems in continuous time.

Example 1 (Differential game) Let W be the set of functions $f : \mathbb{R}_+ \to A$, where A is a set of "actions" (with at least two elements), and $N = \{x_t(g) \mid g \in W,\ t \in \mathbb{R}_+\}$, where $x_t(g) = \{f \in W \mid f(\tau) = g(\tau), \forall \tau \in [0, t)\}$, for any $g \in W$ and $t \in \mathbb{R}_+$. At each point in time $t \in \mathbb{R}_+$ a decision $a_t \in A$ is taken. The "history" of all decisions taken in the past (up to, but exclusive of, time t) is a function $f : [0, t) \to A$, i.e. $f(\tau) = a_\tau$ for all $\tau \in [0, t)$. A node at "time" t is the set of all functions that coincide with f on $[0, t)$, all possibilities still open for their values thereafter. It can be shown that $(N, \supseteq)$ is a game tree.

A convenient class of examples for game trees are the, in many respects, simplest trees of all (see also Example 3 of AR1).

Example 2 (Centipedes) Let $(W, \geq)$ be a *completely* ordered set of arbitrary cardinality, $x_t = \{\tau \in W \mid \tau \geq t\}$ for all $t \in W$, and $N = \{\{t\} \mid t \in W\} \cup \{x_t \mid t \in W\}$. The resulting tree $(N, \supseteq)$ satisfies (GT.ii) from Definition 1. But, because for the play $h_\infty = \{x_t\}_{t \in W}$ the intersection $\cap_{t \in W} x_t$ may be empty (e.g., if W were the natural numbers), the tree may not satisfy the "only if"-part of (GT.i). In this case, the tree can be completed by adding an "infinite" element w_∞ without affecting the order-theoretic structure (see Proposition 8 of AR1). Then, W has a maximum w_∞ and

$w_\infty \in x$ for all $x \in h_\infty$. As a consequence, $(N, \supseteq)$ also satisfies the "only if"-part of (GT.i) and is a game tree. We refer to this as the *W-centipede*.

Nodes in a game tree that are properly followed by other nodes are called *moves*, i.e. $X = \{x \in N \mid (\downarrow x) \setminus \{x\} \neq \emptyset\}$ is the set of all moves (decision points). All other nodes are called *terminal*, and $E = \{x \in N \mid \downarrow x = \{x\}\}$ denotes the set of terminal nodes. It can be shown that a node $x \in N$ is terminal if and only if there is $w \in W$ such that $x = \{w\}$ (see Lemma 3(b) in the Appendix). The following is a more fundamental classification of nodes in a game tree.

Definition 2 Let $(N, \supseteq)$ be a game tree and $x \in N \setminus \{W\}$. Say that x is **finite** if $(\uparrow x) \setminus \{x\}$ has a minimum, **infinite** if $x = \inf (\uparrow x) \setminus \{x\}$, and **strange** if $(\uparrow x) \setminus \{x\}$ has no infimum. Denote the sets of infinite and strange nodes in N by $I(N)$ and $S(N)$, respectively, and by $F(N)$ the set of finite nodes together with the root W.

The three possibilities in this definition are exhaustive, that is, all nodes $x \in N \setminus \{W\}$ are either finite, infinite, or strange. For, if $(\uparrow x) \setminus \{x\}$ has an infimum z, it is either a minimum (and then x is finite), or $z \notin (\uparrow x) \setminus \{x\}$. In the latter case, it follows from the definition of an infimum that $z = x$.

On finite nodes a function p can be defined that assigns to every finite node its *immediate predecessor*. Namely, let $p : F(N) \to X$ be given by

$$p(x) = \min (\uparrow x) \setminus \{x\} \text{ for all } x \in F(N) \setminus \{W\} \tag{2}$$

and $p(W) = W$ by convention. Hence, $x \subset p(x) = \cap_{y \in (\uparrow x) \setminus \{x\}} y$ for $x \in F(N) \setminus \{W\}$.

A node $x \in N$ is infinite if and only if $x = \cap_{y \in (\uparrow x) \setminus \{x\}} y$ by Lemma 4(b) in the Appendix. In this sense, infinite nodes can be reconstructed from the other nodes in a game tree. That is, the set of infinite nodes for a game tree $(N, \supseteq)$ can alternatively be defined as $I(N) = \left\{x \in N \,\middle|\, x = \cap_{y \in (\uparrow x) \setminus \{x\}} y \right\}$.

Example 3 (Infinite centipede) Let $W = \{1, 2, \ldots, \infty\}$ be the natural numbers together with "infinity" ∞ and consider the corresponding W-centipede (Example 2). This example is illustrated in Fig. 1. There is a unique infinite node in this game tree, namely $\{\infty\} \in N$. For, $\cap \{x_t \mid 1 \leq t < \infty\} = \{\infty\}$, but $\{\infty\} \notin \{x_t \in N \mid t = 1, 2, \ldots\} = (\uparrow \{\infty\}) \setminus \{\{\infty\}\}$.

Not all centipedes are so well behaved as the infinite centipede. In particular, it may be the case that *moves* are infinite nodes. Intuitively, an infinite move corresponds to a

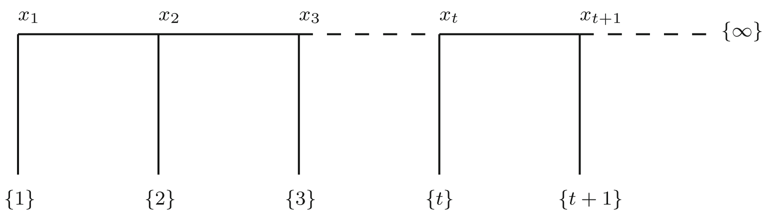

Fig. 1 The infinite centipede. $W = \{1, 2, \ldots, \infty\}$, $x_t = \{t, t+1, \ldots, \infty\}$

decision point that is never actually reached, because it is due after an infinite sequence of other decision points. This is the case, for instance, if decisions are taken at each point in continuous time.

Example 4 (Continuous centipede) If in Example 2 $W = [0, 1]$, all singletons $\{t\}$ are finite, except for the "last" node, $\{1\}$. All moves, though, are infinite, except for the root $[0, 1]$. For, $\cap \{y \,|y \in (\uparrow x_t) \setminus \{x_t\}\} = \cap \{x_\tau \,|\tau < t\} = [t, 1] = x_t$, provided $t > 0$ (otherwise the intersection is empty), implies that x_t is infinite.

Though finite and infinite nodes can show up in centipedes in various combinations, strange nodes cannot. For, in a centipede for a singleton $\{t\} \in N$, the move $x_t \in N$ provides an infimum for the chain $(\uparrow \{t\}) \setminus \{\{t\}\}$ (even if $x_t = \{t\}$, as in the continuous centipede for $t = 1$). A move $x_t \in X$ in a centipede is either infinite or satisfies $x_t \subset \cap_{x\in(\uparrow x_t)\setminus\{x_t\}} x$. In the latter case, there is a unique $\tau \in W$ such that $\tau \in \left(\cap_{x\in(\uparrow x_t)\setminus\{x_t\}} x\right) \setminus x_t$. (For, if $\tau, \tau' \in \left(\cap_{x\in(\uparrow x_t)\setminus\{x_t\}} x\right) \setminus x_t$ are such that, say, $\tau \geq \tau'$, then $x_{\tau'} \supseteq x_\tau \supset x_t$ and $\tau, \tau' \in x$ for all $x \in (\uparrow x_t) \setminus \{x_t\}$ imply $\tau' \in x_\tau$, hence, $\tau' = \tau$ by the properties of a centipede.) Therefore, $x_\tau \in N$ provides a minimum for $(\uparrow x_t) \setminus \{x_t\}$ and $x_t \in X$ is finite. Centipedes are, therefore, examples without strange nodes. To illustrate strange nodes takes a separate example, like the following "Twins" (Example 13 of AR1).

Example 5 (Twins) Let $W = [0, 1]$ and $N = \{\{w\} |\, w \in W\} \bigcup \{x_t\}_{t=1}^{\infty}$, where

$$x_t = \left[0, \frac{1}{t+1}\right] \cup \left[\frac{t}{t+1}, 1\right] \text{ for all } t = 1, 2, \ldots$$

It can be shown that this is a game tree (see AR1). The singletons $\{0\}, \{1\}$ have the same (proper) predecessors

$$(\uparrow \{1\}) \setminus \{\{1\}\} = (\uparrow \{0\}) \setminus \{\{0\}\} = \{x_t \in N \,|t = 1, 2, \ldots\}$$

Therefore, they are both lower bounds for the chain $\{x_t \in N \,|t = 1, 2, \ldots\}$, but they are disjoint. Hence, this chain has no infimum, and $\{0\}, \{1\} \in N$ are strange.

Observe that neither $\{0\}$ nor $\{1\}$ can be removed without affecting the structure of the tree. In fact, any removal of one of the two leads to a violation of condition (GT.ii) from Definition 1 (see Sect. 4.1 below).

2.2 Extensive decision problems

The primary goal of the concept of a game tree is to define extensive forms without any finiteness or discreteness assumptions. Below such a definition is provided (Definition 7 of AR1 and Definition 3.1 of AR2).

Some preparation is needed. For a game tree $(N, \supseteq)$ with set of outcomes W and an arbitrary subset $a \subseteq W$ (not necessarily a node), the *down-set* of a is given by $\downarrow a = \{x \in N \,|x \subseteq a\}$ and the *up-set* of a by $\uparrow a = \{x \in N \,|a \subseteq x\}$. Moreover, the set of *immediate predecessors* of $a \in 2^W$ is defined by

$$P(a) = \{x \in N \,|\, \exists y \in \downarrow a : \uparrow x = (\uparrow y) \setminus (\downarrow a)\} \tag{3}$$

Since nodes in a game tree are sets of outcomes, they too may, but need not, have immediate predecessors. It is easy to see that a node has an immediate predecessor if and only if it is finite (AR2, p. 224).

Definition 3 An **extensive decision problem** (EDP) with player set I is a pair (T, C), where $T = (N, \supseteq)$ is a game tree with set of outcomes W and $C = (C_i)_{i \in I}$ is a system consisting of collections C_i (the sets of players' choices) of nonempty unions of nodes (hence, sets of outcomes) for all $i \in I$ such that

- (EDP.i) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$, then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ for all $i \in I$;
- (EDP.ii) $x \cap \left[\bigcap_{i \in J(x)} c_i\right] \neq \emptyset$ for all $(c_i)_{i \in J(x)} \in A(x)$ and for all $x \in X$;
- (EDP.iii) if $y, y' \in N$ with $y \cap y' = \emptyset$ then there are $c, c' \in C_i$ for some player $i \in I$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$;
- (EDP.iv) if $x \supset y \in N$, then there is $c \in A_i(x)$ such that $y \subseteq c$ for all $i \in J(x)$, for all $x \in X$;

where $A(x) = \times_{i \in J(x)} A_i(x)$, $A_i(x) = \{c \in C_i \,|\, x \in P(c)\}$ are the choices available to $i \in I$ at $x \in X$, and $J(x) = \{i \in I \,|\, A_i(x) \neq \emptyset\}$ is the set of decision makers at x, which is required to be nonempty, for all $x \in X$.

The interpretation is as follows (see AR1, Section 5, for additional details). A choice $c \in C_i$ is *available* at $x \in X$ if $x \in P(c)$. (EDP.i) stands in for information sets: if two distinct choices $c, c' \in C_i$ are ever simultaneously available, then they are disjoint and available at the same moves—at those in the *information set* $g = P(c) = P(c')$. (EDP.ii) requires that simultaneous decisions by different players at a common move do select some outcome. (EDP.iii) states that for any two disjoint nodes, there is a player, who can eventually take a decision that selects among them. Finally, (EDP.iv) states that, if a player takes a decision at a given node, he must be able not to discard any given successor of the node.[3]

An important point about an EDP is that it allows several players to decide at the same move. This, at times, simplifies both the representation of a game and the analysis of equilibrium. We will elaborate on this point in Sect. 3.3 below (Examples 8 and 9).

2.3 Strategies

A *pure strategy* for player $i \in I$ is a function $s_i : X_i \to C_i$, where $X_i = \{x \in X \,|\, i \in J(x)\}$, such that

$$s_i^{-1}(c) = P(c) \text{ for all } c \in s_i(X_i) \tag{4}$$

[3] (EDP.iv) excludes absent-mindedness (Piccione and Rubinstein 1997) by Proposition 13 of AR1, as in the original formulation by Kuhn (1953).

where $s_i(X_i) \equiv \{s_i(x) | x \in X_i\}$. That is, the function s_i assigns to every move $x \in X_i$ a choice $c \in C_i$ such that (a) choice c is *available* at x, i.e. $s_i(x) = c \Rightarrow x \in P(c)$ or $s_i^{-1}(c) \subseteq P(c)$, and (b) to every move x in an *information set* $g = P(c)$ the same choice gets assigned, i.e. $x \in P(c) \Rightarrow s_i(x) = c$ or $P(c) \subseteq s_i^{-1}(c)$, for all $c \in C_i$ that are chosen somewhere, viz. $c \in s_i(X_i)$. Let S_i denote the set of all pure strategies for player $i \in I$. A *pure strategy combination* is an element $s = (s_i)_{i \in I} \in S \equiv \times_{i \in I} S_i$.

Clearly, an EDP ought to satisfy some basic properties, like that every strategy combination induces an outcome/play. This takes a notion of when a pure strategy combination "induces" an outcome. Define, for every $s \in S$, the correspondence $R_s : W \to W$ by

$$R_s(w) = \bigcap \{s_i(x) | w \in x \in X, i \in J(x)\}. \tag{5}$$

Say that strategy combination s *induces* the outcome w if $w \in R_s(w)$, i.e. if it is a fixed point of R_s.

In an arbitrary EDP, the correspondence R_s for a given strategy combination $s \in S$ may not have a fixed point at all or a whole continuum thereof. But neither the empty set nor a (nonsingleton) set of outcomes belongs to the domain of players' preferences. This calls for further restrictions. The two basic desiderata on an EDP, expressed in terms of R_s, are as follows.

(A1) For every $s \in S$, there is some $w \in W$ such that $w \in R_s(w)$.
(A2) If for $s \in S$, there is $w \in W$ such that $w \in R_s(w)$, then R_s has no other fixed point and $R_s(w) = \{w\}$.

(A1) says that for every strategy combination $s \in S$, there is an outcome/play $w \in W$ that is induced by s. Say that an EDP (T, C) is *playable* if (A1) holds. (A2) requires that the induced outcome is unique. (A1) and (A2) define a function $\phi : S \to W$ that associates a unique outcome to each pure strategy combination. (Furthermore, this function is onto by Theorem 4 of AR2.) These two properties are, therefore, necessary and sufficient to define a normal form (without payoffs).

The differential game (Example 1), where the choices at a node $x_t(f)$ are given by $c_t(x_t(f), a) = \{g \in x_t(f) | g(t) = a\}$ for any action $a \in A$, is an example of an EDP that is not playable, and where (A2) is also violated (see Examples 10 and 12 of AR2).[4] Another example that violates (A2) is the "Twins" (Example 5) with a single player and perfect information: there is a strategy that assigns to every move the next move in the row, $s(x_t) = x_{t+1}$ for all $t = 1, 2, \ldots$ This strategy induces the two outcomes $0, 1 \in W$.

The main result of AR2 (Theorem 6 and Corollary 5(b); see also Alós-Ferrer et al. forthcoming) states that (A1) and (A2) are essentially equivalent to two properties of the tree: "regularity" and "up-discreteness." Thus, these two properties represent the appropriate restriction on game trees for a well-founded sequential decision theory.

Definition 4 A game tree $(N, \supseteq)$ is **regular** if there are no strange nodes, i.e. if $S(N) = \emptyset$. It is **up-discrete** if every (nonempty) chain in N has a maximum.

[4] That without restrictions on allowable strategies no outcome may be induced in a continuous-time problem has also been observed by Simon and Stinchcombe (1989) and by Stinchcombe (1992).

3 Discrete extensive form games

3.1 Up-discrete trees

It can be shown (AR2, p. 235) that up-discreteness is equivalent to the chains $\uparrow x$ for $x \in N$ being dually well-ordered (that is, all their subsets have a maximum). This condition is common in order theory and theoretical computer science (see Koppelberg 1989, chp. 6). It implies that the set of *immediate successors*, $p^{-1}(y) = \{x \in F(N) \,|\, p(x) = y\}$, of a move is nonempty and forms a partition of the move by finite nodes.

Proposition 1 *Let $T = (N, \supseteq)$ be an up-discrete game tree. Then, for every $x \in X$*

(a) *the preimage $p^{-1}(x)$ is a partition of x, hence nonempty;*
(b) *for every $y \in N$ with $y \subset x$, there exists $z \in p^{-1}(x)$ such that $y \subseteq z$.*

Of course, up-discreteness excludes continuous-time examples. On the other hand, up-discreteness imposes a tight connection between choices (as sets of outcomes) and nodes, as the following result shows.

Proposition 2 *Let $T = (N, \supseteq)$ be an up-discrete game tree and $A(T)$ the set of all $a \in 2^W$ that are unions of nodes. Define*

$$N(a) = \{x \in N \,|\, x \subseteq a \text{ and } x \text{ is maximal in } a\}. \tag{6}$$

for any $a \in A(T)$. Then,

(a) *$N(a)$ is a partition of a; and*
(b) *its predecessor set is given by $P(a) = \{p(x) \,|\, x \in N(a) \cap F(N)\}$.*

This implies that for any $a \in A(T)$ there exists a coarsest partition by nodes. Applying this result to choices from an EDP yields the following: for any choice $c \in C_i$ for some $i \in I$ there is a set $N(c) \subseteq N$ such that c is a disjoint union over the nodes in $N(c)$ and $P(c) = \{p(x) \,|\, x \in N(c) \cap F(N)\}$. This property is crucial for going back and forth between trees where plays are taken as the primitive and trees where nodes are the primitive (see Sect. 4).

3.2 Discrete trees

To the best of our knowledge, all games that have been studied in the literature—with the exception of differential games—satisfy (A1) and (A2), that is, they are regular and up-discrete. As a matter of fact, this is because they usually satisfy even more stringent conditions. In particular, differential games and the transfinite cheap-talk game by Aumann and Hart (2003) seem to be the only examples which fail the following property that is complementary to up-discreteness.

Definition 5 A game tree $(N, \supseteq)$ is **down-discrete** if the chain $(\uparrow x) \setminus \{x\}$ has an infimum in $E \cup (\uparrow x) \setminus \{x\}$ for every $x \in N \setminus \{W\}$. It is **discrete** if it is up-discrete and down-discrete.

That is, a game tree is down-discrete if for all $x \in N \setminus \{W\}$ the chain $(\uparrow x) \setminus \{x\}$ has either a minimum or an infimum in the set of terminal nodes. Furthermore:

Proposition 3 *A game tree is down-discrete if and only if it is regular and all moves are finite (i.e. $X \subseteq F(N)$).*

Infinitely repeated games, infinite bilateral bargaining games (Rubinstein 1982), or stochastic games (Shapley 1953) all employ discrete trees, as do the infinite extensive form games in Aumann (1964). "Long cheap-talk" à la Aumann and Hart (2003) employs a tree that satisfies (A1) and (A2), but is not down-discrete. It has a time axis of order $\omega + 1$, where ω denotes the first infinite (limit) ordinal and $\omega + 1$ its successor according to the usual well-order. A similar example is obtained as follows.

Example 6 ($\omega + 1$-centipede) Consider the W-centipede (Example 2) with $W = \{1, 2, \ldots, \omega, \omega + 1\}$. There, the infimum of the chain $\{x_t \,|1 \le t < \omega\} = (\uparrow x_\omega) \setminus \{x_\omega\}$ is the move x_ω. Hence, this tree is not down-discrete, even though it is regular and up-discrete.

An EDP defined on the tree of the previous example will satisfy (A1) and (A2), even though the tree is not discrete—in contrast to differential games that fail (A1) and (A2). So, discreteness is more than what is needed. But it has convenient implications. A particularly striking consequence of discrete trees is captured by the next theorem. It characterizes discrete trees, within the class of regular game trees, by the properties of the *immediate predecessor function* $p : F(N) \to X$.

Theorem 1 *For a regular game tree $(N, \supseteq)$ the following statements are equivalent:*

(a) *$(N, \supseteq)$ is discrete;*
(b) *all infinite nodes are terminal ($X \subseteq F(N)$) and the immediate predecessor function $p : F(N) \to X$ satisfies that for all $x \in F(N) \setminus \{W\}$*

$$x \subset p(x) \text{ and if } x \subset y \in N \text{ then } p(x) \subseteq y \subseteq \cup_{t=1}^{\infty} p^t(x) \tag{7}$$

where $p^1 = p$, $p^t = p \circ p^{t-1}$ for all $t = 2, 3, \ldots,$;
(c) *$X \subseteq F(N)$ and the chain $\uparrow x \cap \downarrow y = \{z \in N \,|x \subseteq z \subseteq y\}$ is finite for all $x, y \in F(N)$.*
(d) *the chain $\uparrow x$ is finite for all $x \in X$.*
Furthermore, if any of the statements (a)–(d) hold, the immediate predecessor function is onto.

Proof "(a) implies (b)": Since discreteness entails down-discreteness, that $X \subseteq F(N)$ follows from Proposition 3. Let p be defined as in (2). Then, $x \subset p(x)$ and if $x \subset y \in N$, then $y \in (\uparrow x) \setminus \{x\}$ implies $p(x) \subseteq y$, for any $x \in F(N) \setminus \{W\}$.

Moreover, for $x \in F(N)$ consider the chain $\left\{p^t(x)\right\}_{t=1}^{\infty}$. Since the tree is up-discrete, it has a maximum $z = p^k(x)$. Therefore, $p(z) = p^{k+1}(x) = p^k(x) = z$ which is a contradiction unless $z = W$. Hence, $\cup_{t=1}^{\infty} p^t(x) = W \supseteq y$ for any $y \in (\uparrow x) \setminus \{x\}$.

"(b) implies (c)": That $X \subseteq F(N)$ is immediate. Thus, let $x, y \in F(N)$ and $x \subset y$. Then $p(x) \subseteq y \subseteq \cup_{t=1}^{\infty} p^t(x)$ by (7). It follows that there must be some $k \ge 1$ such that $p^{k-1}(x) \subset y \subseteq p^k(x)$ (where p^0 is the identity). But by (7) applied to $p^{k-1}(x)$ this implies $y = p^k(x)$. Hence, the chain $\uparrow x \cap \downarrow y$ is contained in the finite chain $\{x, p(x), p^2(x), \ldots, p^k(x) = y\}$ and, therefore, itself finite. If $x, y \in F(N)$ are such that $x = y$, then $\uparrow x \cap \downarrow y = \{x\}$ is finite as well. If $x, y \in F(N)$ are such that $y \subset x$ or $x \cap y = \emptyset$, then $\uparrow x \cap \downarrow y$ is empty, thus finite.

"(c) implies (d)": This follows by choosing $y = W$.

"(d) implies (a)": First notice that, if the tree is trivial, $N = \{W\}$, there is nothing to prove. Hence, suppose it is nontrivial. We have that $W \in F(N)$ by Definition 2. Let $x \in X, x \neq W$. Since $\uparrow x$ is finite, it follows that the node x is finite, thus $X \subseteq F(N)$. Proposition 3 implies that the tree is down-discrete. To prove up-discreteness, let $h \in 2^N$ be any chain in N. If h contains no move, then h contains only one element which is its maximum. If there is a move $x \in h$, the chain $\uparrow x$ is finite by hypothesis. Enumerate its elements, $x = x_0 \subset x_1 \subset x_2 \subset \cdots \subset x_m = W$, and let t be the largest integer such that $x_t \in h$. We claim that x_t is a maximum for h. Choose any $z \in h$. If $z \subseteq x$ then $z \subseteq x = x_0 \subseteq x_t$. If $x \subset z$ there is $0 < \tau \leq m$ such that $z = x_\tau$ and, therefore, $z = x_\tau \subseteq x_t$. It follows that x_t is a maximum for h.

This proves the equivalence of (a)–(d). Now, we prove that the immediate predecessor function p is onto. Because discreteness entails up-discreteness, for any $x \in X$, there exists a maximum y for a maximal chain in $(\downarrow x) \setminus \{x\}$ (where the latter is nonempty by $x \in X$).[5] Note that then $x = \min(\uparrow y) \setminus \{y\}$ and hence $y \in F(N)$. Since $y \subset x$, it follows from (b) that $p(y) \subseteq x$. If $p(y) \neq x$, then $p(y) \in (\downarrow x) \setminus \{x\}$ with $y \subset p(y)$ contradicts the construction of y. Therefore, $p(y) = x$ and, because $x \in X$ was arbitrary, p is surjective.

The hypothesis of a regular game tree is necessary for Theorem 1. This is illustrated by Example 5 ("Twins"), which satisfies both (b) and (c) of Theorem 1 but is not regular and, hence, not down-discrete (by the "only if"-part of Proposition 3). The following example illustrates that up-discreteness is responsible for the very last part of (7), viz. that the root can be reached from any move by iterating the immediate predecessor function.

Example 7 (Augmented Inverse Infinite Centipede) Consider the negative integers augmented by a "smallest" element $-\infty$ and call this set W. The corresponding W-centipede is a down-discrete game tree. All nodes are finite (because $(\uparrow x_t) \setminus \{x_t\} = \{x_\tau\}_{\tau=t-1}^{-\infty}$ and $(\uparrow \{t\}) \setminus \{t\} = \{x_\tau\}_{\tau=t}^{-\infty}$) and immediate predecessors are given by $p(\{t\}) = x_t$ and $p(x_t) = x_{t-1}$ for all $t = -1, -2, \ldots,$ But, if the predecessor function is iterated from any node, the root is never reached, i.e., $\cup_{k=1}^{\infty} p^k(x) \subset W$, even though $p(\{-\infty\}) = W$ (that is, even though the immediate predecessor function is surjective). This is so, because no maximal chain in $(\downarrow W) \setminus \{W\}$ has a maximum.

Most textbooks, of course, simply use trees where all plays have finite length (contain finitely many elements). Say that a game tree $(N, \supseteq)$ has *finite height* if every chain in N has a maximum and a minimum. This definition has no implications on the number of choices available at any given move (i.e. on the number of successors of a given move). But it does restrict the lengths of plays. For, if there were an infinite

[5] This only requires "weak up-discreteness" (see AR2, Definition 2.2).

play for a game tree $(N, \supseteq)$ with finite height, then this play would have to have a minimum $x \in N$ by definition. Since a game tree with finite height is clearly discrete, Theorem 1 (c) would then imply that the chain $\uparrow x \cap \downarrow W$ is finite; but since x is the minimum of the play, $\uparrow x \cap \downarrow W$ equals the play—a contradiction. Hence, for a game tree with finite height, all plays have finite length. Conversely, a game tree, where all plays have finite length, is clearly a game tree with finite height.

Moreover, for game trees with finite height, all nodes must be finite. For, if all plays have finite length, then all chains of the form $(\uparrow x) \setminus \{x\}$ (for a node $x \in N$) also do. Hence, all these chains have minima, implying that $N = F(N)$.[6] Consequently, by Theorem 1(b), the immediate predecessor function is defined on *all* nodes for game trees with finite height.

3.3 Discrete extensive forms

In this subsection, we show that if one restricts attention to games on discrete trees, a much simpler definition of an EDP can be obtained that is still equivalent to the original one (Definition 1) for this class of trees. The key step is to replace the three properties (EDP.ii), (EDP.iii), and (EDP.iv) by a single, but equivalent condition.

Definition 6 A **discrete extensive form** (DEF) is a pair (T, C), where $T = (N, \supseteq)$ is a discrete game tree with set of outcomes W and $C = (C_i)_{i \in I}$ a system consisting of collections C_i (the sets of players' choices) of nonempty unions of nodes (hence, sets of outcomes) for all $i \in I$, such that

(DEF.i) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$, then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ for all $i \in I$;

(DEF.ii) $p^{-1}(x) = \left\{x \cap \left(\cap_{i \in J(x)} c_i\right) \,\middle|\, (c_i)_{i \in J(x)} \in A(x)\right\}$, for all $x \in X$;

where $A(x) = \times_{i \in J(x)} A_i(x)$, $A_i(x) = \{c \in C_i \,|x \in P(c)\}$ are the choices available to $i \in I$ at $x \in X$, and $J(x) = \{i \in I \,|A_i(x) \neq \emptyset\}$ is the set of decision makers at x, which is required to be nonempty, for all $x \in X$.

(DEF.i) is identical to (EDP.i), that is, the "information set property" that players cannot deduce from the available choices at which move in the information set they are. The second property is also fairly intuitive. (DEF.ii) says that, at any given move, the combined decisions of the relevant players lead to a new node, and that any (immediate) successor of the move can be selected by an appropriate combination of players' decisions. (The intersection with the move $x \in X$ is needed because of potentially large information sets.)

Theorem 2 *Let $T = (N, \supseteq)$ be a discrete game tree with set of outcomes W and $C = (C_i)_{i \in I}$ a system consisting of collections C_i of nonempty unions of nodes for all $i \in I$. The pair (T, C) is an EDP if and only if it is a DEF.*

Proof What needs to be shown is that (EDP.ii), (EDP.iii), and (EDP.iv) are equivalent to (DEF.ii), because (DEF.i) is precisely (EDP.i).

[6] The converse is not true, not even in the class of discrete trees. All nodes can be finite, even though some plays do not have finite length. This is, for instance, the case in Example 3, when the node $\{\infty\}$, but not the outcome ∞, is removed from the tree.

"if:" Consider some $x \in X$ and $(c_i)_{i\in J(x)} \in A(x)$. By hypothesis there is some $y \in (\downarrow x) \setminus \{x\}$ such that $p(y) = x$ and $y = x \cap \left(\cap_{i\in J(x)} c_i\right)$. As $y \neq \emptyset$, (EDP.ii) holds.

Next, consider $y, y' \in N$ such that $y \cap y' = \emptyset$. We claim that $\uparrow y \cap \uparrow y'$ has a minimum. If either $y \in X$ or $y' \in X$, this follows from Theorem 1(d). Hence, suppose $y, y' \in E$. If there is $z \in (\uparrow y) \setminus \{y\}$ such that $z \notin \left(\uparrow y'\right) \setminus \left\{y'\right\}$, then $z \in X$ and $\uparrow y \cap \uparrow y' \subseteq \uparrow z$ and the desired conclusion follows, because the latter chain is finite by Theorem 1(d). Therefore, we are left with the case $(\uparrow y)\setminus\{y\} \subseteq \left(\uparrow y'\right)\setminus\left\{y'\right\}$. Analogously, we obtain the reverse inclusion, that is, we can assume $(\uparrow y) \setminus \{y\} = \left(\uparrow y'\right) \setminus \left\{y'\right\}$. But that implies $y, y' \in S(N)$ in contradiction to Proposition 3.

Let $x = \min \uparrow y \cap \uparrow y'$, and note that $y \subset x$ and $y' \subset x$. By Proposition 1(b) there are $z, z' \in p^{-1}(x)$ such that $y \subseteq z$ and $y' \subseteq z'$. Let $(c_i)_{i\in J(x)}, \left(c_i'\right)_{i\in J(x)} \in A(x)$ be such that $x \cap \left(\cap_{i\in J(x)} c_i\right) = z$ and $x \cap \left(\cap_{i\in J(x)} c_i'\right) = z'$. Because $z \cap z' = \emptyset$, there is some $i \in J(x)$ such that $c_i \neq c_i'$. By (DEF.i) $c_i \cap c_i' = \emptyset$, which completes the verification of (EDP.iii).

Let $x \supset y \in N$. By Proposition 1(b), there exists $z \in p^{-1}(x)$ such that $y \subseteq z$. Let $(c_i)_{i\in J(x)} \in A(x)$ be such that $z = x \cap \left(\cap_{i\in J(x)} c_i\right)$. Then $y \subseteq c_i$ for all $i \in J(x)$ which verifies (EDP.iv).

"only if:" First, we claim that for any $x \in X$ and $(c_i)_{i\in J(x)} \in A(x)$ the intersection $x \cap \left(\cap_{i\in J(x)} c_i\right)$ is an element of $p^{-1}(x)$.

By (EDP.ii) there is $w \in W$ such that $w \in x \cap \left(\cap_{i\in J(x)} c_i\right)$. By Proposition 1(a), there exists $y \in p^{-1}(x)$ with $w \in y$. By (EDP.iv) there are $c_i'' \in A_i(x)$ such that $y \subseteq c_i''$ for all $i \in J(x)$. Since $w \in y \cap \left(\cap_{i\in J(x)} c_i\right)$, (EDP.i) implies that $c_i'' = c_i$ for all $i \in J(x)$. Therefore, $y \subseteq x \cap \left(\cap_{i\in J(x)} c_i\right)$.

Suppose there is $w' \in \left(x \cap \left(\cap_{i\in J(x)} c_i\right)\right) \setminus y$. Then by the same argument as before there is $y' \in p^{-1}(x)$, disjoint from y by Proposition 1(a), such that $w' \in y' \subseteq x \cap \left(\cap_{i\in J(x)} c_i'\right)$ for some $\left(c_i'\right)_{i\in J(x)} \in A(x)$. Therefore, by (EDP.i) the choices c_i and c_i' must be disjoint for all $i \in J(x)$. This contradicts the construction of w'. Therefore, $y = x \cap \left(\cap_{i\in J(x)} c_i\right)$, completing the proof of the claim.

This shows that $\left\{x \cap \left(\cap_{i\in J(x)} c_i\right) \middle| (c_i)_{i\in J(x)} \in A(x)\right\} \subseteq p^{-1}(x)$. To prove the reverse inclusion, consider some $y \in p^{-1}(x)$. By (EDP.iv) there are choices $c_i \in A_i(x)$ such that $y \subseteq c_i$ for all $i \in J(x)$. Hence, $y \subseteq x \cap \left(\cap_{i\in J(x)} c_i\right)$. Since $x \cap \left(\cap_{i\in J(x)} c_i\right) \in p^{-1}(x)$ by the previous claim, it follows from Proposition 1(a) that $y = x \cap \left(\cap_{i\in J(x)} c_i\right)$.

Theorem 2 captures the following quasi-operational interpretation of an extensive form. At every move $x \in X$ each player $i \in J(x)$ is told (by an "umpire") which choices $c \in C_i$ she has available (in the sense that $x \in P(c), c \in C_i$) and asked to select one of those. No other information is released to players. Given the decisions by all players in $J(x)$, taking the intersection gives a node, which becomes the new "state" of the game.

The simplification associated with the transition from an EDP to a DEF may come at a cost. Clearly, transfinite and continuous-time examples are excluded. But other than that nothing is lost. This is the content of the next result.

Theorem 3 *If (T, C) is a DEF, then (A1) and (A2) hold.*

Proof By Proposition 3, the tree T is regular. Since (T, C) is a DEF, the tree T is up-discrete by definition. Hence, (A1) follows from Theorem 3 in AR2.

Property (A2) will follow from Theorem 5 and Proposition 7 of AR2 if (T, C) fulfills two properties introduced in that paper (see also Alós-Ferrer et al., forthcoming). The first one, (EDP.ii') (see Proposition 7 in AR2) is fulfilled by Corollary 3 and Proposition 9 in AR2. The second, selectiveness (see Definition 6.2 in AR2) follows from Proposition 6(b) and Lemma 5 in AR2. The first states that a regular tree is selective if every history has a continuation with a maximum. The second implies that the latter property follows from up-discreteness.

This theorem shows that DEFs satisfy the basic requirements for a well-founded interpersonal decision theory. Furthermore, the definition of DEF is easier to verify than that of an EDP because the four properties collapse to two.

Another simplification is already implicit in the definition of an EDP (Definition 3). The original definition of extensive forms games by Selten (1975) required that the set of decision makers at any move contains exactly one player. In this case (DEF.ii) reduces to $p^{-1}(x) = \{x \cap c \,|c \in A(x)\}$ for any $x \in X$. For simultaneous decisions this leads to cascading information sets. This is allowed but not necessary in the present framework, which encompasses games where several players may decide at the same move.[7] This, for instance, simplifies the extensive form representation of an infinitely repeated game. Furthermore, consider the following examples.

Example 8 (Nonatomic games) Let the player set be a continuum, e.g., $I = [0, 1]$. All players choose simultaneously from a set A with at least two elements. This can be formalized by taking as the set of outcomes W the set of functions $f : [0, 1] \to A$. Each player $i \in I$ faces the choices $c_i(a) = \{f \in W \,|\, f(i) = a\}$ for all $a \in A$. It is straightforward to check that (DEF.i) and (DEF.ii) are fulfilled. That is, nonatomic games are discrete games. The traditional cascading information sets would give rise to a tree of staggering complexity, when compared to the simplicity of the formalization proposed here.[8]

Example 9 Spence's job-market signalling model (Spence 1973) serves as an example on how the present approach may simplify the analysis of equilibria. In that model chance initially assigns a productivity (type) $\theta \in \{\theta_L, \theta_H\}$, where $0 < \theta_L < \theta_H$, to

[7] An alternative formalization that allows several players to choose at the same move has been introduced by Selten (1999).

[8] More generally, any normal-form game is automatically associated to a natural (trivial) extensive form. For games with a continuum of players (see e.g., Mas-Colell 1984; Carmona 2009), of course, other problems remain. For instance, straightforward aggregation of a continuum of independent mixed strategies is prevented by the failure of the law of large numbers for such a framework. This problem can be bypassed in the very definition of the mixed extension of the game, by e.g., changing the concept of aggregation as in Uhlig (1996). Khan and Sun (1999) proposed abandoning the standard continuum framework in favor of a hyperfinite agent set. More recently, Podczeck (2010) shows that the law of large numbers holds in appropriately extended probability spaces, without the recourse to constructions from nonstandard analysis. A closely related problem is the rigorous mathematical foundation of models involving random matching of a continuum of agents (Alós-Ferrer 1999; Podczeck and Puzzello forthcoming).

a potential "worker". The prospective employee then decides on an education level $e \in \mathbb{R}_+$ in order to signal her productivity. This signal, but not the productivity, is observed by "a competitive industry" which then offers a wage $w \in [\theta_L, \theta_H]$. This wage is equal to the expected productivity, given some beliefs conditional on the observed signal. To complete the specification of an extensive form game, assume two firms, who compete à la Bertrand by offering wages to the worker, who in turn chooses among wage offers. In any subgame perfect equilibrium, the worker will choose the highest offer, and Bertrand competition will imply that both firms offer a wage equal to expected productivity.

This guarantees that the productivities expected by the two firms are equal along the equilibrium path. The natural equilibrium concept for this game, perfect Bayesian equilibrium (PBE), does not force the two firms to hold the same beliefs *off* the equilibrium path, though. The stronger criterion of sequential equilibrium (Kreps and Wilson 1982)—that does not apply to this game, since it is only defined for finite games—by contrast would imply equal beliefs across firms even at unreached information sets—the "common belief property." But PBE does not.[9]

This problem is complicated when simultaneous moves are captured by cascading information sets; see e.g., the graphical representation by Mas-Colell et al. (1995, p. 451). For this example, both firms move simultaneously after observing education level e. For the classical representation, outcomes are ordered four-tuples $(\theta, e, w_1, w_2) \in \{\theta_L, \theta_H\} \times \mathbb{R}_+ \times [\theta_L, \theta_H] \times [\theta_L, \theta_H] = W$, assuming that firm 1 "moves first" and ignoring the last (trivial) decision by the worker. When $x_t(\overline{e}) = \{w \in W \,|\theta = \theta_t, e = \overline{e}\}$ denotes the node where type $t = L, H$ has chosen $\overline{e} \in \mathbb{R}_+$, firm 1's beliefs are $\mu_1(\overline{e}) = (\mu_1(x_L(\overline{e})), \mu_1(x_H(\overline{e}))) \in \Delta^1$, where Δ^1 is the one-dimensional simplex. Firm 2 "moves second" at her (infinite) information set $g_2(\overline{e}) = \{y_t(\overline{e}, \overline{w}_1) \,|t = L, H, \overline{w}_1 \in [\theta_L, \theta_H]\}$, with $y_t(\overline{e}, \overline{w}_1) = \{w \in W \,|\theta = \theta_t, e = \overline{e}, w_1 = \overline{w}_1\}$ denoting the node reached after type $t = L, H$ has chosen $\overline{e} \in \mathbb{R}_+$ and firm 1 has offered $\overline{w}_1 \in [\theta_L, \theta_H]$. Her beliefs are then given by a probability measure $\mu_2(\cdot \,|\overline{e}) : g_2(\overline{e}) \to \mathbb{R}_+$.

This specification is void of economic content, but complicates the "common belief property," because it becomes necessary in a PBE to specify beliefs of firm 2 on the wage offered by firm 1 (and, out of equilibrium, nothing pins down such beliefs). Common beliefs are captured by the statement that the probability mass $\mu_2(\{y_t(\overline{e}, \overline{w}_1) \,|\overline{w}_1 \in [\theta_L, \theta_H]\} \,|\overline{e})$ numerically equals the probability $\mu_1(x_t(\overline{e}))$, for $t = L, H$ for all $\overline{e} \in \mathbb{R}_+$. The beliefs μ_1 and μ_2 are radically different formal objects!

Since the present formulation allows us to replace cascading information sets by players deciding at the same move, the natural representation is much simpler. Outcomes become triplets $(\theta, e, (w_1, w_2)) \in \{\theta_L, \theta_H\} \times \mathbb{R}_+ \times [\theta_L, \theta_H]^2 = W$ and all nodes $x_t(\overline{e})$ are directly followed by terminal nodes (the nodes $y_t(\overline{e}, \overline{w}_1)$ are dispensed with). The information sets $g(\overline{e}) = \{x_t(\overline{e}) \,|t = L, H\}$ (common to both firms) consist

[9] Admittedly, there is no universally accepted definition of PBE for general extensive form games. Mas-Colell et al. (1995, p. 450) avoid this problem by incorporating the equality of beliefs in their definition of PBE for its use in signaling games of the Spence type. From the theoretical point of view this is, of course, unsatisfactory.

of two nodes, beliefs for both firms are $\mu_1(\overline{e}), \mu_2(\overline{e}) \in \Delta^1$, and the common beliefs property is captured by the statement $\mu_1(\overline{e}) = \mu_2(\overline{e})$ for all $\overline{e} \in \mathbb{R}_+$.

Another implication of Theorem 3 is that DEFs can be represented in normal form (without payoffs). This raises issues about equivalences between DEFs.

Thompson (1952) introduced four transformations that define equivalence relations on the set of finite extensive form games (see also Dalkey 1953).[10] He showed that these transformations jointly characterize another equivalence relation, given by agreement of the semi-reduced (or pure-strategy reduced) normal form. An important objective for future research is to extend such a characterization to DEFs.

A first difficulty with such an endeavor is that the proofs use finite recursive algorithms. A second, and probably more severe, obstacle is that one of Thompson's transformations, "deletion of irrelevant decision points," relies on payoffs. Then, the best that one can hope for is a characterization of the equivalence relation on DEFs given by agreement of triplets of the form (S, W', ϕ), where $S = \times_{i \in I} S_i$ is the space of pure strategy combinations, W' is a quotient space of outcome equivalence classes, and $\phi : S \to W'$ a surjection. (Such a triplet may be called a "normal form," deliberately dropping the word "game.") The conceptual difficulty with such an approach is that the equivalence relation on outcomes would have to imply that all players are indifferent between outcomes in one equivalence class. And this is a statement about the players' preferences, and not about the representation. On the other hand, no game is complete without a specification of the players' preferences. Therefore, studying Thompson transformations in the context of games represented by DEFs remains on our agenda.

4 Discrete games when nodes are primitives

Traditional formulations for extensive form games rely on representing the tree by a graph. It turns out that this is encompassed by the present framework. In this section, we aim to show how DEFs translate into a graph representation. However, a few subtleties appear in the transition to a graph representation. Those have to do with the role of terminal nodes.

In a game tree, the set of plays can be identified with the set of underlying outcomes, thus making it the natural domain for preferences or payoff functions. Of course, for trees of finite height, plays are one-to-one with the set of terminal nodes. On the other hand, for infinite-horizon games, this equivalence may break down, since it is not guaranteed that all plays correspond to terminal nodes. For example, in the infinite centipede (Example 3), the infinite terminal node $\{\infty\}$ could be removed without affecting the structure of the game tree. This implies that there is some freedom in how one constructs the graph representation. To find a simple representation, one needs to know how far the graph can be "pruned" without affecting the structure of the game. This is what we study in the next subsection.

[10] One of Thompson's transformations, "inflation/deflation," does not preserve perfect recall. This has prompted Elmes and Reny (1994) to drop it and reformulate another one so that perfect recall is maintained.

4.1 Singletons and plays

Recall that a node is terminal if and only if it is a singleton (Lemma 3(b) in the Appendix). This implies that $X = N \setminus \{\{w\}\}_{w \in W}$ is an alternative definition of (the set of) moves for a game tree. This does not mean, however, that $\{w\} \in N$ for all $w \in W$. The intersection of the elements of an infinite chain of nodes may contain only a single element $w \in W$, even though the singleton $\{w\}$ does not belong to N. It was shown in AR1 (Proposition 10) that the singletons from the underlying set W can be *added* (e.g., for repeated games) to the set of nodes, as "terminal nodes," without affecting the structure of the tree. This is clearly the case in the infinite centipede where, if the node $\{\infty\}$ were absent, it could be added without changing the set of plays/outcomes. Define a *complete* game tree as one where $\{w\} \in N$ for all $w \in W$. That is, any game tree can be "completed" by adding singletons.

In Examples 3, resp. 4 the infinite terminal nodes $\{\infty\}$ resp. $\{1\}$ could be *removed* without affecting the structure of the tree. Example 5, though, features terminal nodes that cannot be removed without changing the tree. This is so, because the terminal nodes $\{0\}$ and $\{1\}$ in Example 5 are not infinite nodes.

Finite and strange nodes cannot be removed. For, any node $x \in F(N) \cup S(N)$, $x \neq W$, satisfies $x \subset \cap_{y \in (\uparrow x) \setminus \{x\}} y$ (by definition if it is finite or by Lemma 4(c) in the Appendix if it is strange), so that there are $w \in x$ and $w' \in \left(\cap_{y \in (\uparrow x) \setminus \{x\}} y\right) \setminus x$. Therefore, if $x \in (S(N) \cup F(N)) \cap E$ were removed, then in the resulting tree $w \in y$ would imply $w' \in y$ (because $y \in (\uparrow x) \setminus \{x\}$ in the original tree) for all $y \in N \setminus \{x\}$ in contradiction to (GT.ii).[11] Hence, if at all, only infinite terminal nodes may be removed without harm.

The following result states that infinite terminal nodes can indeed be *removed* from a game tree without affecting its structure. The resulting tree has the same order-theoretic structure as the original tree and essentially the same plays. Furthermore, if the resulting set tree is completed by adding the now-missing singletons, the original game tree reemerges. In other words, for game trees infinite terminal nodes are *precisely* those, the presence or absence of which is immaterial to the structure of the tree.

Proposition 4 *If $(N, \supseteq)$ is a complete game tree with set of outcomes W, then for every set of infinite terminal nodes $Y \subseteq E \cap I(N) \equiv E_I$ the partially ordered set $(N \setminus Y, \supseteq)$ is a game tree with set of outcomes W', the mapping $\Upsilon : W \to W'$ given by $\Upsilon(w) = w \setminus Y$ is bijective, and if $(N \setminus Y, \supseteq)$ is completed by adding all singletons, the result is the complete game tree $(N, \supseteq)$.*

4.2 Simple trees

Proposition 4 shows that infinite terminal nodes may be suppressed without affecting the structure of the tree. Doing so allows us to go further and give a definition of discrete trees which uses nodes as primitives. Such a definition of a "simple tree" as

[11] In order to recover (GT.ii), one would need to remove also the element w from the underlying set W. But this, of course, changes the structure of the tree by changing the set of outcomes/plays.

an ordered set of nodes relies on two additional facts. First, it is known from AR1 that trees as partially ordered sets give rise naturally to trees as collections of subsets of an underlying set (of outcomes/plays). Second, Theorem 1(d) yields that discreteness can be characterized by finiteness of the sets of predecessors of moves. (In the following definition the symbol $\geq$ denotes the predecessor relation.)

Definition 7 A **simple tree** $T = (N, \geq)$ is a partially ordered set (the elements of which are called nodes) with a maximum $x_0 \in N$ (called the root) such that

(ST.i) for all $x \in N$, the set $\uparrow x = \{y \in N \mid y \geq x\}$ is a finite chain;
(ST.ii) for all $x, y \in N$, if $x > y$ then there exists $z \in N$ such that $x > z$ and neither $y \geq z$ nor $z \geq y$;

where $>$ denotes the strict order derived from $\geq$ (i.e. $x > y \Leftrightarrow x \geq y \neq x$).

For a simple tree, the set of moves is given by $X = \{x \in N \mid \exists\, y \in N : x > y\}$. The following result clarifies in which sense the above definition is equivalent to the concept that has been developed in the previous Section.

Proposition 5 (a) *Let $T = (N, \geq)$ be a simple tree and W the set of plays (maximal chains) for T. If $W(N) = \{W(x)\}_{x \in N}$ is the collection of all sets $W(x) = \{w \in W \mid x \in w\}$ with $x \in N$, then $T' = (W(N), \supseteq)$ is a discrete game tree (with underlying set W) that is order-isomorphic*[12] *to T with all nodes finite. Further, $(W(N) \cup \{\{w\} \mid w \in W\}, \supseteq)$ is a complete game tree (which may have infinite terminal nodes).*
(b) *Let $T' = (N, \supseteq)$ be a discrete game tree, and let $F(N)$ be its set of finite nodes. Then, $T = (F(N), \supseteq)$ is a simple tree.*

In summary, the transition from simple trees as defined above to (complete) discrete game trees is a mere modeling decision. The only relevant difference is that, in simple trees, infinite terminal nodes are not included.

The order relation $\geq$ in Definition 7 cannot directly be replaced by set inclusion, because the resulting object might not be a game tree. The reason is that, with such a definition, nodes may contain elements which are irrelevant in the sense that they do not correspond to any play.[13]

Most textbook expositions use trees where, as in simple trees, nodes are taken as the primitives. An exception is Ritzberger (2002), where discrete trees are used with nodes as sets of outcomes as in Definition 1. A prominent example for simple trees in a textbook treatment are the trees used by Osborne and Rubinstein (1994).

Example 10 (Osborne-Rubinstein trees) Let A be an arbitrary set of "actions" and Z a set of (finite or infinite) sequences from A such that (a) $\emptyset \in Z$, (b) if $(a_\tau)_{\tau=1}^{t} \in Z$ (where t may be infinity) and $k < t$, then $(a_\tau)_{\tau=1}^{k} \in Z$, and (c) if an infinite sequence $(a_\tau)_{\tau=1}^{\infty}$ satisfies $(a_\tau)_{\tau=1}^{t} \in Z$ for every positive integer t, then $(a_\tau)_{\tau=1}^{\infty} \in Z$.

[12] This means that $x \geq y$ if and only if $W(x) \supseteq W(y)$, for all nodes $x, y \in N$.

[13] The resulting tree fails to be its own set representation by plays in the sense of AR1, so that $(W(N), \supseteq)$ might have a simpler structure than T.

The set W of outcomes is given by $W = W_F \cup W_\infty$ where W_F is the set of maximal and finite sequences in Z and W_∞ is the set of infinite sequences in Z.

The ordering on nodes is given by $(a_\tau)_{\tau=1}^{t} \geq \left(a'_\tau\right)_{\tau=1}^{k}$ if $t \leq k$ and $a_\tau = a'_\tau$ for all $\tau = 1, \ldots, t$, and $\emptyset \geq z$ for all $z \in Z$. Then, $(Z, \geq)$ is a (rooted) tree, satisfying (ST.i). It may fail (ST.ii), though, because a given node may only be followed by a single other node.[14] This can be fixed by adding the following condition: (d) If $(a_\tau)_{\tau=1}^{t} \in Z$, then there exists $a'_t \neq a_t$ such that $\left((a_\tau)_{\tau=1}^{t-1}, a'_t\right) \in Z$. Under this condition, $(Z, \geq)$ is a simple tree.

Let N be formed by the singletons $\{\{w\}\}_{w \in W_\infty}$ together with all sets of the form $W(z) = \{w \in W \,|z \subseteq w\}$ for all finite sequences $z \in Z$. Then $(N, \supseteq)$ is order-isomorphic to $(Z, \geq)$ and a discrete game tree. (Notice that $W(\emptyset) = W$.)

For a simple tree $(N, \geq)$ the immediate predecessor of a node $x \in N \setminus \{x_0\}$ is given by $\hat{p}\,(x) = \min\,(\uparrow x) \setminus \{x\}$, in analogy to the definition of immediate predecessor in game trees. The immediate predecessors of a set $c \subseteq N$ of nodes are given by $\hat{p}(c) = \left\{\hat{p}(x)\,|x \in c\right\}$. Accordingly, $\hat{p}^{-1}\,(c) = \cup_{x \in c}\hat{p}^{-1}\,(x)$ for a set c of nodes. Moves $x \in X$ are defined by the condition $\hat{p}^{-1}\,(x) \neq \emptyset$.

For a set of nodes c in a simple tree, $W(c)$ denotes the image of c under the mapping $W(\cdot)$, i.e. $W(c) = \{W(x)\,|x \in c\}$ is a set of nodes in the set representation $(W\,(N)\,, \supseteq)$ as given by Proposition 5(a). In order to distinguish it from the union, $V(c) = \cup_{x \in c} W(x)$ denotes the union.

Definition 8 A **simple extensive form** with player set I is a pair (T, C), where $T = (N, \geq)$ is a simple tree and $C = (C_i)_{i \in I}$ consists of collections C_i of sets of nodes such that

- (SF.i) if $\hat{p}\,(c) \cap \hat{p}\left(c'\right) \neq \emptyset$ and $c \neq c'$, then $\hat{p}\,(c) = \hat{p}\left(c'\right)$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ and all $i \in I$;
- (SF.ii) $\left\{\{y\}\,\middle|y \in \hat{p}^{-1}\,(x)\right\} = \left\{\hat{p}^{-1}\,(x) \cap \left(\cap_{i \in J(x)} c_i\right)\,\middle|(c_i)_{i \in J(x)} \in A\,(x)\right\}$, for all $x \in X$;
- (SF.iii) no choice $c \in C_i$ contains two distinct ordered nodes;
- (SF.iv) for every move $x \in X$ and every choice $c \in C_i$, $\hat{p}^{-1}(x) \setminus c \neq \emptyset$;

where $A\,(x) = \times_{i \in J(x)} A_i\,(x)\,, A_i\,(x) = \left\{c \in C_i\,\middle|x \in \hat{p}\,(c)\right\}$ are the choices available to $i \in I$ at $x \in X$, and $J\,(x) = \{i \in I\,|A_i\,(x) \neq \emptyset\}$ is the set of decision makers at x, which is required to be nonempty, for all $x \in X$.

(SF.i) is the direct translation of (DEF.i), the "information set property." (SF.ii) is the version with nodes of (DEF.ii). Properties (SF.iii) and (SF.iv) are required here because now choices are not sets of outcomes anymore. If they were sets of outcomes (as in Definition 3) such conditions would be unnecessary.

The combination of properties (SF.i) to (SF.iii) implies Kuhn's exclusion of absent-mindedness (Kuhn 1953). Absent-mindedness allows the "state" that obtains (the move in the information set at which the player is called upon to choose) to depend on the decision maker's choice.

[14] In the terminology of AR1, Osborne-Rubinstein trees may not be *decision trees*.

Lemma 1 *Let $T = (N, \geq)$ be a simple tree and let (T, C) be a simple extensive form. If $c \in C_i$ is a choice, then the information set $\hat{p}(c)$ does not contain two distinct ordered nodes.*

A property like (SF.iii) would be redundant if choices were sets of outcomes. For, in that case, if a choice covers the outcomes in a node, then it must cover all outcomes in all successors thereof. This is not the case when choices are sets of nodes.

The proof of Lemma 1 does not use property (SF.iv). The latter is a non-triviality requirement, which forbids that a choice contains all immediate successors of a given move. In such a situation, if choices were sets of outcomes, the choice would have to include that move, and hence the problem cannot arise.

Property (SF.iv) is necessary for the next Lemma, which allows us to translate the choices from a simple extensive form on a simple tree $T = (N, \geq)$ into choices in the set representation by plays $(W(N), \supseteq)$ given by Proposition 5(a). In this representation, nodes are sets of plays (maximal chains) of T, that is, the representation has the set of plays of T as the underlying set. For sets of plays a that are unions of nodes of $W(N)$, Proposition 2 gives the decomposition $N(a)$ that consists of nodes in $W(N)$ that are maximal in a (see (6)).

Lemma 2 *Let $T = (N, \geq)$ be a simple tree, W the associated set of plays (maximal chains), and (T, C) a simple extensive form. If $c \in C_i$ is a choice, then*

(a) $N(V(c)) = \{W(x) \,|x \in c\}$, *where* $V(c) = \cup_{x \in c} W(x)$;
(b) $P(V(c)) = W\left(\hat{p}(c)\right)$, *where* $W(Y) = \{W(y) \,|y \in Y\}$ *for any* $Y \subseteq N$.

The following result states that simple extensive forms are the translation of EDPs from discrete game trees to the corresponding simple trees.

Proposition 6 (a) *Let (T, C) be a simple extensive form with player set I and $T' = (W(N), \supseteq)$ the set representation given by Proposition* 5(a). *If $C' = \left(C'_i\right)_{i \in I}$ with $C'_i = \{V(c) \,|c \in C_i\}$ for all $i \in I$, then $\left(T', C'\right)$ is an EDP on a discrete game tree.*

(b) *Let $\left(T', C'\right)$ be an EDP with player set I on a discrete game tree $T' = (N, \supseteq)$ and $T = (F(N), \supseteq)$ the associated simple tree from Proposition* 5(b). *If $C = (C_i)_{i \in I}$ with $C_i = \left\{N\left(c'\right) \cap F(N) \,\middle|c' \in C'_i\right\}$ for all $i \in I$, where $N(c')$ is as in Proposition 2, then (T, C) is a simple extensive form.*

This final result allows one to go back and forth between an extensive form representation with a simple tree, where nodes are the primitives, and one with a discrete game tree, where plays/outcomes are the primitives.

Example 11 Reconsider Example 9, Spence's job-market signalling model. Now the nodes are the primitives, hence identify plays/outcomes $w \in W$ with terminal nodes $z\,(\theta, e, (w_1, w_2)) \in Z$, denote the root by x_0, and keep the notation $x_t\,(e)$ for the firms' moves. Choices by firms are now sets of nodes, $c_1\,(\overline{w}_1) = \{z \in Z \,|w_1 = \overline{w}_1\}$ and $c_2\,(\overline{w}_2) = \{z \in Z \,|w_2 = \overline{w}_2\}$. Choices for the worker of type $t = L, H$ are the singletons $c_t\,(\overline{e}) = \{x_t\,(\overline{e})\}$. The verification of conditions (SF.i-iv) is straightforward.

5 Conclusions

Many applications of extensive form games to economics involve large action spaces and/or a long time horizon. The traditional definitions of extensive forms, on the other hand, are restricted to finite games. While some of the theoretical constructs, e.g., belief-based refinements of Nash equilibria, depend on finiteness, we argue that the basic objects of extensive form analysis do not. In this paper, we propose a self-contained definition of extensive forms that encompasses all commonly used generalizations (with the exception of transfinite games and continuous-time problems), in particular games with continuum action spaces and/or an infinite time horizon.

We show that this definition satisfies the two basic desiderata that every strategy combination induces an outcome and that it does so uniquely. Moreover, the freedom to take plays or nodes as the primitives is preserved.

The appropriate substitute for finiteness turns out to be a discreteness property "along the time axis." This can be characterized by the existence of an immediate predecessor function on the set of moves in the tree. Intuitively, this means that for every decision point, its history can be uniquely reconstructed in a finite number of steps.

An important consequence of this characterization is a dramatic simplification of the formal framework. An extensive form can be fully specified by two objects, the tree and the players' choices, and two consistency conditions: that the decisions of the relevant players at any given move lead to a new node (new state of the game) and that players cannot deduce from available choices where they are in an information set. Despite its simplicity, a "discrete extensive form" imposes no constraints on its practical applicability.

Appendix: Proofs

Lemma 3 *Let $T = (N, \supseteq)$ be a game tree.*[15]

(a) *If $x \in N$ contains at least two distinct elements, then for each $w \in x$ there is a node $x' \in N$ such that $w \in x' \subset x$;*
(b) *a node $x \in N$ is terminal if and only if there is $w \in W$ such that $x = \{w\}$.*

Proof (a) Let $w, w' \in x \in N$ such that $w' \neq w$. By (GT.ii) there is $x' \in N$ such that $w \in x'$, but $w' \notin x'$. As $w \in x \cap x'$, but $w' \in x \setminus x'$, (GT.i) implies that $x' \subset x$. (b) The "if"-part is trivial. The "only if"-part follows from (a).

Lemma 4 *Let $(N, \supseteq)$ be a game tree. A node $x \in N \setminus \{W\}$ is*

(a) *the infimum of a chain $h \in 2^N$ if and only if $x = \cap_{y \in h} y$;*
(b) *infinite if and only if $x = \cap_{y \in (\uparrow x) \setminus \{x\}} y$;*
(c) *strange if and only if $(\uparrow x) \setminus \{x\}$ has no minimum and $x \subset \cap_{y \in (\uparrow x) \setminus \{x\}} y$;*

[15] This result also holds for all irreducible W-set trees, where irreducibility is (GT.ii) from Definition 1 (see AR1 for a definition of W-set trees). Part (b) corresponds to Lemma 1 in AR2.

(d) *Further, x is not infinite (that is, it is either finite or strange) if and only if*

$$\text{for every chain } h \in 2^N \text{, if } x \supseteq \cap_{y\in h} y \text{ then } x \supseteq y \text{ for some } y \in h. \quad (8)$$

Proof Parts (a), (b), and (c) are proved in AR2 (Lemma 2). To see the "if"-part of (d) let x be an infinite node. Consider the chain $h = (\uparrow x) \setminus \{x\}$. By (b), $x = \cap_{y\in h} y$, but $x \subset y$ for all $y \in h$, so that (8) cannot hold. For the "only if"-part suppose that x fails (8). There exists a chain h in N such that $x \supseteq \cap_{y\in h} y$, but there is no $y \in h$ such that $x \supseteq y$. Since $x \supseteq \cap_{y\in h} y$ implies $x \cap y \neq \emptyset$ (because $\cap_{y\in h} y \neq \emptyset$, by the "only if"-part of (GT.i)) for all $y \in h$, it follows from the "if"-part of (GT.i) that $x \subset y$ for all $y \in h$, i.e. $y \in (\uparrow x) \setminus \{x\}$ for all $y \in h$. Thus, $x \supseteq \cap_{y\in h} y \supseteq \cap_{y\in(\uparrow x)\setminus\{x\}} y \supseteq x$. Hence, x is infinite.

Remark 1 A (meet-)CPO is a poset with a top element (a maximum) such that all "directed" sets have an infimum. Alós-Ferrer and Ritzberger (2005a) show that in a tree the only directed sets are the chains. In a CPO finite elements are defined as those elements such that if $x \supseteq \inf h$, then $x \supseteq y$ for some $y \in h$, whenever h is directed. By Lemma 4(a), for game trees the infimum is actually the intersection, giving rise to part (d) of Lemma 4. In game trees the finite elements of CPOs translate into non-infinite nodes. Interestingly, it is possible to show that if a tree is a CPO, then it must be a complete (join-)semilattice (see Alós-Ferrer and Ritzberger 2005a).

Proof of Proposition 1

(a) For every $w \in x \in X$, the set $M_w(x) = \{y \in N \,|w \in y \subset x\}$ is nonempty by Lemma 3(a) and a chain by the "if" -part of (GT.i). By up-discreteness there is $x_w = \max M_w(x)$ for all $w \in x$. If $w, w' \in x$ are such that $x_w \cap x_{w'} \neq \emptyset$, then by construction and (GT.i) $x_w = x_{w'}$. This shows that $M(x) = \{x_w \,|w \in x\}$ is a partition of x with each element maximal among the nodes in $(\downarrow x) \setminus \{x\}$. Note that, if y is maximal among the nodes properly contained in x then necessarily $x = \min(\uparrow y) \setminus \{y\}$, hence, $y \in F(N)$ and $y \in p^{-1}(x) \neq \emptyset$. Conversely, if $y \in p^{-1}(x)$ then $y = x_w$ for any $w \in y$. We conclude that $M(x) = p^{-1}(x)$.

(b) Let $z = \max(\uparrow y) \setminus (\uparrow x)$, which exists by up-discreteness.

Proof of Proposition 2

(a) For any $w \in a$ let $N_w(a) = \{x \in \downarrow a \,|w \in x\}$. By the "if"-part of (GT.i) $N_w(a)$ is a chain that is nonempty, because $a \in A(T)$. By up-discreteness there is $x_w = \max N_w(a)$ for all $w \in a$. If $w, w' \in a$ are such that $x_w \cap x_{w'} \neq \emptyset$, then by construction and (GT.i) $x_w = x_{w'}$. This shows that $N(a) = \{x_w \,|w \in a\}$ is a partition of a with each element maximal among the nodes in $\downarrow a$.

(b) Let $x \in N$ be such that $\uparrow x = (\uparrow y) \setminus (\downarrow a)$ for some $y \in \downarrow a$, that is, $x \in P(a)$ as in (3). It follows that $x \notin \downarrow a$ and $x \in X$, because $y \in (\downarrow x) \setminus \{x\}$. Let $w \in y$ and consider x_w as in (a). By construction, $p(x_w) = x$, implying that $x_w \in F(N)$. Thus, $P(a) \subseteq \{p(x) \,|x \in N(a) \cap F(N)\}$
Let $x_w \in N(a) \cap F(N)$ and $x = p(x_w)$. Then $x_w \in \downarrow a$ and $\uparrow x = (\uparrow x_w) \setminus \downarrow a$, that is $x \in P(a)$. We conclude that $P(a) = \{p(x) \,|x \in N(a) \cap F(N)\}$.

Proof of Proposition 3 Consider a down-discrete game tree. By definition $(\uparrow x) \setminus \{x\}$ always has an infimum, so there are no strange nodes. Moreover, if $(\uparrow x) \setminus \{x\}$ has no minimum, x must be its infimum and, therefore, x is terminal. If, conversely, a game tree is regular and all moves are finite, then for every $x \in X$ there exists min $(\uparrow x) \setminus \{x\}$ by the definition of finite nodes; furthermore, any non-finite node $x \in N \setminus F(N)$ must then be terminal and the infimum of $(\uparrow x) \setminus \{x\}$ by Lemma 4(a) and (b), as there are no strange nodes by regularity.

Proof of Proposition 4 Recall that, since $(N, \supseteq)$ is a (complete) game tree, one can take the set W both as the set of plays and as the underlying set (Theorem 3 of AR1). W is also the underlying set of $(N \setminus Y, \supseteq)$. The "if"-part of (GT.i) for $(N, \supseteq)$ is trivially inherited by $(N \setminus Y, \supseteq)$. If h is a chain in $N \setminus Y$, then it is a chain in N and, therefore, has a lower bound in W; that is, there is $w \in \cap_{x \in h} x$, because $(N, \supseteq)$ is complete. Thus $(N \setminus Y, \supseteq)$ satisfies also the "only if"-part of (GT.i).

To verify (GT.ii), let $v, v' \in W$ be such that $v \neq v'$. By (GT.ii) for $(N, \supseteq)$ there are $x, x' \in N$ such that $v \in x \setminus x'$ and $v' \in x' \setminus x$. If $x, x' \in N \setminus Y$, we are done. If $x \in Y \subseteq E_I$ then by Lemma 3(b) $x = \{v\}$. Since x is infinite, $\{v\} = \cap_{y \in (\uparrow x) \setminus \{x\}} y$ by Lemma 4(b). Thus, there is $z \in (\uparrow x) \setminus \{x\}$ such that $v \in z$ and $v' \notin z$. If $x' \in N \setminus Y$ then $v \in z \setminus x'$ and $v' \in x' \setminus z$, as required. If $x' \in Y \subseteq E_I$ then, analogously, $x' = \left\{v'\right\}$ and there is $z' \in \left(\uparrow x'\right) \setminus \left\{x'\right\}$ such that $v' \in z'$ and $v \notin z'$. Hence, $(N \setminus Y, \supseteq)$ satisfies (GT.ii) and is a game tree.[16]

Let $w' \in W'$. If $w' \in W$, there is a unique play for $(N, \supseteq)$, namely w' itself, such that $\Upsilon(w') = w'$. If $w' \notin W$ then there is $y \in Y$ such that $w' \cup \{y\}$ is a chain for $(N, \supseteq)$. Because $y \in Y \subseteq E_I$, by Lemma 3(b) $y = \{v\}$ for some $v \in W$. Therefore, $v \in x$ for all $x \in w'$, since $w' \cup \{y\}$ is a chain. Since $(N \setminus Y, \supseteq)$ is a game tree, Theorem 3(c) of AR1 implies that $(\uparrow \{v\}) \setminus \{\{v\}\} \in W'$ is a play (because $\{v\} \notin N \setminus Y$), so that $w' \subseteq (\uparrow y) \setminus \{y\}$ and maximality imply $w' = (\uparrow y) \setminus \{y\}$. Because y is infinite, $y = \cap_{x \in w'} x$ by Lemma 4(b), and there can be no other $y' \in Y \setminus \{y\}$ such that $w' \cup \left\{y'\right\}$ is a chain for $(N, \supseteq)$. Therefore, $w = w' \cup \{y\}$ is the unique play for $(N, \supseteq)$ such that $\Upsilon(w) = w'$. Since this shows that every element of W' has a unique preimage under Υ, this mapping is bijective.

Finally, as all terminal nodes are singletons by Lemma 3(b), adding to $(N \setminus Y, \supseteq)$ all singletons must yield the (complete) game tree $(N, \supseteq)$ with all singletons present.

Proof of Proposition 5 (a) Note that $(W(N), \supseteq)$ is the "image in plays" introduced in AR1 (Section 2.4). Conditions (ST.i) and (ST.ii) imply that T is a decision tree (see AR1, Section 2.3) and that $T' = (W(N), \supseteq)$ is order-isomorphic to T. Next, it is shown that T' is a game tree. To see (GT.i), let h' be a chain in $W(N)$. By order isomorphism, there exists a chain h in T such that $h' = \{W(x) \mid x \in h\}$. Since h is contained in some maximal chain $w \in W$ by the Hausdorff Maximality Principle, it follows that $w \in W(x)$ for all $x \in h$, as required by the "only if"-part of (GT.i). To see the "if"-part, let h' be a subset of $W(N)$ such that there exists some $w \in W$ with $w \in W(x)$ for all $W(x) \in h'$. It follows that the set h in T such that $h' = \{W(x) \mid x \in h\}$ is a chain, thus also h' is a chain.

[16] Since the underlying set has not changed, Theorem 3 of AR1 implies that the sets of plays in the new and the former trees are bijective. The present proof, however, is constructive.

Now turn to (GT.ii). Let $w, w' \in W$ with $w \neq w'$. Since w and w' are maximal chains in T, there exist $x, x' \in N$ such that $x \in w$ and $x' \in w'$, but $x \notin w'$ and $x' \notin w$. It follows that $w \in W(x) \setminus W(x')$ and $w' \in W(x') \setminus W(x)$. Thus, (GT.ii) also holds.

It has been shown that T' is a game tree. By the existence of a maximum in T, T' is rooted. By (ST.i) and order isomorphism $\uparrow W(x)$ is finite and hence $W(x)$ is a finite node for all $W(x) \in W(N)$. Thus T' is regular and rooted, and, by Theorem 1, it is discrete.

(b) The existence of a maximum follows from the fact that T' is rooted. (ST.i) follows from (GT.i) and Theorem 1. Finally consider (ST.ii). This condition is Separability in AR1, which is fulfilled in any game tree (see AR1, Lemma 8 and Definition 4). Hence, for $x, y \in F(N)$ with $x > y$ it follows that there exists $z \in N$ with $x > z$ such that neither $y \geq z$ nor $z \geq y$. If $z \notin F(N)$, then by definition it follows that there must exist $z' \in F(N)$ with $x > z' > z$ such that neither $y \geq z'$ nor $z' \geq y$.

Proof of Lemma 1. Suppose $x, y \in \hat{p}(c)$ with $x > y$ for $c \in C_i$. Note that this implies that $i \in J(x) \cap J(y)$. The chain $(\uparrow y) \setminus (\uparrow x)$ is finite by (ST.i) and, therefore, has a maximum z, which fulfills $x > z \geq y$. By construction, $z \in \hat{p}^{-1}(x)$. Therefore, (SF.ii) implies that there is a choice $c' \in C_i$ such that $c' \in A_i(x)$ and $z \in c'$.

If $c = c'$, then $z \in c$. Since $y \in \hat{p}(c)$, there exists a $y' \in \hat{p}^{-1}(y) \cap c$. But then $z \geq y > y'$ and $z, y' \in c$ contradict (SF.iii). It follows that $c \neq c'$.

Since $x \in \hat{p}(c) \cap \hat{p}(c')$ and $c \neq c'$, (SF.i) implies that $\hat{p}(c) = \hat{p}(c')$ and $c \cap c' = \emptyset$. Then, $y \in \hat{p}(c)$ implies $y \in \hat{p}(c')$. By (SF.ii), there is a $y'' \in \hat{p}^{-1}(y) \cap c'$. Thus, $z \geq y > y''$ with $z, y'' \in c'$ contradicts (SF.iii).

Proof of Lemma 2. (a) We first show that, for any $x \in c$, $W(x) \in N(V(c))$. Since $W(x) \subseteq \cup_{x' \in c} W(x') = V(c)$, it remains to show that it is actually maximal in that set. Suppose not. Then, there exists $y \in N$ such that $y > x$ and $W(y) \subseteq V(c)$. By (SF.iii), $y \notin c$.

Construct a chain of nodes iteratively as follows. Let $z_0 = y$. For every $k = 1, 2, \ldots,$ by (SF.iv), there exists a $z_k \in \hat{p}^{-1}(z_{k-1})$ such that $z_k \notin c$. The chain $\{z_0, z_1, \ldots, \}$ is either infinite or it ends at a terminal node z_K. Let w be a play containing this chain. Then $w \in W(y) \subseteq V(c)$. Therefore there exists $z_w \in c$ such that $w \in W(z_w)$.

We claim that $z_k > z_w$ for all $k = 0, 1, \ldots,$ and prove this inductively. Since $y, z_w \in w$ and w is a chain, either $z_w \geq y$ or $y > z_w$. In the first case, $z_w \geq y > x$ but $x, z_w \in c$, which contradicts (SF.iii). Therefore $z_0 = y > z_w$. By induction, suppose $z_{k-1} > z_w$. Since $z_k, z_w \in w$ and w is a chain, either $z_w \geq z_k$ or $z_k > z_w$. In the first case, $z_{k-1} > z_w \geq z_k$. Since $\hat{p}(z_k) = z_{k-1}$, by definition of $\hat{p}$, we have that $z_w = z_k$, which contradicts $z_k \notin c$. Hence $z_k > z_w$, completing the argument.

It follows that the chain $\{y, z_1, z_2, \ldots, \}$ is contained in $\uparrow z_w$. Since the latter is finite by (ST.i), the chain must end in a terminal node z_K, which contradicts $z_K > z_w$.

This shows that the nodes $W(x)$ with $x \in c$ are maximal in $V(c)$. Conversely, let $z \in W(N)$ be maximal in $V(c) = \cup_{x' \in c} W(x')$. Then, there exists some $x \in c$ such that $z \cap W(x) \neq \emptyset$, which by (GT.i) in the set representation and maximality of z in $W(x)$ implies $z = W(x)$.

(b) By Proposition 2(b), $P(V(c)) = \{P(W(x)) \,|x \in c\} = \left\{W\left(\hat{p}(x)\right) |x \in c\right\} = W\left(\hat{p}(c)\right)$. The second equality follows from $P(W(x)) = W\left(\hat{p}(x)\right)$, which follows from the order isomorphism in Proposition 5(a).

Proof of Proposition 6. (a) By Theorem 2 it only needs to be shown that (DEF.i) and (DEF.ii) hold for $\left(T', C'\right)$.

To see (DEF.i), let $V(c_1), V(c_2) \in C'_i$ for some $i \in I$ be such that $V(c_1) \neq V(c_2)$, hence $c_1 \neq c_2$, and $P(V(c_1)) \cap P(V(c_2)) \neq \emptyset$. We wish to show that $\hat{p}(c_1) \cap \hat{p}(c_2) \neq \emptyset$. To see this, let $W(x) \in P(V(c_1)) \cap P(V(c_2))$. By definition of P, for $k = 1, 2$ there exists $W(y_k) \in \downarrow V(c_k)$ such that $\uparrow W(x) = (\uparrow W(y_k)) \backslash (\downarrow V(c_k))$, which in particular implies $W(x) \nsubseteq V(c_k)$. Let $W(z_k)$ be the maximum of the chain $(\uparrow W(y_k)) \backslash (\uparrow W(x))$, which is finite by Theorem 1(d). It follows that $p(W(z_k)) = W(x)$ which by order isomorphism (Proposition 5(a)) implies $\hat{p}(z_k) = x$.

We claim that $W(z_k)$ is maximal in $V(c_k)$. To see this, suppose there is $W(x') \in W(N)$ such that $W(z_k) \subset W(x') \subseteq V(c_k)$. Since $W(y_k) \subseteq W(z_k)$, it follows that $W(x') \in \uparrow W(y_k)$. If $W(x') \in \uparrow W(x)$, then $W(x) \subseteq W(x') \subseteq V(c_k)$ contradicts that $W(x) \nsubseteq V(c_k)$. Hence $W(x') \in (\uparrow W(y_k)) \setminus (\uparrow W(x))$, contradicting the definition of $W(z_k)$.

By Lemma 2(a) $W(z_k) \in N(V(c_k)) = W(c_k)$, hence $z_k \in c_k$. Since $\hat{p}(z_k) = x$ for $k = 1, 2$, this shows that $x \in \hat{p}(c_1) \cap \hat{p}(c_2)$. Thus it follows from (SF.i) that $\hat{p}(c_1) = \hat{p}(c_2)$ and $c_1 \cap c_2 = \emptyset$.

It remains to show that $V(c_1) \cap V(c_2) = \emptyset$. Suppose there is a play w such that $w \in V(c_1) \cap V(c_2)$. Then there are $x_1 \in c_1$ and $x_2 \in c_2$ such that x_1 and x_2 belong to the play w. This implies that they are ordered, say, $x_1 \geq x_2$. Since $c_1 \cap c_2 = \emptyset$, in fact $x_1 > x_2$. This implies that $x_1 \geq \hat{p}(x_2)$. Since $\hat{p}(c_1) = \hat{p}(c_2)$, there is some $z \in c_1$ such that $\hat{p}(x_2) = \hat{p}(z)$, thus $x_1 \geq z$. If $x_1 = z$ would hold, then $\hat{p}(x_1) = \hat{p}(x_2)$ in contradiction to $x_1 > x_2$. Hence, $x_1 > z$ which contradicts (SF.iii).

Before turning to (DEF.ii), note that a choice (set of plays) $V(c) \in C'_i$ is available at a node $W(x)$ in T' if and only if the choice (set of nodes) c is available at the node x of T. This follows from $W(\hat{p}(c)) = P(V(c))$ (Lemma 2(b)). Therefore we will not distinguish between availability in (T, C) and in (T', C').

To show (DEF.ii), we first claim that, given $y \in \hat{p}^{-1}(x)$ such that $\{y\} = \hat{p}^{-1}(x) \cap \left(\cap_{i \in J(x)} c_i\right)$ for some $(c_i)_{i \in J(x)} \in A(x)$, it follows that $W(y) = W(x) \cap \left(\cap_{i \in J(x)} V(c_i)\right)$. The proof is by double inclusion. First let $w \in W(y)$. Since $y \in \hat{p}^{-1}(x)$, it follows that $x \geq y$ and thus $w \in W(x)$. For each $i \in J(x)$, $y \in c_i$ implies that $w \in V(c_i)$, thus $w \in W(x) \cap \left(\cap_{i \in J(x)} V(c_i)\right)$. Conversely, let $w \in W(x) \cap \left(\cap_{i \in J(x)} V(c_i)\right)$. Suppose $w \notin W(y)$. Since $x \in X$ and w is a play, there is a $z \in w \cap \hat{p}^{-1}(x)$. Further, for each $i \in J(x)$ there exists a $z_i \in c_i \cap w$. Then $\hat{p}(z_i) \in w \cap \hat{p}(c_i)$. Since $x \in w \cap \hat{p}(c_i)$, it follows that $x = \hat{p}(z_i)$. Otherwise, $\hat{p}(c_i)$ would contain two distinct ordered nodes, a contradiction to Lemma 1. Since $x = \hat{p}(z_i) = \hat{p}(z)$ and $z, z_i \in w$, it follows that $z = z_i$. Therefore $z \in \hat{p}^{-1}(x) \cap \left(\cap_{i \in J(x)} c_i\right) = \{y\}$, a contradiction.

To complete the verification of (DEF.ii) we now show that

$$p^{-1}(W(x)) = \left\{W(x) \cap \left(\cap_{i \in J(x)} V(c_i)\right) \middle| (c_i)_{i \in J(x)} \in A(x)\right\}$$

for every $x \in X$. Let $W(y) \in p^{-1}(W(x))$ (again by double inclusion). This implies $y \in \hat{p}^{-1}(x)$ by the order isomorphism in Proposition 5(a). By (SF.ii), there exists $(c_i)_{i \in J(x)} \in A(x)$ such that $\{y\} = \hat{p}^{-1}(x) \cap \left(\cap_{i \in J(x)} c_i\right)$. By the claim above, it follows that $W(y) = W(x) \cap \left(\cap_{i \in J(x)} V(c_i)\right)$. Conversely, let $a = W(x) \cap \left(\cap_{i \in J(x)} V(c_i)\right)$ for some $(c_i)_{i \in J(x)} \in A(x)$. By (SF.ii) there exists $y \in \hat{p}^{-1}(x)$ such that $\{y\} = \hat{p}^{-1}(x) \cap \left(\cap_{i \in J(x)} c_i\right)$. By the claim above, $a = W(y)$. This completes the proof of (DEF.ii).

(b) By Theorem 2 (DEF.i) and (DEF.ii) hold for $\left(T', C'\right)$. It remains to check (SF.i-iv) for (T, C). First notice that, if $c' \in C_i'$ for some $i \in I$ and $c = N(c') \cap F(N)$, it follows that $\hat{p}(c) = P\left(c'\right)$ by Proposition 2 (b). For $P(c') = \left\{p(x) \,\middle|\, x \in N(c') \cap F(N)\right\} = \hat{p}(c)$, because $c = N(c') \cap F(N)$ and $\hat{p}$ is just the notation for the immediate predecessor when $T = (F(N), \supseteq)$ is considered as a simple tree.

To show (SF.i), let $N(c_1') \cap F(N), N(c_2') \cap F(N) \in C_i$ for some $i \in I$ be such that $\hat{p}(N(c_1') \cap F(N)) \cap \hat{p}(N(c_2') \cap F(N)) \neq \emptyset$ and $N(c_1') \cap F(N) \neq N(c_2') \cap F(N)$. This implies $c_1' \neq c_2'$ and $P(c_1') \cap P(c_2') \neq \emptyset$. By (DEF.i), it follows that $P(c_1') = P(c_2')$ and $c_1' \cap c_2' = \emptyset$. Hence $\hat{p}(N(c_1') \cap F(N)) = \hat{p}(N(c_2') \cap F(N))$. It remains only to show that $N\left(c_1'\right) \cap N\left(c_2'\right) \cap F(N) = \emptyset$. By contradiction, suppose there exists a node $x \in N\left(c_1'\right) \cap N\left(c_2'\right) \cap F(N)$. Then, since x is actually also a node in T', it follows that $\emptyset \neq x \subseteq c_1' \cap c_2'$, a contradiction.

Turn to (SF.ii). A choice (set of nodes) $N\left(c'\right) \cap F(N) \in C_i$ is available at a node x in $T = (F(N), \supseteq)$ if and only if the choice (set of plays) c' is available at the node x of $T' = (N, \supseteq)$. This follows from $\hat{p}(N(c) \cap F(N)) = P(c)$. Suppose $y \in p^{-1}(x)$ is such that $y = x \cap \left(\cap_{i \in J(x)} c_i'\right)$ for some $\left(c_i'\right)_{i \in J(x)} \in A(x)$. For each $i \in J(x)$, $x \in P\left(c_i'\right) = \hat{p}(N(c_i') \cap F(N))$ and $y \subseteq c_i'$ implies that $y \in N\left(c_i'\right) \cap F(N)$ by Proposition 2. This shows that $y \in p^{-1}(x) \cap \left(\cap_{i \in J(x)} N\left(c_i'\right)\right) \cap F(N)$. Suppose there is $z \in F(N)$ with $z \in p^{-1}(x) \cap \left(\cap_{i \in J(x)} N\left(c_i'\right)\right)$. It follows that $z \subseteq x \cap \left(\cap_{i \in J(x)} c_i'\right) = y$, hence $z = y$. This shows that (DEF.ii) implies (SF.ii).

Consider (SF.iii). If for any $c' \in C_i'$ for some $i \in I$ there are $x, y \in N\left(c'\right) \cap F(N)$ such that $x \supset y$, this contradicts that $N\left(c'\right)$ is a partition of c' by Proposition 2(a).

It remains to show (SF.iv). Let x be a move in $F(N)$ and $c = N(c') \cap F(N)$ for some $c' \in C_i'$, for some $i \in I$. If $\hat{p}^{-1}(x) \cap c = \emptyset$, (SF.iv) is void. Therefore, we can assume that $\hat{p}^{-1}(x) \cap c \neq \emptyset$. Then there is $y \in c$ such that $\hat{p}(y) = x$. Suppose $\hat{p}^{-1}(x) \setminus c = \emptyset$. Then, by Proposition 2, $z \subseteq c'$ for every $z \in \hat{p}^{-1}(x)$. By Proposition 1(a) $x = \cup_{z \in \hat{p}^{-1}(x)} z$, hence, $x \subseteq c'$, which contradicts $y \in N(c')$ (Proposition 2).

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