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RESEARCH ARTICLE

Pricing rules and Arrow–Debreu ambiguous valuation

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Abstract This paper considers pricing rules of single-period securities markets with finitely many states. Our main result characterizes those pricing rules C that are super-replication prices of a frictionless and arbitrage-free incomplete asset structure with a bond. This characterization relies on the equivalence between the sets of *frictionless securities* and *securities* priced by C. The former captures securities without bid-ask spreads, while the second captures the class of securities where, if some of its delivers is replaced by a higher payoff, then the resulting security is characterized by a higher value priced by C. We also analyze the special case of pricing rules associated with securities markets admitting a structure of basic assets paying one in some event and nothing otherwise. In this case, we show that the pricing rule can be characterized in terms of capacities. This *Arrow–Debreu ambiguous state price* can be viewed as a

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generalization for incomplete markets of Arrow–Debreu state price valuation. Also, some interesting cases are given by pricing rules determined by an integral w.r.t. a *risk-neutral capacity*. For instance, incomplete markets of Arrow securities and a bond are revealed by a Choquet integral w.r.t. a special risk-neutral capacity.

Keywords Pricing rule · Frictionless incomplete market · Ambiguity · State price · Capacity, Lehrer integral · Choquet integral

JEL Classification D52 · D53

1 Introduction

Since the Arrow's Role of Securities seminal paper, the financial general equilibrium models assume that the price of assets satisfies equilibrium conditions in a competitive setting where many agents demand assets profiles in accordance with their preferences and their endowments, providing the foundations for the study of financial markets by a celebrate fundamental result asserting that financial markets must not offer arbitrage opportunities. For instance, in a two-period economy, it implies the impossibility, at equilibrium, to realize positive net financial returns in the second period without spending at the initial period some amount of money in the asset market. Furthermore, the fundamental theorem of asset pricing for frictionless complete markets[1] enforces linear pricing rule: the cost of replication of any security is given by the mathematical expectation of its payoffs stream under the unique state contingent price or risk-neutral probability obtained by the no-arbitrage principle.

Nowadays, a widely studied paradigm says that complete markets assumption becomes the exception rather than the rule in the study of financial markets. Market incompleteness says that not all securities admit a perfect hedge, and therefore, in many cases, the seller of a security should consider a superhedging strategy[2] in order to protect against any possible claims of the buyer of such security.[3] Hence, in a financial economy where agents can trade a finite and potential limited number of frictionless securities, the pricing rule gives the minimum cost of getting a payoff equal to (or larger than) a given contingent claim in any state of nature, which is also known as the super-replication price. Importantly, by no-arbitrage and assuming the presence of

[1] Recall that a financial market is complete if the trading of basic securities reproduce any financial payoff stream, otherwise the financial market is incomplete.

[2] A superhedging strategy or super-replication is a portfolio strategy which generates payoffs across the states that are at least as large as the underlying security.

[3] Some results show that this is typically the case for some important classes of securities markets, for example, a well-known result from Ross (1976) says that whenever the payoff of every *call* or *put option* can be replicated, the securities market must be complete. Also, Aliprantis and Tourky (2002) showed that if the number of securities is less than half the number of states of the world, then generically we have the absence of perfect replication of *any* option. Hence, the approach of finding the value of an option by reference to the prices of the primitive securities breaks down for any option. In another way, Baptista (2007) showed that (generically) if every *risk binary contingent claim* is non-attainable, then every option is non-attainable. For further results, see Polyrakis and Xanthos (2011).

a fair risk-free security, the super-replication price of any security can be determined by its supremum expected value with respect to all risk-neutral probabilities.

Another prominent problem in the study of financial markets is the possibility of frictions affecting tradeable assets.[4] Among others, frictions include bid-ask spreads, short-sales constraints and short-selling costs, and differences between borrowing and lending rates. In such cases, the market might be complete and we still have more than one underlying risk-neutral probability and the pricing rule is also given by the supremum over the expected values with respect to all risk-neutral probabilities. A consequence is that any normalized pricing rule should satisfy a set of mild and intuitive conditions as essentially obtained by a well-known representation theorem for a functional described by a set of probability measures.[5] Next section assumes such conditions as a primitive for pricing rules, as done by Jouini and Kallal (2001), and discusses its intuitive appeal.

It is quite immediate that given a pricing rule, there are many candidates for the corresponding underlying type of financial structure. So, given a non-linear pricing rule, how to identify the type of market imperfection related to it? Our main result identifies the case of pricing rules related to frictionless incomplete markets by finding a special property for pricing rules avoiding market imperfections affecting tradeable securities.

For our main result characterizing those pricing rules C that are super-replication prices of some frictionless incomplete asset structure, we established an equivalence between the set of *frictionless securities* and *undominated securities* priced by C. The set of *frictionless securities* priced by C is defined as

$$F_C := \{Y : C(Y) + C(-Y) = 0\},$$

and the set of *undominated securities* priced by C is defined as

$$L_C := \{Y : X > Y \Rightarrow C(X) > C(Y)\}.$$

While a frictionless security can be bought and sold without any frictions, undominated securities have the property that if a payoff assigned to a state by the security is replaced by a bigger payoff, then the resulting security has a strictly superior super-replication price.[6] So, for an undominated security, there is no gain that can be added while maintaining its super-replication price. On the other hand, for a dominated security X, there is some Y paying never less than X and delivering more in at least one state of nature with same price, i.e., $C(Y) = C(X)$. So, if an agent purchase X instead of Y, then she/he is discarding the positive contingent wealth sure in the event where

4 In this paper the term friction is used for market imperfections related to assets available in the market, so we assume the convention that incompleteness is a market imperfection not qualified as a friction.

5 See, for instance, Huber (1981), Gilboa and Schmeidler (1989), and Chateauneuf (1991). The same characterization is the key for the representation of coherent risk measures as introduced by Artzner et al. (1999).

6 Formally, this definition captures the pricing rule's domain of monotonicity. Also, given an arbitrary pricing rule, any frictionless security is an undominated security.

the first security reveals a worse performance than the second one. Hence, our main result says that C reveals a frictionless securities market if, and only if, every security such that every payoff cannot be improved without additional cost is frictionless.

We also analyze the special case of pricing rules associated with arbitrage-free securities markets admitting a structure of basic assets paying one in some event and nothing otherwise, i.e., a structure of *bets* or $\{0, 1\}$-securities. In this case, we show that the pricing rule can be characterized in terms of a capacity by obtaining the extended set of linear states prices of such markets as the set of probabilities below such capacity. This capacity called *Arrow–Debreu ambiguous state price* can be viewed as a generalization for incomplete markets of Arrow–Debreu price valuation.[7] Also, some interesting cases of $\{0, 1\}$-securities markets are associated with pricing rules determined by an integral w.r.t. a subadditive capacity, and in such case, this ambiguous state price is called a *risk-neutral capacity*. More precisely, many markets of bets are revealed through pricing rules given by a Lehrer integral w.r.t. a risk-neutral capacity. Moreover, the special case of partition markets (i.e., markets of bets where basic assets induce a partition of state space) is always revealed through pricing rules given by a Choquet Integral w.r.t. a risk-neutral capacity given by a plausibility measure.

2 Framework and a fundamental result

We consider a single-period economy where the uncertainty is modeled by a finite state space $S = \{s_1, \ldots, s_n\}$. A mapping $X : S \to \mathbb{R}$ is a security that gives the right to $X(s)$ units of consumption or wealth in the second period in each state of nature $s \in S$. A special class of securities is given by the family of bets on events modeled by characteristic functions of events: Given an event A, we denote its characteristic function by $A^* : S \to \{0, 1\}$ where $A^*(s) = 1$ iff $s \in A$. So, a bet on the event A is given by the security A^*.

We denote by $C : \mathbb{R}^S \to \mathbb{R}$ a pricing rule, i.e., agents have to pay $C(X)$ units of initial wealth in order to guarantee at least $X(s)$ units of wealth in each state $s \in S$. Following well-known works in the literature, we shall make the following assumptions concerning a pricing rule

Definition 1 A pricing rule satisfies :

(i) C is sublinear, i.e.,

$$C(\lambda X) = \lambda C(X), \text{ and}$$
$$C(X + Y) \leq C(X) + C(Y),$$

[7] In a sense, ambiguous state prices can be viewed as the pricing consequence of some results in the general equilibrium literature on no-trade with ambiguity averse agents. For instance, Mukerji and Tallon (2001) analyzed conditions in which ambiguity aversion can be viewed as a foundation for incomplete markets. Recently, de Castro and Chateauneuf (2011) show that under unambiguous endowments, more ambiguity aversion implies less trade. In another way on ambiguity aversion and its equilibrium consequences, de Castro et al. (2011) recapture the Arrow–Debreu's state contingent model in an asymmetric information economy and show that under maximin expected utility agents any "maximin" efficient allocation is incentive compatible.

for all $X, Y \in \mathbb{R}^S$ and all non-negative real number λ;
(ii) C is arbitrage free, i.e., $C(X) > 0$ for any nonzero security $X \geq 0$;
(iii) C is normalized, i.e., $C(S^*) = 1$;
(iv) C is monotonic, i.e., $C(X) \geq C(Y)$ for all $X, Y \in \mathbb{R}^S$ s.t. $X \geq Y$;
(v) C is constant additive, i.e.,

$$C\left(X + kS^*\right) = C(X) + k,$$

for all $X \in \mathbb{R}^S$ and every real number k.

Note that all along the paper, for sake of brevity, we will call indifferently a financial pricing rule or a super-replication pricing rule, any pricing rule satisfying the properties of Definition 1.

Such properties are usual and have been proposed by Jouini (2000), Jouini and Kallal (2001) and Castagnoli et al. (2002), among others. The assumption (i) means that the price of a security is proportional to the quantity purchased and that it is less expensive to purchase a portfolio of securities than to purchase each security separately. We note that subadditivity implies that

$$C(X) \geq -C(-X),$$

that is, the price at which X can be bought is larger than or equal to the price at which it can be sold. The assumption (ii), which was first suggested by Ross (1978), captures the absence of arbitrage opportunities by imposing that there are no free security that are non-negative in every state of nature and strictly positive in at least one. Assumption (iii) means that the riskless asset can be bought and sold without any frictions and that riskless rate is equal to zero. The assumption (iv) is a natural condition saying that any investor will not pay more for less. Finally, the assumption (v) means either to purchase a portfolio composed by a security and the riskless asset or to purchase each of these securities separately leads to the same price.

We recall that in a given financial economy, where in order to transfer wealth from the initial date to the future, agents can trade a finite number of securities, the induced pricing rule C reveals for any security X its minimum cost $C(X)$ of getting a payoff equal to (or larger than) the delivers promised by X across the states of nature. Thus, the pricing rule C is also referred as a *super-replication price* of its underlying securities market.[8]

In terms of representation, every financial pricing rule can be computed by the following representation:

Theorem 2 *For any pricing rule satisfying conditions* (i–v) *there is a closed and convex set $\mathcal{K}$ of probability measures, where at least one element is strictly positive, such*

[8] See Appendix Part A for the precise definition of super-replication price of frictionless securities markets without arbitrage opportunities. This concept will play a major role in next sections. We note that the values $-C(-X)$ and $C(X)$ can also be interpreted as arbitrage bounds on the price of X. Indeed, the normative argument is that investors would not pay more than $C(X)$ for X and would not sell it for less than $-C(-X)$.

that for any security X

$$C(X) = \max_{P \in \mathcal{K}} E_P(X)$$

With such representation in mind, given a pricing rule C, its *extended* set of risk-neutral probabilities $\mathcal{K}$ is the closure of the usual set of risk-neutral probabilities, knowing also as the set of "underlying" linear pricing rules. By this representation, we are motivated to adopt a definition saying that the probability measure $P \in \mathcal{K}$ "prices" X if it satisfies $C(X) = E_P(X)$.

3 Pricing rules and frictionless securities markets

The usual way in the literature in order to obtain a pricing rule starts from a given financial market and consider the arbitrage-free pricing problem related to superhedging strategies. This leads a functional form for the super-replication price describing the market valuation, which satisfies the five conditions as given in Definition 1 of pricing rules. It seems useful to review some cases of arbitrage-free market structures $\mathcal{M}$ and its derived set of risk-neutral probabilities $\mathcal{Q}_{\mathcal{M}:}$, with the induced super-replication price or pricing rule:

(i) if markets are complete and frictionless, then the set $\mathcal{Q}_{\mathcal{M}:}$ has only one element, i.e., C is the well-known linear pricing rule of a complete market;
(ii) If markets are incomplete and without any other imperfection, i.e., the set of tradeable securities is free of any friction, then we will call such market a frictionless incomplete market. In this case, the $\mathcal{Q}_{\mathcal{M}:}$ is the set of probabilities defining all linear valuation pricing every basic asset, or taking into account the multi-period framework with a normalized price, $\mathcal{Q}_{\mathcal{M}:}$ is the set of martingale measures of the traded securities (Jouini and Kallal 1995);
(iii) If the traded securities can be bought at a price (the ask) that is potentially higher than the price (the bid) at which they can be sold, then $\mathcal{Q}_{\mathcal{M}:}$ is the set of martingale measures of any price between the normalized bid and ask price (Jouini and Kallal 1995; see also, Bensaid et al. 1992; Baccara et al. 2006).
(iv) If agents are subject to short-sales constraints, then the set $\mathcal{Q}_{\mathcal{M}:}$ is the set of supermartingale measures of the traded securities normalized price (Dybvig and Ross 1986; Jouini and Kallal 1995).

Since in any case above the induced pricing rule is the supremum expected value with respect to all probability in $\mathcal{Q}_{\mathcal{M}:}$, it illustrate how different market structures share a common form of pricing rules and attest the consistency of the general aspect of Definition 1. Such generality reveals an interesting identification problem; in fact, for a given pricing rule, it is possible that there are many candidates for its underlying market structure type. Of course, if we take a linear pricing rule, it is quite immediate that the underlying market must be complete and frictionless. On the other hand, in the case of a non-linear pricing rule, it seems problematic to identify the respective market imperfection related to it. Traditionally, in a competitive market, the observed price reveals the whole pertinent information to agents. One question that seems interesting to us is whether the knowledge of the pricing rule can reveal the type of incompleteness or else the kind of frictions related to the tradeable securities in the market.

Our main result characterizes those pricing rules C that are super-replication prices of a frictionless incomplete market structure with the riskless bond and without arbitrage opportunities. We perform this resulting by adding a new condition to the list of necessary properties (i–v) shared by any financial pricing rule. Next subsections discuss the necessary concepts for our characterization.

3.1 Frictionless and unambiguously priced securities

Given a pricing rule C, its possible lack of additivity sounds natural to be related to some frictions in the financial market. For instance, there is friction for a security X if the buying price $C(X)$ is not the same as its selling price $-C(-X)$, and in this case, the subadditivity captures the natural intuition that its selling pricing $C(X)$ may be greater than its buying price $-C(-X)$. Thus, the set of *frictionless securities* is defined by

$$F_C := \left\{ X \in \mathbb{R}^S : C(X) + C(-X) = 0 \right\}.$$

A security X belonging to F_C means that it can be bought and sold without any frictions when priced by C. A pricing rule C, thanks to its basic properties, always induces a collection of frictionless securities with a structure of linear space, formally:

Lemma 3 *Let C be a financial pricing rule, the set of frictionless securities F_C is a linear subspace.*

We already know that given a pricing rule C, there is a unique convex and closed set of probabilities $\mathcal{K}$ related to C revealing the set of linear pricing rules compatible with the underlying market. Under multiple linear pricing rules, the market has several securities with many expected values generated by the multiple usual risk-neutral valuation. On the other hand, many securities may be immune to the existence of multiple linear pricing, which motivates to define a security X as an *unambiguously priced security* if for all linear pricing rules $P, Q \in \mathcal{K}$

$$E_P(X) = E_Q(X).$$

Such condition means that all linear pricing rules agree about the price of X, in particular, any risk-neutral probability $P \in \mathcal{K}$ "prices" the security X. A simple and interesting result says that

Lemma 4 *Given a financial pricing rule C, a security X is frictionless if, and only if, X is unambiguously priced.*

3.2 Pricing rules and undominated securities

In the real world, many insure contracts give examples of contingent promises for which by discarding some of promised deliveries it does not affect its price. In fact, this is the essence of incompleteness phenomena in financial markets. Formally, given a pricing rule C, its set of *undominated securities* is defined by[9]

$$L_C := \left\{X \in \mathbb{R}^S : Y > X \ \Rightarrow C(Y) > C(X)\right\}.$$

A undominated security X is a security with the property that if some payoff assigned to a state by the claim is replaced by a better payoff, then the resulting security is strictly more expensive than the original one. On other hand, for a dominated security X, by definition, there is Y such that $Y > X$ and $C(Y) = C(X)$. It means that if an agent purchases X instead of Y as above, then she/he is discarding the wealth $Y(s) - X(s)$ in each state of the event $\{Y > X\}$. We note that any frictionless security X is undominated; in fact, for a pricing rule C, since the set of multiple linear pricing rules $\mathcal{K}$ contains a strictly positive probability P_0, if $Y > X$ from X unambiguously priced, we obtain that[10]

$$C(Y) \geq E_{P_0}(Y) > E_{P_0}(X) = C(X).$$

3.3 Main result

Recalling that a frictionless incomplete market is an incomplete market where tradeable securities do not exhibit, bid-ask spreads one obtains:

Theorem 5 *A pricing rule C is a super-replication price of a frictionless and arbitrage-free incomplete market of tradeable securities F including the riskless bond if, and only if, C is a financial pricing rule satisfying $F_C = L_C$. In such case the marketed space satisfies $F = F_C$.*

In worlds, for a given pricing rule that the investor would observe in the financial market, the underlying complete or incomplete market structure will not exhibit friction in any tradeable security if, and only if, any undominated security is unambiguously priced or, equivalently, frictionless. Also, in this case, the presence of any non-linearity reveals that the corresponding financial market is incomplete. In another way, taking into account the viewpoint of a price taker investor choosing between securities priced by a non-linear pricing rule satisfying our main result, the choice of any friction security will make the investor suboptimal in the sense that it is available a security that improves the former in at least one contingence.[11]

[9] In the context of decision theory under uncertainty, Lehrer (2007) provided a representation for preferences using a similar notion called *fat-free acts*.

[10] Or, in another way, given a security X s.t. $C(X) = -C(-X)$, for any $Y > X$ we obtain that by no-arbitrage that $C(Y) - C(X) = C(Y) + C(-X) \geq C(Y - X) > 0$.

[11] Of course, this reasoning supposes an investor that prefers always increase his/her wealth in any future contingency.

Our characterization allows to recover the underlying frictionless incomplete market associated with any pricing rule C such that $L_C = F_C$. In fact, the market space should be given by $F = F_C$, which is a linear space and containing the bond S^*. By considering an arbitrary basis $\{X_0, X_1, \ldots, X_m\}$ of F with $X_0 = S^*$, we can find the corresponding price by considering $q_j = C\left(X_j\right)$ for any $j \in \{0, 1, \ldots, m\}$. Finally, the pricing rule C should satisfy

$$C(X) = \min\left\{\Sigma_j \theta_j q_j : \Sigma_j \theta_j X_j \geq X\right\}.$$

Also, the pricing rule C is associated with a set of probabilities $\mathcal{Q}$, which is the same as the set of extended set of risk-neutral probabilities, i.e.,

$$\mathcal{Q} = \left\{P \in \Delta : E_P\left(X_j\right) = q_j \text{ for all } j\right\}.$$

So, C is the super-replication price of the arbitrage-free securities market

$$\mathcal{M} = \left(X_j, q_j; 0 \leq j \leq m\right).$$

Now, we present some examples showing how our result can reveal when the underlying market is given by a frictionless incomplete market of securities.

Example 6 Consider the pricing rule $C : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$C(X) = \max\left\{E_{P_1}(X), E_{P_2}(X)\right\},$$

where $P_1 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ and $P_2 = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$. We note that, for all security $X = (x_1, x_2, x_3)$

$$C(X) = \max\left\{\alpha x_1 + \left(\frac{3}{4} - \alpha\right) x_2 + \frac{1}{4} x_3 : \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]\right\}.$$

It is simple to see that $F_C = \left\{X \in \mathbb{R}^3 : x_1 = x_2\right\}$ and $X = (1, 2, 0) \in L_C$ with bid-ask $1/4$. Hence, C is not a super-replication price of a frictionless incomplete market.

An interesting fact is that the pricing rule in Example 6 is a special case of *insurance functional* as studied by Castagnoli et al. (2002). So, in this case, the underlying insurance market must admit frictions for some tradeable securities.

Example 7 Consider $C : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$C(X) = \begin{cases} x_3, & \text{if } x_1 + x_2 - 2x_3 < 0 \\ \frac{1}{2}(x_1 + x_2), & \text{if } x_1 + x_2 - 2x_3 \geq 0 \end{cases}$$

We note that, for any security X

$$C(X) = \max\left\{\alpha x_1 + \alpha x_2 + (1 - 2\alpha) x_3 : \alpha \in \left[0, \frac{1}{2}\right]\right\}$$

We note that in this case, $F_C = L_C = \left\{X \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 = 0\right\}$. Hence, C is the super-replication price of the incomplete market where, *e.g.*, basic assets are given by $(1, 1, 1)$, $(2, 0, 1)$, both with price 1.

Example 8 Given a strictly positive probability Q, let $C : \mathbb{R}^S \to \mathbb{R}$ be a pricing rule defined by

$$C(X) = (1 - \varepsilon) E_Q(X) + \varepsilon \max X(S),$$

that we call an *epsilon-contaminated pricing rule*.

In fact, given a security X

$$C(X) = \max_{P \in (1-\varepsilon)\{Q\} + \varepsilon \Delta} E_P(X).$$

In this case, $F_C = \text{span}\{S^*\}$ and $L_C = \mathbb{R}^S$. Hence, C is not the super-replication price of a frictionless incomplete market. We note hat for any security X, its bid-ask is given by

$$BA(X) := C(X) + C(-X) = \varepsilon(\max X - \min X).$$

4 Markets of bets

Arrow (1953) introduced the notion of contingent markets where agents can trade promises concerning future uncertain realizations. A wide class of assets used is known as Arrow securities characterized by a promise on a particular state of nature $s \in S$, i.e., in a financial market, the set of possible Arrow securities is given by $\mathbb{A} := \left\{\{s\}^* : s \in S\right\}$.[12] Given an event A, recall that the $\{0, 1\}$-security A^* is also often called a *bet on (the event) A*.

Definition 9 We say that the mapping $C : \mathbb{R}^S \to \mathbb{R}$ is the super-replication price of a frictionless market of $\{0, 1\}$ -securities without arbitrage opportunities if C is a super-replication price of an arbitrage-free securities market $\mathcal{M} = \left(X_j, q_j; 0 \leq j \leq m\right)$ where there is a collection of events $B_1, \ldots, B_m$ such that $X_j = B_j^*$ for any $j \in \{1, \ldots, m\}$.

For instance, a simple example is given by a complete securities market revealed by a pricing rule C such that there is a strictly positive probability P such that

$$C(X) = E_P(X).$$

In such case, the value $P(\{s\}) := p_s$ is the Arrow–Debreu state price of the contingence $s \in S$. Also, in this case, the underlying market can be constructed by choosing the whole collection of simple bets $\{s\}^*$ with respective prices p_s. Another simple

[12] Of course, markets with only Arrow securities is a very particular case of markets with $\{0, 1\}$-securities.

example is obtained by considering the pricing rule given by

$$C(X) = \max_{P \in \Delta} E_P(X) = \max_{s \in S} X(s).$$

In this case, the underlying market is given by the arbitrage-free financial market with only one bet given by the bond S^* with normalized price 1.

In this section, we characterize the class of frictionless incomplete markets of bets. Before, we need to recall some mathematical notation and definition about non-additive measure and integration.

4.1 Capacities and non-additive integration

A capacity is a set-function $\mu : 2^S \to [0, 1]$ such that: (i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$; and (ii) $A \supseteq B \Rightarrow \mu(A) \geq \mu(B)$. We say that a capacity μ is concave if for all $A, B \in 2^S$

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B).$$

Of course, any concave capacity is subadditive, in the sense that for all disjoint events $A, B \in 2^S$

$$\mu(A \cup B) \leq \mu(A) + \mu(B),$$

but the converse is not true.[13] The case of convex and subadditive capacities follows in an analogous way by taking the reverse inequalities.

The set of *unambiguous events* induced by the capacity μ is defined by[14]

$$\mathcal{E}_\mu := \left\{A \in 2^S : \mu(A) + \mu(A^c) = 1\right\},$$

which defines the linear subspace

$$F_\mu := \text{span}\left\{A^* : A \in \mathcal{E}_\mu\right\}.$$

Another important concept related to a capacity μ is its anticore defined by

$$\text{acore}(\mu) := \left\{P \in \Delta : P(A) \leq \mu(A),\ \forall A \in 2^S\right\}.$$

The outer capacity of μ, denoted by μ^*, is defined by:

$$A \in 2^S \mapsto \mu^*(A) = \min\left\{\mu(B) : B \in \mathcal{E}_\mu \text{ and } A \subset B\right\},$$

So, given a capacity μ, since $\mu^* \geq \mu$ clearly $\text{acore}(\mu) \subset \text{acore}(\mu^*)$.

13 See, for instance, Schmeidler (1972) and Chateauneuf and Jaffray (1989).

14 This is the natural notion of unambiguous events in our context of pricing rules. For a discussion on the variety of preference-based definitions of unambiguous events see, for instance, Klibanoff et al. (2011) and references therein.

A capacity μ is called a-exact if for every event $A \subset S$, there is a probability $P \in \text{acore}(\mu)$ with $P(A) = \mu(A)$. A capacity μ has no-gap if for every positive measure $\tau : 2^S \to [0, 1]$ that satisfies $\mu \geq \tau$, there is p in the acore of μ such that $p \geq \tau$. Note that it is not imposed that $\tau(S) = 1$.

We note that, given a financial pricing rule C, we define the *price of bets* by

$$\mu_C(A) := C\left(A^*\right) \text{ for any } A \subset S.$$

Of course, an event $B \in \mathcal{E}_{\mu_C}$ iff the corresponding bet B^* is unambiguously priced by C. So, by Lemma 4, B is an unambiguous event iff the bet B^* is frictionless.

Clearly, if the price of bets μ_C is additive, we come back to the usual Arrow–Debreu state prices because it is the case if, and only if, the set of linear valuations $\mathcal{Q}$ is a singleton. On the other hand, if the price of bets is non-additive, we call μ_C an *Arrow–Debreu ambiguous state price*. In fact, if there are two probabilities $P, Q \in \mathcal{Q}$ such that $P \neq Q$, then there is an event $E \subset S$ such that $P(E) \neq Q(E)$, which means that not all linear valuation in $\mathcal{Q}$ agree about the price of the bet E^*. In particular, the same ambiguous valuation holds for some state s in the ambiguous event E.

It is immediate to see that, given a pricing rule C with its related set of multiple linear valuations $\mathcal{Q}$, the ambiguous state price μ_C is a subadditive and a-exact capacity. Moreover, since for all $E \subset S$, μ_C is the "upper probability" w.r.t. $\mathcal{Q}$, that is, μ_C is the upper envelope obtained from the set of linear valuations $\mathcal{Q}$ given by

$$\mu_C(E) = \max_{P \in \mathcal{Q}} P(E),$$

are always true the following inclusion:

$$\mathcal{Q} \subset \text{acore}(\mu_C) \subset \text{acore}\left(\mu_C^*\right).$$

Also, since $\mathcal{Q} \cap \Delta^+ \neq \emptyset$, the set of probabilities measures dominated by μ_C contains a strictly positive probability. Summing up, for a given capacity μ to be an ambiguous state price, it should be subadditive, a-exact, and $\text{acore}(\mu) \cap \Delta^+ \neq \emptyset$.

The "concave integral" was proposed and characterized by Lehrer (2009) for capacities, which differs from the well-known Choquet integral when the capacity is not convex. In a similar way, Lehrer integral can be defined as a "convex integral". A special goal of this paper is the case of pricing rules that are characterized by a convex Lehrer integral. For the next definition, we use the convention saying that a *contingent claimX* is a security with non-negative payoffs.

Definition 10 Let C be a pricing rule over the set of contingent claims $\mathbb{R}_+^S$, then C is a Lehrer integral if

$$C(X) = (\mathcal{L}) \int X \mathrm{d}\mu_C \quad \text{for all} \quad X \in \mathbb{R}_+^S$$

where,

$$(\mathcal{L}) \int X \mathrm{d}\mu_C := \min \left\{ \sum \alpha_i \mu_C (A_i) : \sum \alpha_i A_i^* = X, \alpha_i \geq 0 \right\}.$$

In this case, we call C a “Lehrer pricing rule” and μ_C a “risk neutral capacity”. Finally, we have the following multiple linear valuations representation

$$C(X) = \max_{P \in \mathrm{acore}(\mu_C)} E_P(X).$$

The last equality above follows from Azrieli and Lehrer (2007) because every pricing rule is supposed to be constant additive, and hence, the underlying risk-neutral capacity induced by a Lehrer pricing rule has no-gap.[15] Also, following the Remark 3 of Lehrer (2009), if a pricing rule over contingent claims $X \in \mathbb{R}_+^S$ is, in fact, a Lehrer pricing rule, then the constant additivity property enables us to extend the domain of the “Lehrer pricing rule” from the non-negative securities to all securities. For instance, given $X \in \mathbb{R}^S$ with $\min_S X < 0$, then $X - (\min_S X) S^* \geq 0$ and

$$C(X) := C\left(X - \left(\min_S X \right) S^* \right) + \min_S X.$$

Now, we recall the definition of Choquet integral (Choquet 1954):

Definition 11 Let $C : \mathbb{R}^S \to \mathbb{R}$ be a pricing rule, then C is a Choquet integral if

$$C(X) = (\mathcal{C}) \int X \mathrm{d}\mu_C \quad \text{for all} \quad X \in \mathbb{R}_+^S$$

where,

$$(\mathcal{C}) \int X \mathrm{d}\mu_C := \int_{-\infty}^{0} [\mu_C(\{X \geq t\}) - 1] \, \mathrm{d}t + \int_{0}^{\infty} \mu_C(\{X \geq t\}) \, \mathrm{d}t.$$

In this case, we call C a “Choquet pricing rule” and, again, μ_C a risk neutral capacity. Finally, we obtain the following multiple linear valuations representation (due to Schmeidler 1986)[16]

$$C(X) = \max_{P \in \mathrm{acore}(\mu_C)} E_P(X).$$

[15] In fact, this result was established for core of capacities with large core (the dual concept of no-gap) is also presented in Lehrer (2009). A caveat: by taking a functional given by the maximum over the a-acore of some a-exact capacity, such functional is a Lehrer integral iff such capacity has no-gap.

[16] Recall that every pricing rule considered in this paper is subadditive.

4.2 Pricing rules and frictionless markets of bets

This section will explain why non-additive measures and integration is useful in some important characterizations of incomplete markets. For instance, suppose that a pricing rule C is given by a non-additive integral of Lehrer or Choquet; in this case as we defined in the previous Section, we call μ_C a risk-neutral capacity. Essentially, the main idea of the fundamental Arrow–Debreu valuation in complete markets is that for every security, its arbitrage-free price is the (usual) integral of the state-payoff weighted by its unique state price or risk-neutral probability. By extending the possibilities of Arrow–Debreu valuation through non-additive probabilities, this paper also shows that in many cases, the super-replication price of every security can be computed as an integral of the state-payoff weighted by its risk-neutral capacity. We will see also that not all arbitrage-free market of bets has a pricing rule C related to Choquet or Lehrer integral, but for all arbitrage-free market of bets, its set of risk-neutral probability $\mathcal{Q}$ is revealed by its ambiguous state price μ_C.[17]

The following result characterizes the case of frictionless securities markets admitting a structure of $\{0, 1\}$-assets and shows that the multiple state prices set related to the case of incomplete markets is determined by ambiguous state price of this market. In fact, in every market of bets, the extended set of risk-neutral probabilities is given by the *acore* of the ambiguous state price μ_C. Also, for many markets of bets, the linear complete markets pricing rule can be reformulated by considering a non-additive expected value with respect to the risk-neutral capacity μ_C.

Theorem 12 *A mapping $C : \mathbb{R}^S \to \mathbb{R}$ is a super-replication price of a frictionless and arbitrage-free securities market of bets with the bond if, and only if, C is a pricing rule such that its set $\mathcal{Q}$ of extended risk neutral probabilities satisfies*

$$\mathcal{Q} = \text{acore}\left(\mu_C^*\right).$$

Hence, in this case

$$C(X) = \max_{P \in \text{acore}(\mu_C)} E_P(X).$$

So, $F_{\mathcal{E}_{\mu_C}}$ is the set of tradeable securities and acore (μ_C) is the set of extended risk-neutral probabilities of the underlying market.

This theorem provides a complete characterization of the class of capacities that can be viewed as an ambiguous state price of a frictionless and arbitrage-free market of bets. In fact, a capacity μ is an ambiguous state price of a frictionless and arbitrage-free market of bets if, and only if, μ is an upper probability w.r.t. $\mathcal{Q}$ which contains a strictly positive probability and $\mathcal{Q} = \text{acore}\left(\mu_C^*\right)$. In this case, clearly, we also have that acore $(\mu_C) = \text{acore}\left(\mu_C^*\right)$.

Next, an example that gives a case of pricing rules of a frictionless incomplete market with no structure of assets given by bets.

[17] So, in our terminology, an Arrow–Debreu ambiguous state price μ_C is a risk-neutral capacity when for any security X, $C(X)$ is given by the "integral" of X w.r.t. μ_C, in the sense of Lehrer or Choquet.

Example 13 We consider again the functional as in the Example 7, where for any security X,

$$C(X) = \max\left\{\alpha x_1 + \alpha x_2 + (1-2\alpha)x_3 : \alpha \in \left[0, \frac{1}{2}\right]\right\}$$

We already proved that C is a pricing rule of a frictionless and arbitrage-free financial market. Note that for all non-empty event $E \neq \emptyset$,

$$\mu_C(E) \in \left\{\frac{1}{2}, 1\right\} \text{ with } \mu_C(E) = \frac{1}{2} \text{ iff } A \in \{\{s_1\}, \{s_2\}\},$$

which implies that $\mathcal{E}_{\mu_C} = \{\emptyset, S\}$, hence for any $E \neq \emptyset$, we have that $\mu_C^*(A) = 1$ and acore $\left(\mu_C^*\right) = \Delta$. Since $\delta_{\{s_1\}} \notin$ acore (μ_C), we obtain that

$$\text{acore}(\mu_C) \neq \text{acore}\left(\mu_C^*\right).$$

Hence, any underlying market for the pricing rule C is not a market of bets.

In the next result, we obtain a consequence of the previous theorem on digital markets describing the case of pricing rules of frictionless and arbitrage-free markets of bets given by a Lehrer integral, and in special, the class of risk-neutral capacities for frictionless and arbitrage-free markets.

Corollary 14 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be a pricing rule, then (i) is equivalent to (ii):*

- (i) *C is a super-replication price of a frictionless and arbitrage-free market of bets, and for any contingent claim $X \geq 0$*

$$C(X) = (\mathcal{L})\int X d\mu_C.$$

- (ii) *C is a pricing rule such that its set $\mathcal{Q}$ of extended risk neutral probabilities satisfies*

$$\mathcal{Q} = acore\left(\mu_C^*\right),$$

and μ_C has no-gap.
In another way, we can summarize this result by saying that an ambiguous state price μ_C is a risk neutral capacity iff μ_C has no-gap.

Recalling that every ambiguous state price μ_C is an a-exact capacity, the important results derived by Biwas et al. (1999) finding sufficient conditions which guarantee that a-exact game (or, a-exact capacity) has large core (or, has no-gap) have obviously important consequences for our study of pricing rules. Among such sufficient conditions given by Biwas et al. (1999), the cardinality condition obtained in their Theorem 3 is an useful result for pricing rules. The consequence is,

Corollary 15 *Suppose that the cardinality of the state space is not greater than four, then a pricing rule C is a super-replication price of a frictionless and arbitrage-free market of bets if and only if C is a Lehrer pricing rule.*

Next, we present an example of a super-replication price C of a frictionless and arbitrage-free incomplete market of bets, which is not a Lehrer pricing rule.

Example 16 For the state space $S = \{s_1, \ldots, s_5\}$, consider the financial market $\mathcal{M} = \left(X_j, q_j;\ 0 \le j \le 3\right)$ with securities $X_0 = S^*, X_1 = \{s_1, s_3\}^*,\ X_2 = \{s_1, s_2, s_4\}^*,\ X_3 = \{s_1, s_2, s_5\}^*$ and prices $q_0 = 1, q_1 = \frac{2}{5}, q_2 = \frac{3}{5}$ and $q_3 = \frac{3}{5}$. Since $P_0 = \frac{1}{5}S^*$ is a risk-neutral probability, there is no-arbitrage opportunity. Simple computation shows that the set of extended risk-neutral probabilities is given by

$$\mathcal{Q}_{\mathcal{M}} = \left\{P =: \left(p, \frac{3}{5} - 2p, \frac{2}{5} - p, p, p\right) : p \in \left[0, \frac{3}{10}\right]\right\}.$$

and by our Theorem 12 acore $(\mu_C) = \mathcal{Q}_{\mathcal{M}}$.
Now, choosing the security $X := 2\{s_1\}^* + \{s_2\}^*$, one has

$$C(X) = \max_{P \in \text{acore}(\mu_C)} E_P(X) = \frac{3}{5}.$$

Notice also that any writing of X by $X = \Sigma \alpha_i A_i^*, \alpha_i \ge 0$, is given by

$$X = \alpha\{s_1, s_2\}^* + (1 - \alpha)\{s_2\}^* + (2 - \alpha)\{s_1\}^*,$$

with $\alpha \in [0, 1]$. Since $\mu_C(\{s_1, s_2\}) = \frac{3}{5}$, $\mu_C(\{s_2\}) = \frac{3}{5}$, and $\mu_C(\{s_1\}) = \frac{3}{10}$, we obtain that

$$(\mathcal{L}) \int X \mathrm{d}\mu_C = \min_{\alpha \in [0,1]} \left\{\alpha \frac{3}{5} + (1 - \alpha)\frac{3}{5} + (2 - \alpha)\frac{3}{10}\right\} > \frac{3}{5} = C(X).$$

One immediate consequence is that μ_C has a gap. In fact, by taking $\tau = \frac{3}{10}\{s_3, s_4, s_5\}^*$, it is easy to see that $\tau \le \mu_C$. But, if a probability $P \in$ acore (μ_C) dominates τ, then $\frac{2}{5} - p \ge \frac{3}{10}$ and $p \ge \frac{3}{10}$, so $p \le \frac{1}{10}$ and $p \ge \frac{3}{10}$, a contradiction.

4.3 Pricing rules and frictionless partition markets

This section studies a special case of market of bets. Given a frictionless and arbitrage-free market of bets $\mathcal{M} = \left(B_j^*, q_j\right)_{j=1}^m$. We say that such market is a *frictionless and arbitrage-free partition market of securities* when$\{B_k\}_{k=1}^n$ is a partition of the state space S.

Next result characterizes in an interesting way pricing rules for partition markets:

Theorem 17 *Let* $C : \mathbb{R}^S \to \mathbb{R}$ *be given, then the following assertions are equivalent:*

(i) *C is a Choquet pricing rule of a frictionless and arbitrage-free security market;*
(ii) *C is a super-replication price of a frictionless and arbitrage-free partition market;*
(iii) *C is a pricing rule given by the functional*

$$C(X) = \sum_{j=1}^{n} P_0(B_j) \max_{s \in B_j} X(s), \quad \text{for all} \quad X \in \mathbb{R}^S,$$

where P_0 is a strictly positive probability and $\left\{B_j\right\}_{j=1}^{n}$ is a partition of S.
(iv) *C is a pricing rule of a frictionless securities market such that F_C is a Riesz space containing the riskless bond S^**
In any case, the set of attainable claims F_C is the Riesz vector space generated by the P-atoms of the "Boolean algebra" $\mathcal{E}_{\mu_C}$ of unambiguous events, and the concave risk neutral capacity μ_C satisfies $\mu_C = \mu_C^*$.

Theorem 17 states that in any frictionless partition market, the price valuation is given by a Choquet pricing with respect to a special concave risk-neutral capacity given by plausibility measures as introduced by Shafer (1976). Moreover, this is the unique type of Choquet integral related to frictionless incomplete markets. On the other hand, such special super-replication pricing is related to a "rich" structure of assets in the sense that the marketed space is a Riesz space, which is necessarily generated by a partition markets of bets.

Next, an example that gives a case of pricing rules of a frictionless market of bets with no partition structure of basic bets.

Example 18 Consider a capacity μ over the power algebra generated by the state space $S = \{s_1, s_2, s_3, s_4\}$ defined by[18]

$$\begin{aligned}
\mu_1 = \mu_4 = \mu_{12} = \mu_{34} &= \frac{1}{2}, \\
\mu_2 = \mu_3 = \mu_{23} = 1 - \mu_{14} &= \frac{1}{3}, \\
\mu_{13} = \mu_{123} = \mu_{234} &= \frac{5}{6}, \\
\mu_{24} = \mu_{124} = \mu_{134} &= 1.
\end{aligned}$$

We note that μ is an upper probability w.r.t.

$$\mathcal{Q} = \left\{ \left(p, \frac{1}{2} - p, p - \frac{1}{6}, \frac{2}{3} - p \right) : \frac{1}{6} \leq p \leq \frac{1}{2} \right\},$$

and $\mathcal{Q} = \operatorname{acore}(\mu^*)$. So, by our Lehrer pricing rule characterization, the price functional

$$C(X) := \min \left\{ \sum \alpha_i \mu(A_i) : \sum \alpha_i A_i^* = X, \alpha_i \geq 0 \right\}, \quad \text{for all} \quad X \geq 0,$$

[18] For simplicity, we denote for each $A \subset S$, $\mu(A) = \mu_{i: i \in A}$.

defines a Lehrer pricing rule of a frictionless market of bets. On the other hand, since $\mu^*_{123} = 1$, i.e., $\mu \neq \mu^*$, every possible basic structure of bets cannot induce a partition of the state space. For instance, a possible underlying market is given by the assets $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, and S^* priced by the risk-neutral capacity μ.

We note that the class of super-replication prices of frictionless market that can be written as a Choquet integral is linked to financial markets where *derivative markets* (in the sense of Aliprantis et al. 2000) are complete.[19] A restatement of the result due to Ross (1976), provided by Aliprantis et al. (2000), says that derivative markets are complete if and only if the vector space of attainable securities is a Riesz subspace. Hence, by our characterization of partition markets, the family of Choquet pricing rules of frictionless and arbitrage-free securities markets describe the super-replication prices in markets where derivative markets are complete. For instance, note that in the Example 18, the Arrow security $\{s_2\}^*$ is not replicated and the basic securities $\{s_1, s_2\}^*$ and $\{s_2, s_3\}^*$ induces through the "min" operator the Arrow security paying related to the scenario two. On the other hand, given a partition $\left\{B_j\right\}_{j=1}^{n}$ of the state space S and a strictly positive probability P_0 inducing the risk-neutral capacity given by the plausibility measure

$$\mu\left(A\right) = \sum_{j:B_j \cap A \neq \emptyset} P_0\left(B_j\right).$$

There is no no-replicated bet A^* given through a security induced by the minimum between two bets in this partition market.[20]

4.4 Arrow–Debreu ambiguous valuation

Given a pricing rule C for a financial market, for all event $A \subset S$, its "price of bet" is given by $C\left(A^*\right) = \mu_C\left(A\right)$, and, by considering $A = \{s\}$, we have computed the vector of *Arrow–Debreu (ambiguous) state prices* $(\mu_s)_{s\in S}$ defined by $\mu_s := \mu_C\left(\{s\}\right)$, which can reflect the intuition that investors may be unable to form a single additive probability belief about asset returns.[21] Clearly, the information revealed by a ambiguous state price might not be enough for pricing all bet because the general subadditivity of μ_C implies that the knowledge of each state price μ_s only gives an upper bound on the prices of bets on events.[22]

[19] A *ABW-derivative security* is any security that has the same payoff in states in which the payoffs of all basic assets are the same.

[20] In fact, as we saw in Theorem 17, in this class of markets the family of unambiguous events (unambiguously priced bets) form an algebra (a Riesz linear subspace) of subsets.

[21] In a general equilibrium framework, Ozsoylev and Werner (2011) show the possibility of illiquid market under information transmission in asset markets when agents' probabilistic information is ambiguous and analyze its consequences for asset prices. On the other hand, Condie and Ganguli (2011) show that strong-form efficient equilibrium prices exist even when many ambiguity averse investors in the market make use of information in a way that is substantially different from traditional financial models.

[22] For instance, given the state prices μ_s and μ_{s^*} then $\mu_{ss^*} \leq \mu_s + \mu_{s^*}$.

We saw in the results about markets of bets that a special class of capacities plays a fundamental role. In fact, in such markets pricing, rules can be characterized in terms of an ambiguous state price μ because the related set of linear pricing rules is given by family of all probabilities dominated by μ. That is,

$$C\left(X\right)=\max_{P\leq\mu}E_{P}\left(X\right),$$

and the ambiguous state price μ gives the upper bound for the value of each event.

We also obtained some special cases of pricing rules where an integral with respect to a *risk-neutral capacity* characterizes completely the valuation rule in the entire universe of securities. For such financial markets, for any security X

$$C\left(X\right)=\text{“Integral” of } X \text{ w.r.t } \mu_{C}.$$

For instance, Lehrer pricing rules says that for any contingent claim $X\geq 0$,

$$C\left(X\right)=\min\left\{\sum\alpha_{i}\mu_{C}\left(A_{i}\right):\sum\alpha_{i}A_{i}^{*}=X,\alpha_{i}\geq 0\right\}.$$

Which means that in order to price a claim X, it is enough to consider the unique non-additive *event price* $\{\mu_{C}\left(A\right)\}_{A\subset S}$ and find the cheapest portfolio of bets that “replicates” X.

Thus, the notion of non-linear market evaluation extends the usual way of pricing in complete market setting to many incomplete markets of bets without arbitrage opportunities by taking the introduced Arrow–Debreu ambiguous valuation through a non-additive integral with respect to the *risk-neutral capacity* μ_{C}.

A simple example of pricing rule that captures the previous intuition is given by the functional

$$C_{A}(X)=\sum_{s\in E_{0}}X\left(s\right)Q\left(\{s\}\right)+Q\left(E_{o}^{c}\right)\max_{s\in E_{o}^{c}}X\left(s\right),$$

where $Q\in\Delta^{+}$. Note that the cost of betting on the event E is given by the following concave capacity,

$$\mu_{C_{A}}\left(E\right)=\begin{cases}Q(E), & E\subseteq E_{0}\\ Q(E\cap E_{0})+Q\left(E_{0}^{c}\right), & \text{otherwise.}\end{cases}$$

Clearly,

$$C\left(X\right)=\left(\mathcal{C}\right)\int X\mathrm{d}\mu_{C_{A}}.$$

One possible underlying market of securities revealed by this pricing rule is the potential incomplete market of Arrow securities and one bond with the assets S^{*}, $\left(\{s_{k}\}^{*}\right)_{k=1,\ldots,K}$ and corresponding prices 1, $(q_{k})_{k=1,\ldots,K}$, where E_{0} is the set of all unambiguous states and $q_k = Q(\{s_k\})$. Hence, C_A can be viewed as an *Arrow ambiguous pricing rule* of an incomplete market of Arrow securities.

The result about Choquet pricing shows that even without transaction costs, the valuation in incomplete financial markets can be achieved through the notion of Choquet integration. In fact, in a sense, this result comes in contrast with Bettzüge et al. (2000) that analyzed a general equilibrium model with transaction costs satisfying mild conditions and showed that the Choquet non-linear pricing approach, as proposed by Chateauneuf et al. (1996) for the case of transactions costs, typically does impose restrictive pricing conditions that are incompatible with non-linear equilibrium prices. A positive aspect of this limitation is that Choquet pricing can distinguish special market characteristics that are beyond the condition of a perfect market. For instance, from our results, we can say that every incomplete market given through a basic structure of assets with the riskless bond and only Arrow securities can be constructed by a special case of Choquet pricing. In the case of transaction costs, Choquet pricing also have special implications, as showed by Castagnoli et al. (2004),[23] and this topic represents a starting point of our goal for future research about pricing rules for markets with frictions.

5 Appendix

5.1 Part A : Frictionless securities markets and its pricing rules

Arrow (1953) proposed the approach of contingent markets with the presence of a complete frictionless securities market and used the results from Arrow and Debreu (1954) as well as McKenzie (1954) for the existence of competitive equilibrium. Magill and Quinzii (1996) and Magill and Shafer (1991) are basic references for the case of general equilibrium analysis of frictionless incomplete markets, and such works provided a list of the main contributions in this field. In another way, Föllmer and Schied (2004) provided a treatment of the basic results in frictionless incomplete markets following the lines of standard finance theory.

Next, under no-arbitrage assumption, we describe the case of a competitive securities market without assuming completeness and avoiding the possibility of frictions in any tradeable security.[24] Formally, a pricing rule C is a super-replication price of a frictionless and arbitrage-free securities market if we have the following conditions:

- There is a finite number of frictionless assets $X_j \in \mathbb{R}^S$, $0 \leq j \leq m$, with respective prices $q_j \in \mathbb{R}$, where $X_0 = S^* := (1, \dots, 1)$ is the *riskless bond* with

[23] They have shown that Choquet pricing rules can represent strong frictionalities, in the sense that the existence of any frictionless tradeable security makes the whole market frictionless. The *weakness* of this result in a setup of two-period economy with finitely many states is the required existence of a fully revealing security, i.e., a security for which the whole available information is summarized by its contingent payments.

[24] Carvajal and Weretka (2010) argue that the principle of no-arbitrage asset pricing is also consistent with non-competitive behavior of the arbitragers and extend the fundamental theorem of asset pricing to the non-competitive setting.

the price normalization $q_0 = 1$.[25] We note that there is only one possible deviation from the standard frictionless complete markets setup given by the possibility of incomplete markets when the set of attainable claims, denoted by $F := span\{X_0, X_1, \ldots, X_m\}$, is a proper subspace of $\mathbb{R}^S$.

- This collection of assets and prices characterizes a frictionless market of securities denoted by

$$\mathcal{M} = \left(X_j, q_j;\ 0 \leq j \leq m\right),$$

which is supposed to be without arbitrage opportunities, i.e., for all portfolio $\theta \in \mathbb{R}^{m+1}$,

$$\sum_{j=0}^{m} \theta_j X_j > 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j > 0,$$
$$\sum_{j=0}^{m} \theta_j X_j = 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j = 0.$$

Recall that a financial market $\mathcal{M} = \left(X_j, q_j;\ 0 \leq j \leq m\right)$ offers no-arbitrage opportunity if and only if there is a strictly positive probability[26] $P_0 \in \Delta$ such that $E_{P_0}(X_j) = q_j, 0 \leq j \leq m$ (see, for instance, Theorem 1.6 in Föllmer and Schied 2004). Also, given the financial market $\mathcal{M}$, we denote by

$$\mathcal{Q}_{\mathcal{M}} = \{P \in \Delta^+ : E_P(X_j) = q_j,\ \forall j \in \{0, \ldots, m\}\},$$

the set of *risk-neutral probabilities (or martingale measures)*

- Finally, C is the super-replication price of the frictionless securities market $\mathcal{M}$, i.e., for all security $X \in \mathbb{R}^S$

$$C(X) = \inf\left\{\sum_j \theta_j q_j : \sum_j \theta_j X_j \geq X\right\}.$$

Any $Y = \sum_j \theta_j X_j \geq X$ gives a corresponding super-replication strategy $\theta \in \mathbb{R}^{m+1}$ for the security X, and in our case, the existence of superhedging strategies for all security follows from the existence of the riskless bond.

25 We suppose, *w.l.g.*, that this collection of assets are non-redundant (linearly independent). We do not suppose they are positive.

26 Note that P_0 strictly positive means that $P_0(\{s\}) > 0$ for any $s \in S$. The collection of strictly positive probabilities is denoted by Δ^+. Also, we are denoting $E_P(X)$ as the integral of the random variable X w.r.t. the probability P.

It is worth noticing that for a frictionless securities market $\mathcal{M}$ offering no-arbitrage opportunity, the super-replication prices satisfy[27]

$$C(X) = \sup_{P \in \mathcal{Q}_{\mathcal{M}}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

Hence, by taking the closure of the set of risk-neutral probabilities $\mathcal{Q} := \overline{\mathcal{Q}_{\mathcal{M}}}$, we obtain that C is a pricing rules (Definition 1) determinate by $\mathcal{Q}$,[28]

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

Building on the well-known properties discussed above, a trivial Lemma about super-replication prices is naturally derived:

Lemma 19 *The mapping $C : \mathbb{R}^S \to \mathbb{R}$ is a super-replication price of a frictionless securities market with the riskless bond and without arbitrage opportunities if, and only if:*

(1) *There exist a linearly independent set $\{X_0, X_1, \ldots, X_m\} \subset \mathbb{R}^S$ with $X_0 = S^*$ and a strictly positive probability P_0 such that: $E_{P_0}(X_j) = C(X_j) = -C\left(-X_j\right), 0 \leq j \leq m$; and*

(2) *Denoting $q_j := C(X_j)$, it is true that*

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X), \quad \text{for all} \quad X \in \mathbb{R}^S.$$

So, in this case C is the super-replication price of the market $\mathcal{M} = \left\{X_j, q_j\right\}_{j=0}^m$.

This Lemma 19 summarizes the structure of a frictionless incomplete market with the bond revealed by a pricing rule C. Statement (1) says that each basic security is free of arbitrage and pricing by C without frictions. Statement (2) gives that any security has its super-replication cost computed through the set of *extended* risk-neutral probabilities $\mathcal{Q}$.

5.2 Part B: Proofs of the results in the main text

Proof of Theorem 2 By Proposition 2.1 (Chap. 10) in Huber (1981), conditions (i), (iii), (iv), (v) are necessary and sufficient for the existence of a closed and convex set $\mathcal{K}$ of probability measures such that

$$C(X) = \max_{P \in \mathcal{K}} E_P(X).$$

[27] See, for instance, Föllmer and Schied (2004), Theorem 1.31 and Remark 1.32.

[28] We note that

$$\mathcal{Q} = \{P \in \Delta : E_P(X_j) = q_j, \forall j \in \{0, \ldots, m\}\}.$$

Now, if the pricing rule C is strictly positive, then $C\left(\{s_i\}^*\right) > 0, \forall i \in \{1, \ldots, n\}$. Hence, for every state $s_i \in S$, there is a probability $P_i \in \mathcal{K}$ such that $E_{P_i}\left(\{s_i\}^*\right) > 0$. Since $\mathcal{K}$ is convex, we obtain that it is possible to find a strictly positive probability in $\mathcal{K}$. For the converse, by assumption, there is a strictly positive probability $P_0 \in \mathcal{K}$; hence, if $X > 0$

$$C(X) \geq E_{P_0}(X) \geq \max_{s \in S} P_0(\{s\})\, X(s) > 0.$$

□

Proof of Lemma 3 For the proof, only the sublinearity of C is needed. First, consider $Y \in F_C$ and $\lambda \in \mathbb{R}_+$, since C is positively homogeneous, we have that $C(\lambda Y) = \lambda C(Y)$ and $C(\lambda(-Y)) = \lambda C(-Y)$, then $C(\lambda Y) + C(-\lambda Y) = 0$, i.e., $\lambda Y \in F_C$. If $\lambda < 0$, follows from the definition that $-Y \in F_C$ and then $(-\lambda)(-Y) \in F_C$, i.e., $\lambda Y \in F_C$.

Now, if $Y, Z \in F_C$, since C is subadditive

$$C(Y+Z) \leq C(Y) + C(Z), \text{ and}$$
$$C(-(Y+Z)) \leq C(-Y) + C(-Z),$$

hence, adding these two inequalities

$$0 = C(0) \leq C(Y+Z) + C(-(Y+Z)) \leq 0,$$

i.e., $Y + Z \in F_C$. □

Proof of Lemma 4 Since for all security X, $C(X) = \max_{P \in \mathcal{K}} E_P(X)$, if X is frictionless, then

$$\max_{P \in \mathcal{K}} E_P(X) = -\max_{P \in \mathcal{K}} E_P(-X)$$

which is equivalent to

$$\max_{P \in \mathcal{K}} E_P(X) = \min_{P \in \mathcal{K}} E_P(X),$$

and since $P \to E_P(X)$ is continuous and $\mathcal{K}$ is compact, then $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{K}$. For the converse, if X is such that $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{K}$, and the same is true for $-X$. Hence, $C(X) = E_P(X)$ and $C(-X) = E_P(-X)$ for any $P \in Q$, which entails $C(X) = -C(-X)$. □

In order to prove Theorem **5**, we will show some auxiliary results.

For a given frictionless securities market, we recall the following simple and important result:[29]

[29] See, for instance, the Corollary 1.34 in Föllmer and Schied (2004), which also cover the case with infinitely many states. Note also that in our context, for any X, $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{Q}_{\mathcal{M}}$ iff $E_P(X) = E_Q(X)$ for all $P, Q \in \overline{\mathcal{Q}}_{\mathcal{M}}$.

Lemma 20 *Consider an arbitrage-free financial market* $\mathcal{M} = \{X_j, q_j, 0 \leq j \leq m\}$*, a security* $X \in F := span\{X_j\}_{0 \leq j \leq m}$ *if, and only if,* $E_P(X) = E_Q(X)$ *for all* $P, Q \in \mathcal{Q}_{\mathcal{M}} = \{P \in \Delta^+ : E_P(X_j) = q_j, \forall j \in \{0, \ldots, m\}\}$.

Given a pricing rule C, its induced set of probabilities that agree about the expected value of every frictionless securities is given by,

$$\mathcal{Q}_C := \{P \in \Delta : \; E_P(Y) = C(Y), \quad \text{for all} \quad Y \in F_C\}.$$

A useful characterization of pricing rules of a frictionless securities market says that

Lemma 21 *Let* $C : \mathbb{R}^S \to \mathbb{R}$ *be given, then* (i) *is equivalent to* (ii)*:*

- (i) *C is the pricing rule of a frictionless securities market without arbitrage opportunities;*
- (ii) *C is a strictly positive linear form on* F_C *and*

$$C(X) = \max_{P \in \mathcal{Q}_C} E_P(X).$$

Furthermore, under (i) *and* (ii) F_C *is the set of attainable claims and* $\mathcal{Q}_C$ *is the set of extended risk-neutral probabilities of the underlying market.*

Proof $(i) \Rightarrow (ii)$ By Lemma 19, there exist a collection of linearly independent elements $X_0, \; X_1, \ldots, \; X_m \in \mathbb{R}^S$ with $X_0 = S^*$ and a strictly positive probability P_0 on 2^Ssuch that $E_{P_0}(X_j) = C(X_j) = -C(-X_j), 0 \leq j \leq m$. Moreover, $\forall X \in \mathbb{R}^S$

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : E_P(X_j) = C(X_j) =: q_j; \; 0 \leq j \leq m\}$.

Now, note that no-arbitrage principle implies that C is a strictly positive linear form on $F := \text{span}\{X_j\}_{j=0}^m$; actually, by the no-arbitrage condition, there is a strictly positive probability P_0 such that $\forall Y \in F$, $C(Y) = E_{P_0}(Y)$.

Also, we note that if $C : \mathbb{R}^S \to \mathbb{R}$ is a super-replication price of the frictionless market

$$\mathcal{M} = \{X_j, \; q_j; \; 0 \leq j \leq m\},$$

then $F_C = F$. In fact, since $E_P(X) = C(X)$ for any $X \in F$ and for any $P \in \mathcal{Q}$, clearly $F \subset F_C$. Conversely, let $X \in F_C$. Since for any $P \in \mathcal{Q}$,

$$E_P(X) \leq C(X) \text{ and } E_P(-X) \leq C(-X),$$

and

$$E_P(X) + E_P(-X) = 0 = C(X) + C(-X),$$

we obtain that $E_P(X) < C(X)$ is impossible, i.e., given an asset $X \in F_C$ the mapping $P \mapsto \Phi_X(P) := E_P(X)$ is constant over $\mathcal{Q}$, and by Lemma 20, $X \in F$. Moreover, we obtain that C is a strictly positive linear form on F_C.

Finally, by the definitions of $\mathcal{Q}_C$ and $\mathcal{Q}$ and since $F_C = \operatorname{span}\{X_0, \ldots, X_m\}$, it is easy to see that $\mathcal{Q}_C = \mathcal{Q}$.

$(ii) \Rightarrow (i)$ Since $S^* \in F_C$, let us consider $X_0, X_1, \ldots, X_m$, with $X_0 = S^*$, a basis of the linear subspace F_C. We intend to show that C is a super-replication price for the securities market $\left\{X_j, q_j := C\left(X_j\right)\right\}_{j=0}^m$.

By our assumption, the restriction $C\mid_{F_C}$ of C on the linear subspace F_C of the Euclidian space $\mathbb{R}^S$ is a strictly positive linear form; hence, it admits a strictly positive linear extension $\overline{C}\mid_{F_C}$ on $\mathbb{R}^S$ (see, for instance, Clark 1993, Theorem 6). Clearly, it is true that $\overline{C}\mid_{F_C}(S^*) = 1$; therefore, there is a strictly positive probability P_0 on $\left(S, 2^S\right)$ such that $E_{P_0}(X) = \overline{C}\mid_{F_C}(X)$, for any $X \in \mathbb{R}^S$; in particular, $E_{P_0}\left(X_j\right) = \overline{C}\mid_{F_C}\left(X_j\right) = C\left(X_j\right), 0 \leq j \leq m$. So, the condition (1) of Lemma 19 is satisfied. We choose $F_C =: F$ as the set of attainable securities. The proof of (ii) implies (i) will be completed if we prove that $\mathcal{Q}_C = \mathcal{Q}$, where $\mathcal{Q} := \overline{\mathcal{Q}}_{\mathcal{M}}$. We note that $\mathcal{Q}_C$ is non-empty[30] because we just saw that there is a strictly positive probability $P_0 \in \mathcal{Q}_C$

Since for any $j \in \{0, 1, \ldots, m\}$ the security X_j is frictionless, we obtain that every probability $P \in \mathcal{Q}_C$ is an extended risk-neutral probability for the market $\mathcal{M} = \left\{X_j, q_j := C\left(X_j\right);\ 0 \leq j \leq m\right\}$.[31] For the other necessary inclusion, let $P \in \mathcal{Q}$ and an arbitrary $Y \in F_C$, i.e., P is such that

$$E_P\left(X_j\right) = C\left(X_j\right),\ 0 \leq j \leq m,$$

and Y is such that there exist $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{R}$ where

$$Y = \sum_{j=0}^m \lambda_j X_j.$$

Since $C\mid_{F_C}$ is a linear mapping:

$$\begin{aligned} E_P(Y) &= E_P\left(\sum_{j=0}^m \lambda_j X_j\right) = \sum_{j=0}^m \lambda_j E_P\left(X_j\right) \\ &= \sum_{j=0}^m \lambda_j E_P\left(X_j\right) = \sum_{j=0}^m \lambda_j C\left(X_j\right) = C\left(\sum_{j=0}^m \lambda_j X_j\right) = C(Y). \end{aligned}$$

Hence, $P \in \mathcal{Q}_C$, which completes the proof. □

Proof of Theorem 5 From Lemma 19 and Theorem 2, we already know that necessarily C is a financial pricing rule. So we just confine on the equality $F_C = L_C$.

30 Also, it is clear that $\mathcal{Q}_C$ is compact and convex.

31 By the existence of the strictly positive probability P_0, the financial market $\mathcal{M}$ is a market of securities with no-arbitrage opportunity.

(⇒) We want to show that $L_C = F_C$. In fact, we saw in the main text that for any financial pricing rule C, it is true that $F_C \subset L_C$.
Now, suppose that $X \in L_C$, then by definition $Y > X \Rightarrow C(Y) > C(X)$. Since C is a super-replication price of a frictionless market of securities $F = F_C$. So, by supposing that $X \notin F_C$, since

$$\begin{aligned} C(X) &= \min\{C(Y) : Y \geq X \text{ and } Y \in F_C\} \\ &\overset{(X \notin F_C)}{=} \min\{C(Y) : Y > X \text{ and } Y \in F_C\}, \end{aligned}$$

there is $Z \in F_C$ such $Z > X$ and $C(Z) = C(X)$, a contradiction.

(⇐) Since C is a financial pricing rule, we know that there is a non-empty, closed, and convex set of probabilities measures $\mathcal{K}$ such that $\mathcal{K} \cap \Delta^+ \neq \emptyset$ and for any $X \in \mathbb{R}^S$,

$$C(X) = \max_{P \in \mathcal{K}} E_P(X).$$

By Lemma 21, it is enough to show that C is strictly positive linear form on F_C and $\mathcal{K} = \mathcal{Q}_C$.

The inclusion $\mathcal{K} \subset \mathcal{Q}_C$ is simple. Consider $P \in \mathcal{K}$, if $P \notin \mathcal{Q}_C$, then there is $X \in F_C$ such that $E_P(X) < C(X) = -C(-X)$; hence, $E_P(-X) > C(-X) = \max_{P \in \mathcal{K}} E_P(-X)$, a contradiction.

So, we need to show that $\mathcal{K} \subsetneq \mathcal{Q}_C$ is impossible. Assume that there is $P_1 \in \mathcal{Q}_C$ such that $P_1 \notin \mathcal{K}$. Then, through the classical strict separation theorem (see, for instance, Dunford and Schwartz 1988), there is a security X_0 such that

$$E_{P_1}(X_0) > \max_{P \in \mathcal{K}} E_P(X_0) = C(X_0).$$

If we prove that there is $Y \in F_C,\ Y \geq X_0$ such that $C(X_0) = C(Y)$, this will entail a contradiction, since

$$E_{P_1}(X_0) > C(X_0) = C(Y) = E_{P_1}(Y) \geq E_{P_1}(X_0).$$

So it is enough to show that for any security X, setting

$$E_X := \left\{Y \in \mathbb{R}^S : Y \geq X \text{ and } C(Y) = C(X)\right\},$$

there is $Y \in F_C \cap E_X$.

This result is obvious if $X \in F_C$, so let us assume that $X \notin F_C$. Recall that from Theorem 2, $\mathcal{K}$ contains at least a strictly positive probability P_0.

Let us now prove that E_X is bounded from above, otherwise there would exist a sequence $\{Y_k\}_{k\geq 1}$, $Y_k \in E_X, \forall k \geq 1$ and $s_0 \in S$ such that $\lim_k Y_k(s_0) = +\infty$. But

$$\lim_k C(Y_k) \geq \lim_k E_{P_0}(Y_k) = \lim_k \sum_{s\in S} P_0(s) Y_k(s)$$
$$\geq \sum_{s\neq s_0} P_0(s) X(s) + \lim_k P_0(s_0) Y_k(s_0) = \infty,$$

contradicting $C(Y_k) = C(X), \forall k \geq 1$.

Let us now show that E_X has a maximal element for the partial order $\geq$ on $\mathbb{R}^S$. Thanks to Zorn ´s lemma, we just need to prove that every chain $(Y_\lambda)_{\lambda\in\Phi}$ in E_X has an upper bound. Define Y by

$$Y(s) := \sup_{\lambda\in\Phi} Y_\lambda(s), \forall s \in S.$$

Since E_X is bounded from above, it implies that $Y \in \mathbb{R}^S$. It remains to check that $C(Y) = C(X)$, let $\varepsilon > 0$ be given, and let $s_i \in S$, hence there is $\lambda_i \in \Phi$ such that $Y(s_i) \leq Y_{\lambda_i}(s_i) + \varepsilon$, since $(Y_\lambda)_{\lambda\in\Phi}$ is a chain there is $n \geq 1$ and $\widetilde{\lambda} \in \{\lambda_1, \ldots, \lambda_n\}$ such that $Y_{\widetilde{\lambda}} \leq Y \leq Y_{\widetilde{\lambda}} + \varepsilon S^*$, therefore $C\left(Y_{\widetilde{\lambda}}\right) \leq C(Y) \leq C\left(Y_{\widetilde{\lambda}}\right) + \varepsilon$, since $C\left(Y_{\widetilde{\lambda}}\right) = C(X)$ it turns out that $C(Y) = C(X)$. Let now Y_0 be a maximal element of E_X, the proof will be completed if we show that $Y_0 \in F_C$. From the hypothesis $F_C = L_C$, it is enough to show that $Y_0 \in L_C$. Let Y_1 be an arbitrary security such that $Y_1 > Y_0$, since Y_0 is a maximal element in E_X, it comes that $Y_1 \notin E_X$, but $Y_1 > X$, therefore $C(Y_1) > C(X) = C(Y_0)$, so $Y_0 \in L_C$ which completes the proof of $\mathcal{K} = \mathcal{Q}_C$.

Now, since

$$C(X) = \max_{P\in\mathcal{Q}_C} E_P(X)$$

and $\mathcal{Q}_C \cap \Delta^+ \neq \emptyset$, it is easy to see that C is a strictly positive linear form on F_C. □

Proof of Theorem 12 (⇒) Our assumption says that C is a super-replication price of a frictionless securities market of bets with the riskless bond, that is, there is a list of events $\{S, B_1, \ldots, B_m\}$ such that $X_j = B_j^*$ for all j, and:

$$\mathcal{Q} = \{P \in \Delta : P\left(B_j\right) = \mu_C\left(B_j\right),\ 0 \leq j \leq m\}.$$

We note that it is enough to show that acore$\left(\mu_C^*\right) \subset \mathcal{Q}$. So, assume that $P \in$ acore$\left(\mu_C^*\right)$ and let B_j be a basic bet, which is, of course, an unambiguous event for μ_C. By definition of μ_C^*, one has $\mu_C^*\left(B_j\right) = \mu_C\left(B_j\right)$, therefore $P\left(B_j\right) \leq \mu_C^*\left(B_j\right)$ implies $P\left(B_j\right) \leq \mu_C\left(B_j\right)$. Also, the same for B_j^c holds: $\mu_C^*\left(B_j^c\right) = \mu_C\left(B_j^c\right)$ and

$P\left(B_j^c\right) \leq \mu_C^*\left(B_j^c\right)$ implies $P\left(B_j^c\right) \leq \mu_C\left(B_j^c\right)$. By $P\left(B_j\right) + P\left(B_j^c\right) = 1 = \mu_C\left(B_j\right) + \mu_C\left(B_j^c\right)$, it turns out that $P\left(B_j\right) = \mu_C\left(B_j\right)$.

($\Leftarrow$) We need to prove that there exist $B_0, B_1, \ldots, B_m \in 2^S$ with $B_0 = S$, a strictly positive probability P_0 on 2^S such that $q_j := P_0\left(B_j\right) = C\left(B_j^*\right)$, for any $j \in \{0, 1, \ldots, m\}$, and $\forall X \in \mathbb{R}^S$

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : P\left(B_j\right) = C\left(B_j^*\right),\ 0 \leq j \leq m\}$.

Our assumption says that the set of extended risk-neutral probabilities $\mathcal{Q}$ satisfies

$$\mathcal{Q} = \left\{P \in \Delta : P(E) \leq \mu_C^*(E),\ \text{for all } E \subset S\right\}.$$

First, note that $\mathcal{E}_{\mu_C} \neq \emptyset$ because S is unambiguous. Consider

$$G := \operatorname{span}\left\{B^* : B \in \mathcal{E}_{\mu_C}\right\}.$$

Clearly, there is a basis of G given by a finite set $\left\{B_0^*, B_1^*, \ldots, B_m^*\right\}$ with $B_0 = S$ and $B_j \in \mathcal{E}_{\mu_C}$ for all j. Also, we set $q_j := \mu_C\left(B_j\right) = P_0\left(B_j\right) > 0$ for all j for some P_0 strictly positive (recall that $\mathcal{Q} \cap \Delta^+ \neq \emptyset$).

Now, suppose that $P \in \Delta$ is such that

$$P\left(B_j\right) = \mu_C\left(B_j\right) \quad \text{for all} \quad j.$$

Given an event E, there is an unambiguous event F such that $\mu_C^*(E) = \mu_C(F)$. Also, since $F^* \in G$, there is a set $J_F \subset \{0, 1, \ldots, m\}$ such that $F^* = \Sigma_{j \in J_F} B_j^*$. In fact, $\left\{B_j^*\right\}_{j \in J_F}$ is a collection of disjoint unambiguous events with $F = \cup_{j \in J_F} B_j$, which allows us to obtain that

$$\mu_C^*(E) = \mu_C(F) = P(F) \geq P(E).$$

That is, $P \in \operatorname{acore}\left(\mu_C^*\right)$.

For the converse, suppose that there is a probability P s.t. $P \leq \mu_C^*$, but such that there is some j with

$$P\left(B_j\right) \neq \mu_C\left(B_j\right).$$

Since

$$1 = P\left(B_j\right) + P\left(B_j^c\right) = \mu_C\left(B_j\right) + \mu_C\left(B_j^c\right),$$

we have

$$P\left(B_j\right) - \mu_C\left(B_j\right) = \mu_C\left(B_j^c\right) - P\left(B_j^c\right),$$

that is, $P\left(B_j\right) > \mu_C\left(B_j\right)$ iff $\mu_C\left(B_j^c\right) > P\left(B_j^c\right)$. Hence, $P \notin \text{acore}\left(\mu_C\right)$ and by the always true sentence

$$\mathcal{Q} \subset \text{acore}\left(\mu_C\right) \subset \text{acore}\left(\mu_C^*\right),$$

we conclude that

$$\mathcal{Q} \neq \text{acore}\left(\mu_C^*\right),$$

a contradiction. □

Proof of Corollary 14 By the Theorem 12, we known that a pricing rule of a frictionless market of bets is the upper probability w.r.t. acore (μ_C). On the other hand, by Azrieli and Lehrer (2007) and Lehrer (2009), the Lehrer integral is constant additive iff the capacity has large core (in our case, has no-gap), and in this case, for any contingent claim $X \in \mathbb{R}_+^S$

$$(\mathcal{L}) \int X \mathrm{d}\mu_C = \max_{P \in \text{acore}(\mu_C)} E_P\left(X\right);$$

hence,

$$C\left(X\right) = (\mathcal{L}) \int X \mathrm{d}\mu_C.$$

□

For the proof of Theorem 17, we need some previous results. We will see that the possibility of pricing rules of frictionless securities markets given by a Choquet integral is related to some strong condition on the set of attainable securities. For that we present the next well-known definition,

Definition 22 A Riesz subspace of $\mathbb{R}^S$ is a linear subspace F of $\mathbb{R}^S$ such that $X, Y \in F$ implies that $X \vee Y \in F$ and $X \wedge Y \in F$.

Next result shows that a Choquet pricing rule of a frictionless and arbitrage-free securities markets entails a strong condition on the set of attainable securities.

Lemma 23 *If a pricing rule C of a frictionless and arbitrage-free securities market is a Choquet integral then the induced ambiguous state price μ_C is a concave capacity and the induced subspace $F = F_C$ of attainable securities is a Riesz space.*

Proof First, we note that from Proposition 3 given by Schmeidler (1986) we have that if C is a subadditive Choquet integral with respect to the capacity μ_C, then μ_C is a concave capacity.

Let us now prove that F is a Riesz space.

Let $X, Y \in F$, then by Lemma 20, we have that for any $P \in \mathcal{Q}$, $E_P(X)+E_P(Y) = C(X) + C(Y)$. Since C is a Choquet Integral with respect to a concave capacity, it turns out that[32]

$$C(X) + C(Y) \geq C(X \vee Y) + C(X \wedge Y).$$

Therefore, using the previous equality

$$E_P(X \vee Y) + E_P(X \wedge Y) = E_P(X) + E_P(Y) \geq C(X \vee Y) + C(X \wedge Y).$$

But $E_P(X \vee Y) \leq C(X \vee Y)$ and $E_P(X \wedge Y) \leq C(X \wedge Y)$ for any $P \in \mathcal{Q}$. Hence, $E_P(X \vee Y) = C(X \vee Y)$ and $E_P(X \wedge Y) = C(X \wedge Y)$ for any $P \in \mathcal{Q}$ which implies by Lemma 20 that $X \vee Y$ and $X \wedge Y$ belongs to F. □

Another important result is.[33]

Lemma 24 *Let F be a Riesz subspace of $\mathbb{R}^n$ containing the unit vector $1_{\mathbb{R}^n} := (1, \ldots, 1) \in \mathbb{R}^n$ then F is a "partition" linear subspace of $\mathbb{R}^n$,* i.e.*, up to a permutation* [34]*:*

$$x \in F \textit{ iff } x = \left(x_1, \ldots x_1, \ldots, x_j, \ldots, x_j, \ldots, x_m, \ldots, x_m\right).$$

Proof The proof is by induction on the cardinality n of $S \geq 1$. Clearly, the result is true if $n = 1$; now, assume that the result is true for $n = k$ and let us show that it remains true for $n = k + 1$.

So let F be a subspace of $\mathbb{R}^{k+1}$ containing $1_{\mathbb{R}^{k+1}}$, and let G be defined by:[35]

$$G := \left\{ y = (x_1, \ldots, x_k) \in \mathbb{R}^k : \exists x_{k+1} \text{ s.t. } (y, x_{k+1}) \in F \right\}.$$

It is straightforward to check that G is a Riesz subspace of $\mathbb{R}^k$ containing $1_{\mathbb{R}^k}$, therefore by the induction hypothesis and up to a permutation $y \in G$ is equivalent to $y =$

32 See, for instance, Huber (1981, p. 260, 261).

33 We give a direct proof of this result, which also can be deduced directly from some results obtained by Polyrakis (1996, 1999). But, in order to derive this Lemma from Polyrakis's results we need to introduce a series of concepts that are beyond the scope of this work. See also Proposition 6 in Polyrakis and Xanthos (2011).

34 Now, we use the notation $1_{\mathbb{R}^n}$ and not $\left(\mathbb{R}^n\right)^*$ in order to avoid the confusion with the dual of $\mathbb{R}^n$. The notation A^* is used only for characteristic funtions induced by subsets of the state space S.

35 For $y = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $x_{k+1} \in \mathbb{R}$ we use the following notation:

$$\left(y, x_{k+1}\right) := \left(x_1, \ldots, x_k, x_{k+1}\right) \in \mathbb{R}^{k+1}.$$

$\left(x_1, \ldots, x_1, \ldots x_j, \ldots, x_j, \ldots, x_m, \ldots, x_m\right)$ where $x_j \in \mathbb{R}, 1 \leq j \leq m$. Clearly, if $x \in F$, then $x \in \widetilde{G} \oplus \widetilde{H}$, the direct sum of the linear subspaces of $\mathbb{R}^{k+1}$ given by

$$\widetilde{G} = \left\{(y, 0) \in \mathbb{R}^{k+1} : y \in G\right\}$$
$$\widetilde{H} = \left\{(0, \ldots, 0, x_{k+1}) \in \mathbb{R}^{k+1} : x_{k+1} \in \mathbb{R}\right\}.$$

Therefore, $\dim F \leq \dim \widetilde{G} \oplus \widetilde{H} = m + 1$. It is also immediate to see that $\dim F \geq m$; in fact, $y \in G$ is equivalent to

$$y = \sum_{j=1}^{m} x_j V_j^*,$$

where each $V_j^* \in \mathbb{R}^k$, i.e., $V_j \subset \{1, \ldots, k\}$, and $\left\{V_1^*, \ldots, V_m^*\right\}$ is a basis of G. Let $z_j \in \mathbb{R}$ be such that $(V_j^*, z_j) \in F, 1 \leq j \leq m$; it is immediate to see that $\left\{\left\{V_1^*\right\}, \ldots, \left\{V_m^*\right\}\right\}$ linearly independent in G implies $\left\{\left\{V_1^*, z_1\right\}, \ldots, \left\{V_m^*, z_m\right\}\right\}$ linearly independent in F, hence $\dim F \geq m$.

Two cases have to be examined:

(1) $\dim F = m + 1$: Clearly, since $F \subset \widetilde{G} \oplus \widetilde{H}$, this implies that $F = \widetilde{G} \oplus \widetilde{H}$ and F is a "partition" space.

(2) $\dim F = m$: In such a case, since $\left\{W_j^* := \left\{V_j^*, z_j\right\}, 1 \leq j \leq m\right\}$ is linearly independent in F, $\left\{W_j^* :, 1 \leq j \leq m\right\}$ is a basis of F. Hence, we obtain that $x \in F$ if and only if there are x_j, $1 \leq j \leq m$ such that $x = \sum\limits_{j=1}^{m} x_j W_j^*$, in particular,

$$x_{k+1} = \sum_{j=1}^{m} x_j z_j, \;\; (\Gamma).$$

So, it remains to show that there is $j_0 \in \{1, \ldots, m\}$ such that for any $x \in F$, it is possible to write $x = \sum_{j=1}^{m} x_j V_j^* + x_{j_0}$. Note that is enough to show that all the z_j's are equal to zero except $z_{j_0} = 1$. Since $1_{\mathbb{R}^{k+1}} \in F$ by the above property (Γ), we obtain that $\sum_{j=1}^{m} z_j = 1$.

Now take $j \neq i, \; j, i \in \{1, \ldots, m\}$. Since F is a Riesz space, $W_j^*, W_i^* \in F$ implies that $W_j^* \wedge W_i^* \in F$, but $W_j^* \wedge W_i^* = \left(\left(V_j \cap V_i\right)^*, z_j \wedge z_i\right)$ and $V_j \cap V_i = \emptyset$, hence by property (Γ), we obtain that $0 = \sum_{j=1}^{m} x_j z_j = z_j \wedge z_i$, therefore $z_j \geq 0$. On the other hand, the Riesz space structure implies also that $W_j^* \vee W_i^* \in F$, but $W_j^* \vee W_i^* = \left(1_{\mathbb{R}^{k+1}}, z_j \vee z_i\right)$l; hence, by property (Γ), we obtain that $z_j \vee z_i = z_j + z_i$.

Summing up, we have

$$\sum_{j=1}^{m} z_j = 1, \text{ therefore for any } j \neq i,\ \ j, i \in \{1, \ldots, m\} :$$
$$z_j \wedge z_i = 0 \text{ and } \ z_j \vee z_i = z_j + z_i;$$

this implies that there is a unique $j_0 \in \{1, \ldots, m\}$ such that $z_{j_0} = 1$ and for any $j \in \{1, \ldots, m\} \setminus \{j_0\}$ it is true that $j = 0$, the desired result. □

Proof of Theorem 17

$(i) \Rightarrow (ii)$ Since C is a Choquet pricing rule of a frictionless and arbitrage-free securities market, by Lemma 23, we know that the set of attainable securities F is a Riesz subspace of $\mathbb{R}^S$ containing the riskless bond S^*. Therefore, by Lemma 24, we obtain that F is a partition linear subspace of $\mathbb{R}^S$; hence, C is the super-replication price of a frictionless and arbitrage-free partition market.

$(ii) \Rightarrow (iii)$ By assumption, we have a partition $\{B_1, \ldots, B_m\}$ of the state space S and a strictly positive probability P_0 such that $P_0\left(B_j\right) = C\left(B_j^*\right)$ for any $j \in \{1, \ldots, m\}$.

Recall that,

$$\mathcal{Q} = \left\{P \in \Delta : P\left(B_j\right) = P_0\left(B_j\right), 1 \leq j \leq m\right\}$$

and

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

Clearly, we have that

$$\mathcal{Q} = \left\{P \in \Delta : \exists \left\{A_j\right\}_{j=1}^{m} \text{ s.t. } A_j \subset B_j \text{ and } P\left(A_j\right) = P_0\left(B_j\right), 1 \leq j \leq m\right\}.$$

Hence, given an arbitrary security X, consider $\left\{A_j\right\}_{j=1}^{m}$ where $A_j := \arg\max_{s \in B_j} X(s)$ and take a linear valuation $\widehat{P}$ such that $\widehat{P}\left(A_j\right) = \widehat{P}\left(B_j\right)$ for all j. Note that if $s \notin A_j$ for all j, then $\widehat{P}(\{s\}) = 0$.

Hence,

$$E_{\widehat{P}}(X) = \sum_{j=1}^{m} \sum_{s \in B_j} \widehat{P}(\{s\}) X(s) = \sum_{j=1}^{m} \sum_{s \in A_j} \widehat{P}(\{s\}) X(s) = \sum_{j=1}^{m} \widehat{P}\left(B_j\right) \max_{s \in B_j} X(s).$$

Now, for all $Q \in \mathcal{Q}$, we obtain that for any $j \in \{1, \ldots, m\}$

$$\sum_{s \in B_j} Q(\{s\}) X(s) \leq \sum_{s \in B_j} Q(\{s\}) \max X\left(B_j\right) = Q\left(B_j\right) \max X\left(B_j\right)$$
$$= \widehat{P}\left(B_j\right) \max X\left(B_j\right).$$

Hence,

$$E_Q(X) = \sum_{j=1}^{m} \sum_{s \in B_j} Q(\{s\}) X(s) \leq E_P(X), \forall Q \in \mathcal{Q},$$

which shows that $C(X) = \sum_{j=1}^{m} \widehat{P}\left(B_j\right) \max_{s \in B_j} X(s) = \sum_{j=1}^{m} P_0\left(B_j\right) \max_{s \in B_j} X(s)$.

$(iii) \Rightarrow (i)$ By our assumption, we have that there is a strictly positive probability P_0 and a partition $B_1, \ldots, B_m$ of S and such that $\forall X \in \mathbb{R}^S$

$$C(X) = \sum_{j=1}^{m} P_0(B_j) \max_{s \in B_j} X(s).$$

Hence, the induced ambiguous state price is given by the plausibility measure

$$\mu_C(A) = \sum_{k \in \left\{j : B_j \cap A \neq \emptyset\right\}} P_0\left(B_j\right),$$

and it is well known that

$$C(X) = (\mathcal{C}) \int X \mathrm{d}\mu_C,$$

i.e., C is a Choquet pricing rule. Now, it is easy to see that F_C is given by the partition linear subspace $span\left\{B_1^*, \ldots, B_m^*\right\}$. Now, consider a security Y and suppose that Y is not represented by a linear combination from the set $\left\{B_1^*, \ldots, B_m^*\right\}$. In this case,

$$C(Y) = \sum_{j=1}^{m} P_0(B_j) \max_{s \in B_j} Y(s),$$

and there is j and there are $r, \omega \in B_j$ such that $Y(r) > Y(\omega)$. Now, take Z such that for any $s \neq \omega$, $Y(s) = Z(s)$ and $Z(\omega) = Y(r)$. In this case, it is easy to see that $C(Z) = C(Y)$, that is, if $Y \notin \text{span}\left\{B_1^*, \ldots, B_m^*\right\}$ then $Y \notin L_C$. So, $L_C = F_C$ which implies that C is a Choquet pricing rule of a frictionless and arbitrage-free securities market.

Finally, we note that Lemma 24 given us the equivalence between the items (ii) and (iv), which complete the proofs of the desired equivalence.

Now, the fact that F_C is a Riesz linear subspace it follows from Lemma 23 or Lemma 24, as we see just before. The fact that μ_C is concave follows as in the Lemma 23. From Nehring (1999), since μ_C is concave, we obtain that $\mathcal{E}_{\mu_C}$ is a Boolean algebra. It remains to show that $\mu_C^* \leq \mu_C$. First, let us show that μ_C^* is concave: Let A_1, A_2 be subsets of S, by definition of μ_C^*, there exist $B_1 \supset A_1$ and $B_2 \supset A_2$, $B_i \in \mathcal{E}_{\mu_C}$ such that $\mu_C^*(A_i) = \mu_C(B_i), i = 1, 2$. Hence, $\mu_C^*(A_1) + \mu_C^*(A_2) = \mu_C(B_1) + \mu_C(B_2) \geq \mu_C(B_1 \cup B_2) + \mu_C(B_1 \cap B_2)$. Since $B_1 \cup B_2, B_1 \cap B_2 \in \mathcal{E}_{\mu_C}$, $B_1 \cup B_2 \supset A_1 \cup A_2$ and $B_1 \cap B_2 \supset A_1 \cap A_2$, it turns out that $\mu_C^*(A_1) + \mu_C^*(A_2) \geq \mu_C^*(A_1 \cup A_2) + \mu_C^*(A_1 \cap A_2)$. Let $A \subset S$, μ_C^* concave implies that there is a probability $P \in \text{acore}\left(\mu_C^*\right)$, but Theorem 12 guarantees that $\text{acore}(\mu_C) = \text{acore}\left(\mu_C^*\right)$ hence $P \in \text{acore}(\mu_C)$, therefore

$$\mu_C^*(A) = P(A) \leq \mu_C(A).$$

□

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