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RESEARCH ARTICLE

# Taxation without commitment

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**Abstract** This paper considers a Ramsey model of linear capital and labor income taxation in which a benevolent government cannot commit ex-ante to a sequence of policies for the future. In this setup, if the government is forced to keep budget balance in every period, then it may not be able to sustain zero capital taxes in the long run, as shown in Benhabib and Rustichini (J Econ Theory 77:231–259, 1997) and Phelan and Stachetti (Econometrica 69:1491–1518, 2001). However, (Dominguez in J Econ Theory 135:159–170, 2007) shows that if the government is allowed to borrow and lend to households, the optimal capital income tax still converges to zero in the long run, as long as the value of defaulting is independent of the level of government debt. This paper provides a game theoretic setup with government debt where the value of the worst equilibrium only depends on the initial level of capital and can be determined in advance. This implies that under our assumptions the best sustainable equilibrium has zero capital taxes in the long run, even in the absence of government commitment.

**Keywords** Fiscal policy · Optimal taxation · Debt · Game theory

**JEL Classification** E62 · H21 · H63 · C7

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## 1 Introduction

A long standing question in the optimal taxation literature concerns the extent to which capital taxes should be used to finance public spending. While in the short run, it is optimal to tax capital to collect costless revenue from a sunk investment, in the long run, using this source of taxation will distort the accumulation of capital. Chamley (1986) and Judd (1985) show that, in an economy of infinitely lived agents, the optimal capital income tax is zero in steady state.[1]

It has been believed that the zero capital tax result critically hinges on the ability of the government to commit ex-ante to a sequence of future taxes. Namely, Judd (1985) says that his "results indicate that redistribution of income through capital income taxation is effective only if it is unanticipated and will persist only if policy-makers cannot commit themselves to low taxation in the long run." Later work by Benhabib and Rustichini (1997) and Phelan and Stacchetti (2001) confirms this intuition by finding that when commitment binds, the optimal capital income tax may not converge to zero in the long run.

However, if the government does not have to keep budget balance and is allowed to borrow and lend to households, then it can use asset accumulation as a commitment device, so that in the long run the lack of commitment is no longer binding. Dominguez (2007), who analyzes the model in Benhabib and Rustichini (1997) for the case where government bonds are allowed, shows that capital taxes are zero in steady state as long as the value of defaulting only depends on the level of capital and not on the level of debt.

This paper contributes to the existing literature by setting up a game where government bonds are allowed and showing that the value of the worst sustainable equilibrium is independent of their level, as was assumed by Dominguez (2007). This feature is not trivial to obtain, since we want to allow households to be indebted and repay their debt in equilibrium, but the value of the worst equilibrium cannot depend on the level of their debt. For a discussion of this issue, see Chari and Kehoe (1993a,b) who use a setup without capital to model debt default. They allow the government to default, but they either assume that households can always commit to their debt, or debt repayment can never be enforced. This paper, on the other hand, allows households to default, but it also introduces an enforcement mechanism that makes household default nontrivial.

A similar commitment mechanism was studied in Gobert and Poitevin (2006), who analyze risk sharing contracts with noncommitment and savings. They find that savings can be used as a commitment mechanism that enables agents to overcome the lack of commitment. The same is true in this paper, where the government uses asset accumulation to relax its incentive compatibility constraint.

To model the interactions between the government and households in the absence of commitment, we follow a game theoretic approach that builds on the stream of literature developed by Chari and Kehoe (1990) on sustainable equilibria. This concept allows a more parsimonious definition of subgame-perfect equilibria when some agents are too small to behave strategically.

[1] Chamley (1986) actually shows that if preferences are separable and iso-elastic in consumption, the optimal capital tax is zero in finite time.

On a more technical side, this paper provides a setup where the worst sustainable equilibrium can be determined in advance, which allows a clearer characterization of the set of equilibria of the game. Benhabib and Rustichini (1997) derive the best policy without commitment assuming that the worst punishment is known and given by an exogenous "value of default". Phelan and Stacchetti (2001) argue that this is not always the case since the government's incentive constraint usually binds in the worst equilibrium, which means that the worst punishment has to be determined endogenously. This paper provides sufficient conditions for these two approaches to be equivalent. In particular, if the government is allowed to make lump-sum transfers to consumers, which is a common assumption in most taxation models, then it is credible to give the households the worst possible expectations regarding future capital taxes, since it is incentive compatible for the government to tax the initial sunk capital at expropriatory rates, given that any remaining revenue can be redistributed to households as a lump-sum transfer. Thus, no additional incentives need to be given for the government to act according to the households' expectations, which means that the continuation of a worst equilibrium is still a worst equilibrium in this model.

The paper proceeds as follows. Section 2 sets up the model and derives the optimal policy plan with full commitment, which corresponds to the Chamley (1986) and Judd (1985) solution. Section 3 sets up the game without commitment and characterizes the set of sustainable equilibria, while deriving the main result of the paper that the worst sustainable equilibrium is independent of the level of government debt. Section 4 characterizes the best sustainable equilibrium and derives its steady state properties. Section 5 concludes with a brief summary of the main findings of the paper. Some of the proofs can be found in the "Appendix".

## 2 The Ramsey economy

### 2.1 Model setup

The economy has a continuum of measure one of infinitely lived identical households, an arbitrary number of firms that behave competitively, and a benevolent government. Time is discrete. There is only one good in the economy, which can be used as consumption good, public good or capital.

#### *2.1.1 Households*

Households derive utility from consumption $c_t$, labor $n_t$, and consumption of a public good $g_t$. They discount the future at rate $\beta$, with $0 < \beta < 1$, so that each household's lifetime utility is given by

$$\sum_{t=0}^{\infty} \beta^t [u(c_t, n_t) + v(g_t)].$$

**Assumption 1** The function $u$ is increasing in consumption, decreasing in labor, globally concave and meets the following Inada conditions $u_c(0, n) = -u_n(c, \infty) = \infty$

and $u_c(\infty, n) = u_n(c, 0) = 0$. The utility of the public good $v$ is increasing and concave, with $v'(0) = \infty$ and $v'(\infty) = 0$.

The amount of public good consumed in each period is decided by the government.

For each unit of work, households receive after tax wages of $w_t(1-\tau_t^n)$. Households rent their capital $k_t$ to firms for which they receive an after tax return of $r_t(1-\tau_t^k)$. Assume that capital is fully depreciated. This assumption is made for simplicity and does not affect the qualitative results of the paper. All taxes have to be lower than one, but they are allowed to be negative, so that $\tau_t^s \leq 1$ for $s = n, k$. The government can choose to make lump-sum transfers $T_t$ to the households, which must always be weakly positive.

Each period, the government chooses an upper bound $I_t$ on the investment each household is allowed to make, so that the following investment constraint is met every period

$$k_{t+1} \leq I_t. \tag{1}$$

This assumption does not change the set of attainable allocations, but it guarantees that the worst equilibrium is well defined, as we will see later.

The government can borrow or lend to households using government bonds. A bond that pays one unit of consumption good in period $t+1$ costs $q_t$ units of consumption good in period $t$. The government may decide to default on its bonds. Let $d_t$ be an indicator function for whether the government defaults on its current bonds. Households can also default on bonds, but there is an independent agency (for example, courts) that will enforce repayment as long as the percentage of households defaulting is lower than $\nu$, with $0 < \nu < 1$. If repayment is enforced, households have to pay their initial debt plus an administrative cost $\xi > 0$. Let $\delta_t$ be the percentage of consumers who default in period $t$. If households have positive bond holdings $b_t \geq 0$, they will never default, so $\delta_t = 0$. Otherwise, households will choose to default when a sufficiently large mass of households is defaulting, so that there are only two possible equilibrium values for household default: $\delta_t = 0$, where no one defaults, or $\delta_t = 1$, where everyone defaults. Furthermore, it is always a dominant strategy for households to default when $\delta_t = 1$ and not to default when $\delta_t = 0$, so that individual default decisions do not need to be modeled explicitly.

Thus, households must meet the following budget constraint every period, as well as a no-ponzi condition that guarantees that the value of their debt remains bounded. Since the marginal utility of consumption is always positive, we can assume, without loss of generality, that the budget constraint is always met with equality

$$w_t(1-\tau_t^n)n_t + r_t(1-\tau_t^k)k_t + b_t(1-d_t)(1-\delta_t) + T_t = c_t + k_{t+1} + q_t b_{t+1}. \tag{2}$$

Households choose consumption, labor, and investment in capital and bonds to maximize the present value of utility subject to their budget constraints, the investment constraints and no-ponzi condition for given $k_0$ and $b_0$.

Given that utility is concave and the constraint set is convex, the following first-order conditions, together with the households' budget constraints and no-ponzi condition, are necessary and sufficient for household optimality as long as the investment constraint (1) is not binding

$$u_n(c_t, n_t) + w_t(1 - \tau_t^n)u_c(c_t, n_t) = 0 \quad (3)$$

$$k_{t+1}[u_c(c_t, n_t) - \beta r_{t+1}(1 - \tau_{t+1}^k)u_c(c_{t+1}, n_{t+1})] = 0 \quad (4)$$

$$q_t u_c(c_t, n_t) - \beta(1 - d_{t+1})(1 - \delta_{t+1})u_c(c_{t+1}, n_{t+1}) = 0. \quad (5)$$

This fully characterizes the households' optimal response to a given path of policies and prices.[2]

### 2.1.2 Government

The government is benevolent, which means that it maximizes the utility of a representative household. It receives revenue from proportional taxes on labor income $\tau_t^n$ and capital income $\tau_t^k$ each period. The government needs to finance its chosen amount of the public good $g_t$ and the lump-sum transfers to the households $T_t \geq 0$. It chooses the upper bound $I_t$ on the investment each household is allowed to make in the current period and which can take any nonnegative value. It decides whether to default on its debt $b_t$ and chooses the price of debt for the following period. Given this, the government must meet the following budget constraint every period, as well as a no-ponzi condition that guarantees it cannot perpetually borrow to finance past debt

$$g_t + T_t + b_t(1 - d_t)(1 - \delta_t) = w_t\tau_t^n n_t + r_t\tau_t^k k_t + q_t b_{t+1}. \quad (6)$$

### 2.1.3 Firms and market clearing

Each period, firms maximize profits given the prices for labor $w_t$ and capital $r_t$ and have access to the production function $F(k_t, n_t)$.

**Assumption 2** The function $F$ has constant returns to scale, is increasing and concave in both factors of production and meets the following conditions $F_k(0, 1) = \infty$ and $F_k(\infty, 1) = 0$.

Firm profits are thus given by

$$\pi = F(k_t, n_t) - r_t k_t - w_t n_t.$$

Firm optimality and market clearing imply that factor prices are given by the marginal productivity of each factor

$$w_t = F_n(k_t, n_t) \quad (7)$$

$$r_t = F_k(k_t, n_t). \quad (8)$$

[2] Whenever the investment constraint (1) is binding, replace condition (4) with condition $k_{t+1} = I_t$ and the inequality constraint $u_c(c_t, n_t) \leq \beta r_{t+1}(1 - \tau_{t+1}^k)u_c(c_{t+1}, n_{t+1})$.

Firm ownership is irrelevant since the constant returns to scale technology ensures that profits are zero.

In the goods market, the following resource constraint must be met every period

$$c_t + g_t + k_{t+1} = F(k_t, n_t). \tag{9}$$

Market equilibrium also determines the percentage of households defaulting. As discussed before, this variable can take the following values

$$\delta_t = \begin{cases} 0 & \text{if } b_t > 0 \\ 0 \text{ or } 1 & \text{if } b_t < 0 \end{cases}. \tag{10}$$

## 2.2 Attainable allocations

Let $\pi = (\pi_0, \pi_1, \ldots)$ denote the sequence of government policies $\pi_t = (\tau_t^n, \tau_t^k, T_t, I_t, q_t, d_t, g_t)$, let $x = (x_0, x_1, \ldots)$ denote the sequence of household choices $x_t = (c_t, n_t, k_{t+1}, b_{t+1})$, and let $p = (p_0, p_1, \ldots)$ denote the sequence of market clearing prices $p_t = (r_t, w_t, \delta_t)$.

A *competitive equilibrium* is a triplet $(\pi, x, p)$ that satisfies the following conditions: $(i)$ given $\pi$ and $p$, households choose $x$ to maximize utility subject to their budget constraints (2), the investment constraints (1) and no-ponzi condition; $(ii)$ the government meets its budget constraints (6) and no-ponzi condition with $T_t \geq 0$; $(iii)$ firm optimality and market clearing conditions (7–10) are met.

Let us start by considering the case where the investment constraint (1) is not binding. If this is the case, then a triplet $(\pi, x, p)$ is a competitive equilibrium if it meets Eqs. (2–5) and Eqs. (7–10) with $T_t \geq 0$, as well as a transversality conditions that ensures that the value of assets and debt remain bounded in the long run. We do not need to impose condition (6), since it is always met when conditions (2) and (9) are verified.

Following the approach developed by Lucas and Stokey (1983), we can simplify these conditions by multiplying the households's budget constraint by the marginal utility of consumption and plugging in the households's optimality conditions to get the following implementability condition

$$\begin{aligned} &u_n(c_t, n_t)n_t + u_c(c_t, n_t)c_t + \beta u_c(c_{t+1}, n_{t+1})r_{t+1}(1 - \tau_{t+1})k_{t+1} \\ &\quad + \beta u_c(c_{t+1}, n_{t+1})(1 - d_{t+1})(1 - \delta_{t+1})b_{t+1} \\ &= u_c(c_t, n_t)T_t + u_c(c_t, n_t)r_t(1 - \tau_t)k_t + \beta u_c(c_t, n_t)(1 - d_t)(1 - \delta_t)b_t \end{aligned}$$

Define $m(c_t, n_t) \equiv u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t$ and make a change of variables to replace nominal bonds $b_t$ for the value of consumer assets $a_t \equiv u_c(c_t, n_t)r_t(1 - \tau_t^k)$ $k_t + u_c(c_t, n_t)b_t$ $(1 - d_t)(1 - \delta_t)$ to write the implementability condition as

$$m(c_t, n_t) + \beta a_{t+1} = u_c(c_t, n_t)T_t + a_t. \tag{11}$$

On the other hand, if the investment constraint (1) is binding, then condition (4) is replaced by condition $k_{t+1} = I_t$ and the inequality constraint $u_c(c_t, n_t) \leq \beta r_{t+1}(1 - \tau_{t+1}^k)u_c(c_{t+1}, n_{t+1})$. For this case, the implementability condition is given by

$$m(c_t, n_t) + \beta a_{t+1} \geq u_c(c_t, n_t)T_t + a_t.\text{[3]} \tag{12}$$

Let $y = (y_0, y_1, \ldots)$ denote the sequence of allocations $y_t = (c_t, g_t, n_t, k_t, a_t)$. An allocation $y$ is *attainable* if there are prices and policies such that it will be the outcome of a competitive equilibrium for a given initial level of capital and government debt.

**Lemma 1** *An allocation $y$ is attainable if and only if it meets the following conditions for $t \geq 0$*

$$m(c_t, n_t) + \beta a_{t+1} \geq a_t \tag{13}$$

$$c_t + g_t + k_{t+1} = F(k_t, n_t), \tag{14}$$

with $\lim_{t\to\infty} \beta^t a_{t+1} = 0$ *and given* $k_0$ *and* $a_0 \geq u_c(c_0, n_0)\min(0, b_0)$.

A proof of this lemma can be found in the appendix, where it is shown that any attainable allocation can be implemented using the following taxes

$$\tau_{t+1}^k = 1 - \frac{1}{\beta F_k(k_{t+1}, n_{t+1})} \frac{u_c(c_t, n_t)}{u_c(c_{t+1}, n_{t+1})}$$

$$\tau_t^n = 1 + \frac{1}{F_n(k_t, n_t)} \frac{u_n(c_t, n_t)}{u_c(c_t, n_t)}.$$

This characterization of attainable allocations will be useful to determine the optimal policies with and without commitment. In particular, it will allow us to write the problem recursively using as state variables the level of capital and the value of consumer assets. We can guarantee that the transversality condition is met by constraining $a$ to always be below the natural debt limit $\widehat{a}(k)$,[4] so from now on the implementability condition and the resource constraint will be used as necessary and sufficient conditions for attainability, with the underlying condition that $a \leq \widehat{a}(k)$.

## 2.3 Optimal taxes with commitment

This section derives the optimal policy plan when the government can choose the policies for all future periods at time zero. It introduces a recursive formulation of the problem (that will also be used for the no commitment case) to derive the benchmark Chamley (1986) and Judd (1985) result of zero capital taxes in the long run.

The Ramsey problem selects the attainable allocation that maximizes the welfare of the representative consumer. The outcome of a *Ramsey equilibrium* is a sequence $y$ that maximizes the present value of utility $\sum_{t=0}^{\infty} \beta^t [u(c_t, n_t) + v(g_t)]$ subject to $y$ being attainable and given an initial stock of capital $k_0$ and an initial level of debt $b_0$. Given this, we can write the Ramsey problem using the following sequence formulation

[3] When the investment constraint is binding, the same allocation can be achieved through an increase in capital taxes with the additional revenue redistributed in a lump-sum fashion, which is why this assumption does not change the set of attainable allocations.

[4] For a given level of $k$, the maximum level of $a$ that can be repaid by the government is $\widehat{a}(k) \equiv \max \sum_{t=0}^{\infty} \beta^t m(c_t, n_t)$ subject to $c_t + k_{t+1} = F(k_t, n_t)$ and $k_0 = k$.

$$\begin{aligned}
&\max_{\{x,g\}} \sum_{t=0}^{\infty} \beta^t [u(c_t, n_t) + v(g_t)] \\
\text{subject to}\quad & m(c_t, n_t) + \beta a_{t+1} \geq a_t \\
& c_t + g_t + k_{t+1} = F(k_t, n_t) \\
& a_0 \geq u_c(c_0, n_0) \min(0, b_0).
\end{aligned}$$

This problem can also be written recursively using $a$ and $k$ as state variables. The Bellman equation for this problem is given by[5]

$$\begin{aligned}
V(k, a) &= \max_{x,g} [u(c, n) + v(g) + \beta V(k', a')] \\
\text{subject to}\quad & m(c, n) + \beta a' \geq a \\
& c + g + k' \leq F(k, n).
\end{aligned}$$

If $m(c, n)$ is concave, then the constraint set is convex, which means that the value function $V(k, a)$ will be concave and the first-order conditions are necessary and sufficient for optimality. If this condition is not met, first-order conditions are still necessary for an optimum, but no longer sufficient, since it is also necessary to verify that the second-order conditions are met to make sure we are at a maximum.

Using $\mu$ as the multiplier on the implementability condition and $\rho$ as the multiplier on the resource constraint, we can write the Lagrangian for this problem in the following way

$$L = u(c, n) + v(g) + \beta V(k', a') + \mu[m(c, n) + \beta a' - a] - \rho[c + g + k' - F(k, n)].$$

Combining the first-order conditions for $k'$ and $a'$ with the envelope conditions for $k$ and $a$, we obtain the following equations

$$\begin{aligned}
V_k(k\ , a\ ) &= \beta F_k(k, n) V_k(k', a') \\
V_a(k', a') &= V_a(k, a) = \mu.
\end{aligned}$$

A steady state for the Ramsey economy has constant $c, n, g, k$, and $a$ as well as constant multipliers $\mu$ and $\rho$. From the optimality condition for $k$, it is clear that in steady state $\beta F_k = 1$. This is the well-known Chamley (1986) and Judd (1985) result that capital income taxes converge to zero in the long run, while labor taxes remain positive

$$\begin{aligned}
\tau^k &= 1 - \frac{1}{\beta F_k} \frac{u_c}{u_c'} = 0 \\
\tau_t^n &= 1 + \frac{1}{F_n} \frac{u_n}{u_c} = 1 - \frac{1 + \mu m_c / u_c}{1 + \mu m_n / u_n} > 0.
\end{aligned}$$

[5] For the initial period, the implementability condition is given by $m(c, n) + \beta a' \geq u_c(c, n) \min(0, b_0)$.

## 3 Sustainable equilibria without commitment

This section introduces lack of commitment by modeling the taxation problem as a game where the government and the agents in the economy make sequential decisions every period. The equilibrium concept is defined, and a characterization of equilibrium outcomes is formalized based on a maximum threat point of reversion to the worst equilibrium, in the spirit of Abreu (1988).

### 3.1 Game setup

This section introduces a game where the lack of commitment is modeled explicitly and the value of default that sustains the initial plan is determined endogenously. Since households and firms behave competitively, whereas the government behaves strategically, we will use the notion of sustainable equilibria introduced by Chari and Kehoe (1990), where all strategies only depend on the past history of the government's actions. This equilibrium definition is equivalent to subgame perfection, but does not allow strategies to be contingent on the individual actions of other infinitesimal agents, since they cannot be observed individually. Furthermore, we do not impose renegotiation proofness, since this would imply that all equilibria which are not the best one would be renegotiated, since the households and the government have the exact same preferences, which would destroy all the possibilities for punishing bad behavior by reverting to a worse equilibrium.

The timing of the game is as follows. At the beginning of period $t$, the government decides $\pi_t = (\tau_t^n, \tau_t^k, T_t, I_t, q_t, d_t, g_t)$. After observing this, households make their choices $x_t = (c_t, n_t, k_{t+1}, b_{t+1})$ given market clearing prices $p_t = (r_t, w_t, \delta_t)$.

Let $h_t$ be the history of government decisions until time $t$ so that $h_t = (\pi_0, \ldots, \pi_t)$. Following Chari and Kehoe (1990), all the strategies in the game will be contingent only on this history, since households are infinitesimal and have no power to influence the observable aggregate variables, which means that they will not behave strategically. Thus, knowing the households' strategies and government's actions until time $t$ is enough to characterize all the history until then.

The strategy for the government is given by $\sigma$. The strategy for each period $t$ is a mapping from the history $h_{t-1}$ into the government's decision space $\pi_t$, so that $\pi_t = \sigma_t(h_{t-1})$. When choosing a given strategy, the government anticipates that histories will evolve according to $h_t = (h_{t-1}, \sigma_t(h_{t-1}))$. Let $\sigma^t$ denote the sequence of government strategies from time $t$ onward.

The strategy for a representative household is given by $f$. The strategy for each period $t$ is a mapping from the history $h_t$ into the households' decision space $x_t$, so that $x_t = f(h_t)$. Let $f^t$ denote the sequence of household strategies from time $t$ onward.

Firms and markets jointly work as a third player that has strategy $\phi$ mapping the history $h_t$ into the vector of market clearing prices $p_t$, so that $p_t = \phi(h_t)$. Let $f^t$ denote the sequence of market clearing prices from time $t$ onward.

In the next section, we will specify how each player chooses its strategy in a sustainable equilibrium.

### 3.2 Sustainable equilibria

At time $t$, the government and households choose an action for time $t$ and a contingent plan for the future. This is equivalent to choosing an action for today while anticipating future behavior, since both the government and households have time consistent preferences, which means that the plan they choose today will be optimal tomorrow. The problem solved by the government and households at time $t$ is described later.

For every history $h_{t-1}$, given allocation rule $f$ and pricing rule $\phi$, the government chooses $\sigma^t$ to maximize the present value of utility

$$\sum_{s=t}^{\infty} \beta^{s-t}[u(c_s(h_s), n_s(h_s)) + v(g_s(h_{s-1}))]$$

subject to

$$\begin{aligned} g(h_{s-1}) + T_s(h_{s-1}) &= w_s(h_s)\tau_s^n(h_{s-1})n_s(h_s) + r_s(h_s)\tau_s^k(h_{s-1})k_s(h_{s-1}) \\ &\quad + q_t(h_{s-1})b_{s+1}(h_s) - b_s(h_{s-1})(1-\delta_t(h_s))(1-d_t(h_{s-1})) \\ T_s(h_{s-1}) &\geq 0 \end{aligned}$$

and realizing that future histories are induced by $\sigma^t$ according to $h_s = (h_{s-1}, \sigma_s(h_{s-1}))$.

For every history $h_t$, given policy rule $\sigma^t$ (and the histories it induces) and pricing rule $\phi$, each household chooses $f^t$ to maximize the present value of utility

$$\sum_{s=t}^{\infty} \beta^{s-t} u(c_s^i(h_s), n_s^i(h_s))$$

subject to

$$\begin{aligned} c_s^i(h_s) &= w_s(h_s)(1-\tau_s^n(h_{s-1}))n_s^i(h_s) + T_s(h_{s-1}) \\ &\quad + b_s^i(h_{s-1})(1-\delta_t(h_s))(1-d_s(h_{s-1})) - q_s(h_{s-1})b_{s+1}^i(h_s) \\ &\quad + r_s(h_s)(1-\tau_s^k(h_{s-1}))k_s^i(h_{s-1}) - k_{s+1}^i(h_s). \\ k_{s+1}^i(h_s) &\leq I_t(h_{s-1}). \end{aligned}$$

Market clearing and firm optimality require that for every history $h_t$ firm demand must equal household supply for every production factor, which happens when factor prices equal their marginal productivity. Depending on the history of the game, households may coordinate on different equilibria for default on bonds, so that $\phi_t(h_t)$ is given by

$$\begin{aligned} w_t(h_t) &= F_n(k_t(h_{t-1}), n_t(h_t)) \\ r_t(h_t) &= F_k(k_t(h_{t-1}), n_t(h_t)) \end{aligned}$$

$$\delta_t(h_t) = \begin{cases} 0 & \text{if } \ b_t(h_{t-1}) > 0 \\ 0 \quad \text{or } 1 & \text{if } \ b_t(h_{t-1}) < 0 \end{cases}.$$

A *sustainable equilibrium* is a triplet $(\sigma, f, \phi)$ that satisfies the following conditions: (i) given $f$ and $\phi$, the continuation of contingent policy plan $\sigma$ solves the government's problem for every history $h_{t-1}$; (ii) given $\sigma$ and $\phi$, the continuation of contingent allocation rule $f$ solves the households' problem for every history $h_t$; (iii) given $f$ and $\sigma$, the continuation of the contingent pricing rule $\phi$ is such that markets clear for every history $h_t$.

### 3.3 Worst sustainable equilibrium

Let $W(\sigma, f, \phi \,|k_0, b_0)$ denote the present value of utility that results from a sustainable equilibrium $(\sigma, f, \phi \,|k_0, b_0)$. Sustainable equilibria can be ranked according to the value they induce. In this section, we determine the lower bound on the value of a sustainable equilibrium given initial conditions. This is important because the worst equilibrium can work as a credible threat that dissuades the government from defaulting on its promises, which will allow us to sustain other equilibria.

Consider a *default equilibrium* where households always (for every time $s$ starting today and after every history) expect to face $\tau^k_{s+1} = d_{s+1} = 1$ and $\delta_s = 1$ if $b_s < 0$. Based on these expectations, we will determine the value of the default equilibrium (which is independent of the initial level of bonds), show that it is a sustainable equilibrium and that no other sustainable equilibrium can attain a value lower than the default equilibrium.

When households face expectations $\tau^k_{s+1} = d_{s+1} = 1$ and $\delta_s = 1$ if $b_s < 0$ for $s \geq t$, after every history, their budget constraint for the current period $t$ is

$$c_t = w_t(1-\tau^n_t)n_t + T_t + \max(0, b_t)(1-d_t) - q_t b_{t+1} + r_t(1-\tau^k_t)k_t - k_{t+1},$$

and their expected budget constraint for the future is

$$c_s(h_s) = w_s(h_s)(1-\tau^n_s(h_{s-1}))n_s(h_s) + T_s(h_{s-1}) - q_s(h_{s-1})b_{s+1}(h_s) - k_{s+1}(h_s).$$

Thus, none of the decision variables today affects the constraints in the future, which means that when deciding $x_t = (c_t,\ n_t,\ k_{t+1},\ b_{t+1})$, households only maximize the current utility $u(c_t, n_t)$ subject to today's budget constraint, which implies that it is always optimal to choose $b_{t+1} = k_{t+1} = 0$ regardless of the value of $I_t$. Households thus solve the following problem

$$\max_{c_t, n_t} u(c_t, n_t) \text{ subject to } c_t = w_t(1-\tau^n_t)n_t + T_t + \max(0, b_t)(1-d_t) + r_t(1-\tau^k_t)k_t.$$

The solution to this problem is given by the budget constraint together with the following first-order condition

$$u_n(c_t, n_t) + w_t(1-\tau^n_t)u_c(c_t, n_t) = 0.$$

Market clearing conditions are met with $w_t = F_n(k_t, n_t)$, $r_t = F_k(k_t, n_t)$, $\delta_t = 0$ if $b_t \geq 0$ and $\delta_t = 1$ if $b_t < 0$.

Thus, in any given period, the market outcome only depends on the policies chosen for that period (and not on any past history), and there is no intertemporal link between periods, since both capital and bonds are chosen to be zero.

Since the future welfare is independent of today's actions, in each period, the government will choose actions to maximize the current period's utility by solving the following problem

$$\begin{aligned} & \max u(c_t, n_t) + v(g_t) \\ \text{subject to} \quad & c_t = w_t(1-\tau_t^n)n_t + T_t + \max(0, b_t)(1-d_t) + r_t(1-\tau_t^k)k_t \\ & u_n(c_t, n_t) + w_t(1-\tau_t^n)u_c(c_t, n_t) = 0 \\ & w_t = F_n(k_t, n_t), r_t = F_k(k_t, n_t) \\ & c_t + g_t = F(k_t, n_t). \end{aligned}$$

Notice that in this problem it is always a best response for the government to choose $d_t = \tau_t^k = 1$, since this can always be offset by increasing the lump-sum transfers. Given this, we can summarize the first three constraints using the following implementability condition

$$u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t = T_t u_c(c_t, n_t) \geq 0.$$

The flow utility in each period is thus given by

$$\begin{aligned} U^d(k_t) = & \max_{c_t, n_t, g_t} u(c_t, n_t) + v(g_t) \\ \text{subject to} \quad & u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t \geq 0 \\ & c_t + g_t = F(k_t, n_t). \end{aligned}$$

Since there is no new capital investment, after the first period there is no capital, which means that the present value of this equilibrium is given by

$$V^d(k_t) = U^d(k_t) + \frac{\beta}{1-\beta}U^d(0),$$

which is independent of the initial level of debt.

We must also verify that our initial assumptions for the households' expectations are correct to make sure that the default equilibrium is a sustainable equilibrium. Indeed, along the equilibrium path, it is always a best response for the government to choose $d_t = \tau_t^k = 1$, which means that the households' beliefs were consistent.

So far, we have not specified the values of $q_t$ and $I_t$ in the default equilibrium. That is, because the strategies and beliefs specified are an equilibrium for *any* values of $q_t$ and $I_t$. To see this, notice that the amount invested in bonds and capital is always zero no matter what these values are. In particular, we will choose our default equilibrium to have $q_s = I_s = 0$ for every period $s$ and after every history, which will be useful to prove the more tricky issue that this is indeed the worse equilibrium. As was shown by Phelan and Stacchetti (2001), it need not be the case in general that the worst equilibrium can be determined independently of the rest of the game. However, for our setup, this turns out to be the case, as is shown in the proposition later.

**Proposition 1** *The default equilibrium is the worst sustainable equilibrium, i.e.,* $W(\sigma, f, \phi\,|k_0, b_0) \geq W(\sigma^d, f^d, \phi^d\,\left|k_0, b_0'\right.)$ *for any sustainable equilibrium* $(\sigma, f, \phi\,\left|k_0, b_0'\right.)$ *with the same initial amount of capital.*

*Proof* To prove this lemma, we will show that for any sustainable equilibrium $(\sigma, f, \phi\,|k_0, b_0)$, it is true that $W(\sigma, f, \phi\,|k_0, b_0) \geq W(\sigma^d, f, \phi\,|k_0, b_0)$ and $W(\sigma^d, f, \phi\,|k_0, b_0) = W(\sigma^d, f^d, \phi^d\,\left|k_0, b_0'\right.)$. Then, it must be the case that the default equilibrium is the worst sustainable equilibrium since $W(\sigma, f, \phi\,|k_0, b_0) \geq W(\sigma^d, f^d, \phi^d\,|k_0, b_0)$.

The fact that $W(\sigma, f, \phi\,|k_0, b_0) \geq W(\sigma^d, f, \phi\,|k_0, b_0)$ follows from the fact that in a sustainable equilibrium $(\sigma, f, \phi\,|k_0, b_0)$, the government must be best responding to the households' strategy and market prices. Thus, if the government chooses $\sigma$ when $\sigma^d$ is available, it must be true that $W(\sigma, f, \phi\,|k_0, b_0) \geq W(\sigma^d, f, \phi\,|k_0, b_0)$.

To show that $W(\sigma^d, f, \phi\,|k_0, b_0) = W(\sigma^d, f^d, \phi^d\,\left|k_0, b_0'\right.)$, we will verify that the allocations that result from $(\sigma^d, f, \phi\,|k_0, b_0)$ are exactly the same as the equilibrium outcome from $(\sigma^d, f^d, \phi^d\,\left|k_0, b_0'\right.)$. Thus, if the government plays the default strategy when households are playing any other strategy, the level of welfare is exactly equal to that of the default equilibrium. This is because by choosing $I_t(h_{t-1}) = 0$ and $q_t(h_{t-1}) = 0$, the government ensures that households do not invest in capital or bonds. By choosing $\tau_t^k(h_{t-1}) = 1$ and $d_t(h_{t-1}) = 1$, the government ensures that the value of consumer assets is zero. When households make their choices optimally to maximize current utility given $\tau_t^n$ and $T_t$, they must choose the same allocation as in the default equilibrium, which is the solution to the following problem

$$[c_t(h_t), n_t(h_t)] = \arg\max_{c,n} u(c, n) \text{ st } c = w_t(1 - \tau_t^n)n + T_t,$$

with equilibrium wages given by $w_t(h_t) = F_n(k_t, n_t)$. Thus, $W(\sigma^d, f, \phi\,|k_0, b_0) = W(\sigma^d, f^d, \phi^d\,\left|k_0, b_0'\right.)$.

This establishes the main result of the paper that the value of the worst sustainable equilibrium in this setup can be determined in advance and is independent of the level of government debt.

In this setup, the investment constraint (1) ensures that we are able to reach the outcome of the default equilibrium no matter what the beliefs of the households are. Thus, the upper bound on investment is not necessary for the default strategies to be an equilibrium, but it is necessary to ensure that it is the worst one.

Notice, however, that the value of the default equilibrium may be extremely low. In particular, if we consider functional forms where output is zero when capital is zero and the utility of zero consumption is $-\infty$, then the value of this equilibrium is also $-\infty$. This makes the inability to renegotiate a particularly strong assumption, since the economy is trapped in a situation where everyone would benefit substantially from switching to a new equilibrium.

### 3.4 Characterization of sustainable outcomes

In the spirit of the optimal punishments in Abreu (1988), we will use reversion to the worst sustainable equilibrium as the maximum threat point that allows us to sustain equilibria. The original terminology refers to optimal punishments when firms deviate in a cartel. Here, there is not any kind of collusion *per se*, but we still need to enforce cooperation, since the government's *ex-ante* and *ex-post* incentives are not aligned. Thus, the worst punishment is not inflicted by other firms, but rather by changing the households' expectations, which leads to a different equilibrium that is worse for everyone.

The next lemma characterizes the entire set of sustainable equilibrium outcomes, which are the allocations that can be induced by sustainable equilibria.

**Proposition 2** *An allocation $y$ is the outcome of a sustainable equilibrium $(\sigma, f, \phi\, |k_0, b_0)$ if and only if*

(i) *The allocation $y$ is attainable*

$$m(c_t, n_t) + \beta a_{t+1} \geq a_t \;\; with \;\; a_0 \geq u_c(c_0, n_0) \min(0, b_0).$$
$$c_t + g_t + k_{t+1} = F(k_t, n_t).$$

(ii) *The continuation value of allocation $y$ is always better than the worst sustainable equilibrium*

$$\sum_{t=i}^{\infty} \beta^{t-i}[u(c_t, n_t) + v(g_t)] \geq V^d(k_i) \; for\ every\ i \geq 0.$$

*Proof* To prove this proposition, we will start by showing that the outcome of a sustainable equilibrium must meet the conditions above. Then, we will show that any allocation that meets the conditions above may be attained as the outcome of a sustainable equilibrium.

Suppose that the allocation $y$ is the outcome of a sustainable equilibrium $(\sigma, f, \phi\, |k_0, b_0)$. Consumer optimality requires that $y$ maximizes the households utility at time zero given the policies and prices along the equilibrium path. Government optimality requires that the government's budget constraints be satisfied. Furthermore, in a sustainable equilibrium factor, prices must equal marginal productivity of factors and the resource constraint must hold along the equilibrium path. Thus, the conditions for a competitive equilibrium are satisfied in a sustainable equilibrium, which means that its outcome is attainable and condition (i) must hold.

After any history $h_{i-1}$, the government can get a payoff of at least $V^d(k_i)$ by playing the default strategy $\sigma^d$. Thus, if the government chooses to follow strategy $\sigma$ instead, that has to lead to a present value at least at high as $V^d(k_i)$, which implies that condition $(ii)$ must hold at every time $i$. Thus, if an allocation $y$ is the outcome of a sustainable equilibrium $(\sigma, f, \phi\, |k_0, b_0)$, it must meet conditions (i) and (ii).

Suppose now that an allocation $y$ meets conditions $(i)$ and $(ii)$. Let $\pi$ and $\phi$ be the policies and prices that implement this allocation under commitment. Consider the following strategy for households: as long as the government's action is according to $\pi$, choose allocation $y$; if the government deviates, follow the default equilibrium strategy. Likewise, consider the government's strategy where it acts according to $\pi$ along the equilibrium path and plays the default strategies off equilibrium. Finally, market clearing conditions are given by $\phi$ as long as the government chooses $\sigma$ and are given by $\phi^d$ otherwise. We will show that this is a sustainable equilibrium. First, consider histories where there have been no deviations until time $t$. Since $y$ is attainable, and along the equilibrium path, households will expect to face policies $\pi$ and prices $\phi$, this means that the continuation of $y$ must be optimal for households. The government, on the other hand, can choose to deviate, in which case it would get $V^d(k_t)$ or it can follow the equilibrium path. Since condition (ii) ensures that the payoff along the equilibrium path is always higher than $V^d(k_t)$, it is always incentive compatible for the government not to deviate. Now, consider histories where there has been a deviation before time $t$. Our strategy has specified that in this case both households and the government will play a default equilibrium, which we have shown to be sustainable. Thus, the specified set of strategies is a sustainable equilibrium that leads to outcome $y$.

The idea is that for an allocation to be the outcome of a sustainable equilibrium, it is necessary that households and firms are optimizing given the government's strategy, which means that the resulting allocation must be attainable. Government optimality requires that it is never in the government's best interest to deviate. Since the worst punishment after a deviation is $V^d(k)$, this gives us a lower bound on the utility that can be reached in a sustainable equilibrium at any point in time.

Notice that we were only able to obtain this simple characterization of the set of sustainable equilibria because for this model the value of the worst sustainable equilibrium is given by $V^d(k)$ and can be determined *ex-ante*, which may not be the case for other settings.

## 4 Best sustainable equilibrium

Now that the set of sustainable equilibria has been characterized, we turn to finding, among these, the one that maximizes the initial welfare for the society for given initial conditions for $k_0$ and $b_0$. The program that solves this problem can be written in the following way

$$\max_{y} \sum_{t=0}^{\infty} \beta^t [u(c_t, n_t) + v(g_t)]$$

$$\begin{aligned} \text{subject to} \quad & m(c_t, n_t) + \beta a_{t+1} \geq a_t \\ & c_t + g_t + k_{t+1} = F(k_t, n_t) \\ & \sum_{t=i}^{\infty} \beta^{t-i} [u(c_t, n_t) + v(g_t)] \geq V^d(k_i) \\ & a_0 \geq u_c(c_0, n_0) \min(0, b_0). \end{aligned}$$

This follows directly from lemma 4. Since these restrictions are necessary and sufficient conditions for a sustainable equilibrium, then the allocation that maximizes welfare subject to them must be the outcome of the best sustainable equilibrium.

This formulation is equivalent to the Ramsey problem under commitment, with an additional incentive compatibility condition that ensures that, at each point in time, the government does not want to deviate from the predefined plan.

Barbie and Hermeling (2009), who study the geometry of optimal taxation in models without commitment, show that the additional incentive compatibility condition will in general lead to nonzero capital taxes even though the slope of the indifference curves and the implementability condition is the same in steady state. However, in this particular model, the presence of government debt allows the government to save until this constraint no longer binds, which is why it is optimal to have zero capital taxes in the long run.

Since we were able to determine the worst equilibrium in advance and write it as a function of the capital level only, the problem of finding the best sustainable equilibrium is essentially identical to the one solved by Dominguez (2007), and we will thus reach the same conclusions.

## 4.1 Recursive problem

The program to find the best sustainable equilibrium can also be written recursively for $t > 0$[6] as stated below

$$
\begin{aligned}
V(k, a) &= \max_{y} \left[u(c, n) + v(g) + \beta V(k', a')\right] \\
\text{subject to} \quad & m(c, n) + \beta a' \geq a \\
& c + g + k' = F(k, n) \\
& V(k', a') \geq V^d(k').
\end{aligned}
$$

This problem is well defined when the promised value of $a$ can credibly be repaid by the government, so that $a \leq \overline{a}(k)$,[7] where $\overline{a}(k)$ is the value of household assets that makes the government indifferent between following the predefined plan and defaulting, so that $V(k, \overline{a}(k)) = V^d(k)$.

Define also the first best welfare as the utility level that can be attained when the government does not need to use distortionary taxation and which is the solution to the following recursive problem

$$
V^{fb}(k) = \max_{x,g} \left[u(c, n) + v(g) + \beta V^{fb}(k')\right] \text{ st } c + g + k' = F(k, n).
$$

Define $\underline{a}(k)$ as the highest value of consumer assets for which $V^{fb}(k)$ is attainable, so that $\underline{a}(k) = \max a$ st $V(k, a) = V^{fb}(k)$. Since $V(k, a)$ is a decreasing function of $a$, the fact that $V^{fb}(k) > V^d(k)$ implies that $\underline{a}(k) < \overline{a}(k)$.

[6] For the initial period, the implementability condition is given by $m(c, n) + \beta a' \geq u_c(c, n) \min(0, b_0)$.

[7] Notice that $\overline{a}(k) \leq \hat{a}(k)$, so this constraint also ensures that $a$ remains below the natural borrowing limit.

In the appendix, we show the equivalence of the first-order conditions of this recursive problem and the sequential problem, to provide an easier comparison with the results of Benhabib and Rustichini (1997) and Dominguez (2007), where a sequence formulation was used.

## 4.2 Optimality conditions

As in the full commitment solution, we will use a first-order approach to find the solution to this problem. Since the problem may not be concave, the first-order conditions are necessary, but may not be sufficient for an optimum. The optimality conditions for this problem are equivalent to the first-order conditions for the sequence formulation, as is shown in the appendix.

Associating the implementability condition with the multiplier $\mu$, the resource constraint with the multiplier $\rho$, and the government's incentive compatibility condition with multiplier $\gamma$, we can write the Lagrangian for this problem

$$\begin{aligned} L = u(c, n) + v(g) + \beta V(k', a') + \mu[m(c, n) + \beta a' - a] \\ -\rho[c + g + k' - F(k, n)] + \gamma\beta[V(k', a') - V^d(k')]. \end{aligned}$$

Combining the first-order conditions for $k'$ and $a'$ with the envelope conditions for $k$ and $a$, we obtain the following equations

$$\begin{aligned} V_k(k', a') &= \frac{V_k(k, a)}{\beta F_k(k, n)} + \gamma[V_k^d(k') - V_k(k', a')] \\ V_a(k', a')(1 + \gamma) &= V_a(k, a). \end{aligned}$$

The optimality condition for $k$ shows how the lack of commitment can distort the choice of capital when commitment binds. If this is the case and the value of default reacts more to changes in capital than the value of the optimal sustainable plan, then capital will be distorted downward, since this will help loosen the incentive compatibility constraint. Conversely, if the value of the optimal sustainable plan varies more with capital, then capital will be distorted upward. Thus, if higher capital makes commitment less binding, it will be optimal to subsidize capital. This result is reminiscent of the findings of Benhabib and Rustichini (1997), who show that it could be optimal to either tax or subsidize capital. However, here this will only be true in the short run, since in the long run, the economy will converge to a steady state where commitment does not bind.

The optimality condition for $a$ says that it is optimal for the value of household assets to decrease over time as long as commitment is binding, which leads to an increase in government assets over time. The reason for this is that when the government accumulates assets, it gets a direct benefit of higher utility tomorrow, as well as an additional benefit from loosening the incentive compatibility condition in the future. Thus, to some extent, government assets work as a commitment mechanism that reduces the incentive to default by increasing the welfare of the equilibrium strategy.

The next section describes the long run properties of the economy without commitment.

## 4.3 Steady state

In this section, we show that the steady state of the best sustainable equilibrium of the economy has zero capital taxes, although labor taxes remain positive. This result is consistent with Dominguez (2007) who analyzed a similar model with an exogenous value of default. Since we have now provided a game theoretic setting where the value of the worse equilibrium can be determined in advance and is independent of public debt, the same conclusions must be true.

**Proposition 3** (Zero capital taxes in steady state) *In steady state, the best sustainable equilibrium has zero capital taxes.*

Later follows a sketch of the proof of this proposition. A formal proof is presented in the appendix.

**Sketch of Proof.** Assume the economy is in a steady state with constant $c, n, g, k$, and $a$. The first-order conditions for $c$ and $n$ ($u_c + \mu m_c = \rho$ and $u_n + \mu m_n = -\rho/F_n$) imply that $\rho$ and $\mu$ must also be constant. We can now prove by contradiction that capital taxes cannot be different from zero in the long run.

If capital taxes are not zero, then $\beta F_k(k, n) \neq 1$. From the optimality condition for $k$ derived earlier, this implies that $\gamma > 0$ (recall that $\gamma \geq 0$, since it is the multiplier on an inequality constraint). We can see in the optimality condition for $a$ that when $\gamma > 0$, it must be true that $V_a(k, a) = 0$ in steady state, which means that the implementability condition is not binding. But, then the first best solution is achievable, which means that the incentive compatibility condition cannot bind either, which implies that $\gamma = 0$. Thus, we have reached a contradiction that proves that capital taxes cannot be different from zero in steady state.

Thus, if the economy converges to a steady state solution with $\gamma = 0$, then capital taxes will converge to zero in the long run. Furthermore, there cannot be a long run solution where $\gamma$ does not converge to zero. If $\gamma$ did not converge to zero, then $V_a(k, a)$ would have converge to zero, since $\gamma$ is weakly positive and $V_a(k', a')(1 + \gamma) = V_a(k, a)$. But, we have just seen that when $V_a(k, a)$ is zero, the implementability is not binding, which implies that the first best solution is achievable and the incentive compatibility condition is also not binding, which means that $\gamma$ must converge to zero.

**Proposition 4** (Positive labor taxes in steady state) *In steady state, the best sustainable equilibrium has positive labor taxes as long as $a_0 > \underline{a}(k_0)$.*

*Proof* The first-order conditions for consumption and labor are

$$u_c(c, n) + \mu m_c(c, n) = \rho$$
$$u_n(c, n) + \mu m_n(c, n) = -\rho F_n(k, n),$$

which implies that labor taxes are only zero when $\mu$ is equal to zero and that otherwise they are positive.

In steady state, the incentive compatibility condition is either exactly met or it is slack. If the incentive compatibility is exactly met, then $a = \overline{a}(k) > \underline{a}(k)$, which means that $\mu > 0$, since $\underline{a}(k)$ is the highest value of $a$ for which the implementability condition does not bind. If the incentive compatibility is slack, then given that $\mu' = (1+\gamma)\mu$, it must be the case that $\mu$ is constant since the last time the incentive compatibility condition was binding (or since the initial period, if it was never binding), where it was strictly positive by the argument above. So, labor taxes will be strictly positive in either case.

As long as commitment is binding, increasing government savings not only increases tomorrow's continuation value but also loosens the incentive compatibility constraint. Thus, the government will keep saving until it has achieved enough assets for the incentive compatibility to stop binding. This will happen before the government reaches its asset limit, since $V(\underline{a}(k), k) = V^{fb}(k) > V^d(k) = V(\overline{a}(k), k)$.

Without commitment, the set of steady states to which the economy can converge is (weakly) smaller than in the economy with full commitment, since they need to meet the following incentive compatibility condition

$$V(k, a) = \frac{1}{1-\beta}[u(c, n) + v(g)] \geq V^d(k).$$

However, all steady states without commitment are also steady states with commitment.

## 5 Concluding remarks

If the government is allowed to accumulate assets, then it can reduce its temptation to default on its promises by accumulating wealth, which reduces the need for distortive taxation. This is true as long as the value of default is independent of the government's debt and asset level, because in this case increasing the level of government's assets always loosens the incentive compatibility constraint for the government. In the best sustainable equilibrium, asset accumulation continues until the lack of commitment is no longer binding, so that in the long run, we reach a full commitment solution where capital taxes are zero and the government uses labor taxes to finance the remaining part of public spending.

This paper provides a game theoretic framework that allows government borrowing and lending and is able to generate a worst sustainable equilibrium that is independent of the level of government debt, thus sustaining a best equilibrium where the lack of commitment does not bind in the long run.

Thus, in this setup, the lack of commitment is not able to overturn the Chamley (1986) and Judd (1985) zero capital tax result.

## 6 Appendix

### 6.1 Proof of Lemma 1

A triplet $(\pi, x, p)$ is a competitive equilibrium if it meets Eqs. (2–5) and equations (7–10) with $T_t \geq 0$, as well as a transversality conditions that ensures that the value of assets and debt remain bounded in the long run. We can replace the households's budget constraint (2) with the implementability condition (11).

For the case where the investment condition is binding, the economy must meet the implementability condition (12) instead of condition (11).

We will start by showing that if an allocation $y$ is the outcome of a competitive equilibrium, then it must meet the conditions (13–14). Since $T_t \geq 0$, both condition (11) and condition (12) imply that condition (13) must be met. Condition (14) is the same as condition (9), so it is always met. From the definition of $a_t$, $a_0$ is given by $a_0 = u_c(c_0, n_0) r_t (1-\tau_0) k_0 + u_c(c_0, n_0) b_0 (1-d_0)(1-\delta_0)$. Since $\tau_t^k \leq 1$, the first term of the right hand side of this equation is always positive $u_c(c_0, n_0) r_t (1-\tau_0) k_0 \geq 0$. The second term is either zero or $u_c(c_0, n_0) b_0$ depending on whether there is default or not, so if $b_0$ is positive, the lowest value it can take is zero and if $b_0$ is negative, its lowest possible value is $u_c(c_0, n_0) b_0$, which implies that $u_c(c_0, n_0) b_0 (1 - d_0)(1 - \delta_0) \geq u_c(c_0, n_0) \min(0, b_0)$. Together, these two conditions imply that $a_0 \geq u_c(c_0, n_0) \min(0, b_0)$. The no-ponzi condition requires that the present value of long run debt is zero, which can be written as $\lim_{t\to\infty} \beta^t u_c(c_t, n_t) b_t (1 - d_t)(1 - \delta_t) = 0$. Furthermore, the decreasing marginal returns to accumulable factors imply that capital must remain bounded in the long run, which implies that the transversality condition must be met $\lim_{t\to\infty} \beta^t a_{t+1} = 0$.

Now, we need to show that if an allocation $y$ meets conditions (13–14), then there are prices and policies such that it is the outcome of a competitive equilibrium. These prices and policies may not be unique, so we will focus on the case where there is no household default $\delta_t = 0$, the government only defaults on bonds in period 0 if $b_0 > 0$ and chooses $d_t = 0$ for $t \geq 1$ and there is no constraint on capital investment $I_t = \infty$. Let us now check that all the conditions for a competitive equilibrium can be met. Condition (9) is automatically met, and our default choices meet (10). Let $w_t = F_n(k_t, n_t)$ and $r_t = F_k(k_t, n_t)$ to meet conditions (7–8). Lump-sum transfers are given by $T_t = \left[m(c_t, n_t) + \beta a_{t+1} - a_t\right] / u_c(c_t, n_t)$, so that condition (11) is met. The transversality condition $\lim_{t\to\infty} \beta^t a_{t+1} = 0$ guarantees that the no-ponzi conditions are met. The households' optimality conditions (3–5) can be met with $q_t = \beta u_c(c_{t+1}, n_{t+1}) / u_c(c_t, n_t)$ and the following taxes for $t \geq 0$

$$
\begin{aligned}
\tau_{t+1}^k &= 1 - \frac{1}{\beta F_k(k_{t+1}, n_{t+1})} \frac{u_c(c_t, n_t)}{u_c(c_{t+1}, n_{t+1})} \\
\tau_t^n &= 1 + \frac{1}{F_n(k_t, n_t)} \frac{u_n(c_t, n_t)}{u_c(c_t, n_t)}.
\end{aligned}
$$

Finally, the condition that $a_0 \geq u_c(c_0, n_0) \min(0, b_0)$ is not restrictive, since we can choose the initial tax on capital to satisfy

$$u_c(c_0, n_0)r_0(1 - \tau_0^k)k_0 = a_0 - u_c(c_0, n_0)\min(0, b_0) \geq 0.$$

Thus, all the conditions for a competitive equilibrium have been met.

## 6.2 Equivalence of sequence and recursive approaches

The Lagrangian for the sequence problem to find the best sustainable equilibrium can be written in the following way, where $\widehat{\mu}_t$ is the multiplier on the implementability condition, $\widehat{\rho}_t$ is the multiplier on the resource constraint, and $\widehat{\gamma}_t$ is the multiplier on incentive compatibility condition

$$L = \sum_{t=0}^{\infty} \beta^t \begin{Bmatrix} u(c_t, n_t) + v(g_t) + \widehat{\mu}_t[m(c_t, n_t) + \beta a_{t+1} - a_t] \\ -\widehat{\rho}_t\left[c_t + g_t + k_{t+1} - F(k_t, n_t)\right] \end{Bmatrix}$$
$$+ \sum_{i=1}^{\infty} \beta^i \widehat{\gamma}_i \left[\sum_{t=i}^{\infty} \beta^{t-i}[u(c_t, n_t) + v(g_t)] - V^d(k_i)\right].$$

The first-order conditions for this problem are

$$\begin{aligned} u_{c_t}\left(1 + \sum_{i=1}^{t} \widehat{\gamma}_i\right) + \widehat{\mu}_t m_{c_t} &= \widehat{\rho}_t \\ u_{n_t}\left(1 + \sum_{i=1}^{t} \widehat{\gamma}_i\right) + \widehat{\mu}_t m_{n_t} &= -\widehat{\rho}_t F_{n_t} \\ v'(g_t)\left(1 + \sum_{i=1}^{t} \widehat{\gamma}_i\right) &= \widehat{\rho}_t \\ \beta\widehat{\rho}_{t+1} F_{k_{t+1}} - \beta\widehat{\gamma}_{t+1} V_k^d(k_{t+1}) &= \widehat{\rho}_t \\ \widehat{\mu}_{t+1} &= \widehat{\mu}_t. \end{aligned}$$

The first-order conditions for the recursive problem in Sect. 1.4.2 are

$$\begin{aligned} u_c + \mu m_c &= \rho \\ u_n + \mu m_n &= -\rho F_n \\ v_g &= \rho \\ V_a(a', k')(1 + \gamma) &= -\mu \\ V_k(a', k') &= \rho/\beta + \gamma[V_k^d(k') - V_k(a', k')] \end{aligned}$$

and the envelope conditions for $a$ and $k$ are

$$\begin{aligned} V_a(a, k) &= -\mu \\ V_k(a, k) &= \rho F_k(k, n). \end{aligned}$$

It is easy to see that the equilibrium conditions for the two problems lead to the same allocations for $c, n, k$, and $a$ for as long as the multipliers for the constraints in the recursive formulation $\mu, \rho$ , and $\gamma$ have the following relationship with the multipliers in the sequence approach

$$\begin{aligned} \mu &= \frac{\widehat{\mu}_t}{1+\sum^t \widehat{\gamma}_i} \\ \rho &= \frac{\widehat{\rho}_t}{1+\sum^t \widehat{\gamma}_i} \\ 1+\gamma &= \frac{1+\sum^{t+1} \widehat{\gamma}_i}{1+\sum^t \widehat{\gamma}_i}. \end{aligned}$$

6.3 Impossibility of nonzero capital taxes in steady state

We will now show that any steady state of the best sustainable equilibrium must have zero capital taxes.

The idea is that if commitment were binding in the long run, then the implementability condition would stop binding and labor taxes would converge to zero. But, this can not be an optimal steady state, since it is optimal to take a deviation where labor taxes are increased marginally (with zero first order cost) to increase government assets and make the incentive compatibility constraint less binding (which has a positive first order effect).

More formally, assume that there is a steady state where capital taxes are different from zero. Let $c^{ss}$, $n^{ss}$, $g^{ss}$, $a^{ss}$, and $k^{ss}$ be the allocations in this steady state. In order to meet first-order conditions, these steady states must also have constant multipliers $\rho^{ss}$, $\mu^{ss}$, $\gamma^{ss}$. In steady state, the optimality condition for capital becomes

$$\rho^{ss} F_k(k^{ss}, n^{ss}) = \rho^{ss}/\beta + \gamma^{ss}[V_k^d(k^{ss}) - \rho^{ss} F_k(k^{ss}, n^{ss}).$$

Given this, we can only have capital taxes different from one and $\rho F_k(k^{ss}, n^{ss}) \neq 1$ if $\gamma^{ss} > 0$.

The optimality condition for $a$ implies that

$$\mu^{ss}(1+\gamma^{ss}) = \mu^{ss}.$$

Thus, when $\gamma^{ss} > 0$, we must have $\mu^{ss} = 0$.

But, then the first-order conditions for consumption, labor, and public spending take the following form

$$\begin{aligned} u_c(c^{ss}, n^{ss}) &= \rho^{ss} \\ u_n(c^{ss}, n^{ss}) &= -\rho^{ss} F_n(k^{ss}, n^{ss}) \\ v'(g^{ss}) &= \rho^{ss}. \end{aligned}$$

Furthermore, it must be true that

$$V(k^{ss}, a^{ss}) = V^d(k^{ss}) < V(k^{ss}, \underline{a}(k^{ss})) = V^{fb}(k^{ss}).$$

Now, let us consider a departure from this steady state that will lead to higher welfare than our original candidate for a steady state, thus implying that it could not be an optimal solution to begin with. The departure is as follows. In an initial period (let us call it period 0), starting from our initial steady state, we will choose $a_1 = a^{ss} - \Delta a$ instead of $a^{ss}$ and $k_1 = k^{ss}$. The new levels of consumption, labor, and public spending are the solution to the following problem

$$\begin{aligned} [c_0, n_0, g_0] = \arg\max_{c,n,g} & \; [u(c,n) + v(g)] \\ \text{subject to} \quad & w(c,n) + \beta(a^{ss} - \Delta a) \geq a^{ss} \\ & c + g + k^{ss} = F(k^{ss}, n). \end{aligned}$$

First, let us check that this deviation is feasible. Clearly, the implementability condition and the resource constraint must be met, by construction of the previous problem. The government's incentive compatibility condition now becomes

$$V(k^{ss}, a^{ss} - \Delta a) \geq V^d(k^{ss}).$$

Since $V(k, a)$ cannot be increasing in $a$, this condition must be met since $V(k^{ss}, a^{ss}) = V^d(k^{ss})$ in the original steady state.

Now, we will show that taking this deviation has a zero cost up to a first-order approximation. The new flow utility in period zero is given by

$$\begin{aligned} W(\Delta a | k^{ss}, a^{ss}) = \max_{c,n,g} & \; [u(c,n) + v(g)] \\ \text{subject to} \quad & w(c,n) + \beta(a^{ss} - \Delta a) \geq a^{ss} \\ & c + g + k^{ss} = F(k^{ss}, n). \end{aligned}$$

We can write a first-order Taylor approximation of this expression in the following way

$$W(\Delta a) = W(0) + \Delta a W'(0).$$

Given that the implementability is not binding when $\Delta a = 0$, then $W(\Delta a) = W(0)$, which means that the cost of this deviation is zero to a first-order approximation.

Let us now see if there is any benefit to it. The deviation proceeds as follows. From period 1 to period $T - 1$ (which will be defined shortly), the chosen allocations will be

$$\begin{aligned} & c_t = c^{ss}, n_t = n^{ss}, g_t = g^{ss}, k_t = k^{ss} \;\; for \;\; t = 1, \ldots, T-1 \\ & a_t = a^{ss} - \beta^{1-t} \Delta a \;\; for \;\; t = 1, \ldots, T. \end{aligned}$$

Let us start by checking that the new plan is feasible from periods 1 to $T$. First, the resource constraint must be met since the allocations in this constraint are the same as in the initial steady state. The incentive compatibility must also hold since $V(k, a)$ cannot be increasing in $a$. Finally, the implementability condition will be met since $w(c^{ss}, n^{ss}) \geq a^{ss}(1-\beta)$, which implies that $w(c, n) + \beta(a^{ss} - \beta^{-t}\Delta a) \geq a^{ss} - \beta^{1-t}\Delta a$.

Let $T$ be defined by the following condition

$$\Delta a \frac{1}{\beta^T} = a^{ss} - \underline{a}(k^{ss}).$$

This means that in period $T$, we will reach $a_t = \underline{a}(k^{ss})$ and will be able to increase $V(k^{ss}, \underline{a}(k^{ss})) > V(k^{ss}, a^{ss})$ by switching to the nonconstrained solution. Furthermore, this will lead to a first-order positive welfare increase in the initial period

$$B(\Delta a) = \beta^T[V(k^{ss}, \underline{a}(k^{ss})) - V(k^{ss}, a^{ss})] = \frac{V(k^{ss}, \underline{a}(k^{ss})) - V(k^{ss}, a^{ss})}{a^{ss} - \underline{a}(k^{ss})}\Delta a.$$

Thus, the net benefit of our deviation is strictly positive for a small enough change in $a$, which means that the initial candidate for a steady state was not optimal.

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