Econ Theory (2011) 48:229–242
DOI 10.1007/s00199-011-0646-2

SYMPOSIUM

Informational efficiency with ambiguous information

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Received: 2 September 2010 / Accepted: 19 May 2011 / Published online: 26 June 2011

Abstract This paper proves the existence of fully revealing rational expectations equilibria for almost all sets of beliefs when investors are ambiguity averse and have preferences that are characterized by Choquet expected utility with a convex capacity. The result implies that strong-form efficient equilibrium prices exist even when many investors in the market make use of information in a way that is substantially different from traditional models of financial markets.

Keywords Efficient markets · Ambiguity · Rational expectations equilibrium

JEL Classification D53 · D81 · D82 · G14

We are grateful to Alain Chateauneuf, Ani Guerdjikova, Paolo Siconolfi and Jean-Marc Tallon for helpful comments. We are particularly grateful to an anonymous referee as well as Mark Machina, Klaus Ritzberger, and Nicholas Yannelis as editors of this special issue in honor of Daniel Ellsberg. Finally, we are grateful to Larry Blume and David Easley for advice and suggestions. We also thank several seminar and conference audiences for helpful discussion. Condie acknowledges support from the Solomon Fund for Decision Research at Cornell University.

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1 Introduction

The efficient markets hypothesis is one of the main organizing principles for research in financial economics.[1] It characterizes the ability of prices to accurately reflect traders' information about traded assets. In its strongest form, the efficient markets hypothesis states that all payoff relevant information, whether publicly or privately available, can be inferred from market prices.

The work of Radner (1979), Allen (1981), and Allen (1982) provides conditions under which strong-form efficiency is consistent with market equilibrium in the form of rational expectations equilibrium (REE).[2] They show that in exchange economies where the demand for assets is a differentiable function of asset prices and beliefs and the dimension of the space of private information is less than that of the space of prices, REE prices will reveal all agents' private information almost surely. That is, almost surely there exists an equilibrium that is strong form efficient.[3] These papers model traders whose preferences are smooth functions of information and market prices. In Radner (1979), all traders were modeled as subjective expected utility (SEU) maximizers in the sense of Savage (1954) and Anscombe and Aumann (1963).

The work of Ellsberg (1961) helped to formalize the distinction between risk and uncertainty as first discussed by Knight (1921) and Keynes (1921). His work highlights that individuals may behave differently in situations where they do not know the probability distribution over potential outcomes. These situations without known probabilities have come to be known as uncertain or ambiguous whereas situations where a probability distribution over states is known are called risky.

It has been suggested that the wide variety of sources from which participants receive information in financial markets means that not all information may be of the same quality and some of it may be perceived as ambiguous by participants. For instance, intangible information (Daniel and Titman 2006), which is unrelated to past dividends, or information from a new or relatively unknown source may be perceived as ambiguous (Epstein and Schneider 2008).

In this paper, we examine conditions for informational efficiency in the presence of ambiguity by adapting the model of Radner (1979) to allow for traders whose preferences are represented by Choquet expected utility with a convex capacity Schmeidler (1989). We demonstrate that REE in which prices fully reveal all privately held information exist generically even when traders are sensitive to ambiguity as decribed above. Thus markets may still transmit information effectively, even in the midst of significant ambiguity.

In Condie and Ganguli (2010), we show that with non-smooth ambiguity averse (AA) preferences, there exists a set of economies with positive Lebesgue measure that

[1] See for instance, Fama (1991).

[2] See also Jordan (1983) and Grossman (1978) for closely related work and the survey by Allen and Jordan (1998).

[3] This also assumes there is no 'noise' as described, for instance, in Grossman and Stiglitz (1980). In the case of Allen (1981), the dimension of the space of private information must be half that of the space of prices.

has partially revealing REE.[4] The results of Condie and Ganguli (2010) taken together with the results of the present paper imply that multiple REE with very different informational properties exist for some sets of parameters.

There is a growing body of work that examines the implications of the Schmeidler (1989) representation with a convex capacity in markets. A very incomplete list includes Dow and da Costa Werlang (1992), Mukerji and Tallon (2001), Mukerji and Tallon (2004a), Mukerji and Tallon (2004b), Dana (2004), and Chateauneuf et al. (2000). More generally, models that incorporate traders with ambiguity aversion have recently proved to be a fruitful area of research in theoretical asset pricing.

Epstein and Schneider (2008) find that ambiguous information may lead to asymmetric portfolio reactions, equity premia dependent on idiosyncratic risk, and skewness in returns. Dow and da Costa Werlang (1992) show that ambiguity averse investors may demonstrate portfolio inertia. Chen and Epstein (2002), Maenhout (2004), and Ui (2010) discuss the implications of ambiguity for the equity premium puzzle, while Epstein and Miao (2003) and Uppal and Wang (2003) discuss its possible importance in explaining portfolio home bias. Finally, Cao et al. (2005), Ui (2010), and Easley and O'Hara (2009) show that ambiguity aversion is consistent with limited market participation.[5]

Recent work that studies REE with ambiguity averse traders includes Tallon (1998), Ozsoylev and Werner (2010), Mele and Sangiorgi (2009), Caskey (2009), Condie and Ganguli (2010), and de Castro et al. (2010b). The first four papers study the impact of ambiguity averse investors in financial market economies with asymmetric information. de Castro et al. (2010b) show that max–min utility yields universal existence of REE as defined in their extension and that these REE are incentive compatible and efficient. In closely related work, de Castro et al. (2010a) develop notions of maximin core and maximin Walrasian equilibrium in the private information state contingent market framework studied by Radner (1968).

Ozsoylev and Werner (2010) show that markets with an informed SEU investor and an uninformed Gilboa and Schmeidler (1989) AA investor, where noise traders prevent full revelation by REE prices can be illiquid. Moreover, in such illiquid markets, small informational or supply shocks can have a large impact on asset prices. Their analysis is complementary to the present paper and provides the very interesting insight that if informational efficiency, which we establish as generic within a subclass of AA preferences, is removed due to the presence of noise traders then there can be significant observable implications for financial market variables.

The remainder of the paper proceeds as follows. Section 2 outlines the model and the definition of an REE for the general model. Section 3 contains results on the generic existence of fully revealing REE. Section 4 concludes. Proofs for ancillary results are contained in the appendix.

[4] Condie and Ganguli (2010) use the Gilboa and Schmeidler (1989) representation of AA preferences, which includes those of this paper as a special case.

[5] Epstein and Schneider (2010) provide a recent survey of how models with ambiguity averse (AA) traders have proved useful in studying a variety of financial market phenomena.

2 The model

The market is populated by a finite set $\mathcal{N} = \{1, \ldots, n, \ldots, N\}$ of investors who live for 2 periods labeled 1 and 2.[6] At the end of period 2, one of a finite set Ω of possible states of nature, denoted ω, is realized and investors in the economy consume.

In period 1, some information about asset values is revealed to each investor $n \in \mathcal{N}$ through a private signal s^n from a finite set $\mathcal{S}^n = \{s^n_1, \ldots, s^n_S\}$. The set of all possible collections of private information that might be available to the market is labeled $\Sigma = \times_{n \in \mathcal{N}} \mathcal{S}^n$ with representative element σ and $\mathcal{F}$ denotes the discrete algebra over Σ. The investors' private signals convey information about the likelihood of each outcome $\omega \in \Omega$ in period 2.

Each investor has an endowment $e^n \in \mathbb{R}^{|\Omega|}_{++}$ of the single consumption good and must choose a consumption allocation in $\mathbb{R}^{|\Omega|}_{+}$ for period 2. This allocation is financed by trading state contingent consumption (Arrow securities) over Ω in the market that opens in period 2.[7]

The market opens at the beginning of period 2 and in equilibrium, each investor derives information about the private signals of other investors by observing the prices of the contingent claims that are traded in the market as described in Sect. 2.1. Let $P \subset \mathbb{R}^{|\Omega|}$ be the space of possible prices over contingent claims that can be purchased at the beginning of period 2. The conditions imposed on preferences and endowments ensure that P may be normalized so that its elements are non-negative and sum to one. This normalization will be assumed throughout the paper.

2.1 Preferences and beliefs

Investors in the market are SEU (subjective expected utility) decision-makers or AA (ambiguity averse) decision-makers. The set of SEU maximizing investors is denoted $\mathcal{N}^E$ and has cardinality $N^E \geq 1$ while $\mathcal{N}^A$ denotes the set of AA investors and has cardinality $N^A \geq 1$. Preferences for the AA investors will be described first.

In describing AA investor preferences for this analysis, it is convenient to work with sets of probability measures and so we first introduce some notation to this effect. Let $\mathcal{C}(\Delta^{|\Omega|})$ denote the collection of non-empty, convex, closed subsets of $\Delta^{|\Omega|}$. We denote by $\gamma^n(f) \in \mathcal{C}(\Delta^{|\Omega|})$ the conditional beliefs of an AA investor n about the resolution of uncertainty over Ω when he knows that the joint signal $\sigma \in f$, i.e., it is the information conveyed by f. The tuple $(\gamma^n(f))_{f \in \mathcal{F}}$ in turn is called a *belief system*.

Before proceeding, we recall the definition of a *convex capacity*. A convex capacity is a set function $\nu : \mathcal{F} \to [0, 1]$ such that (i)$B_1 \subseteq B_2 \Rightarrow \nu(B_1) \leq \nu(B_2)$, (ii)$\nu(\emptyset) = 0 = 1 - \nu(\Omega)$, and (iii)$\nu(B_1) + \nu(B_2) \leq \nu(B_1 \cap B_2) + \nu(B_1 \cup B_2)$ for all $B_1, B_2 \in \mathcal{F}$.[8]

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6 This could also be taken to be a finite set of types of investors.

7 The resulting prices produce the state-price density that is often derived in traditional asset pricing models. From this density and a no-arbitrage condition, the prices of other assets can be calculated.

8 The convexity of a capacity tied to requirement (iii).

Using the notion of convex capacities, we make the following assumption about the beliefs of each AA investor. Schmeidler (1989) and Siniscalchi (2006) provide axiomatizations of AA preference representations consistent with this assumption.

Assumption 1 For all $n \in \mathcal{N}^A$ and $\sigma \in \Sigma$, the conditional beliefs $\gamma^n(\sigma)$ are the core of a convex capacity.

We denote by $\hat{\Gamma}_C$ and respectively Γ_C the set of conditional beliefs and beliefs systems, respectively for each AA investor that satisfy the previous assumption.

It is worth noting that convexity of the capacity was suggested as a notion of uncertainty aversion in Schmeidler (1989). It has since been pointed out that convexity of the capacity while sufficient is not necessary for a notion of uncertainty aversion, see for example Epstein (1999) and Ghirardato and Marinacci (2002).

If each AA investor's beliefs can be generated by the core of a convex capacity ν, then each conditional belief can be generated by a set of the form

$$\gamma(\cdot) = \{\pi \in \Delta^{|\Omega|} : \pi(\omega) \geq \nu(\omega)\}. \tag{2.1}$$

where $(\nu(1), \ldots, \nu(|\Omega|)) \in [0, 1]^{|\Omega|}$. Hence beliefs for each joint signal can be described by some vector $(\nu(\omega))_{\omega=1}^{|\Omega|} \in \mathbb{R}_+^{|\Omega|}$ under the restriction that the resulting set must be convex and non-empty. The set of such points is a subset of a Euclidean space, and so it makes sense to impose Lebesgue measure on this set.

While all AA investors have beliefs over $\Sigma \times \Omega$, we make no assumption about whether investors percieve any ambiguity over Σ. This is because we are concerned with the decisions made by the investors after they have received all possible information (from their private signals and from the prices). Hence, the presence or absence of ambiguity over Σ makes no difference to our results.[9]

For each information set $f \in \mathcal{F}$, $\pi^n(f) \in \Delta^{|\Omega|}$ denotes the updated beliefs of an SEU maximizing investor n if she knows that $\sigma \in f$. A belief system for investor n is given by $(\pi^n(f))_{f \in \mathcal{F}}$. The space of belief systems over Ω for AA investor $n \in \mathcal{N}^A$ is denoted Γ_C and that of belief systems over Ω for an investor $n \in \mathcal{N}^E$ is denoted Π.

Investors utilize information from their private signal and from prices. Abusing notation, we let $f(s^n) \in \mathcal{F}$ be the set of joint signals σ that have $\sigma(n) = s^n$, where $\sigma(n)$ is the nth component of σ. Each investor n knows by her private signal that $\sigma \in f(s^n)$.

Note that we have not specified how beliefs are related across information sets beyond assumption 1. For instance, we have not made assumptions about how $\gamma^n(f(s^n))$ relates to $\gamma^n(\sigma)$, when $\sigma(n) = s^n$. While this is an important issue, it is not directly relevant for our analysis since we will establish generic full revelation, i.e. our analysis will be concentrated on beliefs $\gamma^n(\sigma)$ for all $n \in \mathcal{N}$ and $\sigma \in \Sigma$. This is also why we only make assumption 1 for each $\sigma \in \Sigma$ and further assumptions about $\gamma^n(f(s^n))$ for instance are not needed.

[9] However, if one is interested in examining the decisions of investors before and after receiving private signal and price information then introducing ambiguity over Σ may be interesting as it would lead to questions of how beliefs are updated and whether decisions are dynamically consistent.

A price function $\phi : \Sigma \rightarrow P$ defines a price for every joint signal σ. In equilibrium, information is gathered from prices by using the equilibrium price function ϕ, so if the observed price is p, then $\phi^{-1}(p) \in \mathcal{F}$ is the information revealed by price p to all investors $n \in \mathcal{N}$. Combining the information derived from her personal signal and that inferred from prices, investor n in equilibrium has information $f(s^n) \cap \phi^{-1}(p)$, i.e. beliefs $\gamma^n(f(s^n) \cap \phi^{-1}(p))$ for AA investor $n \in \mathcal{N}^A$ (respectively, beliefs $\pi^n(f(s^n) \cap \phi^{-1}(p))$ for SEU investor $n \in \mathcal{N}^E$).

The preferences of the investors in the economy are further specified as follows. We specify the preferences for AA investors for all $f \in \mathcal{F}$ by using the multiple-prior representation of Gilboa and Schmeidler (1989), which is consistent with the convex capacity assumption for all $\sigma \in \Sigma$.[10] It is useful to note that the representation of AA investor n's preferences includes as a special case the situation in which n is an SEU maximizer.

Assumption 2 Given any $f \in \mathcal{F}$,

1. Investor $n \in \mathcal{N}^A$ has preferences over $x^n \in \mathbb{R}_+^{|\Omega|}$ that are represented by the utility function

$$U^n(x^n; f) = \min_{\hat{\pi} \in \gamma^n(f)} \mathbb{E}_{\hat{\pi}}[u^n(x^n)] \tag{2.2}$$

   with $\gamma^n(f) \in \mathcal{C}(\Delta^{|\Omega|})$ and $\hat{\pi} \gg 0$ for all $\hat{\pi} \in \gamma^n(f)$.
2. Investor $n \in \mathcal{N}^E$ has preferences over $x^n \in \mathbb{R}_+^{|\Omega|}$ that are represented by the utility function

$$U^n(x^n; f) = \mathbb{E}_{\pi^n(f)}[u^n(x^n)], \tag{2.3}$$

   with $\pi^n(f) \in \Delta^{|\Omega|}$ and $\pi^n(f) \gg 0$.
3. For *all* $n \in \mathcal{N}$, the von Neumann-Morgenstern utility function $u^n(\cdot)$ satisfies $u^n \in C^2$, $u'^n(\cdot) > 0$, $u''^n(\cdot) < 0$, and $\lim_{x \to 0} u'^n(x) = \infty$.

2.2 Equilibrium

For any price vector $p \in P$, the set of feasible, state contingent consumption bundles, called *the budget set* of investor n is

$$\mathbb{B}(e^n, p) = \{x \in \mathbb{R}_+^{|\Omega|} : p(e^n - x^n) \geq 0\}. \tag{2.4}$$

With this notation at hand, we can now define the equilibrium notion of interest.

Definition 1 A pair (x, ϕ), where $\phi : \Sigma \rightarrow P$ is a price function and $x : \Sigma \rightarrow \mathbb{R}_+^{N|\Omega|}$ is an allocation, is a *rational expectations equilibrium* (REE) if for all n and σ, (x, ϕ) satisfies

[10] See for example Schmeidler (1989) (Proposition, p 582–583).

1. $x^n(\sigma) \in \arg\max U^n(x^n(\sigma); f(\sigma(n)) \cap \phi^{-1}(\phi(\sigma)))$ s.t. $x^n \in \mathbb{B}(e^n, \phi(\sigma))$
2. $\sum_{n \in \mathcal{N}} (e^n(\sigma) - x^n(\sigma)) = 0$

Definition 2 An REE price function ϕ is said to be *fully revealing* if it is injective. It is said to be *partially revealing* if it is not fully revealing.[11] An REE is called fully revealing if the corresponding price function is fully revealing and is called partially revealing otherwise.

Following Radner (1979), we parametrize the space of economies by $\Gamma_C^{N^A} \times \Pi^{N^E}$, the space of belief systems of the investors over Ω. Also, note that $dim(\Sigma) < dim(P)$, where $dim(\Sigma)$ denotes the topological dimension of Σ and $dim(P)$ that of the price space P, as in Radner (1979).

3 Full revelation

In proving the generic existence of fully revealing REE, we will follow the common approach of first assuming that all investors are informed and then demonstrating that under this assumption there exist equilibrium prices that are fully revealing except for a set of parameters that has zero Lebesgue measure. This is the approach followed in Radner (1979), who provides the notions of a full communication economy and a full communication equilbrium (FCE).

A full communication economy is an artificial economy where before trading begins in period 2 every investor knows the joint signal σ. Let $\phi^{FCE}(\sigma) \in P$ denote the equilibrium price vector of this economy. An FCE price function ϕ^{FCE} is a tuple of such equilibrium price vectors, one for each σ.

An FCE price function ϕ^{FCE} is *revealing* if

$$\sigma \neq \sigma' \Rightarrow \phi^{FCE}(\sigma) \neq \phi^{FCE}(\sigma'). \tag{3.1}$$

That is, in a revealing FCE, different joint signals result in different equilibrium price vectors. An FCE that does not have this property is labeled a confounding FCE. It is immediate that a revealing FCE price function will also be a fully revealing REE price function.[12]

Given this observation, we now proceed to show that non-existence of revealing FCE is non-generic. That is, the set of belief systems $((\gamma^n(\sigma))_{n \in \mathcal{N}^A}, (\pi^n(\sigma))_{n \in \mathcal{N}^E})_{\sigma \in \Sigma} \in \Gamma_C^{N^A} \times \Pi^{N^E}$ for which there is no corresponding revealing FCE has measure zero. This establishes generic existence of a fully revealing REE.

The proof proceeds by considering a system of equations for which a solution must exist if an FCE is confounding. We show that the projection of the set of all solutions to the system into the set of belief systems, and hence economies, is of measure zero.

[11] Our notion of partially revealing REE prices includes the case where the prices are non-revealing, i.e., ϕ is a constant function.

[12] This method of proof enables us to impose assumptions only on $\gamma(\sigma)$ (assumption 1) as opposed to the more general space $\gamma(f)$.

The system of equations involves the excess demand functions of AA investors, which may not be differentiable, unlike the model of Radner (1979), and we utilize tools from non-smooth analysis to proceed. The result is restricted to the space of beliefs satisfying assumption 1 because this class of preferences can be shown to lead to demand that is Lipschitz continuous, a requirement for the (non-smooth) implicit function theorem that we employ in the proof.

Before proceeding, we introduce some results from non-smooth analysis used in the proofs. In what follows, for $m \geq 0$, $||\cdot||_m$ denotes the Euclidean metric on $\mathbb{R}^m$.

Definition 3 Let X be a subset of $\mathbb{R}^m$. A function $f : X \to \mathbb{R}^k$ is Lipschitz continuous with Lipschitz constant $K \geq 0$ if for all $x, y \in X$,

$$||f(y) - f(x)||_m \leq K||y - x||_k. \tag{3.2}$$

The next two definitions and the next lemma are from Clarke (1983). By Rademacher's Theorem, a function $F : \mathbb{R}^m \to \mathbb{R}^k$ that is Lipschitz continuous on an open subset of $\mathbb{R}^m$ is differentiable almost anywhere on that subset. Let Λ_F be the set of points in the domain of the function F at which F is not differentiable.

Definition 4 The *generalized Jacobian* of the Lipschitz function F at x, denoted $\partial F(x)$ is given by

$$\partial F(x) = \bar{\text{co}}\,\{\lim DF(x_i) : x_i \to x, x_i \notin \Lambda_F\}. \tag{3.3}$$

where $DF(x_i)$ is the Jacobian of F at the point of differentiability x_i.

The generalized Jacobian is a set of matrices (being a singleton if F is differentiable at x) defined for all x in the domain of a Lipschitz function F.

Definition 5 Let $G : \mathbb{R}^k \to \mathbb{R}^k$ be Lipschitz. The generalized Jacobian $\partial G(x_0)$ at x_0 is said to be of *maximal rank* if every matrix $M \in \partial G(x_0)$ is non-singular.

In order to state the next lemma and proposition, some notation must be clarified. Let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ and suppose $\left(\hat{x}, \hat{y}\right) \in \mathbb{R}^k \times \mathbb{R}^m$ solve $F(x, y) = 0$. Let $\partial_x F(x, y)$ be the set of all $k \times k$ matrices M such that for some $k \times m$ matrix M', the $k \times (k + m)$ matrix $[M, M'] \in \partial F(x, y)$.

Lemma 3.1 (Clarke, p 256, Corollary) *Suppose that $\partial_x F(\hat{x}, \hat{y})$ is of maximal rank. Then there exists a neighborhood Y of $\hat{y}$ and a Lipschitz function $\zeta : Y \to \mathbb{R}^k$ such that $\zeta(\hat{y}) = \hat{x}$ and for all $y \in Y$, $F(\zeta(y), y) = 0$.*

With these results in hand, we now proceed to developing the proof of the main result. Let $Z^E : P \times \hat{\Pi}^{N^E} \to \mathbb{R}_+^{|\Omega|-1}$ represent the excess demand function (in $|\Omega| - 1$ markets) of all SEU investors and $Z^A : P \times \hat{\Gamma}_C^{N^A} \to \mathbb{R}_+^{|\Omega|-1}$ represent the excess demand for the AA investors (in $|\Omega| - 1$ markets). The following lemma is proved as Theorem 3 in Rigotti and Shannon (2008).

Lemma 3.2 $Z^A(\cdot)$ *is Lipschitz continuous in* p.

The price vector p is an equilibrium price vector given beliefs $(\hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\pi}}) \in \hat{\Gamma}_C^{N^A} \times \hat{\Pi}^{N^E}$ if and only if $Z(p, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\pi}}) = Z^A(p, \hat{\boldsymbol{\gamma}}) + Z^E(p, \hat{\boldsymbol{\pi}}) = 0$. To examine whether an FCE is confounding, we attempt to determine the size of the set of beliefs that will generate identical prices under two different joint signals.

Consider the following system of equations, $F : P^2 \times \hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E} \to \mathbb{R}^{3(|\Omega|-1)}$.

$$F(p_1, p_2, \hat{\boldsymbol{\gamma}}_\mathbf{1}, \hat{\boldsymbol{\gamma}}_\mathbf{2}, \hat{\boldsymbol{\pi}}_\mathbf{1}, \hat{\boldsymbol{\pi}}_\mathbf{2}) = \begin{pmatrix} Z(p_1, \hat{\boldsymbol{\gamma}}_\mathbf{1}, \boldsymbol{\pi}_\mathbf{1}) \\ Z(p_2, \hat{\boldsymbol{\gamma}}_\mathbf{2}, \boldsymbol{\pi}_\mathbf{2}) \\ p_1 - p_2 \end{pmatrix} = 0. \tag{3.4}$$

For an FCE to be confounding it must be true that for some distinct $\sigma', \sigma'' \in \Sigma$, there exists a solution to the system $F(p_1, p_2, \boldsymbol{\gamma}(\sigma'), \boldsymbol{\gamma}(\sigma''), \boldsymbol{\pi}(\sigma'), \boldsymbol{\pi}(\sigma'')) = 0$.

Let B be the set of all $\left(p_1, p_2, \hat{\boldsymbol{\gamma}}_\mathbf{1}, \hat{\boldsymbol{\gamma}}_\mathbf{2}, \hat{\boldsymbol{\pi}}_\mathbf{1}, \hat{\boldsymbol{\pi}}_\mathbf{2}\right) \in P^2 \times \hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E}$ that solve the system (3.4). Let $T(B)$ be the projection of this set into $\hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E}$. We first show that the set $T(B)$ has measure zero in $\hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E}$.

This claim is established in the following two propositions, with proposition 1 containing the key result for this proof.

Proposition 1 *For any* $\left(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2\right) \in \hat{\Gamma}_C^{2N^A}$, *let* $\partial F(p_1, p_2, \hat{\boldsymbol{\pi}}_\mathbf{1}, \hat{\boldsymbol{\pi}}_\mathbf{2})$ *be the generalized Jacobian corresponding to* $\left(p_1, p_2, \hat{\boldsymbol{\pi}}_\mathbf{1}, \hat{\boldsymbol{\pi}}_\mathbf{2}\right)$. *Then, every* $M \in \partial F(p_1, p_2, \hat{\boldsymbol{\pi}}_\mathbf{1}, \hat{\boldsymbol{\pi}}_\mathbf{2})$ *is of rank* $3(|\Omega| - 1)$.

Proof To see this, note that for a fixed $\left(\hat{\boldsymbol{\gamma}}_\mathbf{1}, \hat{\boldsymbol{\gamma}}_\mathbf{2}\right) \in \hat{\Gamma}_C^{2N^A}$ the system is differentiable in $(\hat{\boldsymbol{\pi}}_\mathbf{1}, \hat{\boldsymbol{\pi}}_\mathbf{2})$ but because of the possible non-differentiability of $Z^A(\cdot, \hat{\boldsymbol{\gamma}}_\mathbf{1}, \hat{\boldsymbol{\gamma}}_\mathbf{2})$, it need not be differentiable in (p_1, p_2). However, every possible $M \in \partial F(p_1, p_2, \hat{\boldsymbol{\pi}}_\mathbf{1}, \hat{\boldsymbol{\pi}}_\mathbf{2})$ will have the following form,

$$\begin{array}{rcccc} & p_1 & p_2 & \hat{\boldsymbol{\pi}}_\mathbf{1} & \hat{\boldsymbol{\pi}}_\mathbf{2} \\ 1: & A & 0 & C & 0 \\ 2: & 0 & B & 0 & D \\ 3: & I & -I & 0 & 0 \end{array} \tag{3.5}$$

where I is the $(|\Omega|-1) \times (|\Omega|-1)$ identity matrix. The $(|\Omega|-1) \times (|\Omega|-1)$ matrices A and B will vary across elements of $\partial F(\cdot)$, but the identity matrices and the matrices C and D will not. Now consider the $(|\Omega| - 1) \times N^E(|\Omega| - 1)$ matrix C. The form of matrix D is similar. Since only $Z^E(\cdot)$ depends on $\hat{\boldsymbol{\pi}}_\mathbf{1}$ and only SEU investor n's demand is affected by n's beliefs, C will have the form

$$C = \left(C^1, C^2, \ldots, C^{N^E}\right). \tag{3.6}$$

Applying the implicit function theorem to the SEU investor's first-order conditions reveals that each C^n is given by

$$C^n = \begin{pmatrix} -\frac{u^{n\prime}(x(1))}{u^{n\prime\prime}(x(1))} & 0 & \cdots & 0 \\ 0 & -\frac{u^{n\prime}(x(2))}{u^{n\prime\prime}(x(2))} & 0 & \cdots \\ & & \ddots & \\ 0 & \cdots & & -\frac{u^{n\prime}(x(|\Omega|-1))}{u^{n\prime\prime}(x(|\Omega|-1))} \end{pmatrix}. \tag{3.7}$$

By inspection, the matrix C^n spans a space of dimension $(|\Omega|-1)$. Thus the columns corresponding to p_1, C^1, and D^1 will span a space of dimension $3(|\Omega|-1)$, regardless of the entries in the matrix A. Thus, all matrices in $\partial F(p_1, p_2, \boldsymbol{\pi_1}, \boldsymbol{\pi_2})$ span a space of dimension $3(|\Omega|-1)$. □

Proposition 2 *Let $\mu_{\hat{\Pi}^{2N^E}}$ denote the Lebesgue measure on $\hat{\Pi}^{2N^E}$ and let $T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))$ be the set of $(\hat{\boldsymbol{\pi}}_1, \hat{\boldsymbol{\pi}}_2) \in \hat{\Pi}^{2N^E}$ for which system 3.4 has a solution for a fixed $(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2) \in \hat{\Gamma}_C^{2N^A}$. Then for a fixed $\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2$, $\mu_{\hat{\Pi}^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))) = 0$.*

Proof By Proposition 1, and Lemma 3.1, the set of solutions $B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2)$ is a $2(|\Omega|-1) + 2N^E(|\Omega|-1) - 3(|\Omega|-1) = (2N^E-1)(|\Omega|-1)$ dimensional manifold in $P^2 \times \Pi^{2N^E}$.[13] Let $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ be the space corresponding to the exogenous variables in Proposition 1. Let $\{S_i\}_{i=1}^{\infty}$ be a countable covering of the space $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$. Proposition 1 and Lemma 3.1 then tell us that there exist Lipschitz functions $\{\zeta_i\}_{i=1}^{\infty}$ such that $\cup_{i=1}^{\infty}\{\zeta_i(S_i)\}$ covers $T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))$.[14]

By Lemma 4.1, the set $\zeta_i(S_i)$ has measure zero in $\hat{\Pi}^{2N^E}$ since the set $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ is homeomorphic to the Euclidean space of dimension $(2N^E-1)(|\Omega|-1)$ while the dimension of $\hat{\Pi}^{2N^E}$ is $2N^E(|\Omega|-1)$.

13 That is, every solution point is locally homeomorphic to a subset of $\mathbb{R}^{(2N^E-1)(|\Omega|-1)}$.

14 To see how this is done, start with an arbitrary countable covering $\{A_i\}_{i=1}^{\infty}$ of $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$. For each A_i, let B_i be the set of points in B such that the projection of each point into $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ is in S_i. For each of these points b, lemma 3.1 says that there is an open set $V_b \subseteq P^2 \times \hat{\Pi}^2 - (\tilde{P} \times \hat{\Pi}_1 \times \hat{\Pi}_2)$, an open set $W_b \in \tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ and a Lipschitz function ζ_{ib} that maps points in $(\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2) \cap W_b$ into points in $(B - (\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2)) \cap V_b$. By definition $\{W_b\}_{b \in B_i}$ is an open cover of A_i and $\{V_b\}_{b \in B_i}$ is an open cover of B_i. Since A_i and B_i are subsets of Euclidean space, each of these open covers has a countable subcover which we label $\{\mathcal{W}_b\}_{b=1}^{\infty}$ and $\{\mathcal{V}_b\}_{b=1}^{\infty}$ respectively. Let $\mathcal{B}_i$ be the collection of $b \in B_i$ such that either $V_b \in \{\mathcal{V}_b\}_{b=1}^{\infty}$ or $W_b \in \{\mathcal{W}_b\}_{b=1}^{\infty}$. Let us then replace each element A_i with the countable covering $\mathcal{W}_i = \{W_b\}_{b \in \mathcal{B}_i}$. The countable covering over $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ formed in this way is countable since it is a countable collection of countable sets. The collection $\{\zeta_b\}_{b \in \mathcal{B}_i}$ is a countable collection of Lipschitz functions. Define another countable collection of Lipschitz function from $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ into $\hat{\Pi}^2$, $\{\zeta_i\}_{i=1}^{\infty} = \{\{\zeta_{ib}\}_{b \in B_i}\}_{i=1}^{\infty}$ to be $\zeta_{ib}(p, c, d) = (p, c, d, T(\zeta_{ib}(p, c, d)))$ where $(p, c, d) \in W_b$ and $T(\zeta_b(p, c, d))$ is the component projection of $\zeta_{ib}(p, c, d)$ into $\hat{\Pi}^2$. The collection $\{\zeta_i\}_{i=1}^{\infty}$ along with the sets $\{S_i\}_{i=1}^{\infty} = \{\mathcal{W}_i\}_{i=1}^{\infty}$ have the stated properties.

Then since $\{\zeta_i(S_i)\}_{i=1}^{\infty}$ covers $T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))$,

$$\mu_{\hat{\Pi}^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))) \leq \mu_{\hat{\Pi}^{2N^E}}(\cup_{i=1}^{\infty}\zeta_i(S_i)) = \sum_{i=1}^{\infty}\mu_{\hat{\Pi}^{2N^E}}(\zeta_i(S_i)). \quad (3.8)$$

By Lemma 4.1, $\mu_{\hat{\Pi}^2}(\zeta_i(S_i)) = 0$ for all i, which proves the result. □

We now state and prove the main result of this paper.

Theorem 1 *The set of beliefs in* $\Gamma_C^{N^A} \times \Pi^{N^E}$ *for which there is not a fully revealing REE has Lebesgue measure zero.*

Proof Let $\mu_{(\hat{\Gamma}_C^{2N^A}, \hat{\Pi}^{2N^E})}$ be the (product) Lebesgue measure over $\hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E}$, corresponding to the the Lebesgue measure $\mu_{\hat{\Gamma}_C^{2N^A}}$ over $\hat{\Gamma}_C^{2N^A}$ and Lebesgue measure $\mu_{\hat{\Pi}^{2N^E}}$ over $\hat{\Pi}^{2N^E}$. Define

$$T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2)) = \{(\hat{\boldsymbol{\pi}}_1, \hat{\boldsymbol{\pi}}_2) \in \hat{\Pi}^{2N^E} : (\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2, \hat{\boldsymbol{\pi}}_1, \hat{\boldsymbol{\pi}}_2) \in T(B)\}. \quad (3.9)$$

We now employ the properties of the product measure and note that

$$\mu_{\hat{\Gamma}_C^{2N^A}, \hat{\Pi}^{2N^E}}(T(B)) = \int\limits_{\hat{\Gamma}^{2N^A}} \mu_{\hat{P}i^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2)))d\mu_{\hat{\Gamma}^{2N^A}} \quad (3.10)$$

From this, it can be seen that if $\mu_{\hat{\Pi}^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))) = 0$ for $\mu_{\hat{\Gamma}_C^{2N^A}}$-almost all $\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2$, then $\mu_{\hat{\Gamma}_C^{2N^A}, \hat{\Pi}^{2N^E}}(T(B)) = 0$. Thus, using the result of Proposition 2, within $\hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E}$ the set $T(B)$ of confounding beliefs is of $\mu_{\hat{\Gamma}_C^{2N^A}, \hat{\Pi}^{2N^E}}$ measure zero.

As in Radner (1979), one may then extend this result to show that for any finite set of joint signals, the set of beliefs that lead an FCE to be confounding has measure zero, which in turns yields generic existence of fully revealing REE. □

4 Conclusion

We show that when the REE concept is extended to include traders whose preferences are not of the SEU form, generic full revelation in REE is still possible. The result in this paper with non-differentiable utility representations extends the work of Radner (1979), Allen (1981, 1982), which showed that in the lower-dimensional case, smooth preferences imply generic full revelation and provides conditions for informational efficiency under ambiguity.[15]

[15] In particular, Allen (1981, 1982) establish this for the case of smooth price functions under smooth preferences.

Two aspects of our result are worth pointing out. First, while we restricted attention to beliefs satisfying the convex capacity assumption for every signal, our proof essentially applies to any representation of preferences that generate Lipschitz demands that satisfy the maximal rank condition.

Second, we restricted attention to a sub-class of AA preferences since it allowed us to utilize the Lipschitz property. However, as noted in Rigotti and Shannon (2008), the demand generated by AA preferences is in general only approximately pointwise Lipschitz continous. While we believe that generic full revelation will hold for this larger class of preferences also, the proof of this is an item for future research.

Appendix

These results follow from the properties of Lipschitz functions and the Lebesgue measure on $\mathbb{R}^m$.

Lemma 4.1 *A set $A \subset \mathbb{R}^m$ has Lebesgue measure zero iff for each $\epsilon > 0$ there exists a countable set of cubes $\{S_i^\epsilon\}_{i=1}^\infty$ such that for each ϵ, (i) $A \subseteq \cup_{i=1}^\infty S_i^\epsilon$ and (ii) $\sum_{i=1}^\infty \mu(S_i^\epsilon) < \epsilon$.*

Lemma 4.2 *Let $A \subset \mathbb{R}^m$ and suppose that A has Lebesgue measure 0 in $\mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}^k$ is a Lipschitz continuous function then $f(A)$ has Lebesgue measure 0 in $\mathbb{R}^k$.*

Proof Let μ_m represent Lebesgue measure in $\mathbb{R}^m$. Since $\mu_m(A) = 0$, for any $\epsilon > 0$ there exists a set of cubes $\{S_i^\epsilon\}_{i=1}^\infty$ that satisfies (i) $A \subseteq \cup_{i=1}^\infty S_i^\epsilon$ and (ii) $\sum_{i=1}^\infty \mu(S_i^\epsilon) < \epsilon$.

Let δ_i^ϵ be side length of the cube S_i^ϵ. The volume of each cube S_i^ϵ is given by $V(S_i^\epsilon) = (\delta_i^\epsilon)^m$. By the Lipschitz continuity of f, there exists $K > 0$ such that for each $x, y \in A$, $||f(x) - f(y)||_k < K||x - y||_m$. Hence, denoting the center of the cube S_i^ϵ by a_i^ϵ, it follows that $f(S_i^\epsilon)$ must be contained in a cube R_i^ϵ with center $f(a_i^\epsilon)$ and side length less than or equal to $K\delta_i^\epsilon$.

The volume of $f(S_i^\epsilon)$ satisfies $V(f(S_i^\epsilon)) \leq V(R_i^\epsilon) = (K\delta_i^\epsilon)^k = K^k V(S_i^\epsilon)$. Finally, for any $\epsilon > 0$, one may select a set of cubes $\{R_i^\epsilon\}_{i=1}^\infty$ that satisfies the conditions of lemma 4.1 by using the above procedure and selecting a cover of A that has total volume less than or equal to $\epsilon/(K^k)$. □

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