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RESEARCH ARTICLE

Purification, saturation and the exact law of large numbers

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Abstract Purification results are important in game theory and statistical decision theory. We prove a new purification theorem that generalizes several earlier results. The key idea of our proof is to make use of the exact law of large numbers. As an application, we show that every mixed strategy in games with finite players, general action spaces and diffused, conditionally independent incomplete information has many strong purifications.

Keywords Exact law of large numbers · Fubini extension · Incomplete information · Purification · Saturated probability space

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1 Introduction

The idea of purification, i.e., elimination of randomness, is important in game theory and statistical decision theory. Theorem 4 of Dvoretzky et al. (1951a) (DWW Theorem henceforth), which is a generalization of the celebrated theorem of Lyapunov for vector measures, plays a central role. In particular, it says that corresponding to any mixed strategy with finite actions, there exists a pure strategy with identical integrals with respect to a finite set of atomless measures on a measurable space. Here, the pure strategy is called a *purification* of the mixed strategy.

The applications of DWW Theorem to purification problems are also investigated in Dvoretzky et al. (1950, 1951b). In particular, the purification results for statistical decision procedures and for mixed strategies in two person zero-sum games with finite actions are established. The relevance of DWW Theorem to purification results in finite-player games with finite actions and with diffused, incomplete information is suggested by Radner and Rosenthal (1982) and Milgrom and Weber (1985).[1] In these games, each equilibrium in mixed strategy has a payoff equivalent or distribution equivalent purification. A unified approach by applying DWW Theorem to purification problems in games with finite players is presented in Khan et al. (2006). More precisely, Khan et al. establish a stronger purification result that, in the above games with diffused and incomplete information, any mixed strategy (not necessarily an equilibrium) has a strong purification (see Definition 4). In addition, Khan et al. establish the existence of purification for any mixed strategy Nash equilibrium in a large non-anonymous game as in Schmeidler (1973),[2] and the existence of symmetrization for an equilibrium distribution in a large anonymous game as in Mas-Colell (1984) and Khan and Sun (1991).

DWW Theorem has been generalized in several ways. Edwards (1987) shows that DWW Theorem still holds for a countable infinite action space without any additional assumptions, see also Khan and Rath (2009) for an elementary proof. In the context of an uncountable action space, Loeb and Sun (2006) show a generalization of DWW Theorem by working with atomless Loeb measure spaces instead of atomless measure spaces.[3] Moreover, a more general version of DWW Theorem is presented in Podczeck (2009) and Loeb and Sun (2009), where atomless Loeb measure spaces are replaced by *saturated probability spaces* (see Sect. 2.1).

In 1984, Hoover and Keisler introduced the concept of saturated probability spaces in literature. Loosely speaking, a probability space is saturated if its σ-algebra restricted to any set with positive measure is never countably generated modulo all the null subsets (see Definition 2). In comparison, the σ-algebra on the usual Lebesgue unit interval is countably generated (modulo the null sets). Saturated probability spaces

[1] See Radner and Rosenthal (1982, Footnote 3) and Milgrom and Weber (1985, Sect. 5).

[2] See Rath (1992) for a direct proof of the existence of pure strategy Nash equilibria in large games when the payoffs depend on own action and the average response of others.

[3] For the construction of Loeb measure spaces, see Loeb and Wolff (2000).

could serve as a substitute for Lebesgue spaces in situations where the latter fails to work. Keisler and Sun (2009) investigate several properties which are valid on any saturated probability space but invalid on any non-saturated probability space. As a result, such properties can be used to characterize the saturation property of probability spaces. These properties, for instance, include various regularity properties on distributions of correspondences defined on a probability space (such as convexity, closedness, compactness, preservation of upper semi-continuity), and the existence of pure strategy equilibria in games with many players. Recently, there has been a growing literature on applications of saturated probability space in economic theory (see, e.g., Khan et al. 2005; Loeb and Sun 2009; Noguchi 2009; Podczeck 2008, 2009, 2010; Sun and Yannelis 2008).

In this paper, we present a general purification theorem on saturated probability spaces (see Theorem 1), which provides a far reaching generalization of the earlier purification results. In particular, it generalizes the results of Loeb and Sun (2006, 2009) and Podczeck (2009) in the following two ways. First, we work with a general Polish (complete separable metric) action space instead of a compact metric space. Second, we require the payoff functions to be jointly measurable, while in Loeb and Sun (2006, 2009) and Podczeck (2009), the payoff functions should satisfy a more restrictive condition, the *Carathéodory* condition (see Sect. 3).

Our proof is built heavily on the *exact law of large numbers* (ELLN for brevity) systematically studied in Sun (1998, 2006) (see also Sect. 2.2). This ELLN approach is different from the techniques used in Loeb and Sun (2006, 2009) and Podczeck (2009). In particular, Loeb and Sun (2006) make use of the nonstandard analysis. Loeb and Sun (2009) apply techniques of Hoover and Keisler (1984) that certain properties can be transferred from one saturated probability space to another. And in Podczeck (2009), the main result is proved through establishing new results in functional analysis. It is worthwhile to note that in Loeb and Sun (2006, 2009) and Podczeck (2009), the proofs of their purification theorems depend on the setting that the action space is a compact metric space and the payoff functions satisfy the Carathéodory condition. Thus, their methods cannot be applied to the setting of our main result, Theorem 1, directly.

One advantage of this ELLN approach is that one can simultaneously obtain *many* required purification mappings. Specifically, these purification mappings can be indexed by a full subset in an atomless probability space. In comparison, note that in the earlier purification results only the existence of *some* purification mapping has been established. The relevance of the ELLN to the ex post Nash equilibrium of a large game with idiosyncratic uncertainty is already considered in Theorem 7 of Khan and Sun (2002, p. 1792). Further results on ex post Nash equilibrium in large games are established in Khan et al. (2005) and Sun (2007b).

Finally, as an application of Theorem 1, following Khan et al. (2006), we study in Sect. 4 the problem of purification for mixed strategies in game theoretic models as in Milgrom and Weber (1985). We show that every mixed strategy has many strong purifications in such a finite-player game with a general Polish action space (not necessarily compact), and with a diffused, conditionally independent incomplete information structure, and even with discontinuous payoff functions.

The rest of this paper is organized as follows. We present basic results about saturated probability spaces and the ELLN in Sect. 2. The main result, Theorem 1, is presented and discussed in Sect. 3. Section 4 deals with the problem of purification for mixed strategies in finite-player games with incomplete information as in Milgrom and Weber (1985). The proofs of the key results are relegated to Appendix.

2 Saturation and the ELLN

In this section, we introduce basic results about saturated probability spaces in Sect. 2.1, about the ELLN in Sect. 2.2. Section 2.3 deals with one relationship between the saturation property and the existence of rich Fubini extension based on a probability space.

For a Polish (complete separable metric) space X, denote its Borel σ-algebra by $\mathcal{B}_X$, and by $\mathcal{M}(X)$ the space of all Borel probability measures associated with the topology of weak convergence. Given any Borel probability measure $\gamma \in \mathcal{M}(X)$, its support is written as supp γ. It is well-known that, for any measure-valued mapping f from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$, the $\mathcal{I}$-measurability of f with respect to the weak topology is equivalent to the $\mathcal{I}$-measurability of the function $f(\cdot)(B)$ for all $B \in \mathcal{B}_X$. For any $\mathcal{I}$-measurable mapping $g : I \to X$, the induced distribution is defined as λg^{-1} by letting $\lambda g^{-1}(B) := \lambda[g^{-1}(B)]$ for all $B \in \mathcal{B}_X$.

Given two probability spaces based on I, say $(I, \mathcal{I}, \lambda)$ and $(I, \mathcal{I}', \lambda')$, the former is said to be an *extension* of the latter if $\mathcal{I}'$ is a sub-σ-algebra of $\mathcal{I}$, and the restriction of λ to $\mathcal{I}'$ coincides with λ'. Throughout this paper, a set is said to be *countable* if it is finite or countably infinite. Let $\mathbb{N}$ be the set of all the natural numbers.

2.1 Saturated probability space

The notion of saturated probability spaces is introduced into literature by Hoover and Keisler (1984).

Definition 1 A probability space $(I, \mathcal{I}, \lambda)$ is said to be *saturated* if for any two Polish spaces X and Y, any Borel probability measure $\tau \in \mathcal{M}(X \times Y)$ with marginal probability measure τ_X on X, and any measurable mapping g from $(I, \mathcal{I}, \lambda)$ to X with distribution τ_X, there exists a measurable mapping $h : (I, \mathcal{I}, \lambda) \to Y$ such that the measurable mapping $(g, h) : (I, \mathcal{I}, \lambda) \to X \times Y$ has distribution τ.

Given a probability space $(I, \mathcal{I}, \lambda)$, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, denote by $(S, \mathcal{I}^S, \lambda^S)$ the probability space restricted to S. Here $\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}$ and λ^S is the probability measure re-scaled from the restriction of λ to $\mathcal{I}^S$.

As shown in Hoover and Keisler (1984, Corollary 4.5), there is an equivalent definition (Definition 2) for the saturated probability space. To proceed, as in Podczeck (2008, 2009), we first review some concepts related to the *measure algebra* for a probability space.

Let $(I, \mathcal{I}, \lambda)$ be a probability space. Consider a relation '$\sim$' on $\mathcal{I}$ as follows, for any $E, F \in \mathcal{I}$, $E \sim F$ if and only if $\mu(E \triangle F) = 0$, where '$\triangle$' denotes the symmetric difference. It is clear that '$\sim$' is an equivalence relation on $\mathcal{I}$. For any $E \in \mathcal{I}$, let $\hat{E} = \{F \in \mathcal{I} : F \sim E\}$ be the equivalence class of E; it is clear that $E \in \hat{E}$. The pair

$(\hat{\mathcal{I}}, \hat{\lambda})$ is said to be the *measure algebra* of $(I, \mathcal{I}, \lambda)$, here $\hat{\mathcal{I}}$ is the quotient Boolean algebra for the equivalence relation $\sim$, i.e., the set of equivalence classes in $\mathcal{I}$ for $\sim$, and $\hat{\lambda} : \hat{\mathcal{I}} \to [0, 1]$ is given by $\hat{\lambda}(\hat{E}) = \lambda(E)$, for some $E \in \hat{E}$. Also, we can define the operations $\hat{\cup}$, $\hat{\cap}$, $\hat{\setminus}$, $\hat{\triangle}$ and $\hat{\subseteq}$ on $\hat{\mathcal{I}}$ in the following way: For any $\hat{E}, \hat{F} \in \hat{\mathcal{I}}$ with $E \in \hat{E}$ and $F \in \hat{F}$, $\hat{E} \hat{\subseteq} \hat{F}$ if and only if $\lambda(E \setminus F) = 0$, $\hat{E} \hat{\cup} \hat{F} = \widehat{E \cup F}$, and analogously $\hat{\cap}$, $\hat{\setminus}$ and $\hat{\triangle}$ can be well-defined. It is clear that $\hat{\mathcal{I}}$ is a Boolean algebra under $\hat{\setminus}$ and $\hat{\cup}$.

Let $(\hat{\mathcal{I}}, \hat{\lambda})$ be the measure algebra associated to the probability space $(I, \mathcal{I}, \lambda)$. A subset of $\hat{\mathcal{I}}$ is said to be a *subalgebra* of $\hat{\mathcal{I}}$ if it contains $\hat{I}$ (the equivalent class of I) and is closed under $\hat{\cup}$ and $\hat{\setminus}$. A subalgebra $\mathcal{E}$ is *order-closed* with respect to $\hat{\subset}$ if for any non-empty upwards directed subset of $\mathcal{E}$ with supremum in $\hat{\mathcal{I}}$, the supremum belongs to $\mathcal{E}$ as well. A subset A of $\hat{\mathcal{I}}$ is said to *completely generate* $\hat{\mathcal{I}}$ if the smallest order-closed subalgebra in $\hat{\mathcal{I}}$ containing A is $\hat{\mathcal{I}}$ itself. Finally, the *Maharam type* of $\hat{\mathcal{I}}$ (or $(I, \mathcal{I}, \lambda)$) is the least cardinal number of any subset of $\hat{\mathcal{I}}$ which completely generates $\hat{\mathcal{I}}$.

Definition 2 A probability space $(I, \mathcal{I}, \lambda)$ is said to be *countably generated* if the Maharam type of $(I, \mathcal{I}, \lambda)$ is countable. It is said to be *saturated* (or super-atomless) if, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, the Maharam type of the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is uncountable.[4]

By Definition 2, a saturated probability space is an atomless probability space. Suppose $(I, \mathcal{I}, \lambda)$ is a saturated probability space, so is the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$. The Lebesgue unit interval, i.e., the interval $[0, 1]$ associated with the σ-algebra of Lebesgue measurable sets and the Lebesgue measure, is a countably generated probability space; it is thus not a saturated probability space. In comparison, any atomless Loeb probability space is saturated.[5] By Maharam's theorem, a probability space is saturated if and only if its measure algebra is a countable convex combination of measure algebras of uncountable powers of the Borel σ-algebra on $[0, 1]$.[6]

2.2 The ELLN

For any two probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, we write $\mathcal{I} \otimes \mathcal{F}$ as the usual product σ-algebra (including all the null subsets) generated by $\{S \times T : S \in \mathcal{I}, T \in \mathcal{F}\}$, and write $\lambda \otimes P$ as the product probability measure on $\mathcal{I} \otimes \mathcal{F}$. Given any mapping F from $I \times \Omega$ to a Polish space X, for any $i \in I$ and $\omega \in \Omega$, let F_i denote the marginal mapping $F(i, \cdot)$ on Ω, and F_ω the marginal mapping $F(\cdot, \omega)$ on I. As in Sun (1998, 2006), a process F is said to be *essentially pairwise independent* if for λ-almost all $i \in I$, F_i and $F_{i'}$ are independent for λ-almost all $i' \in I$.

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[4] This condition is originally called "$\aleph_1$-atomless" in Hoover and Keisler (1984), "nowhere separable" in Džamonja and Kunen (1995), "rich" in an earlier version of Keisler and Sun (2009), then in Noguchi (2009), "super-atomless" in Podczeck (2009, 2010), and "nowhere countably generatedness" in Loeb and Sun (2009). The authors thank Konrad Podczeck for pointing out the relevance between the saturated property and the "nowhere separable" property.

[5] See Hoover and Keisler (1984).

[6] See Fajardo and Keisler (2002) for details. And see Maharam (1947) for the Maharam's theorem.

We shall construct an essentially pairwise independent process as follows. Let [0, 1] be the unit interval endowed with the Borel σ-algebra $\mathcal{B}_{[0,1]}$ and the uniform distribution. For an atomless probability space $(I, \mathcal{I}, \lambda)$, let $\Omega = [0, 1]^I$ represent the space of all functions from I to the unit interval [0, 1]. By the Kolmogorov's extension theorem, we can consider the continuum product probability space $(\Omega, \mathcal{F}', P')$, where $\mathcal{F}'$ is the σ-algebra generated by cylinders of the form $\{\omega \in \Omega : \omega(i) \in B\}$ for all $B \in \mathcal{B}_{[0,1]}$, and P' is the continuum product probability measure on $(\Omega, \mathcal{F}')$.

Next define π to be a process from $I \times \Omega$ to [0, 1] by letting $\pi(i, \omega) := \omega(i)$ for all $(i, \omega) \in I \times \Omega$. Here, the marginal function π_i is the ith coordinate function on $(\Omega, \mathcal{F}', P')$. It is clear that π_i induces the uniform distribution on [0, 1] for any $i \in [0, 1]$, and π_i, π_j are independent for $i \neq j$. Accordingly, the process π is an essentially pairwise independent process. However, it is well-known that this process π is not $\mathcal{I} \times \mathcal{F}'$-measurable.[7] Indeed, the essentially pairwise independence and the joint measurability of a process with respect to the usual product σ-algebra are never compatible with each other except for the trivial case that almost all random variables are essentially constants.[8]

To overcome the above non-compatibility problem of measurability and independence, we next follow Sun (2006) to work with the framework of *Fubini extension*. It is an enrichment of the usual product probability space on which the Fubini property is retained.

Definition 3 Take as given two probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$:

(A) A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a *Fubini extension* if for any real-valued Q-integrable function F on $(I \times \Omega, \mathcal{W})$,
  (1) F_i is P-integrable on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$, and F_ω is λ-integrable on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$;
  (2) $\int_\Omega F_i \,\mathrm{d}P$ and $\int_I F_\omega \,\mathrm{d}\lambda$ are integrable on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, respectively, in addition, $\int_{I \times \Omega} F \,\mathrm{d}Q = \int_I \left(\int_\Omega F_i \,\mathrm{d}P\right) \mathrm{d}\lambda = \int_\Omega \left(\int_I F_\omega \,\mathrm{d}\lambda\right) \mathrm{d}P$.

(B) A Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ is said to be *rich* if there is a $\mathcal{W}$-measurable process G from $I \times \Omega$ to the interval [0, 1], such that G is essentially pairwise independent, and G_i induces the uniform distribution on [0, 1] for λ-almost all $i \in I$. We say that such a rich Fubini extension is *based on* $(I, \mathcal{I}, \lambda)$, and the process G *witnesses* the richness of the Fubini extension.

In a Fubini extension $(I \times \Omega, \mathcal{W}, Q)$, note that the marginal probability measures of Q on $(I, \mathcal{I})$ and $(\Omega, \mathcal{F})$ are λ and P, respectively. To reflect this property, as in Sun (2006), we denote the Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Next, we introduce the existence of rich Fubini extension based on a saturated probability space.

Lemma 1 *Assume that* $(I, \mathcal{I}, \lambda)$ *is a saturated probability space, then there exists a probability space* $(\Omega, \mathcal{F}, P)$ *extending* $(\Omega, \mathcal{F}', P')$, *such that there exists a rich*

[7] See Doob (1953, p. 67) for the special case that $(I, \mathcal{I}, \lambda)$ is the Lebesgue unit interval.

[8] See Proposition 2.1 of Sun (2006).

Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *on which the process of coordinate functions* π *is* $\mathcal{I} \boxtimes \mathcal{F}$*-measurable and witnesses the richness of the Fubini extension.*

Remark 1 If both $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ are atomless Loeb probability spaces, their Loeb product probability space is a rich Fubini extension as shown in Theorem 6.2 of Sun (1998). Sun (2006, Proposition 5.6) provides another construction, where $I = [0, 1]$ and $(I, \mathcal{I}, \lambda)$ is a probability space obtained from a hyperfinite Loeb counting space via a bijection, and $(\Omega, \mathcal{F}, P)$ is an extension of the usual continuum product probability space $(\Omega, \mathcal{F}', P')$. Based on the construction of Sun (2006), a new rich Fubini extension is presented in Sun and Zhang (2009) where $(I, \mathcal{I}, \lambda)$ is a saturated probability space and an extension of the Lebesgue unit interval. Podczeck (2010) establishes a more general result that $(I, \mathcal{I}, \lambda)$ could be any saturated probability space.

Indeed, a rich Fubini extension satisfies the universality property in the sense that one can construct processes on it with essentially pairwise independent random variables that have any given variety of distributions on a general Polish space. The following result is Proposition 5.3 of Sun (2006).

Lemma 2 *Given a rich Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *and a Polish space* X*. Let* f *be a measurable mapping from* $(I, \mathcal{I}, \lambda)$ *to* $\mathcal{M}(X)$*, then there exists an* $\mathcal{I} \boxtimes \mathcal{F}$*-measurable process* $F : I \times \Omega \to X$ *such that the process* F *is essentially pairwise independent and* $f(i)$ *is the induced distribution by* F_i*, for* λ*-almost all* $i \in I$*.*

The following result is a version of the ELLN in terms of sample means, see Corollary 2.10 of Sun (2006). Namely, in the framework of Fubini extension, if a process F is essentially pairwise independent, then for P-almost every sample function F_ω, its mean is equal to the mean of F.

Lemma 3 *Assume that* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *is a Fubini extension, and* F *is a real-valued, essentially pairwise independent,* $\lambda \boxtimes P$*-integrable process on* $I \times \Omega$*. Then, there exists a* P*-null subset* $N \subseteq \Omega$*, such that* $\int_I F_\omega(i)\, \mathrm{d}\lambda(i) = \int_I \int_\Omega F\, \mathrm{d}P(\omega)\, \mathrm{d}\lambda(i)$*, for all* $\omega \in \Omega \backslash N$*.*

The framework of Fubini extension plays a fundamental role in studying the ELLN. Indeed, this framework is "necessary and sufficient" for the ELLN. First, in such a framework, besides the ELLN in sample means as in Lemma 3, one can establish other forms of ELLN, e.g., the ELLN in terms of sample distributions or coalitional sample distributions, even the converse of the ELLN holds as well (see Sun 2006, Sect. 2.3). Second, if a process is essentially pairwise independent and satisfies the property of coalitional aggregate certainty (i.e., for any $S \subset I$ with $\lambda(S) > 0$, almost every sample function restricted to S has the same distribution as the process restricted to $S \times \Omega$), then there exists a Fubini extension in which the process is measurable, see Sun (2007a).

2.3 Saturation and rich Fubini extension

Given a probability space $(I, \mathcal{I}, \lambda)$, let $\mathcal{C}$ be a countably generated sub-σ-algebra of $\mathcal{I}$. A measurable function f defined on the probability space $(I, \mathcal{I}, \lambda)$ is said to be *essentially $\mathcal{C}$-measurable* if there is a $\mathcal{C}$-measurable function g also defined on I such that $f(i) = g(i)$ for λ-almost all $i \in I$. The following result is Theorem 4.2 of Sun (2006).

Lemma 4 *Let F be an essentially pairwise independent process from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X, and $\mathcal{C}$ a countably generated sub-σ-algebra of $\mathcal{I}$. Then, the set of all $\omega \in \Omega$ such that the function F_ω is essentially $\mathcal{C}$-measurable must have probability zero except for the trivial case that almost all the random variables F_i are constant.*

The next result says that the existence of a rich Fubini extension based on a probability space is a characterization of the saturation property as in Keisler and Sun (2009). It is straightforward from Lemmas 1 and 4, see Appendix for the proof.

Corollary 1 *The probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if there is a rich Fubini extension based on it.*

3 The purification theorem

We first present the main result, Theorem 1, in Sect. 3.1, then we discuss it in Sect. 3.2.

In this section, we fix a saturated probability space $(I, \mathcal{I}, \lambda)$, and a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ as in Lemma 1. Let X be a Polish space associated with $\mathcal{B}_X$ and $\mathcal{M}(X)$ as in Sect. 2. For any $\mathcal{I}$-measurable mapping f from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$, let $f(i; B)$ be the value of the probability measure $f(i)$ for any Borel subset $B \subseteq X$. Denote by $f(i; \mathrm{d}x)$ the integration operator with respect to this probability measure $f(i)$.

Let $\mathcal{H}$ be the collection of real-valued functions ϕ on the product space $I \times X$ such that: (1) ϕ is $\mathcal{I} \otimes \mathcal{B}_X$-measurable, and (2) ϕ is integrally bounded, i.e., there exists a non-negative integrable function α^ϕ from $(I, \mathcal{I}, \lambda)$ to $\mathbb{R}$ with $|\phi(i, x)| \leq \alpha^\phi(i)$ for all $(i, x) \in I \times X$.

3.1 The main result

We are ready to introduce our main result, which is a general purification theorem. See Appendix for the proof.

Theorem 1 *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension based on a saturated probability space $(I, \mathcal{I}, \lambda)$. Assume X is a Polish space, and $\mathcal{D}$ a countable subset of $\mathcal{H}$. Then for any $\mathcal{I}$-measurable mapping $f : I \to \mathcal{M}(X)$, there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F : I \times \Omega \to X$ with the following properties*:

(1) *The process F is essentially pairwise independent and the induced distribution on X of F_i is $f(i)$ for λ-almost all i.*

(2) *For P-almost all $\omega \in \Omega$, the mapping $F_\omega : I \to X$ is a purification for f with respect to $\mathcal{D}$ in the sense that for all $\phi \in \mathcal{D}$,*

$$\int_I \int_X \phi(i, x)\, f(i; \mathrm{d}x)\, \mathrm{d}\lambda(i) = \int_I \phi[i, F_\omega(i)]\, \mathrm{d}\lambda(i). \tag{1}$$

We can interpret Theorem 1 in a decision-making situation. Suppose Ann is the decision maker with the space of uncertainty, $(I, \mathcal{I})$. She can choose an action from the space X and her payoff function is taken from $\mathcal{D}$. Assume further that before making an decision, she has no information about the uncertainty except the distribution λ, which is a probability measure on $(I, \mathcal{I})$. Her objective is to maximize the expected payoff by choosing a *mixed strategy* $f : I \to \mathcal{M}(X)$. That is, her action is a probability measure on the action space X when facing the uncertainty i.

In Theorem 1, what is the role played by the probability space $(\Omega, \mathcal{F}, P)$? This space works as a random device for the decision maker. When facing the uncertainty i, Ann can choose actions with the assistance of this probability space. In particular, she takes the action $F_i(\omega)$ when ω is realized. In this way, she takes a *pure strategy* $F_\omega : (I, \mathcal{I}, \lambda) \to X$ when ω is realized. We call $F : (I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P) \to X$ a *behaviorial strategy*.[9] Assertion (1) in Theorem 1 says that for F_i defined on the probability space $(\Omega, \mathcal{F}, P)$ induces the same distribution as $f(i)$ for λ-almost all i. That is, with the assistance of the random device, the decision maker can implement her mixed strategy f by taking the behaviorial strategy F.

Next, we say something about the property of essentially pairwise independence. Since $(I, \mathcal{I}, \lambda)$ is interpreted the space of uncertainty, the independence condition could model the situation that the decision maker takes actions independently when facing different uncertainty, provided that the information structure is sufficiently "rich". In the theory of large games, $(I, \mathcal{I}, \lambda)$ is used to represent the space of names of the players. The independence property is natural since different players take actions independently.[10]

Under the assumption that the process F is essentially pairwise independent, by the ELLN, Assertion (2) in Theorem 1 implies that almost every F_ω is a required purification. More precisely, for almost any $\omega \in \Omega$, by taking the pure strategy F_ω, Ann can earn the amount of $\int_I \phi[i, F_\omega(i)]\, \mathrm{d}\lambda(i)$, and this amount is exactly the same as what she can expect by taking the mixed strategy f, $\int_I \int_X \phi(i, x)\, f(i; \mathrm{d}x)\, \mathrm{d}\lambda(i)$. In other words, the risk about how much she can earn under different realizations of $\omega \in \Omega$ disappears.

The next result is a generalization of Khan et al. (2006, Corollary 1), which in turn is a generalization of the original DWW Theorem in Dvoretzky et al. (1951a). It follows from Theorem 1 line by line as Loeb and Sun (2006, Corollary 2.4) follows from Loeb and Sun (2006, Theorem 2.2).

Corollary 2 *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension based on a saturated probability space $(I, \mathcal{I}, \lambda)$. Let X be a Polish space. For each k in a countable set K, let μ_k be a finite signed measure on $(I, \mathcal{I})$ that is absolutely continuous with respect to λ. For each j in a countable set J, assume that $\phi_j \in \mathcal{H}$.*

Then for any $\mathcal{I}$-measurable mapping f from I to $\mathcal{M}(X)$, there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F : I \times \Omega \to X$, such that F is essentially pairwise independent,

[9] See Khan et al. (2006) and the references therein for more discussion about mixed strategies and behaviorial strategies.

[10] See Khan and Sun (2002) for a survey on games with many players.

the induced distribution of F_i *is* $f(i)$ *for* λ*-almost all* i; *and for* P*-almost all* $\omega \in \Omega$ *the sample mapping* F_ω *satisfies the following properties*:

1. $\int_I \int_X \phi_j(i,a)\, f(i; \mathrm{d}x)\, \mathrm{d}\lambda(i) = \int_I \phi_j[i, F_\omega(i)]\, \mathrm{d}\lambda(i),\ \forall\, j \in J$;
2. $\int_I f(i; B)\, \mathrm{d}\mu_k(i) = \mu_k[F_\omega^{-1}(B)]$, *for all* $B \in \mathcal{B}_X$ *and* $k \in K$;
3. $F_\omega(i) \in \mathrm{supp}\, f(i)$ *for* λ*-almost all* $i \in I$.

In Corollary 2, let J be an empty set, the existence result of the above corollary is a variation of the DWW Theorem on a saturated probability space $(I, \mathcal{I}, \lambda)$ and a general Polish space X. For another special case, taking J to be empty and the set K contains only one element with $\mu_1 = \lambda$, the existence result of Corollary 2 is Theorem 3.6 **(P6)** of Keisler and Sun (2009).

3.2 Discussion

In this paper, besides establishing the existence result based on saturated probability spaces as in Loeb and Sun (2009) and Podczeck (2009), we can simultaneously obtain *many* required purifications. More precisely, these purifications can be indexed by a full subset in an atomless probability space $(\Omega, \mathcal{F}, P)$.

Recall that the probability space $(\Omega, \mathcal{F}, P)$ can represent all the mappings from I to X (see Sect. 2.2). Assume that the measure-valued mapping f is *nontrivial*, i.e., it is not the case that $f(i)$ is a Dirac measure on X for λ-almost all $i \in I$. Accordingly, for the relevant essentially pairwise independent process F, it is not the case that almost every F_i is essentially a constant. Then, there are many different $\omega \in \Omega$ such that F_ω are different measurable mappings over $(I, \mathcal{I}, \lambda)$. Therefore, if f is nontrivial, we can simultaneously obtain many different purifications for f with respect to $\mathcal{D}$.

We next compare our general purification result, Theorem 1, with the earlier results in Loeb and Sun (2006, 2009) and Podczeck (2009). First, as to the methodology, our result relies heavily on the ELLN. In comparison, the purification theorem of Loeb and Sun (2006) is based on atomless Loeb probability spaces, and the authors make use of techniques in nonstandard analysis. Loeb and Sun (2009) mainly apply techniques as in Hoover and Keisler (1984) that certain types of results over one saturated probability space can be transferred to another. Consequently, the existence result of purifications based on an atomless Loeb probability space, Loeb and Sun (2006, Theorem 2.2), can be transferred to the existence result based on a general saturated probability space. In Podczeck (2009), new results on functional analysis are established to prove the general purification theorem.

Second, in the earlier purification results, i.e., Loeb and Sun (2006, Theorem 2.2) and (2009, Theorem 2.2), Podczeck (2009, Theorem 2), the target space X is a compact metric space. While in our Theorem 1, we take X to be a more general Polish space.

Third, in the earlier results, instead of functions in $\mathcal{H}$, a more restrictive condition is imposed on the functions over the product $I \times X$. Let $\mathcal{H}'$ denote the collection of functions considered in Loeb and Sun (2006, 2009) and Podczeck (2009). Here $\mathcal{H}'$ is the collection of all the functions ϕ on $I \times X$ with the following conditions: (1a) $\phi(\cdot, x)$ is $\mathcal{I}$-measurable on I for each $x \in X$, (1b) $\phi(i, \cdot)$ is continuous on X for each

$i \in I$, and (2) ϕ is bounded by a non-negative λ-integrable function α^{ϕ}. Here, the conditions (1a) and (1b) are the *Carathéodory condition*. It is known that any function satisfying the Carathéodory condition is also jointly measurable. As a result, $\mathcal{H}'$ is a subset of $\mathcal{H}$.

It is worthwhile to note that in Loeb and Sun (2006, 2009) and Podczeck (2009), the proofs of purification theorems therein depend on the setting that the target space is a compact metric space and the functions satisfy the Carathéodory condition. Thus, their methods cannot be applied to our setting in Theorem 1 directly.

Finally, together with Lemma 1, it follows from Theorem 1 that the saturation property of a probability space implies the existence of purification for any measure-valued mapping with a general Polish action space. We note that the converse also holds. Specifically, as illustrated by counterexamples in Loeb and Sun (2009, Remark 2.4) and Podczeck (2009, Theorem 3(B)), if a probability space is not saturated, there exists a measure-valued mapping, a function $\phi \in \mathcal{H}$, such that the purification does not exist.[11]

4 Finite games with incomplete information

In this section, we apply our Theorem 1 to study the problem of purification for games with incomplete information as in Milgrom and Weber (1985). A game Γ with incomplete information consists of a finite set of m players and an information space available to them. Each player n can take actions from X_n, which is a Polish space for $1 \leq n \leq m$; and the Cartesian product $\Pi_{n=1}^{m} X_n$ is written as X. For each player n, a measurable space $(I_n, \mathcal{I}_n)$ represents the set of possible information for her. The information is incomplete in the sense that each player does not know the particulars of the other players' information. The payoff function of player n is $u_n : I_0 \times I_n \times X \to \mathbb{R}$, where $I_0 = \{i_{0k} : k \in \mathbb{N}\}$ is a countable set representing the common state space which affects payoffs of all the players. Thus, player n's payoff function depends on the common states, her own information and the actions of all the players. Denote by $\mathcal{I}_0$ the power set of the countable set I_0. Let $(I, \mathcal{I}) := (\Pi_{j=0}^{m} I_j, \Pi_{j=0}^{m} \mathcal{I}_j)$ be the product measure space and λ be a probability measurable on $(I, \mathcal{I})$. The resulting probability space $(I, \mathcal{I}, \lambda)$ constitutes an information structure of the game. For each player n, assume further that the payoff function $u_n(i_0, i_n, x)$ is an $\mathcal{I}_n \otimes \mathcal{B}_X$-measurable function for each $i_0 \in I_0$; in addition, for all $i \in I$, $|u_n(i_0, i_n, x)| \leq \alpha(i)$, where α is a non-negative integrable function on $(I, \mathcal{I}, \lambda)$.

For $0 \leq j \leq m$, denote by λ_j the marginal probability measure of λ on $(I_j, \mathcal{I}_j)$. Suppose the support of λ_0 is the whole set I_0. As a result, when the common state i_0 is i_{0k}, the conditional probability measure of λ on the space $(\Pi_{j=1}^{m} I_j, \Pi_{j=1}^{m} \mathcal{I}_j)$ exists and is denoted by $\lambda(\cdot\,; i_{0k})$. Moreover, for each player n, let λ_{nk} be the marginal probability measure of $\lambda(\cdot\,; i_{0k})$ on the measurable space $(I_n, \mathcal{I}_n)$. Following Milgrom and Weber

[11] For the counterexample in the special setting that the probability space is a Lebesgue space, see Loeb and Sun (2006, Example 2.7). For a survey about similar counterexamples in the theory of large games, see Khan and Sun (2002, Sect. 5).

(1985), the information structure $(I, \mathcal{I}, \lambda)$ is said to be *conditionally independent*, if $\lambda(\cdot\,; t_{0k}) = \Pi_{n=1}^{m}\lambda_{nk},\ \forall k \in K$.

A *mixed strategy* for player n is a measurable mapping from her information space $(I_n, \mathcal{I}_n)$ to $\mathcal{M}(X_n)$. A *pure strategy* is an $\mathcal{I}_n$-measurable mapping from I_n to X_n, and it can be regarded as a mixed strategy using Dirac measures. A *mixed* (*pure*) *strategy profile* $h = (h_1, \ldots, h_m)$ is a tuple of mixed (pure) strategies, in which h_n specifies a mixed (pure) strategy for player n. Given any mixed strategy profile $f = (f_1, \ldots, f_m)$, the corresponding expected payoff for player n is

$$U_n(f) := \int_I \int_X u_n(i_0, i_n, x) f_1(i_1; \mathrm{d}x_1) \ldots f_m(i_m; \mathrm{d}x_m)\, \mathrm{d}\lambda(i), \tag{2}$$

where for each $i \in I$, the inner integral on X is the iterated integral on $X_m, \ldots, X_1$, respectively. A mixed strategy profile $f = (f_n, f_{-n})$ is called a *Nash equilibrium* for the game Γ if for every player n, $U_n(f_n, f_{-n}) \geq U_n(f_n', f_{-n})$ for any mixed strategy f_n' of player n.

The following definition is proposed in Khan et al. (2006).

Definition 4 A pure strategy profile $g = (g_1, \ldots, g_m)$ is said to be a *strong purification* of a mixed strategy profile $f = (f_1, \ldots, f_m) = (f_n, f_{-n})$ if the following four conditions are satisfied for each player n:

1. $U_n(f) = U_n(g)$.
2. For any given mixed strategy $\tilde{f}_n$ of player n, $U_n(\tilde{f}_n, f_{-n}) = U_n(\tilde{f}_n, g_{-n})$.
3. For each $k \in K$, given $i_0 = i_{0k}$, g_n and f_n have the same conditional distribution on X_n, i.e., $\int_{I_n} f_n(i_n; \cdot)\, \mathrm{d}\lambda_{nk}(i_n) = \lambda_{nk} g_n^{-1}(\cdot)$.
4. For λ_n-almost all $i_n \in I_n$, $g_n(i_n) \in \operatorname{supp} f_n(i_n)$.

Item 1 means that the strong purification yields the same expected payoff as the mixed strategy for all players. Item 2 means that the expected payoff of player n from any mixed strategy is always same irrespective of the opponents play being f_{-n} or g_{-n}. Thus, Items 1 and 2 guarantee that if the mixed strategy f is a Nash equilibrium, so is its strong purification, which is a pure strategy. See Section 3 of Khan et al. (2006) for more discussion.

We are now ready to present our main result for this section, which generalizes Theorem 3.2 of Loeb and Sun (2006). It is about the existence of strong purification for any mixed strategy profile in the game Γ. This result follows from Corollary 2 almost in the same way that Theorem 3.2 of Loeb and Sun (2006) follows from Corollary 2.4 of (2006). The proof is relegated to Appendix.

Theorem 2 *Assume that* (*1*) *the information structure in the game Γ is conditionally independent, and* (*2*) *for each n, $(I_n, \mathcal{I}_n, \lambda_n)$ is a saturated probability space together with a rich Fubini extension $(I_n \times \Omega_n, \mathcal{I}_n \boxtimes \mathcal{F}_n, \lambda_n \boxtimes P_n)$. Then, for any mixed strategy profile $f = (f_1, \ldots, f_m)$ and for each player n, there exists an $\mathcal{I}_n \boxtimes \mathcal{F}_n$-measurable process $F^n : I_n \times \Omega_n \to X_n$, which is essentially pairwise independent and $F^n_{i_n}$ induces the distribution $f_n(i_n)$ on X_n for λ_n-almost all i_n, and $(F^1_{\omega_1}, \ldots, F^m_{\omega_m})$ is a strong purification of f for P_n-almost all $\omega_n \in \Omega_n$, for each n.*

Remark 2 In our model of games with incomplete information, the common state space I_0 is a countable (probably infinite) set and the private information states I_n for each player n can be uncountable, see also Loeb and Sun (2006). In addition, each player's private information space is a saturated probability space, and the action space is a general Polish space. In comparison, in the purification results of Milgrom and Weber (1985) and Khan et al. (2006), the common state space I_0 is finite, while each player's private information space is atomless (not necessarily saturated) and action space is finite. The role of this countable set I_0 in Theorem 2 is similar to that of the countable set K in Corollary 2. In particular, a countable set of probability measures is obtained such that each measure therein is absolutely continuous with respect to some given probability measure, then together with the conditionally independence condition (1), Corollary 2 is applicable.

If there exists a Nash equilibrium in mixed strategy for the game Γ, it is guaranteed by Theorem 2 that there also exists a Nash equilibrium in pure strategy. In general, as discussed in Sect. 3.2, we can simultaneously obtain *many* pure strategy equilibria if the mixed strategy f is nontrivial. In particular, assume that the mixed strategy f is a Nash equilibrium for the game Γ, for each player n, and interpret $(\Omega_n, \mathcal{F}_n, P_n)$ as her random device. Theorem 2 states that for P_n-almost every realization of $\omega_n \in \Omega_n\ \forall n$, the realized pure strategy profile, $(F^1_{\omega_1}, \ldots, F^m_{\omega_m})$, is also a pure strategy Nash equilibrium for Γ.

Next, we note that Theorem 2 generalizes the assumptions of compact metric action spaces and continuous payoff functions in Loeb and Sun (2006, Theorem 3.2) to a general setting of Polish action spaces and measurable payoffs, respectively. In such a general or less demanding setting of game theoretic models with incomplete information, it is stated in our theorem that each mixed strategy has many strong purifications. This reflects the idea of purification that "when information in games is sufficiently disparate among the players and when its distribution is sufficiently diffuse, the players might as well restrict their attention to pure strategies" (see p. 401 of Radner and Rosenthal 1982).

Finally, it is worth mentioning that for the games as in Milgrom and Weber (1985), Radner and Rosenthal (1982) as well, the independence condition for the information structure plays an important role. In particular, it follows from this independence condition that each player's expected payoff depends on the others' strategies only through the induced distributions on their action spaces, and it is then that one can apply Corollary 2 to obtain strong purifications for any mixed strategy profile. In Yannelis and Rustichini (1991), a model of Bayesian game without such an independence condition was introduced where the state space is a probability space and each player's information structure is a general measurable partition of this state space.[12] A positive result on the existence of pure strategy equilibria was established in Yannelis and Rustichini (1991, Theorem 5.2) by modeling the state space as an atomless probability space and by modeling the set of strategies of each player via a $\mathbb{R}^\ell$-valued correspondence which is convex-valued and integrably bounded. However, by modeling the set of

[12] See Yannelis and Rustichini (1991, Sect. 6) for the comparison between this model and those in Milgrom and Weber (1985) and Radner and Rosenthal (1982), see also Yannelis (2009) for a Bayesian game theoretic model with a continuum of players.

strategies for each player via a more general Banach-valued correspondence, one can only obtain an approximate pure strategy Bayesian equilibrium. It is intuitively clear that by formalizing the state space as a saturated probability space, and appealing to the results in Podczeck (2008) and Sun and Yannelis (2008), one can obtain a similar positive result as in the general setting of Yannelis and Rustichini (1991). We hope to take the details up in a subsequent paper.

5 Appendix: Proofs

Proof of Corollary 1 By Lemma 1, the saturation property implies the existence of a rich Fubini extension based on a probability space.

Next, we prove the converse. For the rich Fubini extension based on the probability space $(I, \mathcal{I}, \lambda)$, assume that the process π witnesses the richness. That is, π is an essentially pairwise independent process and the random variable π_i induces the uniform distribution on the interval [0, 1]. Note that this process π is nontrivial. By Lemma 4, $(I, \mathcal{I}, \lambda)$ cannot be countably generated. Otherwise, the Maharam type of $(I, \mathcal{I}, \lambda)$ is countable; i.e., there is a countable set $\hat{\mathcal{E}} = \{\hat{E}_n : n \in \mathbb{N}\}$ of $\hat{\mathcal{I}}$, such that $\hat{\mathcal{E}}$ completely generates $\hat{\mathcal{I}}$. For each equivalent class $\hat{E}_n$ in $\hat{\mathcal{E}}$, we take one element $E_n \in \hat{E}_n$. Then, the σ-algebra $\mathcal{I}$ itself can be completely generated by $\{E_n : n \in \mathbb{N}\}$. Now according to Lemma 4, for P-almost all ω, the sample function π_ω is not $\mathcal{I}$-measurable, which contradicts Assertion (1) of Definition 3(A). Analogously, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, notice that the restriction of the process π to $S \times \Omega$ is also a nontrivial essentially pairwise independent process, then $(I^S, \mathcal{I}^S, \lambda^S)$ is not countably generated either. Therefore, the probability space $(I, \mathcal{I}, \lambda)$ is saturated by Definition 2. □

Proof of Theorem 1 Let us first fix one $\phi \in \mathcal{D}$. Given the saturated probability space $(I, \mathcal{I}, \lambda)$, together with the rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, P)$. Since the measure-valued mapping f is $\mathcal{I}$-measurable, by Lemma 2, there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F : I \times \Omega \to X$, which is essentially pairwise independent and the random variable F_i induces distribution $f(i)$ for λ-almost all $i \in I$. Thus, we prove Assertion (1) in the theorem.

We next show Assertion (2). Notice that F_i induces the distribution $f(i)$ for λ-almost all i, it follows that,

$$\int_X \phi(i, x)\, f(i; \mathrm{d}x) = \int_\Omega \phi[i, F(i, \omega)]\, \mathrm{d}P(\omega); \tag{3}$$

then

$$\int_I \int_X \phi(i, x)\, f(i; \mathrm{d}x)\, \mathrm{d}\lambda(i) = \int_I \int_\Omega \phi[i, F(i, \omega)]\, \mathrm{d}P(\omega)\, \mathrm{d}\lambda(i). \tag{4}$$

Define $G^\phi(i, \omega) = \phi[i, F(i, \omega)]$. We next show in two steps that G^ϕ is a $\lambda \boxtimes P$-integrable function. First, it is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable function on $I \times \Omega$. Towards this end, define H to be a process from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F})$ to $(I \times X, \mathcal{I} \otimes \mathcal{B}_X)$ by letting $H(i, \omega) = [i, F(i, \omega)]$. The mapping H is measurable. Indeed, for any Borel subset $C \subseteq \mathbb{R}, \phi^{-1}(C) \in \mathcal{I} \otimes \mathcal{B}_X$ because ϕ is measurable, then $H^{-1}[\phi^{-1}(C)]$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. Note that $[G^{\phi}]^{-1}(C) = H^{-1}[\phi^{-1}(C)]$, we thus obtain the $\mathcal{I} \boxtimes \mathcal{F}$-measurability of G^{ϕ}. Second, because $\phi(i, x)$ is bounded by $\alpha^{\phi}(i)$ for any $x \in X$, so is $G^{\phi}(i, \omega) = \phi[i, F(i, \omega)]$ for any $\omega \in \Omega$. Therefore, we obtain the $\lambda \boxtimes P$-integrability of G^{ϕ}, because the λ-integrable function α^{ϕ} can also be viewed as a $\lambda \boxtimes P$-integrable function on $I \times \Omega$.

Note that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension of the product space between $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, by Assertion (2), Part A of Definition 3,

$$\int_I \int_\Omega \phi[i, F(i, \omega)] \, \mathrm{d}P(\omega) \, \mathrm{d}\lambda(i) = \int_{I \times \Omega} G^{\phi} \, \mathrm{d}\lambda \boxtimes P. \tag{5}$$

Moreover, we claim that G^{ϕ} is an essentially pairwise independent process. Given any Borel subset C in $\mathbb{R}$, $\phi_i^{-1}(C) \in \mathcal{B}_X$ due to the measurability of $\phi(i, \cdot)$ for λ-almost all $i \in I$. Then for such an i, $F_i^{-1}[\phi_i^{-1}(C)] \in \mathcal{F}$ since F_i is a $\mathcal{F}$-measurable mapping. It is clear that $[G_i^{\phi}]^{-1}(C) = F_i^{-1}[\phi_i^{-1}(C)]$, which implies that G_i^{ϕ} is $\mathcal{F}$-measurable for λ-almost all $i \in I$. Moreover, G_i^{ϕ} and $G_{i'}^{\phi}$ are pairwise independent if F_i and $F_{i'}$ are independent. Accordingly, the process G^{ϕ} is essentially pairwise independent because the process F satisfies this property.

Now we are ready to apply the ELLN for the essentially pairwise independent process G^{ϕ}. By Lemma 3, there exists a P-null subset $N^{\phi} \subseteq \Omega$, such that for any $\omega \in \Omega \backslash N^{\phi}$,

$$\int_{I \times \Omega} G^{\phi} \, \mathrm{d}\lambda \boxtimes P = \int_I G_{\omega}^{\phi}(i) \, \mathrm{d}\lambda(i) = \int_I \phi[i, F_{\omega}(i)] \, \mathrm{d}\lambda(i). \tag{6}$$

Combining the above Eqs. (4)–(6), for any $\omega \in \Omega / N^{\phi}$,

$$\int_I \int_X \phi(i, x) \, f(i; \mathrm{d}x) \, \mathrm{d}\lambda(i) = \int_I \phi[i, F_{\omega}(i)] \, \mathrm{d}\lambda(i). \tag{7}$$

We next fix such a P-null subset for each $\phi \in \mathcal{D}$. Now we can turn to the countable subset $\mathcal{D}$ of $\mathcal{H}$. Following the above procedure, we can construct a countable number of P-null subsets $N^{\phi} \subseteq \Omega, \forall \phi \in \mathcal{D}$, such that Eq. (7) holds for each $\phi \in \mathcal{D}$. Let $N = \bigcup_{\phi \in \mathcal{D}} N^{\phi}$, it is clear that $P(N) = 0$. Hence, by Eq. (7), we obtain that for any $\phi \in \mathcal{D}$ and any $\omega \in \Omega / N$,

$$\int_I \int_X \phi(i, x) \, f(i; \mathrm{d}x) \, \mathrm{d}\lambda(i) = \int_I \phi[i, F_{\omega}(i)] \, \mathrm{d}\lambda(i). \tag{8}$$

Thus, the mapping F_ω, for P-almost all $\omega \in \Omega$, is a required purification of the measure-valued one f with respect to $\phi \in \mathcal{D}$. We complete the proof of Assertion (2) of the theorem. □

Proof of Theorem 2 As in Loeb and Sun (2006, Theorem 3.2), we next apply Corollary 2, the ELLN version of purification theorem, to show Theorem 2.

In what follows, for any n, let $\Pi_{j\neq n}$ denote $\Pi_{1\leq j\leq m, j\neq n}$, which represents the product over all the indices $1 \leq j \leq m$ except for $j = n$. For example, $X_{-n} = \Pi_{j\neq n}X_j$, $I_{-n} = \Pi_{j\neq n}I_j$. And for any $i = (i_0, i_1, \ldots, i_m) \in I$, write i_{-0} to be $(i_1, \ldots, i_m) = (i_n, i_{-n})$.

First fix player n. For each $k \in K$, let $\eta_k = \lambda_0(\{t_{0k}\})$. It is clear that for each $S_n \in \mathcal{I}_n$, $\lambda_n(S_n) = \sum_{k\in K} \eta_k \lambda_{nk}(S_n)$. Thus, each λ_{nk} is absolutely continuous with respect to λ_n. Denote by β_{nk} the Radon-Nikodym derivative of λ_{nk} with respect to λ_n. According to the conditionally independence, we have $\lambda(\cdot\,; i_{0k}) = \Pi_{n=1}^m \lambda_{nk}$ for each k. For any mixed strategy profile $f = (f_1, \ldots, f_m)$, player n's expected payoff $U_n(f)$ defined in Eq. (2) can be written as follows:

$$\sum_{k\in K} \eta_k \int\limits_{i_{-0}\in\Pi_{j=1}^m I_j} \int\limits_{x\in\Pi_{j=1}^m X_j} u_n(i_{0k}, i_n, x) \prod_{j=1}^m f_j(i_j; \mathrm{d}x_j) \prod_{j=1}^m \mathrm{d}\lambda_{jk}(i_j). \tag{9}$$

For the mixed strategy profile f, define

$$\psi_n^f(i_n, x_n) = \sum_{k\in K} \eta_k \beta_{nk}(i_n) \int\limits_{I_{-n}} \int\limits_{X_{-n}} u_n(i_{0k}, i_n, x_n, x_{-n}) \prod_{j\neq n} f_j(i_j; \mathrm{d}x_j) \prod_{j\neq n} \mathrm{d}\lambda_{jk}(i_j). \tag{10}$$

Then, by Eq. (9), the expected payoff for player n is

$$U_n(f) = \int\limits_{I_n} \int\limits_{X_n} \psi_n^f(i_n, x_n) f_n(i_n; \mathrm{d}x_n) \mathrm{d}\lambda_n(i_n). \tag{11}$$

For each $j = 1, \ldots, m$, denote by $\gamma_{jk}^{f_j}$ the induced probability distribution on X_j of $\int_{I_j} f_j(i_j, \cdot) \mathrm{d}\lambda_{jk}(i_j)$. Then, from Eq. (10), we obtain that

$$\psi_n^f(i_n, x_n) = \sum_{k\in K} \eta_k \beta_{nk}(i_n) \int\limits_{X_{-n}} u_n(i_{0k}, i_n, x_n, x_{-n}) \prod_{j\neq n} \mathrm{d}\gamma_{jk}^{f_j}(x_j). \tag{12}$$

Equations (11) and (12) imply that, given $i_0 = i_{0k}$, player n's expected payoff depends on the actions of the other players only through the induced conditional distributions of their strategies on their action spaces.

Recall that α is a λ-integrable function that dominates all the payoff functions. Let α_n be the function from I_n to $\mathbb{R}_+$ such that for each $i_n \in I_n$,

$$\alpha_n(i_n) = \sum_{k \in K} \eta_k \beta_{nk}(i_n) \int\limits_{I_{-n}} \alpha(i_{0k}, i_n, i_{-n}) \prod_{j \neq n} \mathrm{d}\lambda_{jk}(i_j). \tag{13}$$

It is clear that α_n is λ_n-integrable and that $\int_I \alpha(i)\mathrm{d}\lambda(i) = \int_{I_n} \alpha_n(i_n)\mathrm{d}\lambda_n(i_n)$ by the classical Fubini theorem. Recall that for any $x \in X$ and $i \in I$, $|u_n(i_0, i_n, x)| \leq \alpha(i_0, i_n, i_{-n})$. Consequently, Eqs. (10) and (13) imply that for any $i_n \in I_n, x_n \in X_n$, $|\psi_n^f(i_n, x_n)| \leq \alpha_n(i_n)$.

Given the saturated probability space $(I_n, \mathcal{I}_n, \lambda_n)$, together with the rich Fubini extension $(I_n \times \Omega_n, \mathcal{I}_n \boxtimes \mathcal{F}_n, \lambda_n \boxtimes P_n)$, we can apply Corollary 2. The function ψ_n^f here corresponds to ψ_j thereof, and λ_{nk} for $k \in K$, X_n and f_n to η_k, X, and f therein, respectively. By Corollary 2, there exists a process $F^n : I_n \times \Omega_n \to X_n$, which is essentially pairwise independent and $F^n_{i_n}$ induces the distribution $f_n(i_n)$ on X_n for λ_n-almost all $i_n \in I_n$; moreover, there exists a P_n-null subset $M_n \subseteq \Omega_n$ such that for each $\omega_n \notin M_n$, the sample mapping $F^n_{\omega_n} : I_n \to X_n$ is an $\mathcal{I}_n$-measurable mapping and satisfies the following properties:

(i) $\int_{I_n} \int_{X_n} \psi_n^f(i_n, x_n) f_n(i_n; \mathrm{d}x_n)\, \mathrm{d}\lambda_n(i_n) = \int_{I_n} \psi_n^f[i_n, F^n_{\omega_n}(i_n)]\, \mathrm{d}\lambda_n(i_n)$;

(ii) for all Borel set B_n in X_n,

$$\int\limits_{I_n} f_n(i_n; B_n)\mathrm{d}\lambda_{nk}(i_n) = \lambda_{nk}[F^n_{\omega_n}]^{-1}(B_n) = \gamma_{nk}^{f_n}(B_n);$$

(iii) $F^n_{\omega_n}(i_n) \in \operatorname{supp} f_n(i_n)$ for λ_n-almost all $i_n \in I_n$.

Similarly, considering the saturated probability spaces $(I_j, \mathcal{I}_j, \lambda_j)$ together with the rich Fubini extension $(I_j \times \Omega_j, \mathcal{I} \boxtimes \mathcal{F}_j, \lambda_j \boxtimes P_j)$, we can apply the above procedure for each player $j = 1, \ldots, m$. In particular, we can construct F^j, M_j, such that $F^j_{\omega_j}$ satisfies the above (i)–(iii) for player j for all $\omega_j \notin M_j$. Let $\omega = (\omega_1, \ldots, \omega_m)$ be the sample profile, and $F_\omega = (F^1_{\omega_1}, \ldots, F^m_{\omega_m}) = (F^n_{\omega_n}, [F_\omega]_{-n})$.

We next claim that F_ω is a strong purification of f, for any sample profile ω with $\omega_j \notin M_j$ for each j; i.e., it satisfies Items 1–4 in Definition 4. It is clear that the Items 3 and 4 are the above Assertions (ii) and (iii), respectively. We only need to prove Items 1 and 2 in the definition.

Towards this end, fix one such sample profile ω, i.e., $\omega_j \notin M_j$ for all j. For any mixed strategy $\tilde{f}_n$ of player n, let $\tilde{f} = (\tilde{f}_n, f_{-n})$, and $\tilde{F}_\omega = \left(\tilde{f}_n, [F_\omega]_{-n}\right)$. By Eq. (11), the expected payoff of player n with $\tilde{f}$, F_ω and $\tilde{F}_\omega$ are, respectively, given by

$$U_n(\tilde{f}) = \int\limits_{I_n}\int\limits_{X_n} \psi_n^{\tilde{f}}(i_n, x_n)\ \tilde{f}_n(i_n; \mathrm{d}x_n)\ \mathrm{d}\lambda_n(i_n), \tag{14}$$

$$U_n([F_\omega]) = \int\limits_{I_n} \psi_n^{F_\omega}[i_n, F_{\omega_n}^n(i_n)]\ \mathrm{d}\lambda_n(i_n), \tag{15}$$

$$U_n(\tilde{F}_\omega) = \int\limits_{I_n}\int\limits_{X_n} \psi_n^{\tilde{F}_\omega}(i_n, x_n)\ \tilde{f}_n(i_n; \mathrm{d}x_n)\ \mathrm{d}\lambda_n(i_n). \tag{16}$$

Since Assertion (ii) above holds for all players, it is obvious that for $j \neq n$, $\gamma_{jk}^{f_j} = \gamma_{jk}^{F_{\omega_j}^j}$. By Eq. (12), ψ_n^f only depends on the probability distributions $\gamma_{jk}^{f_j}$, $j \neq n$. Hence, we have $\psi_n^f = \psi_n^{F_\omega} = \psi_n^{\tilde{f}} = \psi_n^{\tilde{F}_\omega}$. By Assertion (i) above, it follows that,

$$U_n(f) = \int\limits_{I_n}\int\limits_{X_n} \psi_n^f(i_n, x_n) f_n(i_n; \mathrm{d}x_n)\mathrm{d}\lambda_n(i_n) = \int\limits_{I_n} \psi_n^f[i_n, F_{\omega_n}^n(i_n)]\ \mathrm{d}\lambda_n(i_n)$$
$$= \int\limits_{I_n} \psi_n^{F_\omega}[i_n, F_{\omega_n}^i(i_n)]\mathrm{d}\lambda_n(i_n) = U_n(F_\omega).$$

We thus proved Item 1. Similarly, Item 2 also holds because,

$$U_n(\tilde{f}) = \int\limits_{I_n}\int\limits_{X_n} \psi_n^{\tilde{f}}(i_n, a_n)\tilde{f}_n(i_n; \mathrm{d}a_n)\mathrm{d}\lambda_n(i_n)$$
$$= \int\limits_{I_n}\int\limits_{X_n} \psi_n^{\tilde{F}_\omega}(i_n, a_n)\tilde{f}_n(i_n; \mathrm{d}a_n)\mathrm{d}\lambda_n(i_n) = U_n(\tilde{F}_\omega).$$

□

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