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RESEARCH ARTICLE

Multi-activity contests

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Abstract In many contests, players can influence their chances of winning through multiple activities or "arms". We develop a model of multi-armed contests and axiomatize its contest success function. We then analyze the outcomes of the multi-armed contest and the effects of allowing or restricting arms. Restricting an arm increases total effort directed to other arms if and only if restricting the arm balances the contest. Restricting an arm tends to reduce rent dissipation because it reduces the discriminatory power of the contest. But it also tends to increase rent dissipation if it balances the contest. Less rent is dissipated if an arm is restricted as long as no player is excessively stronger than the other with that arm. If players are sufficiently symmetric in an arm, both players are better off if that arm is restricted. Nevertheless, players cannot agree to restrict the arm if their costs of using the arm are sufficiently low.

Keywords Multi-dimensional competition · Axiomatization · Rent dissipation · Discriminatory power · Comparative advantage · Pareto improvement

JEL Classification C72 · D72

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1 Introduction

In many real-life contests, players have more than one way to influence their chances of winning. For example, firms may be able to obtain rents from the government not only by improving their efficiency, but also by lobbying or even bribing government officials (Tullock 1980; Krueger 1974). Within a firm, employees can attempt to influence the promotion decisions of managers or executives by means other than quality or quantity of work (Milgrom 1988; Eguchi 2005). Job seekers can invest in expensive suits and engage in networking (Granovetter 1974; Montgomery 1991). In sports, players can use drugs to enhance their athletic performance (Berentsen 2002; Maennig 2002). Researchers can increase their chances of publication by improving their paper's exposition as well as its content. And so on.

In this paper, we analyze contests in which players can affect their probability of winning through engaging in multiple activities or multiple "arms" of activity. We develop a simple model of two-player multi-armed contests, and provide an axiomatization of the model's contest success function. Our axiomatization of the multi-armed contest follows and extends the axiomatization by Skaperdas (1996) and Clark and Riis (1998) of the standard Tullock one-armed contest. The Tullock one-armed contest is obtained as a special case of the multi-armed contest by imposing a tight common cap on all arms but one. We analyze the outcomes of the multi-armed contest, including equilibrium effort levels and rent dissipation, and the effects on these outcomes of allowing or restricting arms.

Many interesting results emerge. In the case of symmetric players, rent dissipation is always higher if another arm is allowed into play. In general, if neither player is significantly stronger than the other with an arm, then more rent is dissipated in the contest with that arm than in the contest without it. Intuitively, an extra arm has two potentially opposing effects. First, it raises total effort costs because it increases the discriminatory power of the contest. But, second, it lowers total effort costs if it unbalances the contest, and raises total effort costs if it balances the contest. As long as an arm is not too asymmetric between players, it cannot sufficiently unbalance the contest, and therefore it raises total effort costs. In this case, players' total payoff increases if use of the arm is restricted. Restricting an arm increases players' efforts with other arms if and only if restricting the arm balances the contest. Restricting an arm yields a Pareto improvement if the arm is sufficiently symmetric between players. However, players have a unilateral incentive to break any agreement to restrict an arm if their costs of using the arm unrestricted are sufficiently low.

These findings suggest that rent dissipation might be a greater problem in contests in which participants have more than one activity by which to influence the outcome. For example, firms seeking rents from the government may be able to obtain them not only by improving their own efficiency but also by lobbying government officials, which may be a source of considerable rent dissipation for firms. The additional competition over lobbying would increase the discriminatory power of the contests between the firms, which would tend to increase total effort expenditures. Moreover, if the firms with superior means of improving efficiency also have superior means of lobbying, but their superiority in lobbying is not exceedingly great, then allowing lobbying would unbalance the contests, which would reduce total effort spent on improving efficiency, but would not unbalance the contests enough to counter the effect of the additional competition over lobbying; and as a result, total effort expenditures would rise overall. Restricting lobbying would then not only reduce rent dissipation but also increase total effort spent on efficiency improvements. However, government officials may have little incentive to restrict competition in lobbying because lobbying directly transfers rents to them. Firms could mutually agree to restrict competition on their own, but this is unlikely if some firms have greater means of lobbying than others. Efficient restrictions on lobbying may then be implementable only through the courts.

The rest of the paper is organized as follows. Section 2 briefly discusses related literature. In Sect. 3, we axiomatize the contest success function of a multi-armed contest model. In Sect. 4, we derive the equilibrium outcomes of the multi-armed contest. In Sect. 5, we analyze the effects of allowing or restricting an arm on rent dissipation and total efforts directed to other arms. We also derive conditions under which both players are better off if an arm is banned, and conditions under which players would not unilaterally violate a cap on the use of an arm. Section 6 summarizes and suggests avenues for future research.

2 Related literature

Starting with the seminal contributions of Tullock (1980), Krueger (1974), Posner (1975), and Bhagwati (1982), an extensive economics literature has formally analyzed the properties of various types of contests. Nitzan (1994) and Tollison (1997) provide detailed surveys. Among the more recent contributions, Skaperdas and Syropoulos (1998) analyze contests with complementarity between effort and prize; Nti (1999) analyzes contests with asymmetric prize valuations; Stein (2002) studies multi-player asymmetric contests with a constant returns success function; Baik (2004) studies two-player asymmetric contests where the success function depends on the ratio of the players' efforts and an exogenous measure of their relative ability; and Cornes and Hartley (2005) examine multi-player asymmetric contests with general success functions.

Our paper extends the contest literature by considering the possibility that players might exert effort in more than one activity to increase their probability of winning the contest. Most existing studies assume that players can affect their chances of winning through only one arm. Important exceptions include the studies by Konrad (2000), Chen (2003), Krakel (2005), and Amegashie and Runkel (2007) on sabotage in contests. In their models, players exert effort to improve their own performance, but also exert effort to sabotage their opponent's performance. In our model, players exert effort in multiple activities, each of which can improve their own performance (e.g., investment and lobbying by firms seeking government contracts). In the pioneering conflict models of Hirshleifer (1991) and Skaperdas and Syropoulos (1997), competitors allocate their endowments between two activities, production and appropriation, but their probability of winning the final product depends only on their relative efforts in appropriation.

A paper more closely related to ours is Epstein and Hefeker (2003). They analyze a model of contests with two instruments, both of which can influence the outcome.

However, our model is different from theirs in several crucial respects. The contest success function that we employ is different, and we provide an axiomatic justification for it. Moreover, and perhaps most importantly, we allow players to be asymmetric in every arm. Most of the results in our paper can only be obtained within this more general framework. We also analyze players' incentives to agree not to use an arm. And our results apply more generally to contests with more than two arms.

In our paper, we analyze the effects of restricting use of an arm in contests with more than one arm. Several papers analyze the effects of capping activity in contests, with interesting results. Che and Gale (1998) study the effect of exogenous lobbying caps on campaign expenditures in political contests, finding that caps can perversely increase expenditures. Gavious et al. (2002) study endogenous bid caps in all-pay contests, finding that caps increase the organizers' profits if bidders' marginal costs are sufficiently increasing. These papers analyze caps in one-armed contests only.

3 Axiomatization of K-armed contests

Consider two players, each having several arms (or activities) by which to influence the outcome of an all-pay contest. Player $i \in \{1, 2\}$ exerts effort $x_{ik} \in \mathbb{R}_+$ in activity $k \in \{1, \ldots, K\}$ to increase her chances of winning a prize that each player values at $v > 0$.[1] The marginal cost of effort for player i in the kth arm is constant and denoted by $c_{ik} > 0$. Players can be asymmetric in each arm. Let $\mathbf{x}_i$ denote the vector of player i's efforts with each arm in a K-armed contest, where $\mathbf{x}_i = (x_{i1}, \ldots, x_{iK})$.

Player i's payoff in a K-armed contest with constant marginal costs of effort is then:

$$\Pi_i (\mathbf{x}_1, \mathbf{x}_2) = p_i (\mathbf{x}_1, \mathbf{x}_2)\, v - \sum_{k=1}^{K} c_{ik} x_{ik}, \tag{1}$$

where $p_i (\mathbf{x}_1, \mathbf{x}_2)$ is player i's *contest success function* (CSF), or player i's winning probability as a function of both players' efforts in each of the activities, $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_+^{2K}$, $i = 1, 2$.

Skaperdas (1996) and Clark and Riis (1998) axiomatize the CSF for one-armed symmetric and asymmetric contests, respectively. Extending the work by Skaperdas and Clark and Riis, we axiomatize the CSF for multi-armed contests. Our approach to axiomatizing the CSF for two-player multi-armed contests is to consider a contest with three potential players, $i \in I \equiv \{1, 2, 3\}$, and examine subcontests between any two of these players. Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{R}_+^{3K}$ denote the vector of three players' efforts in all arms and $\mathbf{x}_{-i}$ denote the vector of all but player i's efforts in all arms. Let $p_i^{(2)} (\mathbf{x}_1, \mathbf{x}_2)$ and $p_i^{(3)} (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ be player i's winning probabilities in a two-player and a three-player contest, respectively. To simplify, we will use the same notation p_i to denote player i's winning probabilities in a two-player and a three-player

[1] We use the standard notations $\mathbb{R}_+$ and $\mathbb{R}_{++}$ for nonnegative and positive real number sets, respectively.

contest, and distinguish between them by the number of arguments, that is, $p_i^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = p_i(\mathbf{x}_1, \mathbf{x}_2)$ and $p_i^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = p_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. Let $p_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})$ be player i's winning probability when player 3's vector of efforts $\mathbf{x}_3$ is replaced with $\mathbf{x}_3 = \mathbf{0}$. To derive results for the two-player contest from the three-player contest, we will assume that the ratio of the winning probabilities of any two players is independent of the efforts of the third player (independence from irrelevant alternatives) and that player i's CSF in a two-player contest is equal to player i's CSF in a three-player contest where the third player is inactive, i.e., $p_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0}) = p_i(\mathbf{x}_1, \mathbf{x}_2)$ for $i = 1, 2$. We will formally state these properties as axioms below.

We first introduce the following basic axioms for multi-armed contests:

Axiom 1 (i) *For all* $i \in I$ *and* $\mathbf{x} \in \mathbb{R}_+^{3K}$, $p_i(\mathbf{x}) \geq 0$ *and* $\sum_{i \in I} p_i(\mathbf{x}) \leq 1$.
(ii) *For all* $i \in I$, *if* $\mathbf{x}_i \in \mathbb{R}_{++}^K$ *and* $\mathbf{x}_{-i} \in \mathbb{R}_+^{2K}$, *then* $p_i(\mathbf{x}) > 0$.
(iii) *For all* $i \in I$, *if* $\mathbf{x}_i = \mathbf{0}$ *and* $\mathbf{x}_{-i} \in \mathbb{R}_+^{2K}$, *then* $p_i(\mathbf{x}) = 0$.
(iv) *If* $\mathbf{x}_i \in \mathbb{R}_{++}^K$ *and* $\mathbf{x}_{-i} \in \mathbb{R}_+^{2K}$ *for some* $i \in I$, *then* $\sum_{i \in I} p_i(\mathbf{x}) = 1$.

According to Axiom 1 (i), no matter what efforts players are choosing, any player's probability of winning is nonnegative, and the sum of all players' probabilities of winning cannot exceed one because there is only one prize to allocate. According to Axiom 1 (ii), if player i puts in positive effort in all arms, then no matter what efforts other players are choosing, player i's probability of winning is positive. Axiom 1 (iii) states that if a player puts no effort in all arms, i.e., is inactive, then the player's probability of winning is zero no matter what efforts other players are choosing. Axiom 1 (iv) states that, as long as there is a player who puts positive efforts in all arms, the players' winning probabilities sum to one.

Additionally, we introduce the following two choice axioms:

Axiom 2 *For all* $i \neq j \neq k \in I$, *the odds ratio* $\dfrac{p_i(\mathbf{x})}{p_j(\mathbf{x})}$ *does not depend on* $\mathbf{x}_k$ *for* $\mathbf{x}_i \in \mathbb{R}_+^K$, $\mathbf{x}_j \in \mathbb{R}_{++}^K$, *and* $\mathbf{x}_k \in \mathbb{R}_+^K$.

Axiom 3 *For all* $i = 1, 2$ *and* $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_+^{2K}$, $p_i(\mathbf{x}_1, \mathbf{x}_2) = p_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})$.

Axiom 2 is a version of the independence from irrelevant alternatives property for contests. It says that the ratio of any two players' winning probabilities is independent of the third player's efforts. Axiom 3 says that in a two-player contest, the CSFs of players 1 and 2 are the same as their CSFs in a three-player contest in which the third player is inactive.

Note that Axiom 1 states basic properties of CSFs for three-player contests, $p_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. By Axiom 3, CSFs for two-player contests are the same as the CSFs for corresponding three-player contests with an inactive third player, i.e., $p_i(\mathbf{x}_1, \mathbf{x}_2) = p_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})$ for $i = 1, 2$ and $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_+^{2K}$. Setting $\mathbf{x}_3 = \mathbf{0}$ in the statements of Axiom 2, and using Axiom 3, we obtain the corresponding statements for CSFs in two-player contests. More precisely, if Axiom 3 holds, then Axiom 1 (ii) holds for two-player contests, that is, for all $i = 1, 2$, if $\mathbf{x}_i \in \mathbb{R}_{++}^K$ and $\mathbf{x}_{-i} \in \mathbb{R}_+^K$, then $p_i(\mathbf{x}_1, \mathbf{x}_2) > 0$, and Axiom 1 (iii) also holds for two-player contests, that is, for all $i = 1, 2$, if $\mathbf{x}_i = \mathbf{0}$ and $\mathbf{x}_{-i} \in \mathbb{R}_+^K$, then $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$. If Axiom 3 holds and

Axiom 1 (iii) hold, then Axiom 1 (i) holds for two-player contests, that is, for all $i = 1, 2$ and $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_+^{2K}$, $p_i(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ and $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) \leq 1$, and Axiom 1(iv) also holds for two-player contests, that is, if $\mathbf{x}_i \in \mathbb{R}_{++}^K$ and $\mathbf{x}_{-i} \in \mathbb{R}_+^K$ for some $i = 1, 2$, then $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) = 1$. Thus, under Axiom 3, if the CSF properties stated in Axiom 1 hold for three-player contests, then they also hold for two-player contests.

We now show that Axioms 1 through 3 imply the logit (additive) representation of players' CSFs in two-player K-armed contests:

Lemma 1 *Assume Axioms 1 through 3 hold. Then player i's CSF in a two-player K-armed contest has the logit form*

$$p_i(\mathbf{x}_1, \mathbf{x}_2) = \frac{f_i(\mathbf{x}_i)}{f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2)} \tag{2}$$

if $\mathbf{x}_1 \in \mathbb{R}_{++}^K$ *or* $\mathbf{x}_2 \in \mathbb{R}_{++}^K$*; and if* $\mathbf{x}_1 \in \mathbb{R}_+^K \backslash \mathbb{R}_{++}^K$ *and* $\mathbf{x}_2 \in \mathbb{R}_+^K \backslash \mathbb{R}_{++}^K$*, then* $p_i(\mathbf{x}_1, \mathbf{x}_2)$ *is such that* $p_i(\mathbf{x}_1, \mathbf{x}_2) \geq 0$, $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) \leq 1$, *and* $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$ *if* $\mathbf{x}_i = \mathbf{0}$. *Player i's production function* $f_i(\mathbf{x}_i)$ *satisfies* $f_i(\mathbf{x}_i) \geq 0$ *for* $\mathbf{x}_i \in \mathbb{R}_+^K$, $f_i(\mathbf{0}) = 0$, *and* $f_i(\mathbf{x}_i) > 0$ *for* $\mathbf{x}_i \in \mathbb{R}_{++}^K$, $i = 1, 2$.

Proofs of all results are presented in the "Appendix". Two additional axioms are formulated for two-player K-armed contests:

Axiom 4 *For all* $i = 1, 2$ *and* $k \in \{1, 2, \ldots, K\}$, $p_i(\mathbf{x}_1, \mathbf{x}_2)$ *is nondecreasing in* x_{ik} *for* $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_+^{2K}$ *and continuous and strictly increasing in* x_{ik} *for* $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_{++}^{2K}$.

Axiom 5 *For all* $i = 1, 2$, $k \in \{1, 2, \ldots, K\}$, *and* $\lambda > 0$,

$$p_i(x_{11}, \ldots, \lambda x_{1k}, \ldots, x_{1K}, x_{21}, \ldots, \lambda x_{2k}, \ldots, x_{2K}) = p_i(\mathbf{x}_1, \mathbf{x}_2)$$

if $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_{++}^{2K}$.

Axiom 4 says that each player's success probability is continuous and strictly increasing with that player's efforts in each activity if the efforts by both players in every activity are positive; and otherwise, each player's success probability is nondecreasing with that player's efforts in each activity. Axiom 5 is the homogeneity property, which states that an equiproportionate change in both players' efforts in an activity does not affect players' success probabilities. Note that if a CSF is homogeneous in each arm, it is invariant to changes in units of measurement for each arm. That is, the probability of a player's winning is the same regardless of whether an arm's effort is measured in, say, hours or minutes.

Proposition 1 below shows that if Axioms 1 through 5 hold, players' CSFs have the logit representation with Cobb-Douglas (CD-type) production functions:

Proposition 1 *Assume Axioms 1 through 5 hold. Then in a two-player K-armed contest, player i's CSF is* $p_i(\mathbf{x}_1, \mathbf{x}_2) = f_i(\mathbf{x}_i) / (f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2))$ *with production functions of CD-type* $f_i(\mathbf{x}_i) = \gamma_i \prod_{k=1}^K x_{ik}^{\alpha_k}$ *if* $\mathbf{x}_1 \in \mathbb{R}_{++}^K$ *or* $\mathbf{x}_2 \in \mathbb{R}_{++}^K$, *where* $\gamma_i > 0$ *and* $\alpha_k > 0$*; and if* $\mathbf{x}_1 \in \mathbb{R}_+^K \backslash \mathbb{R}_{++}^K$ *and* $\mathbf{x}_2 \in \mathbb{R}_+^K \backslash \mathbb{R}_{++}^K$, *then* $p_i(\mathbf{x}_1, \mathbf{x}_2)$ *is such that*

$p_i(\mathbf{x}_1, \mathbf{x}_2) \geq 0$, $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) \leq 1$, $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$ if $\mathbf{x}_i = \mathbf{0}$, and $p_i(\mathbf{x}_1, \mathbf{x}_2)$ is nondecreasing in x_{ik}; $i = 1, 2$; $k \in \{1, 2, \ldots, K\}$.

To further specify the winning probabilities at the boundaries, one can assume that each player's probability of winning is zero when both players have zero effort in at least one arm.

Axiom 6 *For all $i = 1, 2$, $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$ if $\mathbf{x}_1 \in \mathbb{R}^K_+ \backslash \mathbb{R}^K_{++}$ and $\mathbf{x}_2 \in \mathbb{R}^K_+ \backslash \mathbb{R}^K_{++}$.*

The symmetry or anonymity assumption on CSFs further restricts the form of the CSF.

Axiom 7 *For any $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{2K}_+$, $p_1(\mathbf{x}_1, \mathbf{x}_2) = p_2(\mathbf{x}_2, \mathbf{x}_1)$.*

CSFs are symmetric if players' efforts, but not their identities, determine their chances of winning. With CD-type CSFs, symmetry requires $\gamma_i = \gamma$, $i = 1, 2$. If $\gamma_1 \neq \gamma_2$, then for the same effort levels in each of the arms, the two players have different probabilities of winning. Then players' success probabilities depend on their identities.

If Axioms 6 and 7 hold in addition to Axioms 1 through 5, then player i's CSF is

$$p_i(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} \dfrac{f_i(\mathbf{x}_i)}{f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2)} & \text{if} \quad \mathbf{x}_1 \in \mathbb{R}^K_{++} \quad \text{or} \quad \mathbf{x}_2 \in \mathbb{R}^K_{++} \\ 0 & \text{if} \quad \mathbf{x}_1 \in \mathbb{R}^K_+ \backslash \mathbb{R}^K_{++} \quad \text{and} \quad \mathbf{x}_2 \in \mathbb{R}^K_+ \backslash \mathbb{R}^K_{++} \end{cases}, \tag{3}$$

where $f_i(\mathbf{x}_i) = \gamma \prod_{k=1}^K x_{ik}^{\alpha_k}$, $i = 1, 2$.

In the case of two arms, $f_i(\mathbf{x}_i) = \gamma x_{i1}^{\alpha_1} x_{i2}^{\alpha_2}$. Player i's payoff in the two-armed contest is

$$\Pi_i(x_{11}, x_{12}, x_{21}, x_{22}) = \left(\frac{x_{i1}^{\alpha_1} x_{i2}^{\alpha_2}}{x_{11}^{\alpha_1} x_{12}^{\alpha_2} + x_{21}^{\alpha_1} x_{22}^{\alpha_2}} \right) v - c_{i1} x_{i1} - c_{i2} x_{i2} \tag{4}$$

for $(x_{11}, x_{12}) \in \mathbb{R}^2_{++}$ or $(x_{21}, x_{22}) \in \mathbb{R}^2_{++}$, and $\Pi_i(x_{11}, x_{12}, x_{21}, x_{22}) = -c_{i1}x_{i1} - c_{i2}x_{i2}$ otherwise.

A standard Tullock one-armed contest is obtained as the limit of the two-armed contest with the second arm set equal to a common arbitrarily small value $\overline{x}_2$:

$$\begin{aligned} \overline{\Pi}_i(x_{11}, x_{12}) &= \lim_{\overline{x}_2 \to 0} \Pi_i(x_{11}, \overline{x}_2, x_{21}, \overline{x}_2) \\ &= \lim_{\overline{x}_2 \to 0} \left(\frac{x_{i1}^{\alpha_1} \overline{x}_2^{\alpha_2}}{x_{11}^{\alpha_1} \overline{x}_2^{\alpha_2} + x_{21}^{\alpha_1} \overline{x}_2^{\alpha_2}} \right) v - c_{i1} x_{i1} - c_{i2} \overline{x}_2 \\ &= \left(\frac{x_{i1}^{\alpha_1}}{x_{11}^{\alpha_1} + x_{21}^{\alpha_1}} \right) v - c_{i1} x_{i1}. \end{aligned} \tag{5}$$

In general, a Tullock one-armed contest is obtained from the K-armed contest by restricting $K - 1$ arms to common arbitrarily small caps $\overline{x}_k$, $k \in \{2, \ldots, K\}$.

4 Solution to K-armed contest

Suppose Axioms 1–7 hold. Then player i's payoff in the two-player K-armed contest is

$$\Pi_i(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} \dfrac{f(\mathbf{x}_i)}{f(\mathbf{x}_1)+f(\mathbf{x}_2)} v - \mathbf{c}_i \mathbf{x}_i & \text{if } \mathbf{x}_1 \in \mathbb{R}^K_{++} \text{ or } \mathbf{x}_2 \in \mathbb{R}^K_{++} \\ -\mathbf{c}_i \mathbf{x}_i & \text{if } \mathbf{x}_1 \in \mathbb{R}^K_{+} \backslash \mathbb{R}^K_{++} \text{ and } \mathbf{x}_2 \in \mathbb{R}^K_{+} \backslash \mathbb{R}^K_{++} \end{cases}, \quad (6)$$

where $f(\mathbf{x}_i) = \prod_{k=1}^K x_{ik}^{\alpha_k}$ is a homogeneous of degree $\alpha = \sum_{k=1}^K \alpha_k$ CD-type influence production function, $\mathbf{x}_i = (x_{i1}, \ldots, x_{iK})$ is the vector of player i's efforts in all activities, and $\mathbf{c}_i = (c_{i1}, \ldots, c_{iK})$ is the vector of player i's marginal costs of effort; $\mathbf{c}_i \mathbf{x}_i = \sum_{k=1}^K c_{ik} x_{ik}$; $c_{ik} > 0$, $\alpha_k > 0$, $i = 1, 2$, $k \in \{1, \ldots, K\}$.

Let θ_k be player 2's *relative strength* in activity k, $\theta_k \equiv (c_{1k}/c_{2k})^{\alpha_k}$, and Θ_K be player 2's *overall strength* in the K-armed contest, $\Theta_K \equiv \prod_{k=1}^K \theta_k$. Player 1's relative strength in activity k and overall strength are then θ_k^{-1} and Θ_K^{-1}, respectively. Player 1 has a stronger arm k (is more efficient in activity k) than player 2 when $\theta_k < 1$. Player 1 is stronger overall in the K-armed contest when $\Theta_K < 1$. For example, if $\theta_k < 1$ for all $k \in \{1, \ldots, K\}$, player 1 is stronger in each arm (player 1 has an *absolute advantage*), and stronger overall, $\Theta_K < 1$.

We say that the Kth arm *balances* the contest when the favorite in the $(K-1)$-armed contest has a higher overall strength than the favorite in the K-armed contest. That is, $\max\left\{\Theta_{K-1}, \Theta_{K-1}^{-1}\right\} > \max\{\Theta_K, \Theta_K^{-1}\}$. When the inequality is reversed, the introduction of the Kth arm *unbalances* the contest. Addition of the Kth arm tends to balance the contest if one player is stronger overall in the $(K-1)$-armed contest and the other player is stronger in the additional Kth arm.

Players $i = 1, 2$ simultaneously and independently decide on effort levels and the Nash equilibrium is a vector of players' efforts in all K activities, $\mathbf{x}^* \equiv \left(\mathbf{x}_1^*, \mathbf{x}_2^*\right)$, which maximize each player's payoff given the equilibrium effort levels chosen by the opposing player; $\mathbf{x}_i^* \in \mathbb{R}^K_+$, $i = 1, 2$. Let X_k^* denote the equilibrium total effort by the two players in arm k: $X_k^* \equiv x_{1k}^* + x_{2k}^*$. Denote by C^* the total cost of effort expended in the equilibrium by the two players with all of their arms: $C^* \equiv \sum_{i=1}^2 \sum_{k=1}^K c_{ik} x_{ik}^*$. Rent dissipation is then defined as $D^* \equiv C^*/v$. Player i's equilibrium probability of winning and payoff in the K-armed contest are denoted by p_i^* and Π_i^*. The total equilibrium payoff is $\Pi^* \equiv \Pi_1^* + \Pi_2^* = v(1 - D^*)$.

As shown by Baye et al. (1994, 1996), a pure-strategy Nash equilibrium does not exist in Tullock one-armed contests if the discriminatory power of the contest is greater than two, and the solution has to be sought in mixed strategies in this case.[2] Similarly, in the K-armed contest, there is no pure-strategy equilibrium for $\alpha > 2$, where $\alpha = \sum_{k=1}^K \alpha_k$. We restrict our attention to cases where a pure-strategy equilibrium exists.

Proposition 2 *There exists a unique Nash equilibrium of the two-player K-armed contest for* $\alpha \in (0, 1]$ *and for* $\alpha \in (1, 2]$ *and* $\Theta_K \in [(\alpha - 1), (\alpha - 1)^{-1}]$. *In the*

2 Cornes and Hartley (2005) provide general conditions for the existence and uniqueness of pure-strategy equilibrium in rent-seeking games.

equilibrium, player i's effort in activity k is $x^*_{ik} = \frac{\alpha_k}{c_{ik}} v\Theta_K (1 + \Theta_K)^{-2}$*, player i's cost of effort in activity k is* $c_{ik}x^*_{ik} = \alpha_k v\Theta_K (1 + \Theta_K)^{-2}$*, the total effort in activity k is* $X^*_k = x^*_{1k} + x^*_{2k} = \alpha_k v \left(\frac{1}{c_{1k}} + \frac{1}{c_{2k}}\right) \Theta_K (1 + \Theta_K)^{-2}$*, and rent dissipation is* $D^* = C^*/v = 2\alpha\Theta_K (1 + \Theta_K)^{-2} \in (0, 1]$*;* $i = 1, 2$*,* $k \in \{1, \ldots, K\}$.

The discriminatory power, $\alpha = \sum_{k=1}^{K} \alpha_k$, increases as additional arms are introduced. Thus, the restriction on the relative overall strength of players that guarantees the existence of the unique pure-strategy equilibrium becomes more stringent with more arms. By focusing on cases where a pure-strategy equilibrium exists, we implicitly focus on multi-activity contests in which the number of activities over which players compete is not too large.

From Proposition 2's proof, player i's equilibrium payoff is $\Pi^*_i = p^*_i v - \mathbf{c}_i\mathbf{x}^*_i$ with $p^*_i = \left(1 + \prod_{k=1}^{K} \left(\frac{c_{ik}}{c_{jk}}\right)^{\alpha_k}\right)^{-1}$ and $\mathbf{c}_i\mathbf{x}^*_i = \alpha v\Theta_K (1 + \Theta_K)^{-2}$, $i = 1, 2$. Clearly, Π^*_i, x^*_{ik}, and C^* are proportional to v , $i = 1, 2$, $k \in \{1, \ldots, K\}$. Corollary 1 provides other comparative statics.[3]

Corollary 1 *For two contests with the same overall players' strength, the one with the higher discriminatory power, α, has higher equilibrium total cost of effort and rent dissipation. For two contests with the same α, the one that is more balanced (i.e., in which the stronger player has a smaller overall strength) has higher total cost and rent dissipation. For any given α, total cost and rent dissipation are maximized, and the total payoff is minimized, when players are symmetric overall,* $\Theta_K = 1$*. Player i's effort with each arm decreases with i's marginal cost of that arm. Player i's cost of effort with each arm and payoff depend only on the overall relative strength of players and not on the levels of costs.*

We now turn to the effects of allowing or restricting arms in multi-armed contests.

5 *K*-armed versus (*K* − 1)-armed contest

Consider the multi-armed contest analyzed above. Suppose the use of the Kth arm is restricted by a common arbitrary small cap $\overline{x}_K$, which is binding for both players.[4] We compare the outcomes with and without the restriction, that is, we analyze the effects of allowing or restricting the Kth arm, holding the other $K - 1$ arms the same (that is, their marginal costs are the same and they are equally influential with or without the restriction).

5.1 Rent dissipation and total payoff

The addition of arm K has two effects on the total cost of effort and rent dissipation. First, it increases the discriminatory power of the contest, $\alpha = \sum_{k=1}^{K} \alpha_k$, which

[3] Nti (1997) provides comparative statics for the standard one-armed contest.

[4] With a common cap on an arm, both players still choose positive efforts with every arm, because if any player were to choose zero effort with at least one arm, then the player could increase her payoff by increasing by epsilon her effort with each of those arms for which she had chosen zero effort.

tends to increase effort cost and rent dissipation (see Corollary 1). Second, it can balance or unbalance the contest. If no player has an absolute advantage and the overall favorite in the K-armed contest has a lower relative strength than the favorite in the $(K-1)$-armed contest, then arm K balances the contest. In this case, arm K unambiguously increases effort cost and rent dissipation. But if the overall favorite in the K-armed contest has a higher relative strength than the favorite in the $(K-1)$-armed contest, as would be the case, for example, when one of the players has an absolute advantage, then arm K unbalances the contest, which tends to reduce effort cost and rent dissipation.

The K-armed and $(K-1)$-armed contests cannot be ranked unambiguously with respect to the total effort cost, rent dissipation, or the total payoff. However, in the following proposition, we present the necessary and sufficient conditions for the total effort cost and rent dissipation to be higher, and the total payoff to be lower, in the K-armed contest. In the proposition, we use notation $\alpha_{(K)} = \sum_{k=1}^{K} \alpha_k$ and $\alpha_{(K-1)} = \sum_{k=1}^{K-1} \alpha_k$ to distinguish between the discriminatory power of K-armed and $(K-1)$-armed contests.

Proposition 3 *The equilibrium total payoff is lower while the total cost of effort and rent dissipation are higher in the K-armed contest than in the $(K-1)$-armed contest if and only if the asymmetry in the players' relative strength in the Kth arm is sufficiently small: $\theta_K \in (g_1(\Theta_{K-1}), g_2(\Theta_{K-1}))$, where $0 < g_1(\Theta_{K-1}) < \frac{\alpha_{(K-1)}}{\alpha_{(K)}} < 1$ and $1 < \frac{\alpha_{(K)}}{\alpha_{(K-1)}} < g_2(\Theta_{K-1}) < \infty$.*

The exact expressions for the boundaries $g_1(\Theta_{K-1})$ and $g_2(\Theta_{K-1})$ are presented in the "Appendix". The total payoff is lower in the K-armed than in the $(K-1)$-armed contest as long as neither player is excessively stronger than the other with arm K, that is, for intermediate values of players' relative strength in arm K, $\theta_K \in (g_1(\Theta_{K-1}), g_2(\Theta_{K-1}))$. Arm K then reduces the total payoff because it cannot sufficiently unbalance the contest to counter its effect on the discriminatory power of the contest. In this case, the total payoff unambiguously increases if the players cannot use arm K.

In special cases, we can obtain further results on the extent of rent dissipation in K-armed contests. According to Proposition 2, the equilibrium rent dissipation in the K-armed contest is $D^* = 2\alpha\Theta_K (1+\Theta_K)^{-2}$, where $\alpha = \sum_{k=1}^{K} \alpha_k$ and $\Theta_K = \prod_{k=1}^{K} (c_{1k}/c_{2k})^{\alpha_k}$. If players are symmetric in arm K, rent dissipation is necessarily higher in the contest with arm K than in the one without it. If players in a K-armed contest have equal overall strength, $\Theta_K = 1$, then $D^* = \alpha/2 \leq 1$, where the inequality is satisfied since $\alpha \leq 2$ is necessary for the existence of the pure-strategy equilibrium. As additional symmetric arms are introduced, the discriminating power of the contest, $\alpha = \sum_{k=1}^{K} \alpha_k$, grows until it approaches 2, in which case complete rent dissipation occurs.

In the standard symmetric two-player one-armed contest, full dissipation occurs only if the given discriminatory power of the contest is equal to (or exceeds) 2.[5] In the

[5] See Baye et al. (1999) for an analysis of the conditions for full or over-dissipation in the symmetric Tullock one-armed contest.

symmetric two-player multi-armed contest, the discriminatory power of the contest depends on the number of arms, and full dissipation occurs if the number of arms is large enough for the discriminatory power of the contest to reach 2.

When players are symmetric in the costs of all arms, $c_{1k} = c_{2k} = c_k$ for $k \in \{1, \ldots, K\}$, the total equilibrium effort with arm k is $X_k^* \equiv x_{1k}^* + x_{2k}^* = \frac{\alpha_k v}{2c_k}$. In the symmetric contest, total effort with an arm is unaffected by the introduction of other arms. In the special case where all arms are also equally influential, $\alpha_k = \alpha_0$ for $k \in \{1, \ldots, K\}$, the pure-strategy equilibrium exists if and only if $K \leq 2/\alpha_0$. The amount of rent dissipation in the K-armed contest is $D^* = K\alpha_0/2 \leq 1$. Rent dissipation is growing in proportion with the number of arms. For example, rent dissipation in the two-armed contest is twice as high as rent dissipation in the one-armed contest, which has the lowest rent dissipation of all.

5.2 Productive and unproductive arms

The analysis so far has not distinguished arms according to whether their use is more or less socially desirable. One can think of many real-world contests in which one of the influencing activities is less socially desirable than the others. For example, employees can influence their chances of promotion by paying court to managers (socially wasteful), as well as by working hard along various dimensions (socially desirable). An interesting question for the organizers of such contests, and society more broadly, is whether restricting the less productive activity increases effort in the other more productive activities.

Letting the Kth arm be the least productive arm, we can ask whether total effort with the other more productive $K - 1$ arms decreases when arm K is allowed into play. Proposition 3 states that the answer depends only on whether arm K balances the contest.

Proposition 4 *A restriction on the Kth arm reduces total effort with the $K - 1$ arms if and only if the Kth arm balances the contest,* $\max\left\{\Theta_{K-1}, \Theta_{K-1}^{-1}\right\} > \max\{\Theta_K, \Theta_K^{-1}\}$.

If arm K balances the contest (because the overall favorite with K arms has a lower relative strength than the favorite with $K - 1$ arms), allowing arm K increases total effort with the other $K - 1$ arms. If arm K unbalances the contest (because the overall favorite with K arms has a higher relative strength than the favorite with $K - 1$ arms), then it reduces total effort with the other $K - 1$ arms. If arm K is symmetric between players, then it neither balances nor unbalances the contest and therefore does not affect total effort with the other $K - 1$ arms. As a matter of policy, if an arm is unproductive while other arms are productive, and the goal is to maximize total effort with productive arms, then the unproductive arm should not be restricted if and only if it balances the contest.

5.3 Pareto optimality

We have compared contest outcomes with and without the restriction on the use of the Kth arm. But under what conditions would both players prefer to ban arm K?

Intuitively, if a player has a big advantage with an arm, she may not be better off if that arm is banned. Would there ever be a case for a Pareto improvement? Proposition 5 provides the necessary and sufficient conditions for banning the Kth arm to be a Pareto improvement.

Proposition 5 *A player prefers to ban the Kth arm if the player's relative strength with the Kth arm is not too great. For* $\theta_K \in (g_3(\Theta_{K-1}), g_4(\Theta_{K-1}))$*, the* $(K-1)$*-armed contest Pareto dominates the K-armed contest, where* $0 < g_3(\Theta_{K-1}) < 1$ *and* $1 < g_4(\Theta_{K-1}) < \infty$.

The exact expressions for $g_4(\Theta_{K-1})$ and $g_3(\Theta_{K-1}) = \left(g_4(\Theta_{K-1}^{-1})\right)^{-1}$ are derived in the "Appendix". Both players favor the $(K-1)$-armed contest if they are symmetric in arm K. In this case, arm K does not balance or unbalance the contest in favor of any one of the players, so that neither player is worse off if play is restricted to the $K-1$ arms. Moreover, arm K increases the players' effort costs, so both players are strictly better off if play is restricted to the $K-1$ arms. A player is only better off when the Kth arm is allowed if the arm sufficiently shifts the balance of power in her favor, which requires arm K to be sufficiently asymmetric between the players. In general, if the asymmetry between players in an arm is sufficiently small, then they both prefer not to use the arm.

5.4 Prisoners' dilemma

We have shown that both players are better off when the Kth arm is banned if the Kth arm is sufficiently symmetric between the players. But even in this case, an agreement by the players to restrict arm K to a common cap $\overline{x}_K$ is not guaranteed. Proposition 6 provides conditions under which both players have incentive to unilaterally deviate from such an agreement.

Proposition 6 *Consider an agreement to restrict the Kth arm to a common cap* $\overline{x}_K > 0$. *Player i has a unilateral incentive to deviate from this agreement by choosing a higher effort level with the Kth arm if* $c_{iK} < v\frac{\alpha_K}{\overline{x}_K}\frac{\Theta_{K-1}}{[1+\Theta_{K-1}]^2}$.

Players tend to break the agreement to restrict arm K when the value of the prize is high, the cap is tight, the capped arm is influential, players' costs of effort with the capped arm are low, and there is balance of power in the $(K-1)$-armed contest.

Consider player i's incentive to break the agreement to cap arm K. Given that player j adheres to the cap, i might do strictly better by choosing an effort level with arm K that is higher than permitted by the cap. Given that j's effort levels are fixed, i's success probability strictly increases if she increases her effort level with arm K above the cap (by Axiom 4). Player i also incurs a cost to increase her effort level with arm K, but as long as the value of the prize is sufficiently high or her cost of effort with arm K is sufficiently low, her benefit of increasing effort with arm K above the cap outweighs her cost, in which case she strictly prefers to break the agreement. Moreover, by Proposition 5, both players' payoffs are strictly lower in the K-armed than in the $(K-1)$-armed contest if players are sufficiently symmetric in arm K.

Therefore, if arm K is sufficiently symmetric between the players, both players end up using arm K to their own detriment in equilibrium.

6 Conclusion

In many contests, including those between firms for government contracts, between employees for promotion, between job candidates for employment, between academics for publication, and between athletes for championships, participants can engage in more than one activity to increase their chances of winning. To analyze the implications of this important aspect of many contests, we developed and axiomatized a simple model of multi-armed contests. We then studied how allowing or restricting arms affects equilibrium outcomes. Our main findings may be summarized as follows.

Allowing an additional arm into play reduces effort put into arms that are already in play if and only if the additional arm unbalances the contest. An additional arm tends to increase rent dissipation because it increases the discriminatory power of the contest, but it also tends to reduce rent dissipation if it unbalances the contest. If the additional arm is symmetric between players, it cannot unbalance the contest, and therefore it always increases rent dissipation. In general, an additional arm increases rent dissipation unless players are very asymmetric in that arm. As long as this is not the case, the additional arm increases total effort costs and reduces the players' joint payoffs. If players are sufficiently symmetric in an arm, each one is better off if the arm is not allowed into play, but they nevertheless end up using the arm in equilibrium.

These results have policy implications. In many contests, players can influence the outcome through more than one arm, but one of the arms (e.g., lobbying) is less socially productive than others. According to our results, it is entirely possible that the unproductive arm would unbalance the contests, reducing effort with the more productive arms, but not enough to counter its effect on the contest's discriminatory power, so that rent dissipation would increase and total payoffs would decline. And players could not even agree not to use the unproductive arm, creating a substantial welfare loss that could not be avoided without market intervention.

Interesting avenues for further research include axiomatization and analysis of a wider class of contest success functions for multi-armed contests by relaxing the homogeneity assumption (Axiom 5) employed in the paper. We axiomatized the general additive representation for two-player multi-armed contests in Lemma 1, and this axiomatization could easily be extended to multi-player contests. Analyzing multi-player multi-armed contests with general success functions is a promising avenue for further inquiry.

7 Appendix

Proof of Lemma 1 Assume that either $\mathbf{x}_1 \in \mathbb{R}^K_{++}$ or $\mathbf{x}_2 \in \mathbb{R}^K_{++}$. Without loss of generality, let player 2 be the player with positive efforts in all arms. We therefore consider $\mathbf{x}_1 \in \mathbb{R}^K_{+}$, $\mathbf{x}_2 \in \mathbb{R}^K_{++}$, and $\mathbf{x}_3 \in \mathbb{R}^K_{+}$. From Axiom 1 (ii), $p_2(\mathbf{x}) > 0$. The independence from irrelevant alternatives (Axiom 2) implies that $\dfrac{p_1(\mathbf{x})}{p_2(\mathbf{x})}$ does not depend on $\mathbf{x}_3$. That is, the odds ratio $\dfrac{p_1(\mathbf{x})}{p_2(\mathbf{x})}$ is the same whether we replace $\mathbf{x}_3$ with $\mathbf{x}_3 = \mathbf{0}$, $\mathbf{x}_3 = \mathbf{1}$, or any other vector $\mathbf{x}_3 \in \mathbb{R}_+^K$. Hence,

$$\frac{p_1(\mathbf{x})}{p_2(\mathbf{x})} = \frac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})}{p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})} = \frac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}, \tag{B1}$$

where $p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0}) > 0$ and $p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1}) > 0$ by Axiom 1 (ii) since $\mathbf{x}_2 \in \mathbb{R}_{++}^K$.

We can rewrite the odds ratio as

$$\frac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})} = \frac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})} \frac{p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}, \tag{B2}$$

where $p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1}) > 0$ by Axiom 1 (ii) since $\mathbf{x}_3 = \mathbf{1} \in \mathbb{R}_{++}^K$.

Furthermore, we can again use Axiom 2 by setting $\mathbf{x}_2 = \mathbf{1}$ in the odds ratio $\dfrac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}$ and $\mathbf{x}_1 = \mathbf{1}$ in the odds ratio $\dfrac{p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}$ to obtain

$$\begin{aligned}\frac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})} &= \frac{p_1(\mathbf{x}_1, \mathbf{x}_2 = \mathbf{1}, \mathbf{x}_3 = \mathbf{1})}{p_3(\mathbf{x}_1, \mathbf{x}_2 = \mathbf{1}, \mathbf{x}_3 = \mathbf{1})} \frac{p_3(\mathbf{x}_1 = \mathbf{1}, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_2(\mathbf{x}_1 = \mathbf{1}, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})} \\ &= \frac{f_1(\mathbf{x}_1)}{f_2(\mathbf{x}_2)},\end{aligned} \tag{B3}$$

where

$$f_1(\mathbf{x}_1) \equiv \frac{p_1(\mathbf{x}_1, \mathbf{x}_2 = \mathbf{1}, \mathbf{x}_3 = \mathbf{1})}{p_3(\mathbf{x}_1, \mathbf{x}_2 = \mathbf{1}, \mathbf{x}_3 = \mathbf{1})} \geq 0$$

and

$$f_2(\mathbf{x}_2) \equiv \frac{p_2(\mathbf{x}_1 = \mathbf{1}, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})}{p_3(\mathbf{x}_1 = \mathbf{1}, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{1})} > 0,$$

and all the denominators are again positive by Axiom 1 (ii).

When player 3 is inactive, that is, $\mathbf{x}_3 = \mathbf{0}$, then by Axiom 1 (iii), $p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0}) = 0$, and by Axiom 1 (iv),

$$p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0}) + p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0}) = 1. \tag{B4}$$

Therefore,

$$\frac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})}{p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})} = \frac{p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})}{1 - p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})} = \frac{f_1(\mathbf{x}_1)}{f_2(\mathbf{x}_2)} \tag{B5}$$

by (B1), (B3), and (B4). Solving for $p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})$, we obtain $p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0}) = \dfrac{f_1(\mathbf{x}_1)}{f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2)}$, and thus $p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0}) = \dfrac{f_2(\mathbf{x}_2)}{f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2)}$ by

Axiom 1 (iv), for any $\mathbf{x}_1 \in \mathbb{R}_+^K$ and $\mathbf{x}_2 \in \mathbb{R}_{++}^K$, where $f_1(\mathbf{x}_1) \geq 0$ and $f_2(\mathbf{x}_2) > 0$. From Axiom 1 (iii), if $\mathbf{x}_1 = \mathbf{0}$, then $p_1(\mathbf{x}) = 0$ and therefore $f_1(\mathbf{0}) = 0$.

When $\mathbf{x}_1 \in \mathbb{R}_+^K$ and $\mathbf{x}_2 \in \mathbb{R}_+^K \backslash \mathbb{R}_{++}^K$, $p_i(\mathbf{x}_1, \mathbf{x}_2)$ is such that $p_i(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ and $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) \leq 1$ by Axiom 1 (i) and $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$ if $\mathbf{x}_i = \mathbf{0}$ by Axiom 1 (iii). Finally, by Axiom 1, $p_i(\mathbf{x}_1, \mathbf{x}_2) = p_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{0})$ for any $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_+^{2K}$. To summarize, if Axioms 1–3 hold, then player i's CSF in a two-player K-armed contest has the logit form if $\mathbf{x}_1 \in \mathbb{R}_{++}^K$ or $\mathbf{x}_2 \in \mathbb{R}_{++}^K$; and if $\mathbf{x}_1 \in \mathbb{R}_+^K \backslash \mathbb{R}_{++}^K$ and $\mathbf{x}_2 \in \mathbb{R}_+^K \backslash \mathbb{R}_{++}^K$, $p_i(\mathbf{x}_1, \mathbf{x}_2)$ is such that $p_i(\mathbf{x}_1, \mathbf{x}_2) \geq 0$, $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) \leq 1$, and $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$ if $\mathbf{x}_i = \mathbf{0}$. Player i's production function $f_i(\mathbf{x}_i)$ satisfies $f_i(\mathbf{x}_i) \geq 0$ for $\mathbf{x}_i \in \mathbb{R}_+^K$, $f_i(\mathbf{0}) = 0$, and $f_i(\mathbf{x}_i) > 0$ for $\mathbf{x}_i \in \mathbb{R}_{++}^K$, $i = 1, 2$. □

Proof of Proposition 1 First assume that $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_{++}^{2K}$. Suppose players' contest success functions are homogeneous of degree 0 in every activity (Axiom 5). Then, the odds ratio of players' winning probabilities is homogeneous of degree 0 in every activity as well, that is,

$$\frac{p_1(\lambda_1 x_{11}, \ldots, \lambda_K x_{1K}, \lambda_1 x_{21}, \ldots, \lambda_K x_{2K})}{p_2(\lambda_1 x_{11}, \ldots, \lambda_K x_{1K}, \lambda_1 x_{21}, \ldots, \lambda_K x_{2K})} = \frac{p_1(\mathbf{x}_1, \mathbf{x}_2)}{p_2(\mathbf{x}_1, \mathbf{x}_2)} \tag{B6}$$

for any $\lambda_k > 0$. Note that the denominators are positive by Axiom 1(ii) since $\mathbf{x}_2 \in \mathbb{R}_{++}^K$.

According to Lemma 1, Axioms 1–3 guarantee the logit representation of the CSF, which implies that the odds ratio can be written as a ratio of players' production functions:

$$\frac{p_1(\mathbf{x}_1, \mathbf{x}_2)}{p_2(\mathbf{x}_1, \mathbf{x}_2)} = \frac{f_1(\mathbf{x}_1)}{f_2(\mathbf{x}_2)}, \tag{B7}$$

where $f_2(\mathbf{x}_2) > 0$. By Axiom 4, player 1's CSF is continuous and strictly increasing in x_{1k} for $\mathbf{x}_1 \in \mathbb{R}_{++}^K$. Axiom 4 and Axiom 1(iv) imply that player 2's CSF is strictly decreasing in x_{1k} for $\mathbf{x}_1 \in \mathbb{R}_{++}^K$. Hence, the odds ratio $\frac{p_1(\mathbf{x}_1, \mathbf{x}_2)}{p_2(\mathbf{x}_1, \mathbf{x}_2)}$ is strictly decreasing in x_{1k} for $\mathbf{x}_1 \in \mathbb{R}_{++}^K$. From (B7), the production function $f_1(\mathbf{x}_1)$ is continuous, strictly increasing in x_{1k} for $\mathbf{x}_1 \in \mathbb{R}_{++}^K$ as well.

Employing the homogeneity of the odds ratio, we obtain

$$\begin{aligned}\frac{f_1(\lambda_1 x_{11}, \ldots, \lambda_K x_{1K})}{f_2(\lambda_1 x_{21}, \ldots, \lambda_K x_{2K})} &= \frac{p_1(\lambda_1 x_{11}, \ldots, \lambda_K x_{1K}, \lambda_1 x_{21}, \ldots, \lambda_K x_{2K})}{p_2(\lambda_1 x_{11}, \ldots, \lambda_K x_{1K}, \lambda_1 x_{21}, \ldots, \lambda_K x_{2K})} \\ &= \frac{p_1(\mathbf{x}_1, \mathbf{x}_2)}{p_2(\mathbf{x}_1, \mathbf{x}_2)} = \frac{f_1(\mathbf{x}_1)}{f_2(\mathbf{x}_2)}\end{aligned} \tag{B8}$$

for any $\lambda_k > 0$. It follows that

$$f_1(\lambda_1 x_{11}, \ldots, \lambda_K x_{1K})\, f_2(\mathbf{x}_2) = f_2(\lambda_1 x_{21}, \ldots, \lambda_K x_{2K})\, f_1(\mathbf{x}_1). \tag{B9}$$

Setting $\mathbf{x}_2 = \mathbf{1}$ and defining $\boldsymbol{\lambda} \equiv (\lambda_1, \ldots, \lambda_K)$ and $\phi_1(\boldsymbol{\lambda}) \equiv \frac{f_2(\boldsymbol{\lambda})}{f_2(\mathbf{1})}$, we arrive at the functional equation

$$f_1(\lambda_1 x_{11}, \ldots, \lambda_K x_{1K}) = \phi_1(\boldsymbol{\lambda}) f_1(\mathbf{x}_1). \tag{B10}$$

The unique strictly increasing solution to this functional equation is $f_1(\mathbf{x}_1) = \gamma_1 \prod_{k=1}^{K} x_{1k}^{\alpha_k}$ and $\phi_1(\boldsymbol{\lambda}) = \prod_{k=1}^{K} \lambda_k^{\alpha_k}$, where γ_1 and α_k are positive constants; $k \in \{1, 2, \ldots, K\}$. See Corollary 3 in Aczél (1987, p. 55). From $\phi_1(\boldsymbol{\lambda}) = \dfrac{f_2(\boldsymbol{\lambda})}{f_2(\mathbf{1})}$ and $\phi_1(\boldsymbol{\lambda}) = \prod_{k=1}^{K} \lambda_k^{\alpha_k}$, it follows that, $f_2(\mathbf{x}_2) = f_2(\mathbf{1}) \prod_{k=1}^{K} x_{2k}^{\alpha_k}$. Therefore, both players' production functions belong to a class of CD-type production functions: $f_i(\mathbf{x}_i) = \gamma_i \prod_{k=1}^{K} x_{ik}^{\alpha_k}$ for $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_{++}^{2K}$, where $\alpha_k, \gamma_i > 0$; $i \in \{1, 2\}$, and $k \in \{1, 2, \ldots, K\}$.

Next, consider the case where one player has positive efforts in all arms and the other has zero efforts in some arms. Without loss of generality, consider $\mathbf{x}_1 \in \mathbb{R}_{+}^{K} \backslash \mathbb{R}_{++}^{K}$ and $\mathbf{x}_2 \in \mathbb{R}_{++}^{K}$. For $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_{++}^{2K}$, player 1's production function is $f_1(\mathbf{x}_1) = \gamma_1 \prod_{k=1}^{K} x_{1k}^{\alpha_k}$. Holding $\mathbf{x}_2 \in \mathbb{R}_{++}^{K}$ fixed, if for any arm k, $x_{1k} \to 0$, then $f_1(\mathbf{x}_1) \to 0$ and therefore $p_1(\mathbf{x}_1, \mathbf{x}_2) \to 0$. According to Axiom 4, player 1's CSF is nondecreasing in x_{1k} for $\mathbf{x}_1 \in \mathbb{R}_{+}^{K}$ and from Axiom 1 (i), $p_1(\mathbf{x}_1, \mathbf{x}_2)$ is nonnegative. Hence, $p_1(\mathbf{x}_1, \mathbf{x}_2) = 0$, and therefore $p_2(\mathbf{x}_1, \mathbf{x}_2) = 1$ from Axiom 1 (iv), when player 1 has zero effort in some arm, i.e., there exists $k \in \{1, 2, \ldots, K\}$ such that $x_{1k} = 0$. Lastly, if both players have zero efforts in some arms, then $p_i(\mathbf{x}_1, \mathbf{x}_2)$ is such that $p_i(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ and $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) \leq 1$ by Axiom 1(i), $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$ if $\mathbf{x}_i = \mathbf{0}$ by Axiom 1 (iii), and $p_i(\mathbf{x}_1, \mathbf{x}_2)$ is nondecreasing in x_{ik} by Axiom 4, $i = 1, 2$, $k \in \{1, 2, \ldots, K\}$.

To summarize, if Axioms 1 through 5 hold, then for $\mathbf{x}_1 \in \mathbb{R}_{++}^{K}$ or $\mathbf{x}_2 \in \mathbb{R}_{++}^{K}$, player i's CSF has the logit form with CD-type production functions $f_i(\mathbf{x}_i) = \gamma_i \prod_{k=1}^{K} x_{ik}^{\alpha_k}$, where $\alpha_k, \gamma_i > 0$; and if $\mathbf{x}_1 \in \mathbb{R}_{+}^{K} \backslash \mathbb{R}_{++}^{K}$ and $\mathbf{x}_2 \in \mathbb{R}_{+}^{K} \backslash \mathbb{R}_{++}^{K}$, then $p_i(\mathbf{x}_1, \mathbf{x}_2)$ is such that $p_i(\mathbf{x}_1, \mathbf{x}_2) \geq 0$, $p_1(\mathbf{x}_1, \mathbf{x}_2) + p_2(\mathbf{x}_1, \mathbf{x}_2) \leq 1$, $p_i(\mathbf{x}_1, \mathbf{x}_2) = 0$ if $\mathbf{x}_i = \mathbf{0}$, and $p_i(\mathbf{x}_1, \mathbf{x}_2)$ is nondecreasing in x_{ik}; $i = 1, 2$; $k \in \{1, 2, \ldots, K\}$. □

Proof of Proposition 2 We are looking for the equilibrium $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ in the K-armed contest with marginal costs $\mathbf{c}_i = (c_{i1}, \ldots, c_{iK})$, $i = 1, 2$. By Axiom 6, it is never a best response for a player to put zero efforts in some but not all arms, and it can be easily shown that having zero efforts in all arms cannot be part of an equilibrium either. Therefore, we can restrict our attention to $\mathbf{x}_i \in \mathbb{R}_{++}^{K}$, $i = 1, 2$. The optimal interior solution for player i maximizing payoff (6) can be found by first deriving the cost function $C_i^*(z_i)$ as $min\{\mathbf{c}_i \mathbf{x}_i\}$ subject to the constraint $f(\mathbf{x}_i) = z_i$, and then solving the reduced one-armed contest with the derived cost function

$$\pi_i(z_1, z_2) = \left(\frac{z_i}{z_1 + z_2}\right) v - C_i^*(z_i). \tag{B11}$$

The cost function for the CD-type production function $f(\mathbf{x}_i)$ is $C_i^*(z_i) = \sigma_i z_i^{\frac{1}{\alpha}}$, where $\sigma_i = \alpha \prod_{k=1}^{K} \left(\frac{\alpha_k}{c_{ik}}\right)^{-\frac{\alpha_k}{\alpha}}$ and $\alpha = \sum_{k=1}^{K} \alpha_k$. The conditional demand for activity k is $x_{ik}^* = \partial C_i^*(z_i)/\partial c_{ik} = \frac{\alpha_k}{c_{ik}} \frac{\sigma_i}{\alpha} z_i^{\frac{1}{\alpha}}$.

Player i's payoff in the reduced one-armed contest is then

$$\pi_i(z_1, z_2) = \left(\frac{z_i}{z_1 + z_2}\right) v - \sigma_i z_i^{\frac{1}{\alpha}}. \tag{B12}$$

Using substitution $\widetilde{x}_i = z_i^{\frac{1}{\alpha}}$, the payoff can also be written as

$$\widetilde{\Pi}_i(\widetilde{x}_1, \widetilde{x}_2) = \left(\frac{\widetilde{x}_i^{\alpha}}{\widetilde{x}_1^{\alpha} + \widetilde{x}_2^{\alpha}}\right) v - \sigma_i \widetilde{x}_i. \tag{B13}$$

The reduced one-armed contest has a discriminatory power $\alpha = \sum_{i=1}^{K} \alpha_i$ and the relative strength of players in the one-armed contest equals players' overall relative strength in the original K-armed contest, $\left(\frac{\sigma_i}{\sigma_j}\right)^{\alpha} = \prod_{k=1}^{K} \left(\frac{c_{ik}}{c_{jk}}\right)^{\alpha_k}$, $j \neq i = 1, 2$.

The first-order condition for player 1's maximization of payoff (B12) yields $z_2 v (z_1 + z_2)^{-2} = (\sigma_1/\alpha)\, z_1^{\frac{1}{\alpha}-1}$. Dividing both sides of the expression by the similar condition for player 2, we obtain $z_2/z_1 = (\sigma_1/\sigma_2)\,(z_1/z_2)^{\frac{1}{\alpha}-1}$. Hence, $z_2/z_1 = (\sigma_1/\sigma_2)^{\alpha} = \Theta_K$, which together with the first-order condition for player 1 implies that $z_1^{\frac{1}{\alpha}} = (\alpha v/\sigma_1)\, \Theta_K\, (1 + \Theta_K)^{-2}$. Similarly, $z_2^{\frac{1}{\alpha}} = (\alpha v/\sigma_2)\, \Theta_K^{-1} \left(1 + \Theta_K^{-1}\right)^{-2} = (\alpha v/\sigma_2)\, \Theta_K\, (1 + \Theta_K)^{-2}$. The interior equilibrium in the one-armed contest (B12) is

$$\widetilde{x}_i^* = z_i^{\frac{1}{\alpha}} = \alpha \frac{v}{\sigma_i} \frac{(\sigma_i/\sigma_j)^{\alpha}}{\left(1 + (\sigma_i/\sigma_j)^{\alpha}\right)^2} = \alpha \frac{v}{\sigma_i} \frac{\Theta_K}{(1 + \Theta_K)^2}, \tag{B14}$$

where $\Theta_K = \prod_{k=1}^{K} \left(\frac{c_{1k}}{c_{2k}}\right)^{\alpha_k}$; $i = 1, 2$. Thus, player i's effort with arm k is

$$x_{ik}^* = \frac{\alpha_k}{\alpha} \frac{\sigma_i}{c_{ik}} z_i^{\frac{1}{\alpha}} = \frac{\alpha_k}{c_{ik}} v \frac{\Theta_K}{(1 + \Theta_K)^2}. \tag{B15}$$

Therefore,

$$c_{ik} x_{ik}^* = \alpha_k v \frac{\Theta_K}{(1 + \Theta_K)^2} \tag{B16}$$

for any $k \in \{1, \ldots, K\}$, $i = 1, 2$. The total equilibrium effort with arm k is

$$X_k^* \equiv x_{1k}^* + x_{2k}^* = \alpha_k v \left(\frac{1}{c_{1k}} + \frac{1}{c_{2k}}\right) \frac{\Theta_K}{(1 + \Theta_K)^2}. \tag{B17}$$

In the equilibrium, the cost of effort is the same across players, $\mathbf{c}_i \mathbf{x}_i^* = \alpha v \frac{\Theta_K}{(1+\Theta_K)^2}$ for $i = 1, 2$. The total cost of effort and rent dissipation are $C^* = \sum_{i=1}^{2} \sum_{k=1}^{K} c_{ik} x_{ik}^* = 2\alpha v \Theta_K\, (1 + \Theta_K)^{-2}$ and $D^* = 2\alpha \Theta_K\, (1 + \Theta_K)^{-2}$. Players' total payoff is Π^*

$= v - C^* = v\left(1 - 2\alpha\Theta_K (1+\Theta_K)^{-2}\right)$. Player i's probability of winning is $p_i^* = \left(1 + \left(\widetilde{x}_j^*/\widetilde{x}_i^*\right)^{\alpha}\right)^{-1} = \left(1 + \left(\sigma_i/\sigma_j\right)^{\alpha}\right)^{-1}$, where $j \neq i = 1, 2$. That is, $p_1^* = \left(1 + \Theta_K\right)^{-1}$ and $p_2^* = \left(1 + \Theta_K^{-1}\right)^{-1}$, and thus, the overall stronger player has a higher than 50 percent chance of winning. Player i's equilibrium payoff is $\Pi_i^* = p_i^* v - \mathbf{c}_i \mathbf{x}_i^*$; that is, $\Pi_1^* = v\,(1 + (1-\alpha)\,\Theta_K)\,(1+\Theta_K)^{-2}$ and $\Pi_2^* = v\left(1 + (1-\alpha)\,\Theta_K^{-1}\right)\left(1 + {\Theta_K}^{-1}\right)^{-2}$. Both players' payoffs are nonnegative for $\alpha \in (0, 1]$, and for $\alpha \in (1, 2]$ when $\Theta_K \in \left[(\alpha-1), (\alpha-1)^{-1}\right]$.

Lastly, note that the second-order sufficient condition for player 1 in the reduced one-armed contest (B12) is satisfied if and only if $\partial^2\pi_1/\partial z_1^2 < 0$, or equivalently, $-2vz_2\,(z_1+z_2)^{-3} - \sigma_1\left((1-\alpha)/\alpha^2\right) z_1^{\frac{1}{\alpha}-2} < 0$. In the equilibrium, $z_2 = \Theta_K z_1$ and $z_1^{\frac{1}{\alpha}} = (\alpha v/\sigma_1)\,\Theta_K\,(1+\Theta_K)^{-2}$, and the second-order condition for player 1 can be written as $-2\,(1+\Theta_K)^{-1} - (1-\alpha)\,\alpha^{-1} < 0$. The inequality is always satisfied for $\alpha \in (0, 1]$, and it is equivalent to inequality $\Theta_K < \frac{\alpha+1}{\alpha-1}$ for $\alpha > 1$. Nonnegativity of player 1's payoff implies $\Theta_K \leq \frac{1}{\alpha-1} < \frac{\alpha+1}{\alpha-1}$ for $\alpha > 1$. Similarly, the second-order sufficient condition for player 2 always holds for $\alpha \in (0, 1]$, and for $\alpha > 1$ whenever $\Theta_K^{-1} < \frac{\alpha+1}{\alpha-1}$. Nonnegativity of player 2's payoff implies $\Theta_K^{-1} \leq \frac{1}{\alpha-1} < \frac{\alpha+1}{\alpha-1}$ for $\alpha > 1$. Hence, the second-order conditions are satisfied for both players when their payoffs are nonnegative. □

Proof of Corollary 1 Equilibrium total cost of effort and rent dissipation are given by $D^* = C^*/v = 2\alpha\Theta_K\,(1+\Theta_K)^{-2}$; they increase with α when Θ_K is held constant. Since $\partial C^*/\partial\Theta_K = 2\alpha v\,(1-\Theta_K)\,(1+\Theta_K)^{-3} \gtreqless 0$ if and only if $\Theta_K \lesseqgtr 1$, C^* and D^* are maximized (and Π^* is minimized) at $\Theta_K = 1$. Taking the derivative $\partial x_{1k}^*/\partial c_{1k}$ and simplifying, we obtain that $\partial x_{1k}^*/\partial c_{1k} < 0$ whenever $-1 + (1-\Theta_K)\,(1+\Theta_K)^{-1}\,\alpha_k < 0$, or $\Theta_K > \frac{\alpha_k-1}{\alpha_k+1}$. This inequality holds because $\Theta_K \geq \alpha - 1 > \frac{\alpha_k-1}{\alpha_k+1}$ under the parameter conditions that guarantee the existence of the equilibrium in Proposition 2. By similar arguments, $\partial x_2^*/\partial c_2 < 0$. Player i's cost of effort with each arm depends only on the prize value v, the arm's influence parameter α_k, and the overall relative strength of players Θ_K. Lastly, player i's equilibrium payoff depends only on the prize value v, the discriminating power of the contest α, and the overall relative strength of players Θ_K. □

Proof of Proposition 3 The total cost of effort is $C^*(K-1) = 2\alpha_{(K-1)} v\Theta_{K-1}\,(1+\Theta_{K-1})^{-2}$ in the $(K-1)$-armed contest and $C^*(K) = 2\alpha_{(K)} v\Theta_K\,(1+\Theta_K)^{-2}$ in the K -armed contest. Thus, $C^*(K) > C^*(K-1)$ if and only if $\alpha_{(K)}\theta_K\,(1+\Theta_{K-1}\theta_K)^{-2} > \alpha_{(K-1)}\,(1+\Theta_{K-1})^{-2}$, where $\Theta_K = \Theta_{K-1}\theta_K$. The inequality is equivalent to $\theta_K - b\theta_K^{1/2} + \Theta_{K-1}^{-1} < 0$, where $b = \sqrt{\alpha_{(K)}/\alpha_{(K-1)}}\left(1 + \Theta_{K-1}^{-1}\right) > 0$. This inequality can be stated as $\theta_K \in (g_1(\Theta_{K-1}), g_2(\Theta_{K-1}))$ since it is equivalent to $\Theta_{K-1}(\theta_K^{1/2} - \sqrt{g_1(\Theta_{K-1})})(\theta_K^{1/2} - \sqrt{g_2(\Theta_{K-1})}) < 0$, where $g_1(\Theta_{K-1}) \equiv \left(\frac{1}{2}\left(b - \sqrt{b^2 - 4\Theta_{K-1}^{-1}}\right)\right)^2 > 0$, $g_2(\Theta_{K-1}) \equiv \left(\frac{1}{2}\left(b + \sqrt{b^2 - 4\Theta_{K-1}^{-1}}\right)\right)^2 < \infty$, and $b^2 - 4\Theta_{K-1}^{-1} \geq 0$. Inequality $C^*(K) > C^*(K-1)$ always holds if $\theta_K \in \left(\frac{\alpha_{(K-1)}}{\alpha_{(K)}}, \frac{\alpha_{(K)}}{\alpha_{(K-1)}}\right)$. The comparison with respect to rent dissipation is exactly the same since rent dissipation is $D^* = C^*/v$. And, the conditions for the total payoff comparison are reversed since the total payoff is inversely related to rent dissipation, $\Pi^* = v(1 - D^*)$. □

Proof of Proposition 4 Total effort with arm k is $\alpha_k v \left(\frac{1}{c_{1k}} + \frac{1}{c_{2k}}\right) \Theta_K (1 + \Theta_K)^{-2}$ in the K-armed contest, and total effort with arm k is $\alpha_k v \left(\frac{1}{c_{1k}} + \frac{1}{c_{2k}}\right) \Theta_{K-1} (1 + \Theta_{K-1})^{-2}$ in the $(K-1)$-armed contest, where $k \in \{1, \ldots, K-1\}$. Hence, total effort with arm k is higher in the K-armed contest if and only if $\theta_K (1 + \Theta_{K-1}\theta_K)^{-2} > (1 + \Theta_{K-1})^{-2}$, where we use $\Theta_K = \Theta_{K-1}\theta_K$. The inequality is equivalent to $(1-\theta_K)\left(\Theta_{K-1}^2\theta_K - 1\right) > 0$. When the Kth arm is symmetric, $\theta_K = 1$, the total effort with arm k is not affected by the introduction of the Kth arm. If $\theta_K < 1$, the total effort with arm k is higher in the K-armed contest if and only if $\theta_K > \Theta_{K-1}^{-2}$. If $\theta_K > 1$, the condition is reversed: $\theta_K < \Theta_{K-1}^{-2}$. Therefore, total effort with arm k is higher in the K-armed contest whenever θ_K is between 1 and Θ_{K-1}^{-2}. This is equivalent to the condition that Θ_K is between Θ_{K-1} and Θ_{K-1}^{-1}, and it holds whenever the Kth arm balances the contest, $\max\left\{\Theta_{K-1}, \Theta_{K-1}^{-1}\right\} > \max\{\Theta_K, \Theta_K^{-1}\}$. □

Proof of Proposition 5 From the proof of Proposition 2, player 1's equilibrium payoff in the K-armed contest is $\Pi_1^*(K) = v\left(1 + \left(1 - \alpha_{(K)}\right)\Theta_K\right)(1 + \Theta_K)^{-2}$. Let $A \equiv \Pi_1^*(K-1) = v\left(1 + \left(1 - \alpha_{(K-1)}\right)\Theta_{K-1}\right)(1 + \Theta_{K-1})^{-2}$ be player 1's equilibrium payoff in the $(K-1)$-armed contest, which does not depend on the Kth arm's parameters. $\Pi_1^*(K)$ is a decreasing function of θ_K. To see this, note that $\frac{d\Pi_1^*(K)}{d\theta_K} = -v(1 + \Theta_K)^{-3}\left(\left(1 - \alpha_{(K)}\right)(\Theta_K - 1) + 2\right)\frac{\Theta_K}{\theta_K}$. The inequality $\left(1-\alpha_{(K)}\right)(\Theta_K - 1) + 2 > 0$ holds for $\alpha_{(K)} \in (0, 1]$, and it holds for $\alpha_{(K)} \in (1, 2]$ since $\Theta_K < \frac{1}{\alpha_{(K)}-1} < \frac{\alpha_{(K)}+1}{\alpha_{(K)}-1}$. Since $\Pi_1^*(K)$ is decreasing in θ_K, there exists a critical level for θ_K, $g_4(\Theta_{K-1})$, such that $\Pi_1^*(K) \geq \Pi_1^*(K-1)$ if and only if $\theta_K \leq g_4(\Theta_{K-1})$.

Inequality $\Pi_1^*(K) \geq \Pi_1^*(K-1)$ is equivalent to $v\left(1 + \left(1 - \alpha_{(K)}\right)\Theta_K\right) - A(1 + \Theta_K)^2 \geq 0$. The quadratic inequality can be written as $\Theta_K^2 - 2b\Theta_K - c \leq 0$, where $b = \frac{v}{A}\frac{1-\alpha_{(K)}}{2} - 1$ and $c = \frac{v}{A} - 1 > 0$, and it holds for $\Theta_K \leq b + \sqrt{b^2 + c}$. Hence, $\Pi_1^*(K) \geq \Pi_1^*(K-1)$ whenever $\theta_K \leq g_4(\Theta_{K-1})$, where the critical level $g_4(\Theta_{K-1})$ is defined as $g_4(\Theta_{K-1}) = \Theta_{K-1}^{-1}\left(b + \sqrt{b^2 + c}\right) > 0$ with $b = \frac{(1-\alpha_{(K)})(1+\Theta_{K-1})^2}{2(1+(1-\alpha_{(K-1)})\Theta_{K-1})} - 1$ and $c = \frac{(1+\Theta_{K-1})^2}{1+(1-\alpha_{(K-1)})\Theta_{K-1}} - 1$. Similarly, $\Pi_2^*(K) \geq \Pi_2^*(K-1)$ whenever $\theta_K^{-1} \leq g_4(\Theta_{K-1}^{-1})$, or $\theta_K \geq g_3(\Theta_{K-1}) \equiv \frac{1}{g_4(\Theta_{K-1}^{-1})}$. Therefore, for the range of players' relative strength in arm K, $\theta_K \in (g_3(\Theta_{K-1}), g_4(\Theta_{K-1})) \subset (0, \infty)$, the $(K-1)$-armed contest Pareto dominates the K-armed contest. □

Proof of Proposition 6 If players honor an agreement to restrict the use of the Kth arm to a common cap $\overline{x}_K > 0$, the contest reduces to the $(K-1)$-armed contest.

Suppose both players continue to choose the optimal efforts with the $(K-1)$ arms, and player 2 honors the agreement regarding arm K by choosing effort $\overline{x}_K$ with the Kth arm. Then, player 1 chooses x_{1K} to maximize

$$\frac{x_{1K}^{\alpha_K}}{x_{1K}^{\alpha_K}+\Theta_{K-1}\overline{x}_K^{\alpha_K}}v - c_{1K}x_{1K} = \left(1+\Theta_{K-1}\overline{x}_K^{\alpha_K}x_{1K}^{-\alpha_K}\right)^{-1}v - c_{1K}x_{1K}. \tag{B18}$$

Player 1 marginally benefits from breaking the agreement and increasing her effort with arm K above the agreed upon level $\overline{x}_K$ (holding efforts with all other arms at $x_{1k}^*, k = 1, \ldots, K-1$) whenever

$$\frac{\partial \Pi_1}{\partial x_{1K}}|_{x_{1K}=\overline{x}_K} = \alpha_K \left(1+\Theta_{K-1}\right)^{-2}\Theta_{K-1}\overline{x}_K^{-1}v - c_{1K} > 0. \tag{B19}$$

Therefore, player 1 prefers to unilaterally break the agreement if $c_{1K} < v\frac{\alpha_K}{\overline{x}_K}\frac{\Theta_{K-1}}{[1+\Theta_{K-1}]^2}$. Similarly, for player 2. □

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