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RESEARCH ARTICLE

Strategy-proofness of the plurality rule over restricted domains

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Abstract We give a complete characterization of preference domains over which the plurality rule is strategy-proof. In case strategy-proofness is required to hold under all tie-breaking rules, strategy-proof domains coincide with top-trivial ones where the range of the plurality rule admits at most two alternatives. This impossibility virtually prevails when strategy-proofness is weakened so as to hold under at least one tie-breaking rule: unless there are less than five voters, the top-triviality of a domain is equivalent to the (weak) non-manipulability of the plurality rule. We also characterize the cases with two, three or four voters.

Keywords Plurality rule · Strategy-proofness · Domain restrictions

JEL Classification D7

1 Introduction

When strategic voting is a concern, the implementation of social choice rules is surrounded by a major impossibility established by Gibbard (1973) and Satterthwaite

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(1975): there exists no strategy-proof and non-dictatorial social choice function.[1] This impossibility rests on the full domain assumption which allows any logically possible ordering of alternatives as an individual preference. On the other hand, there are restricted domains—such as those satisfying the single-peakedness condition of Black (1948)—that admit non-dictatorial and strategy-proof social choice functions.

The possible escape from the Gibbard–Satterthwaite impossibility through domain restrictions gives rise to two natural directions of research: One is the search for conditions under which this impossibility fails or prevails. For example, Aswal et al. (2003) introduce a linkedness condition and show that every linked domain exhibits the Gibbard–Satterthwaite impossibility.[2] The second direction of research is to take a domain over which the Gibbard-Satterthwaite impossibility fails and characterize the strategy-proof social choice functions defined over this domain. For example, Moulin (1980) characterizes strategy-proof social choice rules defined over single-peaked domains.[3]

More recently, Barbie et al. (2006) propose a third direction of research by picking some social choice rule, namely the Borda Count, and exploring the domains over which this given social choice rule is strategy-proof.[4] They show that the Borda count is strategy-proof over a domain Π if and only if Π is what they call a *cyclic permutation domain*. We follow the same path for the plurality rule by giving a complete characterization of the domains over which it is strategy-proof. As is in the analysis of Barbie et al. (2006), we distinguish between strategy-proofness under any tie-breaking rule (which we simply call "strategy-proofness") and strategy-proofness under some tie-breaking rule (which we call "essential strategy-proofness"). Nevertheless, our negative findings make this distinction virtually insignificant: The plurality rule fails essential strategy-proofness (hence strategy-proofness) on almost every interesting domain.

Section 2 introduces the basic notions. Section 3 states our results. Section 4 makes some concluding remarks.

2 Basic notions

Consider a finite set of alternatives A with $\#A \geq 3$. Let Θ be the set of complete, transitive and antisymmetric binary relations over A. We refer to the elements of Θ as *orderings* and for any ordering $T \in \Theta$, we write $h(T) \in A$ for the alternative which

[1] By a social choice function, we mean a resolute social choice rule that assigns a single outcome to each profile of preferences. The Gibbard–Satterthwaite impossibility is for social choice functions whose ranges admit at least three outcomes.

[2] On the other hand, not every domain that exhibits the Gibbard–Satterthwaite impossibility has to be linked. In fact, the literature seems to be missing a full characterization of domains which are dictatorial in the Gibbard–Satterthwaite sense. Nevertheless, Nehring and Puppe (2005) characterize dictatorial domains among all generalized single-peaked domains.

[3] As a non-exhaustive list of research in the same direction, one can cite Demange (1982) and Barberà et al. (1991, 1993).

[4] A pioneer of this approach is Dasgupta and Maskin (2004) who ask for the domains over which the simple majority rule satisfies certain desirable properties.

is highest ranked by T, i.e., $h(T)\ T\ x$ for all $x \in A$. Any non-empty $\Pi \subseteq \Theta$ is called a *domain*. Let $h(\Pi) = \{x \in A : x = h(T)$ for some $T \in \Pi\}$ be the set of alternatives which are highest ranked by at least one ordering in Π. We say that Π is *top-trivial* iff $\#h(\Pi) = 1$ or 2. So a top-trivial domain admits at most two alternatives which are top-ranked by some ordering within the domain. This is a severe restriction which rules out almost all interesting domains.

Let N be a finite set of voters with $\#N = n \geq 2$, confronted to make a collective choice from A. The *preference* of any $i \in N$ is denoted as $P_i \in \Theta$ while any $P = (P_1, \ldots, P_n) \in \Theta^N$ is a *preference profile*. Given some domain (of admissible preferences) $\Pi \subseteq \Theta$, we conceive a *social choice rule* as a mapping $f : \Pi^N \longrightarrow 2^A \backslash \{\varnothing\}$. Picking some tie-breaking rule $T \in \Theta$, we write $f_T : \Pi^N \longrightarrow A$ for the refinement of f through T.[5]

We say that $i \in N$ *manipulates* $f_T : \Pi^N \longrightarrow A$ at $P \in \Pi^N$ iff there exists $P' \in \Pi^N$ with $P_j = P'_j$ for all $j \in N \backslash \{i\}$ such that $f_T(P)\ P_i\ f_T(P')$ fails. We say that $f_T : \Pi^N \longrightarrow A$ is *strategy-proof* iff it is manipulated by no $i \in N$ at no $P \in \Pi^N$. We qualify $f : \Pi^N \longrightarrow 2^A \backslash \{\varnothing\}$ as *strategy-proof* iff $f_T : \Pi^N \longrightarrow A$ is strategy-proof for every $T \in \Theta$. We call $f : \Pi^N \longrightarrow 2^A \backslash \{\varnothing\}$ *essentially strategy-proof* iff there exists $T \in \Theta$ such that $f_T : \Pi^N \longrightarrow A$ is strategy-proof.

For each $P \in \Pi^N$ and each $z \in A$, we let $n(z; P) = \#\{i \in N : h(P_i) = z\}$. A social choice rule $f : \Pi^N \longrightarrow 2^A \backslash \{\varnothing\}$ is the *plurality rule* iff $f(P) = \{x \in A : n(x; P) \geq n(y; P)$ for all $y \in A\}$ at each $P \in \Pi^N$. Throughout the paper, we restrict our attention to the plurality rule. So f will always stand for the plurality rule and f_T will stand for its refinement through $T \in \Theta$.

3 Results

We first show the equivalence between the top-triviality of a domain Π and the (essential) strategy-proofness of the plurality rule. The result holds when there are at least five individuals.

Theorem 3.1 *Let $n \geq 5$. Consider the plurality rule $f : \Pi^N \longrightarrow 2^A \backslash \{\varnothing\}$. The following three statements are equivalent*:

(i) Π *is top-trivial.*
(ii) f *is strategy-proof.*
(iii) f *is essentially strategy-proof.*

Proof We prove the theorem by establishing (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). As (ii) $\Longrightarrow$ (iii) immediately follows from the definitions, we will show (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (i).

To see (i) $\Longrightarrow$ (ii), take any top-trivial Π. In case $\#h(\Pi) = 1$, the range of f contains a unique alternative and strategy-proofness is vacuously satisfied. Now let $\#h(\Pi) = 2$, i.e., $h(\Pi) = \{x, y\}$ for some distinct $x, y \in A$. Take any $T \in \Theta$. Let

[5] This refinement is made in the usual way: for each $P \in \Pi^N$, we have $f_T(P)\ T\ x$ for all $x \in f(P)$. As Sanver and Zwicker (2006) discuss in more details, such a refinement violates neutrality.

$x\ T\ y$ without loss of generality. Take any $P \in \Pi^N$, any $i \in N$ and any $P' \in \Pi^N$ with $P_j = P'_j\ \forall j \in N\backslash\{i\}$. As $h(\Pi) = \{x, y\}$, we have $n(z; P) = 0\ \forall z \in A\backslash\{x, y\}$. First consider the case $n(x; P) > n(y; P)$. So $f_T(P) = x$. If $h(P_i) = x$, then $f_T(P)\ P_i\ f_T(P')$ holds. If $h(P_i) = y$, then $n(x; P') > n(y; P')$ hence $f_T(P') = x$ implying $f_T(P)\ P_i\, f_T(P')$. Thus no $i \in N$ manipulates f_T at P. The arguments of the case $n(x; P) > n(y; P)$, mutatis mutandis, establish the non-manipulability of f_T at P when $n(y; P) > n(x; P)$. Finally, consider the case $n(x; P) = n(y; P)$. So $f(P) = \{x, y\}$ and $f_T(P) = x$. If $h(P_i) = x$, then $f_T(P) P_i\ f_T(P')$ holds. If $h(P_i) = y$, then $n(x; P') \geq n(y; P')$ hence $f_T(P') = x$ implying $f_T(P)\ P_i\ f_T(P')$. Thus no $i \in N$ manipulates f_T at P, showing (i) $\Longrightarrow$ (ii).

We now show (iii) $\Longrightarrow$ (i). Let Π fail top-triviality. Hence there exist $P^x, P^y, P^z \in \Pi$ with $h(P^x) = x$, $h(P^y) = y$, $h(P^z) = z$ for distinct $x, y, z \in A$. We will show that f fails essentially strategy-proofness. Take any (tie-breaking rule) $T \in \Theta$ and let, without loss of generality, $x\ T\ y\ T\ z$. We complete the proof by showing that f_T fails strategy-proofness under the following four exhaustive cases:

Case 1: n is even (hence $n \geq 6$) and $x\ P^z\ y$.

Pick $P \in \Pi^N$ with $P_1 = P^z$, $P_i = P^x\ \forall i \in \{2, \ldots, \frac{n}{2}\}$ and $P_i = P^y\ \forall i \in \{\frac{n}{2} + 1, \ldots, n\}$. As $n \geq 6$, $f(P) = \{y\}$ thus $f_T(P) = y$. Now consider the profile $P' \in \Pi^N$ where $P_i = P'_i\ \forall i \in N\backslash\{1\}$ while $P'_1 = P^x$. Note that $f(P') = \{x, y\}$ and $f_T(P') = x$. As $x\ P^z\ y$, thus $x\ P_1\ y$, $1 \in N$ manipulates f_T at P.

Case 2: n is even (hence $n \geq 6$) and $y\ P^z\ x$.

Pick $P \in \Pi^N$ with $P_1 = P_2 = P^z$, $P_i = P^y\ \forall i \in \{3, \ldots, \frac{n}{2} + 1\}$ and $P_i = P^x\ \forall i \in \{\frac{n}{2} + 2, \ldots, n\}$. Remark that $n(z; P) = 2$ and $n(x; P) = n(y; P) = \frac{n}{2} - 1$. As $n \geq 6$, $n(x; P) = n(y; P) \geq n(z; P)$, hence $\{x, y\} \subseteq f(P)$ and $f_T(P) = x$. Now consider the profile $P' \in \Pi^N$ where $P_i = P'_i\ \forall i \in N\backslash\{1\}$ while $P'_1 = P^y$. So $n(y; P') > n(x; P') > n(z; P')$ and $f_T(P') = y$. As $y\ P^z\ x$, thus $y\ P_1\ x$, $1 \in N$ manipulates f_T at P.

Case 3: n is odd and $x\ P^z\ y$.

Pick $P \in \Pi^N$ with $P_1 = P_2 = P^z$, $P_i = P^x \forall i \in \{3, \ldots, \frac{n+1}{2}\}$ and $P_i = P^y\ \forall i \in \{\frac{n+3}{2}, \ldots, n\}$. Remark that $n(z; P) = 2$, $n(x; P) = \frac{n-3}{2}$ and $n(y; P) = \frac{n-1}{2}$, thus $n(y; P) = n(x; P) + 1$. Moreover, as $n \geq 5$, $n(y; P) \geq n(z; P)$, hence $y \in f(P)$ but $x \notin f(P)$, implying $f_T(P) = y$. Now consider the profile $P' \in \Pi^N$ where $P_i = P'_i\ \forall i \in N\backslash\{1\}$ while $P'_1 = P^x$. Note that $f(P') = \{x, y\}$ and $f_T(P') = x$. As $x\ P^z\ y$, thus $x\ P_1\ y$, $1 \in N$ manipulates f_T at P.

Case 4: n is odd and $y\ P^z\ x$.

Pick $P \in \Pi^N$ with $P_1 = P^z$, $P_i = P^x\ \forall i \in \{2, \ldots, \frac{n+1}{2}\}$ and $P_i = P^y\ \forall i \in \{\frac{n+3}{2}, \ldots, n\}$. Remark that $n(z; P) = 1$ and $n(x; P) = n(y; P) = \frac{n-1}{2}$. Moreover, as $n \geq 5$, $n(x; P) = n(y; P) > n(z; P)$, hence $f(P) = \{x, y\}$ and $f_T(P) = x$. Now consider the profile $P' \in \Pi^N$ where $P_i = P'_i\ \forall i \in N\backslash\{1\}$ while $P'_1 = P^y$. Note that $f(P') = \{y\}$ and $f_T(P') = y$. As $y\ P^z\ x$, thus $y\ P_1\ x$, $1 \in N$ manipulates f_T at P. □

Theorem 3.1 announces that in a society with at least five individuals, strategy-proofness and essential strategy-proofness are equivalent conditions for the plurality rule. Moreover, these conditions are satisfied by the plurality rule if and only if it is defined over a domain that admits at most two alternatives that are top-ranked by some admissible ordering—hence allowing at most two alternatives within the range of the plurality rule.

We now turn to societies which consist of three or four individuals. We first show that over a domain Π with $\#h(\Pi) \geq 4$, no refinement of the plurality rule is strategy-proof.

Proposition 3.1 *Let $n \in \{3, 4\}$. If $\#h(\Pi) \geq 4$ then $f_T : \Pi^N \longrightarrow A$ fails strategy-proofness for any $T \in \Theta$.*

Proof Let $\#h(\Pi) \geq 4$. Take any distinct $a, b, c, d \in h(\Pi)$ and any $T \in \Theta$ where, without loss of generality, $a\ T\ bT\ c\ T\ d$. We will show that f_T fails strategy-proofness for the cases of $n = 3$ and $n = 4$ separately.

First let $n = 3$. Pick some $P = (P_1, P_2, P_3) \in \Pi^N$ with $h(P_1) = b$, $h(P_2) = c$, $h(P_3) = d$. So $f(P) = \{b, c, d\}$ and $f_T(P) = b$. For f_T to be strategy-proof, we must have $bP_2\ a$, as otherwise $2 \in N$ manipulates f_T at P by pretending some $Q_2 \in \Pi$ with $h(Q_2) = a$. Now take $P' = (P'_1, P'_2, P'_3) \in \Pi^N$ with $h(P'_1) = a$, $h(P'_3) = b$ and $P'_2 = P_2$ (hence $h(P'_2) = c$ and $bP'_2\ a$). So $f(P') = \{a, b, c\}$ and $f_T(P') = a$. However, $2 \in N$ manipulates f_T at P' by pretending some $P''_2 \in \Pi$ with $h(P''_2) = b$, showing that f_T fails strategy-proofness.

Now let $n = 4$. Pick some $P = (P_1, P_2, P_3, P_4) \in \Pi^N$ with $h(P_1) = a$, $h(P_2) = b$, $h(P_3) = c$, $h(P_4) = d$. So $f(P) = \{a, b, c, d\}$ and $f_T(P) = a$. For f_T to be strategy-proof, we must have $a\ P_2\ c$, as otherwise $2 \in N$ manipulates f_T at P by pretending some $Q_2 \in \Pi$ with $h(Q_2) = c$. Now take $P' = (P'_1, P'_2, P'_3, P'_4) \in \Pi^N$ with $h(P'_1) = a$, $h(P'_3) = h(P'_4) = c$ and $P'_2 = P_2$ (hence $h(P'_2) = b$ and $a\ P'_2\ c$). So $f(P') = \{c\}$ and $f_T(P') = c$. However, $2 \in N$ manipulates f_T at P' by pretending some $P''_2 \in \Pi$ with $h(P''_2) = a$, showing that f_T fails strategy-proofness. □

Proposition 3.1 paves the way to a full characterization of the domains over which the plurality rule is (essentially) strategy-proof when $n = 3$ or 4. We first give a couple of definitions: We say that a domain Π is *restricted* for $x \in h(\Pi)$ iff $\#h(\Pi) = 3$ and for each $T \in \Pi$ we have $x\ T\ y$ for some $y \in h(\Pi)\backslash\{x\}$. When $\#A = 3$, a restricted domain is one where each alternative is top-ranked by some admissible ordering while some $x \in A$ is ranked last by no admissible ordering.[6] The condition becomes more demanding when $\#A > 3$, as it requires precisely three alternatives, say a, b, and c, which are top-ranked by some admissible ordering. Moreover, there must exist $x \in \{a, b, c\}$ such that no admissible ordering ranks x last among $\{a, b, c\}$.

In case Π is restricted for $x \in h(\Pi)$ while $y\ T\ z \iff y\ T'\ z\ \forall y, z \in h(\Pi)\backslash\{x\}$ and $\forall T, T' \in \Pi$ with $h(T) = h(T') = x$, we qualify Π as *strongly restricted.*

Theorem 3.2 *Let $n = 4$. Consider the plurality rule $f : \Pi^N \longrightarrow 2^A\backslash\{\varnothing\}$.*

(i) *f is essentially strategy-proof iff Π is top-trivial or strongly restricted.*
(ii) *f is strategy-proof iff Π is top-trivial.*

Proof We first prove the "if" part of (i). It is straightworward to see that the top-triviality of Π implies the strategy-proofness of f_T at each $T \in \Theta$. Now let Π be

[6] This is a particular case of the value-restriction condition that Sen (1966) identifies—hence our nomenclature. As Sen (1966) shows, value-restricted domains ensure the existence of a Condorcet winner.

strongly restricted. Thus $h(\Pi) = \{x, y, z\}$ for some distinct $x, y, z \in A$. Let, without loss of generality, Π be strongly restricted for x while $y\ T\ z\ \forall T \in \Pi$ with $h(T) = x$. Pick some $\tau \in \Theta$ such that $z\ \tau\ y\ \tau\ x$. We establish the essential strategy-proofness of f by showing the strategy-proofness of f_τ. Remark that f_τ is non-manipulable at any $P \in \Pi^N$ with $\#\{h(P_i) : i \in N\} \leq 2$. So we complete the proof by showing that f_τ is non-manipulable at any $P = (P_1, P_2, P_3, P_4) \in \Pi^N$ with $\#\{h(P_i) : i \in N\} = 3$. Let, without loss of generality, $h(P_1) = x$, $h(P_2) = y$ and $h(P_3) = z$. Note that we have $x P_1\ y\ P_1\ z$ and as Π is strongly restricted for x we have $y\ P_2\ x\ P_2\ z$ and $z\ P_3\ x\ P_3 y$. We establish the non-manipulability of f_τ at P under the following three exhaustive cases:

Case 1: $h(P_4) = z$.

So $f(P) = \{z\}$ and $f_\tau(P) = z$. As $f_\tau(P) = h(P_3) = h(P_4)$, neither $3 \in N$, nor $4 \in N$ manipulates f_τ at P. As $\{f_\tau(Q_1, P_2, P_3, P_4) \in A : Q_1 \in \Pi\} = \{f_\tau(P_1, Q_2, P_3, P_4) \in A : Q_2 \in \Pi\} = \{z\}$, $1, 2 \in N$ do not manipulate f_τ at P.

Case 2: $h(P_4) = y$

So $f(P) = \{y\}$ and $f_\tau(P) = y$. As $f_\tau(P) = h(P_2) = h(P_4)$, $2, 4 \in N$ do not manipulate f_τ at P. As $\{f_\tau(Q_1, P_2, P_3, P_4) \in A : Q_1 \in \Pi\} = \{y, z\}$ while $x\ P_1\ y\ P_1\ z$, $1 \in N$ does not manipulate f_τ at P. As $\{f_\tau(P_1, P_2, Q_3, P_4) \in A : Q_3 \in \Pi\} = \{y\}$, $3 \in N$ does not manipulate f_τ at P.

Case 3: $h(P_4) = x$

So $f(P) = \{x\}$ and $f_\tau(P) = x$. As $f_\tau(P) = h(P_1) = h(P_4)$, $1, 4 \in N$ do not manipulate f_τ at P. As $\{f_\tau(P_1, Q_2, P_3, P_4) \in A : Q_2 \in \Pi\} = \{x, z\}$ while $y\ P_2\ x\ P_2\ z$, $2 \in N$ does not manipulate f_τ at P. As $\{f_\tau(P_1, P_2, Q_3, P_4) \in A : Q_3 \in \Pi\} = \{x, y\}$ while $z P_3\ x\ P_3\ y$, $3 \in N$ does not manipulate f_τ at P.

Thus f_τ is strategy-proof, showing the essential strategy-proofness of f.

To prove the "only if" part of (i), let Π be neither top-trivial nor strongly restricted. So either $\#h(\Pi) > 3$ or $\#h(\Pi) = 3$ while Π is strongly restricted for no $a \in h(\Pi)$. In the former case, we know by Proposition 3.1 that f_T is manipulable for any $T \in \Theta$, hence f fails essential strategy-proofness. Now consider the latter case where $\#h(\Pi) = 3$ and Π is strongly restricted for no $a \in h(\Pi) = \{x, y, z\}$. Take any $\tau \in \Theta$ and let, without loss of generality, $x\tau\ y\ \tau\ z$. Recalling that Π is not strongly restricted for x, we will show that f_τ fails strategy-proofness under the following three exhaustive cases:

Case 1: There exist $T, T' \in \Pi$ with $h(T) = h(T') = x$ while $y\ T\ z$ and $z\ T'\ y$.

Take some $P = (P_1, P_2, P_3, P_4) \in \Pi^N$ with $h(P_1) = x\ P_1\ y\ P_1\ z$, $h(P_2) = y$, $h(P_3) = h(P_4) = z$. So $f(P) = \{z\}$ and $f_\tau(P) = z$. However, $1 \in N$ manipulates f_τ at P by pretending some $P_1' \in \Pi$ with $h(P_1') = y$.

Case 2: $y\ T\ z \iff y\ T'\ z\ \forall T, T' \in \Pi$ with $h(T) = h(T') = x$ and $x\ T\ yT\ z\ \forall T \in \Pi$ with $h(T) = x$.

Take some $P = (P_1, P_2, P_3, P_4) \in \Pi^N$ with $h(P_1) = x$, $h(P_2) = y$ and $h(P_3) = h(P_4) = z$. So $f(P) = \{z\}$ and $f_\tau(P) = z$. However, $1 \in N$ manipulates f_τ at P by pretending some $P_1' \in \Pi$ with $h(P_1') = y$.

Case 3: $y\ T\ z \iff y\ T'\ z\ \forall T, T' \in \Pi$ with $h(T) = h(T') = x$ and $x\ T\ zT\ y\ \forall T \in \Pi$ with $h(T) = x$.

As Π is not strongly restricted for x, ($\exists\ P^y \in \Pi$ with $h(P^y) = y$ while $z\ P^y\ x$) or ($\exists P^z \in \Pi$ with $h(P^z) = z$ while $y\ P^z\ x$). So let, without loss of generality, there exist some $P^y \in \Pi$ with $h(P^y) = y$ while $z\ P^y\ x$. Note that Π is not strongly restricted for $z \in h(\Pi)$. Thus, $(\exists T \in \Pi$ with $h(T) = y$ while $x\ T\ z)$ or $(\exists T, T' \in \Pi$ with $h(T) = h(T') = z$ while $x\ T\ y$ and $y\ T'x)$. In the former case, take some $P = (P_1, P_2, P_3, P_4) \in \Pi^N$ with $h(P_1) = x$, $h(P_2) = y\ P_2\ x\ P_2\ z$, $h(P_3) = h(P_4) = z$. So $f_\tau(P) = z$ and $2 \in N$ manipulates by pretending some $P_2' \in \Pi$ with $h(P_2') = x$. In the latter case, take some $P = (P_1, P_2, P_3, P_4) \in \Pi^N$ with $h(P_1) = x$, $h(P_2) = h(P_3) = y$, $h(P_4) = zP_4\ x\ P_4\ y$. So $f_\tau(P) = y$ and $4 \in N$ manipulates by pretending some $P_4' \in \Pi$ with $h(P_4') = x$.

Thus f_τ fails strategy-proofness, showing that f fails essential strategy-proofness.

We now show (ii). We leave the "if" part to the reader. To see the "only if" part, let Π fail top-triviality. If Π is not strongly restricted, then we know by part (i) that f fails essential strategy-proofness hence f fails strategy-proofness. Now let Π fail top-triviality but be strongly restricted. We write $h(\Pi) = \{x, y, z\}$ and let Π be restricted for x, without loss of generality. Pick some $\tau \in \Theta$ with $x\ \tau y\ \tau\ z$ and consider some $P = (P_1, P_2, P_3, P_4) \in \Pi^N$ with $h(P_1) = x$, $h(P_2) = y$, $h(P_3) = h(P_4) = z$. So $f(P) = \{z\}$ and $f_\tau(P) = z$. Note that $\{f_\tau(P_1, Q_2, P_3, P_4) \in A : Q_2 \in \Pi\} = \{x, z\}$. Moreover, Π is strongly restricted for x, thus $x\ P_2\ z$. So, $2 \in N$ manipulates f_τ at P by pretending some $P_2' \in \Pi^N$ with $h(P_2') = x$. Hence f_τ fails strategy-proofness showing that f is not strategy-proof. □

Theorem 3.3 *Let* $n = 3$. *Consider the plurality rule* $f : \Pi^N \longrightarrow 2^A\backslash\{\varnothing\}$.

(i) *f is essentially strategy-proof iff Π is top-trivial or restricted.*
(ii) *f is strategy-proof iff Π is top-trivial.*

Proof We first prove the "if" part of (i). Check that when Π is top-trivial, f_T is strategy-proof at any $T \in \Theta$. Now let Π be restricted for $x \in h(\Pi) = \{x, y, z\}$. Take some $T \in \Theta$ with $x\ T\ y\ T\ z$. We establish the essential strategy-proofness of f by showing the strategy-proofness of f_T. Remark that f_T is non-manipulable at any $P \in \Pi^N$ with $\#\{h(P_i) : i \in N\} \leq 2$. So we complete the proof by showing that f_T is non-manipulable at any $P = (P_1, P_2, P_3) \in \Pi^N$ with $\#\{h(P_i) : i \in N\} = 3$. Let, without loss of generality, $h(P_1) = x$, $h(P_2) = y$, $h(P_3) = z$. So $f(P) = \{x, y, z\}$ and $f_T(P) = x$. As $f_T(P) = h(P_1)$, $1 \in N$ does not manipulate f_T at P. Note that $\{f_T(P_1, Q_2, P_3) \in A : Q_2 \in \Pi\} = \{x, z\}$. Moreover, Π is restricted for x, thus $x\ P_2\ z$. Hence, $2 \in N$ does not manipulate f_T at P. Note also that $\{f_T(P_1, P_2, Q_3) \in A : Q_3 \in \Pi\} = \{x, y\}$. As Π is restricted for x, we have $x\ P_3\ y$. Hence, $3 \in N$ does not manipulate f_T at P. Thus f_T is strategy-proof, showing that f is essentially strategy-proof.

To prove the "only if" part of (i), let Π be neither top-trivial nor restricted. So either $\#h(\Pi) > 3$ or $\#h(\Pi) = 3$ but Π is restricted for no $a \in h(\Pi)$. In the former case, we know by Proposition 3.1 that f_T is manipulable for any $T \in \Theta$, hence f fails essential strategy-proofness. Now consider the latter case where $\#h(\Pi) = 3$ and Π is restricted for no $a \in h(\Pi) = \{x, y, z\}$. Take any $T \in \Theta$ and let, without loss of generality, $x\ T\ y\ T\ z$. As Π is not restricted for x, $(\exists\ P^y \in \Pi$ with $h(P^y) = y$ while $z\ P^y\ x)$ or $(\exists\ P^z \in \Pi$ with $h(P^z) = z$ while $y\ P^z\ x)$. Assume, without loss of generality, the former case and take some $P = (P_1, P_2, P_3) \in \Pi^N$ with $h(P_1) = x$, $h(P_3) = z$ and $P_2 = P^y$. So $f(P) = \{x, y, z\}$ and $f_T(P) = x$. However, $2 \in N$ manipulates f_T at P

by pretending some $P_2' \in \Pi$ with $h(P_2') = z$. Thus, f_T is not strategy-proof, showing that f is not essentially strategy-proof.

We now show (ii). To see the "if" part, we refer to Theorem 3.1 where we show for $n \geq 5$ that the top-triviality of Π implies the strategy-proofness of f. The arguments of that proof are independent of n , thus apply to $n = 3$ as well. To see the "only if" part of (ii), let Π fail top-triviality. If Π is not restricted, then we know by part (i) that f fails essential strategy-proofness hence f fails strategy-proofness. Now let $\#h(\Pi) = 3$ and Π be restricted for $x \in h(\Pi) = \{x, y, z\}$. Pick some $T \in \Theta$ with $z\ T\ y\ T\ x$. Consider some $P = (P_1, P_2, P_3) \in \Pi^N$ with $h(P_1) = x, h(P_2) = y, h(P_3) = z$. So $f(P) = \{x, y, z\}$ and $f_T(P) = z$. Note that $\{f_T(P_1, Q_2, P_3) \in A : Q_2 \in \Pi\} = \{x, z\}$. Moreover, Π is restricted for x, thus $x P_2\ z$. So, $2 \in N$ manipulates f_T at P by pretending some $P_2' \in \Pi^N$ with $h(P_2') = x$. Hence f_T fails strategy-proofness showing that f is not strategy-proof. □

Our last result is for the case of two individuals. We first give a definition: We say that a domain Π is *semi-single-peaked* with respect to $\tau \in \Theta$ iff $x\ T\ y$ holds for every $T \in \Pi$ and for every $x, y \in h(\Pi)\backslash\{h(T)\}$ with $y\ \tau\ x\ \tau\ h(T)$. Notice that semi-single-peakedness restricts the ordering of the alternatives in $h(\Pi)$ that are on one side of the peak. This is a two-fold weakening of the single-peakedness condition of Black (1948) which is concerned with **all** alternatives and *both sides* of the peak.

Theorem 3.4 *Let $n = 2$. Consider the plurality rule $f : \Pi^N \longrightarrow 2^A\backslash\{\varnothing\}$.*

(i) *f is essentially strategy-proof iff Π is semi-single-peaked.*
(ii) *f is strategy-proof iff Π is top-trivial.*

Proof We first prove (i). To show the "if" part, let Π be semi-single-peaked with respect to $\tau \in \Theta$. We show the essential strategy-proofness of f by showing the strategy-proofness of f_τ. Take any $P = (P_1, P_2) \in \Pi^N$. The reader can check that if $h(P_1) = h(P_2)$, then no $i \in N$ manipulates f_τ at P. Now let $h(P_1) = x$ and $h(P_2) = y$ for some distinct $x, y \in A$. Let $x\ \tau\ y$, without loss of generality. Hence $f_\tau(P) = x = h(P_1)$ and $1 \in N$ does not manipulate f_τ at P. We now show that $2 \in N$ does not manipulate f_τ at P either. Take any $z \in \{f_\tau(P_1, Q_2) \in A : Q_2 \in \Pi\}$. We have $z\ \tau\ x$, thus $z\ \tau\ x\ \tau\ y$. As $y = h(P_2)$ for $P_2 \in \Pi$ and Π is semi-single-peaked with respect to τ, we have $x\ P_2\ z$, showing that $2 \in N$ does not manipulate f_τ at P.

To show the "only if" part of (i), let Π fail semi-single-peakedness. Take any $\tau \in \Theta$. We will show that f_τ is not strategy-proof, hence f fails essential strategy-proofness. As Π fails semi-single-peakedness, there exists $P = (P_1, P_2) \in \Pi^N$ such that $x\ \tau h(P_1)\ \tau\ h(P_2)$ and $x\ P_2\ h(P_1)$ for some $x \in h(\Pi)\backslash\{h(P_1), h(P_2)\}$. Note that $f_\tau(P) = h(P_1)$ and $2 \in N$ manipulates f_τ at P by pretending $P_2' \in \Pi$ with $h(P_2') = x$.

We now prove (ii). To show the "if" part, let Π be top-trivial. Take any $T \in \Theta$. We will show that f_T is strategy-proof. Take any $P = (P_1, P_2) \in \Pi^N$. If $h(P_1) = h(P_2)$, then $f_T(P) = h(P_i)$ for both $i \in N$, hence no $i \in N$ manipulates f_T at P. Now let $h(P_1) = x$ and $h(P_2) = y$ for some distinct $x, y \in A$. Let $x\ T\ y$, without loss of generality. Hence $f_T(P) = x = h(P_1)$ and $1 \in N$ does not manipulate f_T at P. As Π is top-trivial, we have $h(\Pi) = \{x, y\}$. So $\{f_T(P_1, Q_2) \in A : Q_2 \in \Pi\} = \{x\}$.

Hence $2 \in N$ does not manipulate f_T at P either, establishing the strategy-proofness of f.

To show the "only if" part of (ii), let Π fail top-triviality. If Π fails semi-single-peakedness as well, then we know by part (i) that f fails essential strategy-proofness hence f fails strategy-proofness. Now let Π be semi-single-peaked with respect to some $T \in \Theta$. Let, without loss of generality, $x\ T\ y\ T\ z$ for some distinct $x, y, z \in h(\Pi)$. Pick some $\tau \in \Theta$ such that $y\ \tau\ x\ \tau\ z$. Consider some $P = (P_1, P_2) \in \Pi^N$ with $h(P_1) = x$, $h(P_2) = z$. So $f(P) = \{x, z\}$ and $f_\tau(P) = x$. Note that $y \in \{f_\tau(P_1, Q_2) \in A : Q_2 \in \Pi\}$. Moreover, Π is semi-single-peaked with respect to T, thus $y P_2\ x$. So, $2 \in N$ manipulates f_τ at P by pretending some $Q_2 \in \Pi^N$ with $h(Q_2) = y$. Hence f_τ fails strategy-proofness showing that f is not strategy-proof.

□

We close the section by thanking an anonymous referee who remarked that when $n = 2$, f_T is a generalized median voter scheme à la Moulin (1980) with $h(T)$ being the fictitious voter. Of course, part (i) of Theorem 3.4 neither implies nor is implied by Moulin (1980). For, while Moulin (1980) shows the strategy-proofness of all generalized median voter schemes over single-peaked domains, Theorem 3.4 establishes the strategy-proofness of a given generalized median voter scheme over larger domains, namely those which are semi-single-peaked.

4 Concluding remarks

We give a complete characterization of the domains over which the plurality rule is (essentially) strategy-proof. Our results are strongly negative: No matter how many alternatives or individuals the social choice problem admits, the plurality rule is strategy-proof over a domain Π if and only if its range over Π admits at most two alternatives. Moreover, weakening strategy-proofness to essential strategy-profness does not bring a noticeable improvement of the situation: as long as the social choice problem admits at least five individuals, essential strategy-proofness of the plurality rule again coincides with the top-triviality of the domain over which it is defined.[7] The case of two, three or four individuals is slightly more permissive: With three or four individuals, the range of the plurality rule can be extended to three outcomes without violating essential strategy-proofness as long as it is defined over a (strongly) restricted domain. This is quite an insignificant relaxation of top-triviality. On the other hand, the divergence between strategy-proofness and essential strategy-proofness is stronger in the case of two individuals: Domains over which the plurality rule is essentially strategy-proof are characterized by a semi-single-peakedness condition satisfied by a variety of domains -including the single-peaked ones. In any case, the general picture is negative: With no restictions over the size of the social choice problem, the (essential) strategy-proofness of the plurality rule is equivalent to the top-triviality of the domain over which it is defined. In other words, the plurality rule can be rendered

[7] Thus, for social choice problems that admit at least five individuals, strategy-proofness and essential strategy-proofness are equivalent conditions for the plurality rule.

strategy-proof through a domain restriction if and only if its range contains at most two outcomes.

As a case in point, the plurality rule is never strategy-proof over rich domains.[8] On the other hand, as Barbie et al. (2006) show, (rich) cyclic permutation domains ensure the strategy-proofness of the Borda count. Thus, one can claim that over rich domains the Borda count is less manipulable than the plurality rule. This claim is compatible with Saari (1990) who shows, within a particular environment, that among all scoring rules, the Borda Count is least susceptible to manipulation. Nevertheless, by abandoning richness, one can construct a top-trivial domain over which the Borda count is manipulable but the plurality rule is not. After all, as Aleskerov and Kurbanov (1999) suggest, determining criteria for degree of strategy-proofness is not a simple task.[9]

To what extent our findings about the plurality rule apply to the more general class of *tops-only* rules, i.e., social choice rules that depend only on the top preferences of voters? As strategy-proofness implies tops-onlyness in a variety of environments,[10] an answer to this question can pave the way to more general results in characterizing the domains over which a given social choice rule is strategy-proof.

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[8] I.e., a domain at which every alternative is ranked at the top by some ordering.

[9] In fact, there is a variety of treatments of the matter, such as (Kim and Roush 1996; Smith 1999; Favardin et al. 2002; Campbell and Kelly 2004; Slinko 2002; Ju 2005; Sanver and Zwicker 2006).

[10] see Weymark (2004) and Chatterji and Sen (2007).

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