Econ Theory (2009) 39:291–306
DOI 10.1007/s00199-008-0334-z

RESEARCH ARTICLE

Existence and efficiency of a stationary subgame-perfect equilibrium in coalitional bargaining models with nonsuperadditive payoffs

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Received: 19 September 2006 / Accepted: 25 January 2008 / Published online: 15 February 2008

Abstract We provide the existence theorem of stationary subgame-perfect equilibrium (SSPE) in a noncooperative coalitional bargaining game model with random proposers. Our model contains a bargaining situation where the coalitional game is nonsuperadditive. We also provide a necessary and sufficient condition for the existence of a pure-strategy SSPE satisfying the efficiency property when the discount factor is close to one. Furthermore, we provide examples where the delay in agreement occurs, even in a random-proposers model, when the game is nonsuperadditive.

Keywords Existence of stationary subgame-perfect equilibrium · Delay · Nonsuperadditive game · Coalitional bargaining

JEL Classification C72 · C78

1 Introduction

This paper examines the existence and efficiency of a stationary subgame-perfect equilibrium (SSPE) in a noncooperative coalitional bargaining game with random proposers. A noncooperative coalitional bargaining model with random proposers was first provided by Okada (1996). A bargaining model with random proposers is common in the literature on legislative bargaining. Starting with Baron and Ferejohn's (1989) seminal work, a considerable number of studies have been undertaken concerning legislative bargaining and a number of variants of the Baron–Ferejohn model have

I am grateful to Akira Okada and an anonymous referee for their useful comments and helpful suggestions.

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been provided (see, for example, Banks and Duggan 2000; Jackson and Moselle 2002; Snyder et al. 2005; Eraslan and McLennan 2006).

In a noncooperative coalitional bargaining model, the bargaining situation is described by an n-person coalitional game (N, v), where N is the set of players and v is the characteristic function. A key feature in our model is that the characteristic function v is not necessarily superadditive, unlike the superadditive game (N, v) was assumed in Okada's work. The superadditive game means that the coarsest coalition structure consisting of the grand coalition $\{N\}$ can generate the maximum total feasible payoff for all players in N. Some economic environments, however, could not be described by a class of superadditive games. Guesnerie and Oddou (1979, 1981) and Greenberg and Weber (1986) have clearly shown that the coalitional games corresponding to the local public good economy are not necessarily superadditive when the local public good is financed through a proportional income tax or a poll tax. The coalitional form game of the local public good economy with congestion effects is not superadditive, even if the tax system is flexible enough to adjust the contributions of each individual, as shown in Mutuswami et al. (2004).[1] Demange (2004) has examined the hierarchical structure of organizations like firms, political parties, and transportation and telecommunication networks. She has pointed out that in some environments, the union of coalitions may generate some inefficiencies due to congestion, increasing the marginal cost of the dissemination of information, or increasing the marginal cost of control. In these cases, the problem is represented by a nonsuperadditive game.

The problem of two-sided matching markets, the so-called roommate or the marriage problem (see, for example, Gale and Shapley 1962; Roth and Sotomayor 1990), can also be considered as a nonsuperadditive game. If we define the coalitional form game such that only the worth of the coalitions consisting of two members is positive and the worth of all other coalitions is zero, the game becomes a nonsuperadditive game. However, since Shapley and Shubik (1972), the game corresponding to the two-sided matching has been given by the superadditive cover of the above coalitional form game, where the worth of large coalitions is determined entirely by the worth of the pairwise combinations that the coalitions members can form. This is the well-known *assignment game*.

Slikker and van den Nouweland (2000) have studied network formation when the game is not necessarily superadditive. They focused on the effects that the costs of establishing communication links has on the networks formed and showed that the pattern of network structures as costs increase depends on whether the underlying coalitional game is superadditive. Recent studies of the legislative bargaining game also include cases of nonsuperadditive games (Jackson and Moselle 2002; a legislative voting game over the division of a pie and a one-dimensional policy, Snyder et al. 2005; a weighted majority voting game, Eraslan and McLennan 2006; a voting game with arbitrary winning coalitions).[2]

[1] Mutuswami et al. (2004) did not mention congestion effects explicitly. They provided a model where the utility function and cost function of the local public good depend on the coalition to which the members belong. This model can incorporate congestion effects.

[2] I am grateful to an anonymous referee for identifying this work on bargaining in legislatures. As a result, we find that voting games contain cases with a nonsuperadditive game.

A noncooperative coalitional bargaining model is represented as an infinite-length extensive game with perfect information and chance moves. Gomes (2005) showed that the coalitional bargaining game is considered as a stochastic game with infinite state spaces and asserted that the existence of SSPEs for stochastic games with infinite state spaces is not trivial. Therefore, we first prove the existence of an SSPE in a noncooperative coalitional bargaining model with random proposers. Moreover, because the bargaining game is not of finite-length but an infinite-length extensive form game with perfect information, the existence of *pure-strategy* Nash equilibria is even unclear.[3] We next clarify a necessary and sufficient condition for a *pure-strategy* efficient SSPE to exist in the random-proposers bargaining model with a coalitional form game containing the nonsuperadditive case. We show that the necessary and sufficient condition is related to the notion of a C-stable solution (or the core of cooperative games with coalition structures) introduced by Aumann and Dreze (1974), Guesnerie and Oddou (1979, 1981) and Greenberg and Weber (1986).

Chatterjee et al. (1993) show that a delay in agreement may occur in a stationary equilibrium for an n-person coalitional bargaining model. This is in contrast to the Rubinstein (1982) two-person alternating-offer model. In their model, a proposer is determined in a fixed order over the players, and the first rejecter becomes the next proposer. Okada (1996) points out that the delay in agreement is caused by their fixed bargaining protocol and shows that no delay in agreement occurs in the bargaining model where the proposer is randomly selected in every round (the random-proposers model). In Okada's paper, an n-person coalitional game (N, v) is assumed to be superadditive. We provide examples where the delay in agreement occurs, even in the random-proposers model when a coalitional form game is not superadditive. We then discuss the delay in agreement in nonsuperadditive games.

The paper is organized as follows. In Sect. 2, we present a noncooperative coalitional bargaining game model with random proposers and provide examples of the nonsuperadditive coalitional form game. In Sect. 3, we provide two existence theorems for a stationary subgame-perfect equilibrium point of the bargaining game. In Sect. 4, we show that the delay in agreement occurs because of nonsuperadditivity. Section 5 concludes the paper.

2 Random-proposers model

The bargaining situation is described by an n-person coalitional form game (N, v) with transferable utility. Here, $N = \{1, \dots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function. The characteristic function v is assumed to be 0-*normalized* ($v(\{i\}) = 0$ for all $i \in N$) and satisfies $\sum_{j=1}^{J} v(S_j) > 0$ for some partition $\{S_1, \dots, S_J\}$ of N. We allow the characteristic function to be *nonsuperadditive*, i.e., there exist two disjoint coalitions S and $T \in 2^N$ such that $v(S \cup T) < v(S) + v(T)$. We state that the grand coalition is *universally efficient* if $v(N) \geq \sum_{k=1}^{K} v(S_k)$ for any partition $\{S_1, \dots, S_K\}$ of N. If the game is superadditive, the grand coalition N is universally

[3] If the game is a *finite-length* extensive form game with perfect information, a *pure-strategy* Nash equilibrium always exists. See, Corollary 1 in Kuhn (1953), p. 61.

efficient. However, the grand coalition N is not necessarily universally efficient in nonsuperadditive games.

Let us next explain the *random-proposers* model of bargaining. A payoff vector for a coalition S is denoted by $y^S = (y_i^S)_{i \in S} \in \mathbb{R}^{|S|}$. A payoff vector y^S for S is called *feasible* if:

$$\sum_{i \in S} y_i^S \leq v(S).$$

We denote by Y^S the set of all feasible payoff vectors for S.

Our noncooperative bargaining model proceeds as follows. In every round $t = 1, 2, \ldots$, one player is selected as a proposer with equal probability among all players which is still active in bargaining. Let N^t be the set of all active players in round t. The bargaining starts with all players in round 1, i.e., $N^1 = N$. The proposer i chooses a coalition S (with $i \in S \subseteq N^t$) and a payoff vector $y^S \in Y^S$. All other players in S sequentially accept or reject the proposal. If all the other players in the coalition accept the proposal, then it is agreed upon and enforced. The remaining players outside S can continue negotiations in the next round. Thus, $N^{t+1} = N^t \backslash S$. If some player in S rejects the proposal, then negotiations go on to the next round and a new proposer is randomly selected by the same rule, i.e., $N^{t+1} = N^t$. The bargaining continues until there is no coalition S of active players such that $v(S) > 0$. Note that players can offer unacceptable proposals or reject any proposals. The delay of agreements occurs in this case. The game could last forever under the possibility of delay.

When a proposal (S, y^S) is agreed upon in round t, the payoff of every member $i \in S$ is $\delta^{t-1} y_i^S$, where δ is a discount factor, and $0 \leq \delta < 1$. For players who do not belong to any coalitions, their payoffs are assumed to be zero. Every player has perfect information.

Our model is formally represented as an infinite-length extensive form game with perfect information and chance moves. We denote by $\Gamma^S(\delta)$ the bargaining model with the player set $S \subseteq N$. Γ^S is used when the discount factor δ converges to one. Let $\sigma_i = \{\sigma_i^t\}_{t=1}^{\infty}$ be a strategy for player i in $\Gamma^N(\delta)$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a strategy combination. Here, σ_i^t is the tth round strategy for player i. A history h_i^t, for $t \geq 1$, is a sequence of all past actions before player i's move in round t in $\Gamma^N(\delta)$. When player i is a proposer in round t, h_i^t is a complete listing of actions in the previous $t-1$ rounds, and when player i is a responder in round t, the proposals by other players in round t is also included in h_i^t. The tth round strategy σ_i^t depends on the history h_i^t of the play of game up to round t and prescribes a proposal (S, y^S) and a response function assigning "yes" or "no". The solution concept that we apply to our bargaining model is a *stationary subgame-perfect equilibrium point* (SSPE).

Definition 1 (i) A strategy combination σ^* of the game $\Gamma^N(\delta)$ is called a *stationary subgame-perfect equilibrium point* (SSPE) if it is a subgame perfect equilibrium point with the property that for every $t = 1, 2, \ldots$, the tth round strategy of every player depends only on the set N^t of all active players and the proposal in round t. (ii) A strategy combination σ^* of the game Γ^N is called a *limit* SSPE if it is a limit point of SSPEs of $\Gamma^N(\delta)$ as $\delta \to 1$.

For an SSPE σ of $\Gamma^N(\delta)$ and every coalition $S \subseteq N$, let $v^S = (v_i^S)_{i\in S}$ denotes the expected payoff vector of players for σ in the subgame $\Gamma^S(\delta)$, and $\theta^S = (T_i^S)_{i\in S}$ be the collection of coalitions T_i^S proposed by every player i on the plays of σ in $\Gamma^S(\delta)$. We denote the collection $\{(v^S, \theta^S) \mid S \subseteq N\}$ the *configuration* of the SSPE σ.

3 Existence theorems

3.1 Existence and C-stable solution

Let us establish the existence theorem for an SSPE in the general random-proposers bargaining model. If mixed strategies with respect to the proposal of a coalition are allowed, the existence of a SSPE is guaranteed. We can prove the existence of SSPEs without mixing randomly the strategies with respect to the proposals of feasible payoff allocations and the responses, {accpet, reject}, to a proposal. By allowing the mixed strategies with respect to only the proposal of a coalition S, we can ensure the continuous correspondences of proposals by each player through Berge's maximum theorem.

Theorem 1 *If mixed strategies with respect to the choice of a coalition by each proposer are allowed, there exists a stationary subgame-perfect equilibrium point of the game* $\Gamma^N(\delta)$.

Proof This existence theorem can be proved in the same line as the proof of Theorem 2.1 in Ray and Vohra (1999) by Berge's maximum theorem and Kakutani's fixed point theorem. Therefore, we omit the proof. □

Denote the set of all partitions of S by:

$$\Pi(S) = \left\{ \{S_1, \ldots, S_K\} \,\middle|\, \bigcup_{k=1}^{K} S_k = S, \text{ and } S_i \cap S_j = \emptyset, i \neq j \right\}.$$

An element $\pi^S = \{S_1, \ldots, S_K\} \in \Pi(S)$ is called a *coalition structure* of S. The function on $\Pi(S)$ is defined by:

$$V(\pi^S; S) = \sum_{k=1}^{K} v(S_k).$$

Definition 2 A coalition structure π is called an *efficient coalition structure* of S if $V(\pi; S) \geq V(\pi'; S)$ for all $\pi' \in \Pi(S)$.

We now provide some examples of nonsuperadditive games.

Example 1 (the local public good economy) Let us consider an economy with one local public good and one private good. We assume that one unit of the private good can be transformed into one unit of the public good. Each individual $i \in N$ is endowed with the same amount I of the private good and has a quasilinear utility function with congestion effects: $u_i(g) - c(|S|) + x_i$, where $u_i(g)$ is the utility from the local public good g and x_i is the consumption of the private good, whereas $c(|S|)$ is the disutility from congestion (here, $|S|$ denotes the cardinality of the coalition S). The coalitional form game (N, v) associated with the local public good economy is defined by, for each $S \subset N$:

$$v(S) = \max_{g \in \mathbb{R}} \left\{ \sum_{i \in S} u_i(g) - |S|c(|S|) + |S|I - g \right\}.$$

The value $v(S)$ denotes the maximum total payoff for the members of coalition S by producing the local public good. If $c(|S|)$ is substantially increasing with respect to $|S|$, forming the grand coalition will not necessarily maximize the total payoff and v is not superadditive.

Example 2 (cost sharing in Demange 2004) A firm needs to allocate various common costs among its units and solve the problem of coordination. We consider a service to be provided to units. The game v is defined by $v(S) = \max(0, \sum_{i \in S} b_i - c(S))$, where $\sum_{i \in S} b_i$ is the aggregate benefit of coalition S for the service and $c(S)$ is the cost of provision. Demange (2004) argues that if congestion exists, and there is an increasing marginal cost of the dissemination of information, or an increasing marginal cost of control, v may be nonsuperadditive.

Example 3 (network formation in Slikker and van den Nouweland 2000) In the network formation game with costs for establishing links, the worth of coalition S of players is defined by:

$$v^{g,c}(S) = \sum_{C \in \pi(S,g)} v(C) - c|g(S)|,$$

where $v(C)$ is the value of component C, $g(S)$ is the restriction of graph g to S, $\pi(S, g)$ is the set of all components of network $g(S)$ and $|g(S)|$ is the number of communication links. In a symmetric three-player game with $v(C) = w_{|C|}$ (where $|C|$ is the number of players in component C), the game belongs to the class of nonsuperadditive games if $w_2 > w_3$.

Example 4 (roommate problem in Gale and Shapley 1962)[4] The set N consists of n people who can be matched in pairs (as roommates in a college dormitory). Each person ranks the $n - 1$ others in order of preference. The situation is considered as a game in which only the worth of the coalitions of two people exists and the worth of other coalitions containing more than three people is zero. Thus, the coalitional form game v is defined by:

$$\begin{aligned} v(\{i, j\}) &= \alpha_{ij} \quad \text{if } \{i, j\} \subset N, i \neq j, \\ v(S) &= 0 \quad\;\; \text{if } |S| \neq 2. \end{aligned}$$

[4] We would like to thank an anonymous referee for providing useful hints concerning the roommate problem. In addition, the referee also suggest one of the examples of the delay in agreements (Example 6).

Then, the game becomes nonsuperadditive. However, the coalitional form game $\tilde{v}$ corresponding to the roommate problem has been commonly given by the superadditive cover of v; $\tilde{v}(\{i, j\}) = \alpha_{ij}$ and $\tilde{v}(S) = \max[\alpha_{i_1 j_1} + \alpha_{i_2 j_2} + \cdots + \alpha_{i_k j_k}]$ for S such that $|S| \neq 2$, with the maximum to be taken over all arrangements of $2k$ distinct pairs in S. See, for example, Shapley and Shubik (1972).

Let us define the efficiency of a SSPE for Γ^N.

Definition 3 A SSPE σ of the game $\Gamma^N(\delta)$ is called *subgame coalitional efficient* if, for every subgame $\Gamma^S(\delta)$, every player $i \in S$ proposes the coalition which is a component of the efficient coalition structure of S in σ. A *limit subgame coalitional efficient* SSPE of Γ^N is defined to be a limit of subgame coalitional efficient SSPEs of $\Gamma^N(\delta)$ as $\delta \to 1$.

The notion of subgame coalitional efficiency requires that the efficient coalition structure is formed in all subgames $\Gamma^S(\delta)$. This notion is stronger than the Pareto efficiency of the expected payoff vector for n players in $\Gamma^N(\delta)$. Note that, by Lemma 1 in the Appendix, the proposal by every player is accepted immediately in every subgame coalitional efficient SSPE of $\Gamma^N(\delta)$. Therefore, the subgame coalitional efficiency implies no-delay of agreements in the bargaining game.

The next theorem (Theorem 2) characterizes the situation where there exists a *pure strategy* and limit subgame coalitional efficient SSPE in Γ^N. To characterize, we use the notion of a Nash bargaining solution. Let us define the Nash bargaining solution of the bargaining problem with coalition S and its worth $v(S)$. Because $v(\{i\}) = 0$ for every player i, the disagreement point is the origin of $\mathbb{R}^{|S|}$. The Nash bargaining solution of $(S, v(S))$ is defined as a solution of the maximization problem:

$$\max_{(y_i)_{i\in S}} \prod_{i\in S} y_i \text{ subject to } \sum_{i\in S} y_i \leq v(S).$$

In transferable utility games (N, v), the Nash bargaining solution of $(S, v(S))$ implies an equal share allocation among the members of coalition S. Thus, player $i \in S$ receives the payoff of $v(S)/|S|$.

Definition 4 Given a coalition structure $\pi(S) = \{S_1(S), \ldots, S_K(S)\}$, a payoff allocation $v^{S*} = (v_i^{S*})_{i\in S}$ is called the *Nash-bargaining-solution payoff allocation under coalition structure* $\pi(S)$ if $(v_j^{S*})_{j\in S_\ell(S)}$ is generated by the Nash bargaining solution of $(S_\ell(S), v(S_\ell(S)))$, $\ell = 1, \ldots, K$.

Note that an efficient coalition structure of each S is expressed by $\pi^*(S) = \{S_1^*(S), \ldots, S_{K^S}^*(S)\}$ in Theorem 2.

Theorem 2 *(i) There exists a pure strategy and limit subgame coalitional efficient SSPE of* Γ^N *if and only if:*

for all $S \subseteq N$ *and* $i \in S$, *the component* $S_\ell^*(S)$ *of the efficient coalition structure and the Nash-bargaining-solution payoff allocation* v^{S*} *constitute a solution of:*

$$\max_{y, T\subset S, i\in T} y_i \text{ subject to } y \in v(T) \text{ and } y_j \geq v_j^{S*} \text{ for } j \in T, j \neq i, \quad (1)$$

where $i \in S^*_\ell(S)$.

(ii) The expected equilibrium payoff vector in Γ^S *is given by the Nash-bargaining-solution payoff allocation under the efficient coalition structure* $\pi^*(S)$ *for all* $S \subseteq N$.

Proof See Appendix. □

Theorem 2 shows that a pure strategy and limit subgame coalitional efficient SSPE exists if and only if each individual obtains the maximum payoff by forming coalition $S^*_k(S)$ under the condition that other individual j must be guaranteed to get his or her payoff $v(S^*_\ell(S))/|S^*_\ell(S)|$, where $S^*_\ell(S) \in \pi^*(S)$ and $j \in S^*_\ell(S)$.

We introduce the notion of a *C-stable solution* considered in Aumann and Dreze (1974), Guesnerie and Oddou (1979, 1981) and Greenberg and Weber (1986).[5]

Definition 5 A *C-stable solution* of (S, v) is a payoff vector y such that y is feasible for some coalition structure $\pi \in \Pi(S)$ and is not blocked by a coalition $T \subseteq S$.

The notion of a C-stable solution is an extension of the core concept. If a coalition structure π is assumed to be $\{N\}$, then the C-stable solution belongs to the core. As shown in Aumann and Dreze (1974), the set of C-stable solutions coincides with the core of the superadditive cover game of v if the set of C-stable solution is not empty.

There is a relationship between Theorem 2 and a C-stable solution. It is easy to see that the condition in Theorem 2 states that for every coalition $S \subseteq N$, the Nash-bargaining-solution payoff allocation under coalition structure $\pi^*(S)$ belongs to the set of C-stable solutions of (S, v). Therefore, Theorem 2 (i) can be restated as the following corollary:

Corollary 1 *There exists a pure strategy and limit subgame coalitional efficient SSPE of* Γ^N *if and only if for every coalition* $S \subseteq N$*, the Nash-bargaining-solution payoff allocation under the efficient coalition structure* $\pi^*(S)$ *belongs to the set of C-stable solutions of the underlying game* (S, v).

4 Delay of agreements

The delay of agreements in equilibrium may occur when the coalitional game is not superadditive, even if the random-proposers bargaining model is adopted. We provide two examples.

Example 5 We consider a four-person game; $N = \{1, 2, 3, 4\}$:

$$v(\{1,2,3,4\}) = 120, \;\; v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = 50,$$
$$v(\{i,j\}) = 100, \quad \text{for } i, j = 2, 3, 4,$$

and $v(\text{others}) = 0$. This game is not superadditive. Assume that the discount factor is almost one. Now consider the following strategies for the players. If the set of

[5] In Aumann and Dreze (1974), the set of C-stable solutions is called *the core of cooperative games with coalition structures*.

active players is $N = \{1, 2, 3, 4\}$, player 1 proposes $(\{1, 2\}, (50, 0))$, player 2 does $(\{2, 3\}, (100 - 125/3, 125/3))$, player 3 does $(\{3, 4\}, (100 - 125/3, 125/3))$, and player 4 does $(\{2, 4\}, (125/3, 100 - 125/3))$. The response rule for players is as follows. Player 1 accepts a proposal y_1 if and only if $y_1 \geq 25$. Player 2, 3, 4 accepts a proposal if and only if $y_j \geq 125/3$ for $j = 2, 3, 4$. When the set of active players is $\{1, i\}$, $i = 2, 3, 4$, a player selected as a proposer proposes $(\{1, i\}, (25, 25))$, and accepts any proposal if he or she is offered a payoff equal to or greater than 25. When the set of active players is $\{i, j\}, i, j = 2, 3, 4$, every player proposes $(\{i, j\}, (50, 50))$ and accepts any proposal if his or her payoff is equal to or greater than 50. When this strategy is employed, the expected payoff of player 1 is 25 and the expected payoff of player 2, 3, 4 is 125/3. It is easy to see that the strategies construct an SSPE in the bargaining game model. First, according to a two-person bargaining game with random proposers, these strategies clearly compose a subgame perfect equilibrium point in the subgame when only two players are still active. Next consider a bargaining game where four players are active. We can check the optimality of the response rule for every player. If player 1 rejects an offer in the four-person bargaining, negotiations go to the next round, and his or her expected payoff is 25. Thus, it is optimal for him or her to accept any offer in the four-person bargaining if he or she receives at least 25. Similarly, it is optimal for player 2, 3, 4 to accept the offer in the four-person bargaining if he or she can get at least 125/3. Given the response rule of the other players, player 2 can obtain (100−125/3) by proposing coalition $\{2, 3\}$ and $\{2, 4\}$. He or she then can receive 25 by proposing $\{1, 2\}$. If he or she proposes a four-person coalition $\{1, 2, 3, 4\}$, he or she obtain only 35/3 (= (120−25−250/3)). Thus, it is optimal to propose $\{2, 3\}$ while demanding (100−125/3). We can check the optimality of the proposals of players 3 and 4 in the same manner. If player 1 proposes $(\{1, 2\}, (50, 0))$, then player 2 rejects the proposal, and negotiations go to the next round. Then, player 1 gets the expected payoff of 25. In order to form a four-person coalition $\{1, 2, 3, 4\}$, player 1 has to guarantee the continuation payoff 125/3 for players 2, 3 and 4 respectively. Because $v(\{1, 2, 3, 4\}) = 120 < 125/3 + 125/3 + 125/3$, player 1 could not make a feasible proposal for $\{1, 2, 3, 4\}$. Moreover, he or she obtains at most (50−125/3) (< 25) by proposing an acceptable offer for $\{1, i\}, i = 2, 3, 4$. Therefore, it is optimal for player 1 to have his or her proposal rejected in round 1. Thus, a delay in agreement occurs when player 1 is selected as a proposer in round 1.

Let us give an intuitive explanation for the delay of agreement in the above example. If a game is superadditive, every player can get at least the expected payoff by proposing the grand coalition when he or she is a proposer. This leads no delay in agreement in a noncooperative coalitional bargaining model with random proposers. However, when a game is not superadditive, forming the grand coalition does not ensure the expected payoff for a proposer. In the example, the sum of expected equilibrium payoffs of all players (25+125/3+125/3+125/3) (=150) could not be feasible in the grand coalition because $v(\{1, 2, 3, 4\}) = 120$. Each player benefits from forming a smaller coalition than the grand coalition in order to avoid congestion. Moreover, it is important for each player when and with whom to form a coalition. In our example, if player i ($i \neq 1$) and player 1 remain at bargaining after player j and k ($j, k \neq 1$) form a two-person coalition, then player i can obtain only 25. On the other hand, players

j and k can get at least 125/3. Thus, player i, $i = 2, 3, 4$, has an incentive to form a two-person coalition $\{i, j\}$ with player $j = 2, 3, 4$ in the first round. Because player 1 is in a weaker position in the bargaining than players 2, 3 and 4, he or she cannot benefit in the bargaining with the three strong players in round 1. Therefore, he or she optimally waits for a two-person coalition of strong players to be formed.

Example 6 Let us recall the roommate problem. There are four people: a, b, c, and d, with the following preferences: person a prefers b to c and does c to d as a roommate, person b prefers c to a and does a to d, person c prefers a to b and does b to d, and person d is indifferent between the three other people. Note that person d is the last choice of everyone else. It is well known that no stable matching exists in this example; see Gale and Shapley (1962). Assume that person a, b, c obtains 30 if he or she is matched with a person at the first rank in his or her preferences, 20 if he or she is matched with a person at the second rank, and 10 if he or she is matched with a person at the third rank. Person d obtains the payoff of 20 whichever persons he chooses. Then, if the coalitional form game (N, v) is defined as follows:

$$v(\{a, b\}) = v(\{b, c\}) = v(\{a, c\}) = 50,$$
$$v(\{i, d\}) = 30, \quad i = a, b, c,$$

and $v(\text{others}) = 0$, the game is nonsuperadditive. It is easy to see that the set of C-stable solutions (equivalently, the core of the assignment game) is empty. Let us consider similar strategies to those in Example 5, assuming that the discount factor is almost one. When four people are active, person a, b, c, as a proposer offers to be matched with his or her most favorite person and $65/3$ as a payoff to his or her partner. Person d proposes $(\{a, d\}, (0, 30))$. Moreover, person a, b, c accepts a proposal if and only if his or her offered payoff is equal to or greater than $65/3$, and person d accepts a proposal if his or her payoff is equal to or greater than 15. When two people remain in the game, every person as a proposer offers half of its worth of the two-persons-coalition and accepts a proposal if and only if his or her offered payoff is equal to or greater than a half of its worth. We can check the above strategy combination is an SSPE of Γ^N in the same manner as in Example 5. In the SSPE, person d's proposal is always rejected in the first round and the delay in agreement occurs with positive probability. We should mention that the argument in the example crucially depends on the preferences of person d, although Gale and Shapley (1962) assumed that the preferences of person d are arbitrary.

Lemma 1 in the Appendix and Corollary 1 have some implications in the delay of agreements. First, we can interpret Lemma 1 as a sufficient condition under which no delay in agreements occurs in every SSPE of the coalitional bargaining game.

Lemma 1′ *If payoff configurations in every SSPE of $\Gamma^N(\delta)$ are feasible in the efficient coalition structure $\pi^*(S)$, then no delay in agreements arises in the SSPE.*

If the game is superadditive, the efficient coalition structure of S is $\{S\}$ itself, and all proposed payoff vectors are feasible in $v(S)$, regardless of the coalition chosen by any proposer. The payoff configuration in every SSPE is then trivially feasible in the efficient coalition structure $\{S\}$. Therefore, no delay occurs in every SSPE of the coalitional bargaining model with random proposers when the game is superadditive.

In Example 5, the equilibrium expected payoff vector $(v_1^*, v_2^*, v_3^*, v_4^*)$ is given by $(25, 125/3, 125/3, 125/3)$. Because the efficient coalition structure of N is $\{\{1, 2\}, \{3, 4\}\}$, the expected payoff vector is not feasible in the efficient coalition structure: $v_1^* + v_2^* = 25 + 125/3 > 50 = v(\{1, 2\})$. In Example 6, the equilibrium payoff vector $(65/3, 65/3, 65/3, 15)$ for person a, b, c, and d is also not feasible in the efficient coalition structure.

Next let us consider Corollary 1 in relation to the delay of agreements. Note that, by Lemma 1, delay of agreement never occurs in every (limit) subgame coalitional efficient SSPE. Therefore, a sufficient condition for the existence of a limit subgame coalitional efficient SSPE can be regarded as a sufficient condition for the existence of a SSPE in which no delay in agreement arises. Thus, Corollary 1 can be restated as follows:

Corollary 1′ *If the discount factor δ is almost one and the Nash-bargaining-solution payoff allocation under the efficient coalition structure of S is in the set of C-stable solutions of the game (S, v) for every coalition $S \subseteq N$, there exists at least one pure strategy SSPE of Γ^N in which no delay in agreement occurs.*

Note that, even if the conditions in Corollary 1′ are satisfied, (other) SSPEs with delayed agreements may exist together. If the SSPE payoff allocation is unique, then there is no inefficient SSPE of Γ^N involving the delay under the conditions in Corollary 1′.

5 Conclusion

We have extended a random-proposers bargaining model to nonsuperadditive games and shown that a SSPE exists in the bargaining model. We provided a necessary and sufficient condition for the existence of a pure strategy and limit subgame coalitional efficient SSPE. The condition is that for each coalition S, the Nash bargaining solution payoff allocation under the efficient coalition structure is in the set of C-stable solutions of the game (S, v). When the set of C-stable solutions of the game is empty, our result shows that there is no pure strategy and efficient SSPE in the bargaining model.

The uniqueness of SSPE payoff allocations is an open question in the random-proposers model with nonsuperaddtive games. In addition, we need to clarify the relationship between the C-stable solution and the delay in agreements in equilibrium. It is noteworthy that the set of C-stable solutions is empty in our examples of the delay in agreement (Examples 5 and 6).

Appendix

Proof of Theorem 2

We provide two lemmas before giving the proof of Theorem 2. In addition, we focus on a class of payoff configurations in these lemmas.

Definition 6 Payoff configurations $\{v^S \mid S \subseteq N\}$ are called *feasible in the efficient coalition structure* $\pi^*(S) = \{S_1^*(S), \ldots, S_{K^S}^*(S)\}$ if, for every S:

$$\sum_{j \in S_\ell^*(S)} v_j^S \leq v(S_\ell^*(S)), \quad \ell = 1, \ldots, K^S.$$

The first lemma shows that in every pure strategy SSPE whose payoff configuration is feasible in the efficient coalition structure, an agreement is made in the first round.

Lemma 1 *In every pure strategy SSPE* σ *of* $\Gamma^N(\delta)$ *with* $\{(v^S, \theta^S) \mid S \subseteq N\}$ *such that the payoff configuration is feasible in the efficient coalition structure, every player* $i \in N$ *proposes in round* 1 *a solution* (S_i, y^{S_i}) *to the maximization problem:*

$$\max_{y^i, S; i \in S} \left(v(S) - \sum_{j \in S, j \neq i} y_j^i \right)$$
$$\text{subject to } y_j^i \geq \delta v_j^N, \quad \text{for all } j \in S, j \neq i. \tag{2}$$

Moreover, the proposal (S_i, y^{S_i}) *is accepted in* σ.

Proof Let $x^i = (x_1^i, \ldots, x_n^i)$ be the expected equilibrium payoff vector when player i becomes the proposer at round 1. By definition of $\Gamma^N(\delta)$, $v_i^N = \sum_{k \in N} x_i^k / n$ for all $i \in N$. We denote by m^i the maximum value of (2). We will prove that $x_i^i = m^i$.

($x_i^i \leq m^i$): Suppose that player i proposes (S, y^S) in round 1 such that $y_i^S > m^i$. Since m^i is the maximum value of (2), for some $j \in S$ with $j \neq i$, $y_j^S < \delta v_j^N$. Let j^* be the last responder of such a kind. Two cases can happen in equilibrium: (i) some responder after j^* rejects i's proposal, and (ii) otherwise. In the case of (ii), since his or her continuation payoff is $\delta v_{j^*}^N$, it is optimal for j^* to reject i's proposal. Therefore, whichever cases happens, i's proposal is rejected and the game goes on to round 2. Then, player i obtains the discounted payoff δv_i^N.

Because we are focusing on a SSPE with payoff configurations that are feasible in the efficient coalition structure, we have:

$$\sum_{j \in S_\ell^*(N)} v_j^N \leq v(S_\ell^*(N)), \quad \text{for } \ell = 1, \ldots, K^N.$$

Thus, the pair $(S_\ell^*(N), (v_j^N)_{j \in S_\ell^*(N)})$ satisfies the constraints of the problem (2). This implies that $v_i^N \leq m^i$. Since $v(\{i\}) \geq 0$ for all $i \in N$, every player i surely obtains more than zero as a payoff when i becomes the proposer. Therefore, $x_i^i \geq 0$. The responder also rejects the proposal where his or her payoff is less than zero. Thus, $x_k^i \geq 0$. We have $v_i^N \geq 0$. Hence, $\delta v_i^N \leq v_i^N \leq m^i$. Player i obtains only δv_i^N, even if he or she proposes a payoff greater than m^i. This implies $x_i^i \leq m^i$.

($x_i^i \geq m^i$): Because $(S_\ell^*(N), (v_j^N)_{j \in S_\ell^*(N)})$ is a feasible solution of the problem (2), $(S_\ell^*(N), (\delta v_j^N)_{j \in S_\ell^*(N)})$ is also a feasible solution. Therefore, $m^i \geq v_i^N \geq \delta v_i^N$. Suppose that $m^i = 0$. Then, $v_i^N = 0$, and the payoff combination $(0, (\delta v_j^N)_{j \in S_\ell^*(N) \setminus \{i\}})$ is feasible for S. Two cases are possible: (i) If $v_j^N = 0$ for all $j \in S_\ell^*(N)\backslash\{i\}$, then there exists a feasible payoff combination $(y_i, (\delta v_j^N)_{j\in S_\ell^*(N)\backslash\{i\}})$ such that $y_i > 0$, (ii) if $v_j^N > 0$ for some $j \in S_\ell^*(N)\backslash\{i\}$, then $\delta v_j^N < v_j^N$. Thus, some $(y_j, (\delta v_j^N)_{j\in S_\ell^*(N)\backslash\{i\}})$ such that $y_j > 0$ become feasible solution. Because m^i is the maximum value of (2), we must have $m^i > 0$. Any solution (S, y^S) satisfies $y_i^S = m^i$ and $y_j^S = \delta v_j^N$ for $j \in S$, $j \neq i$. For any $\varepsilon > 0$, define z^S such that:

$$z_i^S = m^i - \varepsilon, \quad z_j^S = y_j^S + \frac{\varepsilon}{|S| - 1}, \quad j \in S, \ \ j \neq i.$$

If player i proposes (S, z^S), then it is accepted by all $j \in S$, $j \neq i$. Therefore, $x_i^i \geq z_i^S = m^i - \varepsilon$. Since ε is arbitrary, we conclude $x_i^i \geq m^i$.

Finally, we show that i's proposal is accepted in round 1. It is sufficient to prove $\delta v_i^N < m^i$. Suppose that $\delta v_i^N = m^i$. It follows from $\delta v_i^N \leq v_i^N \leq m^i$ that $m^i = v_i^N = 0$. This contradicts to $m^i > 0$. □

We next present a necessary and sufficient condition for the existence of a pure strategy SSPE of $\Gamma^N(\delta)$.

Lemma 2 *For $\psi = \{(v^S, \theta^S) \mid S \subseteq N\}$ such that $v^S = (v_i^S)_{i\in S}$, $\theta^S = (T_i^S)_{i\in S}$ and the payoff configuration $\{v^S \mid S \subseteq N\}$ is feasible in the efficient coalition structure, there exists a pure strategy SSPE σ of $\Gamma^N(\delta)$ with ψ if and only if, for every $S \subseteq N$ and for every $i \in S$,*

(i) the coalition T_i^S constitutes a solution of:

$$\max_{T; i\in T, y^i} \left(v(T) - \sum_{j\in T, j\neq i} y_j^i \right)$$
$$\text{subject to } y_j^i \geq \delta v_j^S, \quad \text{for all } j \in S, \ \ j \neq i, \tag{3}$$

and

(ii) the expected payoff vector $v^S = (v_i^S)_{i\in S}$ satisfies:

$$v_i^S = \frac{1}{|S|}\left\{ v(T_i^S) - \delta \sum_{j\in T_i^S, j\neq i} v_j^S \right\} + \frac{1}{|S|}\delta \sum_{k: i\in T_k^S, k\neq i} v_i^S + \frac{1}{|S|}\delta \sum_{m: i\notin T_m^S} v_i^{S\backslash T_m^S}, \tag{4}$$

where v_i^T is defined to be zero when $T = \emptyset$.

Proof (only if): We can apply Lemma 1 to every subgame $\Gamma^S(\delta)$. Then (i) in Lemma 2 is proved. In the subgame $\Gamma^S(\delta)$, every player i makes a proposal for the payoff allocation $x^i = (x_j^i)_{j\in T_i^S}$ such that:

$$x_i^i = v(T_i^S) - \sum_{j \in T_i^S, j \neq i} \delta v_j^S, \quad x_j^i = \delta v_j^S, \quad j \in T_i^S, \ j \neq i. \tag{5}$$

Since this proposal is accepted in round 1, we can obtain (4) by the definition of $\Gamma^S(\delta)$.

(if): Define the strategy combination σ of $\Gamma^N(\delta)$ such that, in every subgame $\Gamma^S(\delta)$, every player $i \in S$ proposes a solution (T_i^S, x_i) of the problem (3) satisfying (5), and accepts any proposal (T, y^T) if and only if $y_i^T \geq \delta v_i^S$. It is easy to see that σ is a local optimal strategy. □

Let us now turn to the proof of Theorem 2. By the definition of subgame coalitional efficient SSPE, the equilibrium payoff configuration is feasible in the efficient coalition structure. Therefore, we can use Lemmas 1 and 2 to prove Theorem 2.

Proof of Theorem 2 (i) (only if). Assume that a pure strategy and limit subgame coalitional efficient SSPE of Γ^N exists. By (ii) in Lemma 2 and $T_i^S = S_\ell^*(S)$, the expected equilibrium payoff of subgame $\Gamma^S(\delta)$ satisfies that for every coalition $S \subset N$ and for every $i \in S_\ell^*(S)$:

$$v_i^S = \frac{1}{|S|}\left[v(S_\ell^*(S)) - \delta \sum_{k \in S_\ell^*(S), k \neq i} v_k^S\right] + \frac{|S_\ell^*(S)| - 1}{|S|}\delta v_i^S$$
$$+ \left[\sum_{j=1}^{K_S} \frac{|S_j^*(S)|}{|S|}\delta v_i^{S \setminus S_j^*(S)} - \frac{|S_\ell^*(S)|}{|S|}\delta v_i^{S \setminus S_\ell^*(S)}\right].$$

Note that v_i^S depends on $v_i^{S \setminus S_h^*(S)}$, $h = 1, \ldots, K^S$. On the equilibrium plays of the SSPE, a player selected as a proposer proposes coalition $S_\ell^*(S)$ (a component of the efficient coalition structure) and a solution of the maximization problem (3) in Lemma 2, and the proposal is accepted. Players in $S_\ell^*(S)$ then leave the game and the remaining players continue negotiations in the next round. Finally, some $S_m^*(S)$ remains as active players and a proposer offers $S_m^*(S)$ and a solution of the maximization problem in the subgame $\Gamma^{S_m^*(S)}(\delta)$. This proposal is accepted and the game ends. Subgames where active players consist of the components of the efficient coalition structure emerge on the equilibrium path. Therefore, we can derive the expected equilibrium payoff vector $(v_i^S)_{i \in S}$ explicitly by following a procedure such as backward induction of the subgames.

Let us start with the subgame $\Gamma^{S_\ell^*(S)}$, where $S_\ell^*(S) \in \pi^*(S)$. The expected equilibrium payoff $v_i^{S_\ell^*(S)}$ satisfies that for $i \in S_\ell^*(S)$:

$$v_i^{S_\ell^*(S)} = \frac{1}{|S_\ell^*(S)|}\left[v(S_\ell^*(S)) - \sum_{j \in S_\ell^*(S), j \neq i} \delta v_j^{S_\ell^*(S)}\right] + \frac{|S_\ell^*(S)| - 1}{|S_\ell^*(S)|}\delta v_i^{S_\ell^*(S)}. \tag{6}$$

It is easily seen that (6) has a solution $v^{S_\ell^*(S)} = (v(S_\ell^*(S))/|S_\ell^*(S)|, \ldots, v(S_\ell^*(S))/|S_\ell^*(S)|)$ for any δ. The solution is equal to the Nash bargaining solution of $(S_\ell^*(S), v(S_\ell^*(S)))$.

Stepping back one round to the subgame $\Gamma^{S_\ell^*(S)\cup S_m^*(S)}$, we have that for $i \in S_\ell^*(S)$:

$$v_i^{S_\ell^*(S)\cup S_m^*(S)} = \frac{1}{|S_\ell^*(S)\cup S_m^*(S)|}\left\{\left[v(S_\ell^*(S)) - \sum_{j\in S_\ell^*(S), j\neq i}\delta v_j^{S_\ell^*(S)\cup S_m^*(S)}\right]\right.$$
$$\left. + (|S_\ell^*(S)|-1)\delta v_i^{S_\ell^*(S)\cup S_m^*(S)} + |S_m^*(S)|\delta v_i^{S_\ell^*(S)}\right\}.$$

From the above equation system and $v_i^{S_\ell^*(S)} = v_j^{S_\ell^*(S)}$ for every $i, j \in S_\ell^*(S)$, we obtain that:

$$v_i^{S_\ell^*(S)\cup S_m^*(S)} = v_j^{S_\ell^*(S)\cup S_m^*(S)} \quad \text{for all } i, j \in S_\ell^*(S).$$

By repeatedly applying the same procedure, we can see that $v_i^S = v_j^S$ for all $i, j \in S_\ell^*(S)$.

Moreover, as $\delta \to 1$, $v_i^{S_\ell^*(S)\cup S_m^*(S)}$ converges to $v_i^{S_\ell^*(S)}$ for all $i \in S_\ell^*(S)$. It is then easy to see that for every $i \in S_\ell^*(S)$, as δ goes to 1:

$$v_i^S \;\to\; v_i^{S_\ell^*(S)} = v(S_\ell^*(S))/|S_\ell^*(S)|.$$

This implies that the expected equilibrium payoff vector converges to the Nash-bargaining-solution payoff allocation under the efficient coalition structure.

By (i) of Lemma 2 and the above discussion ((ii) of Lemma 2 as $\delta \to 1$), we have the (only if) part of Theorem 2 (i).

(if). Suppose that the condition (1) holds. From (ii) of Lemma 2, the expected payoff vector $(v_i^S)_{i\in S}$ satisfies the equations system (4). We can easily see that each δv_i^S is monotone increasing with δ and converges to the Nash bargaining solution $v(S_\ell^*(S))/|S_\ell^*(S)|$ for $i \in S_\ell^*(S)$ as δ goes to 1. Moreover, δv_i^S is continuous in δ. Therefore, for any δ sufficiently close to 1, we have the following inequalities; for $i \in S_\ell^*(S)$:

$$v(S_\ell^*(S)) - \delta \sum_{k\in S_\ell^*(S), k\neq i} v_k^S \geq \max_{T; i\in T}\left\{v(T) - \delta\sum_{k\in T, k\neq i} v_k^S\right\}. \tag{7}$$

Let us define the strategy combination σ^* of $\Gamma^N(\delta)$. In every subgame $\Gamma^S(\delta)$, every player $i \in S_\ell^*(S)$ proposes $S_\ell^*(S)$ and the payoff vector y^i such that $y_i^i = v(S_\ell^*(S)) - \sum_{k\in S_\ell^*(S), k\neq i} y_k^i$ and $y_j^i = \delta v_j^S$ for $j \in S_\ell^*(S)$. He or she accepts any proposal y_i^T if and only if $y_i^T \geq \delta v_i^S$. From the above inequalities (7) and Lemma 2, σ^* becomes an SSPE of $\Gamma^N(\delta)$. Thus, as $\delta \to 1$, we have a pure strategy and limit subgame coalitional efficient SSPE of Γ^N with the payoff configurations generated by the Nash-bargaining-solution payoff allocation under $\pi^*(S)$.

Proof of Theorem 2 (ii) As shown in the proof of the (only if) part of Theorem 2 (i), the expected equilibrium payoff vector converges to the Nash-bargaining-solution payoff allocation under the efficient coalition structure for each S. This implies Theorem 2 (ii). □

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