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SYMPOSIUM

On purification of measure-valued maps

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Abstract This paper presents new methods to obtain purification results for continuum games, which don't make use of the "many more players than strategies" assumption (Yannelis in Econ Theory (in press) 2007) or of Loeb spaces (Loeb and Sun in Illinois J Math 50, 747–762, 2006). The approach presented doesn't use non-standard analysis; it is based on standard measure theory and in particular on the super-nonatomicity notion introduced in Podczeck (J Math Econ (in press) 2007).

Keywords Games · Purification · Measure-valued maps

JEL Classification C60 · C70

1 Introduction

In rather general settings, games possess equilibria in mixed strategies. This is not so for equilibria in pure strategies, unless some convexity assumptions are made on the action sets and the preferences of players. The reason is that fixed point theorems are needed to prove equilibrium existence.

Based on the convexifying effect of large numbers (as being manifested in Liapounoff's theorem), positive results about the existence of equilibria with pure strategies were established, e.g., by Schmeidler (1973), Radner and Rosenthal (1982), Milgrom and Weber (1985), Rustichini and Yannelis (1991), Yannelis and Rustichini (1991), and Yannelis (2007). In Schmeidler (1973), Rustichini and Yannelis (1991),

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and Yannelis (2007), a convexifying effect is obtained by specifying the set of players as an atomless measure space. In Radner and Rosenthal (1982), Milgrom and Weber (1985), and Yannelis and Rustichini (1991), games with finitely many players are considered, but the environment in the game involves some uncertainty or randomness, and a convexifying effect is obtained by modelling the domain of the randomness (i.e. the state space) as an atomless probability space.

One approach to prove the existence of a pure strategy equilibrium in a game is a two step procedure: first establish the existence of a mixed strategy equilibrium, and then, using some convexifying effect of large numbers, "purify" this equilibrium, i.e. construct a pure strategy equilibrium from the mixed strategy equilibrium obtained in the first step. This approach was recently investigated by Khan et al. (2006), using the classical Dvoretsky–Wald–Wolfowitz theorem on purification of maps to spaces of probability measures on finite sets (see Dvoretsky et al. 1951, Theorem 4), a theorem which in turn is based on Liapounoff's theorem. In fact, as a corollary of the Dvoretsky–Wald–Wolfowitz theorem, Khan et al. (2006, Corollary 1) established an abstract result on purification of measure valued maps (see Remark 2 near the end of this section for a statement) and showed it to provide a unifying tool for dealing with the problem of purification of mixed strategies in finite action games, under both the setting with incomplete information as formulated by Milgrom and Weber (1985) and the setting with a continuum of players in the sense Schmeidler (1973), as well as for dealing with the related problem of symmetrization of equilibria in finite action games in distributional form in the sense of Mas-Colell (1984) and Khan and Sun (1991).

A natural question is whether the purification approach via the Dvoretsky–Wald–Wolfowitz theorem is indeed limited to the context of finite action games. In regard to this, Loeb and Sun (2006, Corollary 2.4) recently showed that this theorem and the corollary of it presented by Khan et al. (2006) remain valid for maps to spaces of Borel probability measures on arbitrary compact metric spaces, provided the domain is an atomless Loeb probability space. (See Remark 1 below for a formal statement of this result by Loeb and Sun 2006.)

By means of a counter example, it was also shown by Loeb and Sun (2006) that their purification result does not hold when, instead of being an atomless Loeb probability space, the domain space is specified to be the unit interval with Lebesgue measure. In view of this, they concluded that their results, to quote the authors, "fail without the use of Loeb measures."

In this paper, we show that the purification result of Loeb and Sun (2006) extends, in fact, to a class of probability spaces that is much larger then the class of atomless Loeb probability spaces. We call the probability spaces that make up this larger class "super-atomless." See Sect. 3 for the actual definition and for examples as well as for some equivalent notions. Here we remark the following three points. First, the definition of "super-atomless" involves only the measure algebra of a probability space. In particular, our result will show that the purification results of Loeb and Sun (2006) do not depend on any special property of an atomless Loeb probability space, but only on the properties of its measure algebra. Second, our proofs do not rely on any nonstandard analysis. Third, any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure (see the Appendix). In particular, the class of super-atomless probability spaces contains countably separated probability spaces, i.e. probability spaces that admit injective measurable mappings to any uncountable Polish space. On the other hand, it is known that no atomless Loeb probability space has the property of being countably separated (see Keisler and Sun 2002). Thus, there are super-atomless probability spaces that substantially differ from atomless Loeb probability spaces.

We also show that in order for a purification result in the sense of Loeb and Sun (2006) to hold, it is necessary that the probability space domain be super-atomless. Thus the class of super-atomless probability spaces is exactly the class of probability spaces for which such a result holds.

Remark 1 Here is the formal statement of the purification result established by Loeb and Sun (2006, Corollary 2.4) (slightly reformulated here concerning notation).

Let $(T, \mathcal{T}, \mu)$ *be an atomless Loeb probability space, let* A *be a compact metric space, and let* $M_P(A)$ *denote the set of all Borel probability measures on* A*. For each* k *in a countable set* K *let* μ_k *be a finite signed measure on* $(T, \mathcal{T})$ *that is absolutely continuous with respect to* μ*, and for each* j *in a countable set* J *let* $\varphi_j : T \times A \to \mathbb{R}$ *be a mapping such that* $\varphi_j(\cdot, a)$ *is measurable for each* $a \in A$*,* $\varphi_j(t, \cdot)$ *is continuous for each* $t \in T$*, and such that for some integrable function* $\rho_j : T \to \mathbb{R}_+$*,* $|\varphi_j(\cdot, a)| \leq \rho_j$ *for all* $a \in A$*. Then given any mapping* $f : T \to M_P(A)$ *such that* $f(\cdot)(B)$ *is measurable for each Borel subset* B *of* A*, there is a mapping* $g : T \to A$*, measurable for* $\mathcal{T}$ *and the Borel sets of* A*, such that for all* $k \in K$*,* $j \in J$*, and all Borel subsets* B *of* A*,*

(1) $\int_T \int_A \varphi_j(t, a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t) = \int_T \varphi_j(t, g(t))\mu(\mathrm{d}t)$,
(2) $\int_T f(t)(B)\mu_k(\mathrm{d}t) = \mu_k(g^{-1}(B))$,
(3) $g(t) \in \operatorname{supp} f(t)$ *for almost all* $t \in T$ *(where* $\operatorname{supp} f(t)$ *denotes the support of the measure* $f(t)$*).*

Remark 2 Corollary 1 in Khan et al. (2006) is the result quoted in the previous remark with $(T, \mathcal{T}, \mu)$ allowed to be any atomless probability space but with A required to be a finite set and with J and K both being finite sets. (Of course, if A is finite, then it suffices to assume for the functions φ_j—as in the original statement of Khan et al. (2006, Corollary 1)—that $\varphi_j(\cdot, a)$ is integrable for each $a \in A$ and each $j \in J$. We also remark in this connection that given any countable set of finite atomless signed measures μ_k on the measurable space $(T, \mathcal{T})$, there exists an atomless probability measure μ on $(T, \mathcal{T})$ such that all the μ_k's are absolutely continuous with respect to μ.) A further reduction, taking for J the empty set and dropping conclusions (1) and (3), gives the original Dvoretsky–Wald–Wolfowitz theorem.

The rest of this note is organized as follows. In the next section some notation and terminology is introduced. In Sect. 3, the notion of a "super-atomless" probability space is presented. Section 4 contains a preliminary result about weak*-measurable mappings, and Sect. 5 our purification results. In the Appendix it is shown, using a result due to Fremlin (2005), that every atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure.

2 Notation and terminology

2.1 General notation

(a) If X is a Banach space then:
- X^* denotes the dual space of X, endowed with the dual norm;
- $\langle x, x^* \rangle$ denotes the value of $x^* \in X^*$ at $x \in X$;
- (X^*, weak^*) means X^* in its weak*-topology.

(b) $\mathcal{B}(Z)$ denotes the Borel σ-algebra of a topological space Z.

(c) Let $(T, \mathcal{T}, \mu)$ be a (positive) measure space.
- For $E \in \mathcal{T}$,
  - 1_E denotes the characteristic function of E;
  - $\mathcal{T}_E$ denotes the trace σ-algebra on E, i.e. $\mathcal{T}_E \equiv \{E \cap F \colon F \in \mathcal{T}\}$;
  - μ_E denotes the subspace measure on E, i.e. the restriction of μ to $\mathcal{T}_E$.
- "$\mathcal{T}$–$\mathcal{B}(Z)$-measurable" for a mapping f from T to a topological space Z means $f^{-1}(B) \in \mathcal{T}$ for each $B \in \mathcal{B}(Z)$.

(d) A function f from a topological space Y to a topological space Z is said to be a Borel function if $f^{-1}(B) \in \mathcal{B}(Y)$ for each $B \in \mathcal{B}(Z)$.

2.2 Measure algebras[1]

Let $(T, \mathcal{T}, \mu)$ be a (positive) measure space.

(a) The *measure algebra* of μ (or of $(T, \mathcal{T}, \mu)$) is the pair $(\mathfrak{A}, \hat{\mu})$ where $\mathfrak{A}$ is the quotient Boolean algebra of $\mathcal{T}$ for the equivalence relation on $\mathcal{T}$ given by $E \sim F$ if and only if $\mu(E \vartriangle F) = 0$, and $\hat{\mu} \colon \mathfrak{A} \to [0, \infty]$ is the functional given by $\hat{\mu}(a) = \mu(E)$ where E is any element of $\mathcal{T}$ determining a, i.e. any element of $\mathcal{T}$ whose equivalence class under $\sim$ is a. In the following, for $a \in \mathfrak{A}$, the notation $a = E^\bullet$ means a is determined by $E \in \mathcal{T}$.

(b) For $a \in \mathfrak{A}$, we denote by $\mathfrak{A}_a$ the *principal ideal* in $\mathfrak{A}$ generated by a, i.e.

$$\mathfrak{A}_a = \{b \in \mathfrak{A} \colon \mu(F \setminus E) = 0 \text{ for } E, F \in \mathcal{T} \text{ with } a = E^\bullet \text{ and } b = F^\bullet\}.$$

Note that if $a = E^\bullet$ then, writing $(\mathfrak{A}_E, \hat{\mu}_E)$ for the measure algebra of the subspace measure μ_E, $\mathfrak{A}_a$ may be identified with $\mathfrak{A}_E$. In particular, $\mathfrak{A}_a$ may be viewed as a Boolean algebra in its own right.

(c) The *Maharam type* of $\mathfrak{A}$ is the least cardinal number of any subset $\mathcal{B} \subset \mathfrak{A}$ which completely generates $\mathfrak{A}$, i.e. of any $\mathcal{B} \subset \mathfrak{A}$ such that the smallest order closed subalgebra in $\mathfrak{A}$ containing $\mathcal{B}$ is $\mathfrak{A}$ itself.[2] Similarly, for any $a \in \mathfrak{A}$, viewing $\mathfrak{A}_a$ as a Boolean algebra in its own right, the Maharam type of $\mathfrak{A}_a$ is the least cardinal number of any subset $\mathcal{B} \subset \mathfrak{A}_a$ which completely generates $\mathfrak{A}_a$.

[1] For the material in this subsection, we refer to Fremlin (2002).

[2] A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is *order-closed* if, for the partial ordering $\subset^\bullet$ of $\mathfrak{A}$ given by $a \subset^\bullet b$ if and only if $\mu(E \setminus F) = 0$ for $E, F \in \mathcal{T}$ with $a = E^\bullet$ and $b = F^\bullet$, any non-empty upwards directed subset of $\mathfrak{B}$ has its supremum in $\mathfrak{B}$ in case the supremum is defined in $\mathfrak{A}$.

(d) The Maharam type of the measure space $(T, \mathcal{T}, \mu)$, or of the measure μ, is defined to be the Maharam type of $\mathfrak{A}$.

(e) Given $E \in \mathcal{T}$, the Maharam type of the subspace measure μ_E is the same as the Maharam type of $\mathfrak{A}_a$ if $a \in \mathfrak{A}$ is determined by E. This is so because, writing $(\mathfrak{A}_E, \hat{\mu}_E)$ for the measure algebra of μ_E, $\mathfrak{A}_E$ can be identified with $\mathfrak{A}_a$ if $a \in \mathfrak{A}$ is determined by E, and the Maharam type of μ_E is, by the previous definition applied to μ_E, just the Maharam type of $\mathfrak{A}_E$.

(f) $(T, \mathcal{T}, \mu)$, or the measure μ, is said to be *Maharam-type-homogeneous* if for each non-zero $a \in \mathfrak{A}$ (i.e. for each $a \in \mathfrak{A}$ with $\hat{\mu}(a) > 0$), the Maharam type of the principal ideal $\mathfrak{A}_a$ is equal to the Maharam type of μ.

(g) In terms of Maharam types, the measure μ is atomless if and only if for each non-zero $a \in \mathfrak{A}$ the Maharam type of $\mathfrak{A}_a$ is infinite. Indeed, by definition of the sets $\mathfrak{A}_a$, μ is atomless if and only if for each non-zero $a \in \mathfrak{A}$ the set $\mathfrak{A}_a$ is infinite. Now given $a \in \mathfrak{A}$, if a subset $\mathcal{B} \subset \mathfrak{A}_a$ is finite, then so is the smallest subalgebra of $\mathfrak{A}_a$ containing $\mathcal{B}$ (which follows by induction, see Fremlin (2002) 312M, p. 24), and a finite subalgebra of $\mathfrak{A}_a$ is automatically order closed in $\mathfrak{A}_a$. Thus $\mathfrak{A}_a$ is infinite if and only if its Maharam type is infinite.

3 Super-atomless probability spaces

As noted in 2.2(g) in the previous section, a measure space $(T, \mathcal{T}, \mu)$ is atomless if and only if for each non-zero element a of its measure algebra $(\mathfrak{A}, \hat{\mu})$ (i.e. each $a \in \mathfrak{A}$ with $\hat{\mu}(a) > 0$) the Maharam type of the principle ideal $\mathfrak{A}_a$ is infinite. Thus, a natural way to strengthen the condition that a measure space be atomless is to require that for each non-zero element a of its measure algebra the Maharam type of the principle ideal $\mathfrak{A}_a$ be uncountable. We call a measure space that satisfies this strengthening of non-atomicity "super-atomless."

Definition Let $(T, \mathcal{T}, \mu)$ be a measure space, with measure algebra $(\mathfrak{A}, \hat{\mu})$. The measure μ (or the measure space $(T, \mathcal{T}, \mu)$) is said to be *super-atomless* if for each non-zero $a \in \mathfrak{A}$ the principal ideal in $\mathfrak{A}$ generated by a has an uncountable Maharam type.

Remark 3 An equivalent definition of "super-atomless" (as stated in Podczeck 2007) is to say that a measure space $(T, \mathcal{T}, \mu)$ is super-atomless if for any $E \in \mathcal{T}$ with $\mu(E) > 0$ the Maharam type of the subspace measure μ_E is uncountable. (See 2.2(e) in the previous section.)

Here are some examples of super-atomless probability spaces.

- Let κ be an uncountable cardinal and let ν_κ be the usual measure on the product space $\{0, 1\}^\kappa$, i.e. ν_κ is the product measure on $\{0, 1\}^\kappa$, each of whose factors is the coin flipping measure. According to Fremlin (2002, Theorem 331K, p. 129), ν_κ is Maharam-type-homogeneous with Maharam type κ, which implies that ν_κ is super-atomless since κ is uncountable.
- Let κ be an uncountable cardinal and let λ_κ be the product measure on $[0, 1]^\kappa$, each of whose factors is Lebesgue measure on $[0, 1]$. Then λ_κ is Maharam-type-homogeneous with Maharam type κ (see Fremlin 2002, 334Y, p. 161, and note for

this reference that the Maharam type of Lebesgue measure on [0, 1] is $\aleph_0$). Thus λ_κ is super-atomless.

– Any atomless Loeb probability space is super-atomless. This fact can be deduced from the material in Jin and Keisler (2000, Sects. 2,3, and 5).

In the Appendix, we shall show, based on a result due to Fremlin (2005), that any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure. That is to say, given a Polish space Z and an atomless Borel probability measure λ on Z, there is a super-atomless measure μ on Z such that, writing $\mathcal{T}$ for the domain of μ, $\mathcal{T} \supset \mathcal{B}(Z)$ and μ agrees with λ on $\mathcal{B}(Z)$. Thus, in particular, Lebesgue measure on [0, 1] can be extended to a super-atomless measure. Note that a super-atomless probability space obtained in this way is countably separated, i.e. admits injective measurable functions to the set of real numbers and hence to any uncountable Polish space. To put this in contrast with Loeb spaces, we remark that it is shown in Keisler and Sun (2002) that if $(T, \mathcal{T}, \mu)$ is any atomless Loeb measure space, Z any Polish space, $f: T \to Z$ any $\mathcal{T} - \mathcal{B}(Z)$-measurable mapping, and ν denotes the image measure of μ under f on $\mathcal{B}(Z)$, then for ν-almost every $z \in Z$ the inverse image $f^{-1}(\{z\})$ has a cardinality at least as large as that of the continuum. In particular, there is no atomless Loeb probability space $(T, \mathcal{T}, \mu)$ so that T may be identified with the unit interval [0, 1] in such a way that the domain of μ includes the Borel σ-algebra of [0, 1]. Thus, the class of super-atomless probability spaces contains members that differ in a substantial way from atomless Loeb probability spaces.

In the proof of Theorem 1 we shall make use of the follow fact.

Fact *Let $(T, \mathcal{T}, \mu)$ be a probability space. Then the following are equivalent.*

(i) *The measure μ is super-atomless.*
(ii) *For every $E \in \mathcal{T}$ with $\mu(E) > 0$, the subspace of $L^1(\mu)$ consisting of the elements of $L^1(\mu)$ vanishing off E is non-separable.*

Proof Note first that for each $E \in \mathcal{T}$ the subspace of $L^1(\mu)$ as specified in (ii) can be identified, in terms of the subspace measure μ_E, with the space $L^1(\mu_E)$. By hypothesis, for each $E \in \mathcal{T}$, μ_E is a finite measure, so by Fremlin (2002, 365X(p), p. 349) $L^1(\mu_E)$ is non-separable if and only if the Maharam type of μ_E is uncountable. Thus the equivalence (i)⇔(ii) follows by Remark 3 above. □

Remark 4 The notions of a saturated probability space and of an $\aleph_1$-atomless probability space, which can be found in Hoover and Keisler (1984), are equivalent to the notion of a super-atomless probability space. A probability space $(T, \mathcal{T}, \mu)$ is called $\aleph_1$-*atomless* by Hoover and Keisler (1984) if $\mathcal{T}$ is atomless over any of its countably generated sub-σ-algebras, where $\mathcal{T}$ being atomless over a sub-σ-algebra $\mathcal{A} \subset \mathcal{T}$ means that given any $D \in \mathcal{T}$ with $\mu(D) > 0$ there is $D_0 \in \mathcal{T}$, with $D_0 \subset D$, such that—denoting by $\mu(D|\mathcal{A})$ and $\mu(D_0|\mathcal{A})$ the conditional probabilities given $\mathcal{A}$ of D and D_0, respectively—for some $A \in \mathcal{A}$ with $\mu(A) > 0$, $0 < \mu(D_0|\mathcal{A})(t) < \mu(D|\mathcal{A})(t)$ for almost all $t \in A$.

The probability space $(T, \mathcal{T}, \mu)$ is called *saturated*, or *rich*, if it is atomless and if given any two Polish spaces X and Y, any Borel probability measure τ on $X \times Y$, and any Borel measurable mapping $f: T \to X$ whose distribution is equal to the marginal of τ on X, there is a Borel measurable mapping $g\colon T \to Y$ such that τ is equal to the distribution of the mapping $(f, g)\colon T \to X \times Y$.

It is not hard to see that $\aleph_1$-atomless and super-atomless are equivalent properties of a probability space, and according to Hoover and Keisler (1984, Corollary 4.5), a probability space is $\aleph_1$-atomless if and only if it is saturated.

In our view, among these three equivalent notions that of a super-atomless probability space, a notion which is defined solely in measure algebraic terms, is the most transparent one, and the one which leads most easily to examples (see in particular the Appendix).

4 An abstract purification result

We find it convenient to prepare out treatment of purification of measure-valued maps by establishing some general result about weak*-measurable functions from a probability space to the dual of an arbitrary separable Banach space. Given a probability space $(T, \mathcal{T}, \mu)$ and a Banach space X, recall that a function $f\colon T \to X^*$ is said to be weak*-measurable if for each $x \in X$ the function $\langle x, f(\cdot)\rangle$ is measurable. The following lemma recalls a fact that is basic for our treatment and is needed in particular for Theorem 1 below.

Lemma 1 *Let $(T, \mathcal{T}, \mu)$ be a probability space, X a Banach space, $h\colon T \to X$ a Bochner integrable function, and $f\colon T \to X^*$ a weak*-measurable function such that* $\operatorname{ess\,sup}_{t\in T} \|f(t)\| < \infty$. *Then the function $\langle h(\cdot), f(\cdot)\rangle$ is integrable.*

Proof Recall that h being a Bochner integrable function means that h is the pointwise limit almost everywhere of a sequence of measurable simple functions and that $\int_T \|h(t)\|\,\mu(\mathrm{d}t) < \infty$. The former fact implies, by weak*-measurability of f, that $\langle h(\cdot), f(\cdot)\rangle$ is measurable for the μ-completion of $\mathcal{T}$. Now if for some $K > 0$, $\|f(t)\| \le K$ for almost all $t \in T$, then $\int_T |\langle h(t), f(t)\rangle|\,\mu(\mathrm{d}t) \le \int_T K\|h(t)\|\,\mu(\mathrm{d}t)$ whence $\int_T |\langle h(t), f(t)\rangle|\,\mu(\mathrm{d}t) < \infty$ by the second of the facts noted above. □

The following theorem can be viewed as an abstract purification result for mappings to the dual of an arbitrary separable Banach space. The fact stated in this theorem is the core of our result about measure-valued mappings.

Theorem 1 *Let $(T, \mathcal{T}, \mu)$ be a probability space, let X be a separable Banach space, let C be a convex and weak*-compact subset of X^*, and let $f\colon T \to X^*$ be a weak*-measurable function such that $f(t) \in C$ for almost all $t \in T$. Let I be a countable set, and for each $i \in I$ let $h_i\colon T \to X$ be a Bochner integrable function. Assume that μ is super-atomless. Then there exists a weak*-measurable function $\widetilde{f}\colon T \to X^*$ such that*

- (i) *$\widetilde{f}(t)$ is an extreme point of C for almost all $t \in T$;*
- (ii) $\int_T \langle h_i(t), \widetilde{f}(t)\rangle\,\mu(\mathrm{d}t) = \int_T \langle h_i(t), f(t)\rangle\,\mu(\mathrm{d}t)$ *for all $i \in I$ (all these integrals being well defined).*

Proof Note first that C is a norm-bounded subset of X^* by hypothesis. Hence, if for a function $u\colon T \to X^*$, $u(t) \in C$ for almost all $t \in T$, then this means

$\operatorname{ess\,sup}_{t\in T}\|u(t)\| < \infty$. Also note that, since h_i is Bochner integrable for each $i \in I$, if $u: T \to X^*$ is weak*-measurable with $\operatorname{ess\,sup}_{t\in T}\|u(t)\| < \infty$, then according to Lemma 1, the integral $\int_T \langle h_i(t), u(t)\rangle\, \mu(\mathrm{d}t)$ is well-defined. Thus, in particular, the integrals in (ii) are well-defined.

In the following, functions from T to X^* that agree μ-almost everywhere will not be distinguished. In particular, we identify the given function f with its μ-equivalence class. Let $L^{\infty,w^*}(\mu, X^*)$ be the vector space of all μ-equivalence classes of weak*-measurable functions $u: T \to X^*$ with $\operatorname{ess\,sup}_{t\in T}\|u(t)\| < \infty$, and let

$$B = \{u \in L^{\infty,w^*}(\mu, X^*) : u(t) \in C \text{ for almost all } t \in T\}.$$

Then $f \in B$ and we let

$$H = \left\{ u \in B\colon \int_T \langle h_i(t), u(t)\rangle\, \mu(\mathrm{d}t) = \int_T \langle h_i(t), f(t)\rangle\, \mu(\mathrm{d}t) \text{ for all } i \in I \right\}.$$

Note that since C is convex, B and hence H are convex subsets of $L^{\infty,w^*}(\mu, X^*)$. We claim that H has an extreme point. To see this, let $L^1(\mu, X)$ be the Banach space of all μ-equivalence classes of Bochner integrable functions from $(T, \mathcal{T}, \mu)$ into X, the norm being given by $\|h\|_1 = \int_T \|h(t)\|\, \mu(\mathrm{d}t)$. According to Dinculeanu (1973, Theorem 3 with $F = \mathbb{R}$), $L^1(\mu, X)^*$ can be (linearly) identified with $L^{\infty,w^*}(\mu, X^*)$ so that $u \in L^{\infty,w^*}(\mu, X^*)$ corresponds to $v \in L^1(\mu, X)^*$ if and only if $\langle g, v\rangle = \int_T \langle g(t), u(t)\rangle \mu(\mathrm{d}t)$ for all $g \in L^1(\mu, X)$ and the dual norm of v is equal to $\operatorname{ess\,sup}_{t\in T}\|u(t)\|$. Thus, as C is a norm-bounded subset of X^*, B and hence H can be considered as norm-bounded subsets of $L^1(\mu, X)^*$. We assert that H can also be viewed as a weak*-closed subset of $L^1(\mu, X)^*$. Indeed, by the hypothesis that X is separable, select a countable dense subset D of X, and for each $d \in D$ set $r_d = \sup\{\langle d, x^*\rangle : x^* \in C\}$. Then, by the Hahn-Banach theorem, since C is a convex and weak*-compact subset of X^*, and since D is dense in X, an element $x^* \in X^*$ belongs to C if and only if $\langle d, x^*\rangle \le r_d$ for all $d \in D$. Note also that given $u \in L^{\infty,w^*}(\mu, X^*)$ and $d \in D$, we have $\langle d, u(t)\rangle \le r_d$ for almost all $t \in T$ if and only if $\int_E \langle d, u(t)\rangle \mu(\mathrm{d}t) \le r_d \mu(E)$ for every $E \in \mathcal{T}$. Consequently, because D is countable, we have

$$B = \bigcap_{d\in D} \bigcap_{E\in\mathcal{T}} \left\{ u \in L^{\infty,w^*}(\mu, X^*)\colon \int_E \langle d, u(t)\rangle \mu(\mathrm{d}t) \le r_d \mu(E) \right\}.$$

This expression displays B as intersection of sets that can be considered as weak*-closed subsets of $L^1(\mu, X)^*$ (by identifying the functions $1_E d$ with the corresponding elements of $L^1(\mu, X)$), thus showing that B can be considered as a weak*-closed subset of $L^1(\mu, X)^*$. A glance at the definition of H now reveals that H can be considered as a weak*-closed subset of $L^1(\mu, X)^*$ (by identifying the functions h_i with the corresponding elements of $L^1(\mu, X)$). Summarizing, H can be (linearly) identified with a weak*-compact subset of $L^1(\mu, X)^*$, and therefore, being convex and non-empty, H has an extreme point, say $\widetilde{f}$.

We claim that $\widetilde{f}$ is an extreme point of B, too. Indeed, suppose this would not be the case. Then there is a non-zero $f_1 \in L^{\infty,w^*}(\mu, X^*)$ such that $\widetilde{f} + f_1 \in B$ as well as $\widetilde{f} - f_1 \in B$.

Since X, being separable, contains a countable subset separating the points of X^*, $f_1 \neq 0$ implies, by weak*-measurability of f_1, that we can find an $E \in \mathcal{T}$ with $\mu(E) > 0$ such that $f_1(t) \neq 0$ for almost all $t \in E$. Fix such an E and let $L^1_E(\mu)$ and $L^\infty_E(\mu)$ be the subspaces of $L^1(\mu)$ and $L^\infty(\mu)$, respectively, consisting of the elements vanishing off E. Note that the dual of $L^1_E(\mu)$ can be identified with $L^\infty_E(\mu)$. Let the subset $S \subset L^1_E(\mu)$ be defined by

$$S = \{1_E(\cdot)\, \langle h_i(\cdot), f_1(\cdot)\rangle : i \in I\}.$$

By the fact stated at the end of Sect. 3, the hypothesis that the measure μ is super-atomless implies that $L^1_E(\mu)$ is non-separable. Hence since I is countable, S cannot separate the points of $L^\infty_E(\mu)$. That is, there is a $g \in L^\infty_E(\mu)$ such that $g \neq 0$ but $\int_T \langle h_i(t), f_1(t)\rangle\, g(t)\mu(\mathrm{d}t) = 0$ for all $i \in I$. We may assume $\|g\|_\infty = 1$. Consider the element gf_1 of $L^{\infty,w^*}(\mu, X^*)$. We have $gf_1 \neq 0$ since $f_1(t) \neq 0$ for almost all $t \in E$ and since $g \in L^\infty_E(\mu)$ with $g \neq 0$. Also, since $\|g\|_\infty = 1$ and since both $\widetilde{f} + f_1 \in B$ and $\widetilde{f} - f_1 \in B$, we have $\widetilde{f} + gf_1 \in B$ as well as $\widetilde{f} - gf_1 \in B$.[3] Now for any $i \in I$,

$$\begin{aligned}\int_T \big\langle h_i(t), \widetilde{f}(t) + g(t) f_1(t)\big\rangle\, \mu(\mathrm{d}t) &= \int_T \big\langle h_i(t), \widetilde{f}(t)\big\rangle\, \mu(\mathrm{d}t) + \int_T \langle h_i(t), g(t) f_1(t)\rangle\, \mu(\mathrm{d}t) \\ &= \int_T \big\langle h_i(t), \widetilde{f}(t)\big\rangle\, \mu(\mathrm{d}t)\end{aligned}$$

because

$$\int_T \langle h_i(t), g(t) f_1(t)\rangle\, \mu(\mathrm{d}t) = \int_T \langle h_i(t), f_1(t)\rangle\, g(t)\mu(\mathrm{d}t) = 0.$$

Since $\widetilde{f} + gf_1 \in B$ and $\widetilde{f} \in H$, it follows that $\widetilde{f} + gf_1 \in H$. Similarly we may see that $\widetilde{f} - gf_1 \in H$. Thus, since $gf_1 \neq 0$, we get a contradiction to the fact that $\widetilde{f}$ is an extreme point of H, establishing the claim that $\widetilde{f}$ is an extreme point of B.

Now by Castaing and Valadier (1977, Theorem IV.15, p. 108), the fact that $\widetilde{f}$ is an extreme point of B implies that $\widetilde{f}(t)$ is an extreme point of C for almost all $t \in T$.[4] (Actually, the result just referred to is stated in terms of extreme points

[3] Recall that C is convex and note that whenever $\widetilde{f}(t) \in C$ and both $\widetilde{f}(t) + f_1(t) \in C$ and $\widetilde{f}(t) - f_1(t) \in C$, then for any number α with $0 \le |\alpha| \le 1$ (and not just for α with $0 \le \alpha \le 1$), $\widetilde{f}(t) + \alpha f_1(t) \in C$ as well as $\widetilde{f}(t) - \alpha f_1(t) \in C$.

[4] More precisely, any version of $\widetilde{f}$ has the property that at almost every $t \in T$ the value taken is an extreme point of C.

of the set of all equivalence classes, modulo null sets, of measurable selections of a correspondence from a measure space to a locally convex Suslin space. However, since X is separable, the closed unit ball in X^* is weak*-metrizable in addition to being weak*-compact, which implies that (X^*, weak^*) is a Suslin space (see Schwartz 1973, p. 96, Theorem 3). In particular, by Thomas (1975, Theorem 1), a function $u: T \to X^*$ is weak*-measurable if and only if it is $\mathcal{T} - \mathcal{B}(X^*, \text{weak}^*)$-measurable, so the set B can be considered as the set of all μ-equivalence classes of $\mathcal{T} - \mathcal{B}(X^*, \text{weak}^*)$-measurable selections of the constant-valued correspondence $t \mapsto C$ from T to 2^{X^*}. In view of these facts, as C is a convex and weak*-compact subset of X^*, Castaing and Valadier (1977, Theorem IV.15, p. 108) indeed applies, showing that $\widetilde{f}(t)$ is an extreme point of C for almost all $t \in T$.) Thus, since $\widetilde{f} \in H$, the proof of the theorem is complete. □

5 Purification of measure-valued maps

We first settle some notation.

Notation. For a compact metric space A,

- $C(A)$ denotes the Banach space of real-valued continuous functions on A with the sup-norm;
- $M(A)$ denotes the Banach space of bounded signed Borel measures on A with the variation norm;
- $M_P(A)$ denotes the subset of $M(A)$ consisting of the Borel probability measures on A;
- δ_a denotes the Dirac measure at $a \in A$.

We identify $M(A)$ with $C(A)^*$ by the Riesz representation theorem; thus for $u \in C(A)$ and $v \in M(A)$, $\langle u, v\rangle$ means $\int_A u(a)v(\mathrm{d}a)$, and the dual norm of $M(A)$ is just the variation norm. We write $M(A) \equiv C(A)^*$ whenever we wish to indicate this identification of $M(A)$ and $C(A)^*$.

The following lemma presents some equivalent notions of measurability for a mapping from a probability space to $M_P(A)$.

Lemma 2 *Let $(T, \mathcal{T}, \mu)$ be a probability space and A a compact metric space. Then for a mapping $f: T \to M_P(A)$ the following are equivalent.*

(i) *For each $B \in \mathcal{B}(A)$, the function $f(\cdot)(B)$ is measurable.*
(ii) *f (viewed as mapping to $M(A)$) is weak*-measurable, i.e., for each $u \in C(A)$ the mapping $\langle u, f(\cdot)\rangle$ is measurable.*
(iii) *f (viewed as mapping to $M(A)$) is $\mathcal{T}$–$\mathcal{B}(M(A), \text{weak}^*)$-measurable.*

Proof (i)⇒(ii) and (iii)⇒(ii) are immediate. For (ii)⇒(i), note that the set of all real-valued bounded Borel functions h on the compact metric space A such that $t \mapsto \int_A h(a) f(t)(\mathrm{d}a)$ is measurable is a vector space that is closed with respect to pointwise limits of bounded sequences (by the Lebesgue dominated convergence theorem) and thus, if it contains $C(A)$, coincides with the set of all real-valued bounded Borel functions on A; in particular, then, it contains the characteristic function of any Borel subset of A. Thus (ii)⇒(i) holds. For (ii)⇒(iii), see Thomas (1975, Theorem 1). □

In the following, if a mapping from a probability space to $M_P(A)$ is simply called "measurable," then this means measurable in the sense of the equivalent conditions as stated in Lemma 2.

Here is our main result on purification of measure-valued maps. It extends Theorem 2.2 of Loeb and Sun (2006) from the class of domain spaces that is given by the atomless Loeb probability spaces to the more general class of domains that is given by the super-atomless probability spaces.

Theorem 2 *Let $(T, \mathcal{T}, \mu)$ be a probability space, let A be a compact metric space, and let $f: T \to M_P(A)$ be a measurable function. Let J be a countable set and for each $j \in J$ let $\varphi_j: T \times A \to \mathbb{R}$ be a mapping such that $\varphi_j(\cdot, a)$ is measurable for each $a \in A$, $\varphi_j(t, \cdot)$ is continuous for each $t \in T$, and such that for some integrable function $\rho_j: T \to \mathbb{R}_+$, $|\varphi_j(\cdot, a)| \leq \rho_j$ for all $a \in A$. Suppose μ is super-atomless. Then there is a function $g: T \to A$ such that*

(a) *g is $\mathcal{T}$–$\mathcal{B}(A)$-measurable;*
(b) $\int_T \int_A \varphi_j(t, a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t) = \int_T \varphi_j(t, g(t))\mu(\mathrm{d}t)$ *for all $j \in J$*

(*all these integrals being well defined; see Remark 5 below*).

We will prove Theorem 2 by application of Theorem 1. For this, the following lemma is needed, which provides a translation of the context of Theorem 2 into that of Theorem 1.

Lemma 3 *Let $(T, \mathcal{T}, \mu)$ be a probability space, let A be a compact metric space, and let $\varphi: T \times A \to \mathbb{R}$ be a mapping such that (i) $\varphi(\cdot, a)$ is measurable for each $a \in A$, (ii) $\varphi(t, \cdot)$ is continuous for each $t \in T$, and (iii) for some μ-integrable $\rho: T \to \mathbb{R}_+$, $|\varphi(\cdot, a)| \leq \rho$ for all $a \in A$. Then the function $h: T \to C(A)$ given by $h(t) = \varphi(t, \cdot)$ is Bochner integrable.*

Proof Note first that A being a compact metric space, $C(A)$ is separable. Recall also that the closed unit ball of the dual of a separable Banach space is weak*-metrizable and that, consequently, a weak*-sequentially closed linear subspace of the dual of a separable Banach space is actually weak*-closed by the Krein-Šmulian theorem.

Let Z be the set of all $\upsilon \in M(A) \equiv C(A)^*$ for which $\langle h(\cdot), \upsilon \rangle$ is measurable. Then Z is a weak*-sequentially closed linear subspace of $M(A)$. According to condition (i), Z contains all Dirac measures on A (because $\langle h(\cdot), \delta_a \rangle = \varphi(\cdot, a)$ if δ_a is the Dirac measure at $a \in A$). Thus Z separates the points of $C(A)$, i.e. Z is weak*-dense in $M(A)$. Hence, by the previous paragraph, we must have $Z = M(A)$. That is, the function h is weakly measurable. As $C(A)$ is separable, Pettis's measurability theorem shows that h is, in fact, strongly measurable. Now if $\rho: T \to \mathbb{R}_+$ is a function chosen according to condition (iii), then we have $\|h(t)\| \equiv \sup_{a \in A} |\varphi(t, a)| \leq \rho(t)$ for almost all $t \in T$, and therefore $\int_T \|h(t)\| \mu(\mathrm{d}t) \leq \int_T \rho(t)\mu(\mathrm{d}t) < \infty$. (Since h is strongly measurable, the term $\int_T \|h(t)\| \mu(\mathrm{d}t)$ is well-defined.) But the fact that h is strongly measurable and that $\int_T \|h(t)\| \mu(\mathrm{d}t) < \infty$ means that h is Bochner integrable. □

Remark 5 Combining Lemmas 1 and 3 shows that, given a probability space $(T, \mathcal{T}, \mu)$ and a compact metric space A, if $\varphi: T \times A \to \mathbb{R}$ is as in the statement of Lemma 3 and $f: T \to M(A)$ is a weak*-measurable function such that $\operatorname{ess\,sup}_{t \in T} \|f(t)\| < \infty$, then the function $t \mapsto \int_A \varphi(t,a) f(t)(\mathrm{d}a)$ is integrable; hence, in particular, if $g: T \to A$ is $\mathcal{T}$–$\mathcal{B}(A)$-measurable, then the function $t \mapsto \varphi(t, g(t))$ is integrable. To see this latter fact, observe the following points: First, $\varphi(t, g(t))$ means the same as $\int_A \varphi(t,a)\delta_{g(t)}(\mathrm{d}a)$. Second, the function $t \mapsto \delta_{g(t)}$ is weak*-measurable since for each $u \in C(A)$, $\langle u, \delta_{g(t)}\rangle = u(g(t))$ and $u(g(\cdot))$ is measurable being the composition of two measurable mappings.

Proof of Theorem 2 Noting that $C(A)$ is separable and that $M_P(A)$ is a convex and weak*-compact subset of $M(A)$, it follows from Theorem 1, in conjunction with Lemma 3, that there is a weak*-measurable function $\widetilde{f}: T \to M(A)$ such that

(i) $\widetilde{f}(t)$ is an extreme point of $M_P(A)$ for almost all $t \in T$;
(ii) $\int_T \int_A \varphi_j(t,a)\widetilde{f}(t)(\mathrm{d}a)\mu(\mathrm{d}t) = \int_T \int_A \varphi_j(t,a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t)$ for all $j \in J$.

According to a well known fact, (i) means that for almost every $t \in T$, $\widetilde{f}(t) = \delta_a$ for some $a \in A$. Modifying $\widetilde{f}$ on some null set $N \in \mathcal{T}$ if necessary, we may assume that "$\widetilde{f}(t) = \delta_a$ for some $a \in A$" holds actually for each $t \in T$. We thus have a function $g: T \to A$ such that $\delta_{g(t)} = \widetilde{f}(t)$ for each $t \in T$. The function g is $\mathcal{T} - \mathcal{B}(A)$-measurable. Indeed, given any $B \in \mathcal{B}(A)$, we have

$$\{t \in T: g(t) \in B\} = \left\{t \in T: \delta_{g(t)}(B) = 1\right\} = \left\{t \in T: \widetilde{f}(t)(B) = 1\right\}.$$

Thus, by Lemma 2(ii)⇒(i), as $\widetilde{f}$ is weak*-measurable, the set $\{t \in T: g(t) \in B\}$ is in $\mathcal{T}$. Thus (a) of Theorem 2 holds. Finally, since

$$\varphi_j(t, g(t)) = \int_A \varphi_j(t,a)\delta_{g(t)}(\mathrm{d}a) \equiv \int_A \varphi_j(t,a)\widetilde{f}(t)(\mathrm{d}a)$$

for all $j \in J$ and all $t \in T$, (ii) implies that (b) of Theorem 2 holds. This completes the proof. □

Theorem 2 has the following corollary, which corresponds to Corollary 2.4 in Loeb and Sun (2006), the result of these authors that was quoted in the introduction, extending that result from the setting of atomless Loeb probability spaces to the setting of super-atomless probability spaces.

Corollary *Let $(T, \mathcal{T}, \mu)$ be a super-atomless probability space and let A be a compact metric space. For each k in some countable set K let μ_k be a finite signed measure on $(T, \mathcal{T})$ that is absolutely continuous with respect to μ, and for each j in some countable set J let $\varphi_j: T \times A \to \mathbb{R}$ be a mapping such that $\varphi_j(\cdot, a)$ is measurable for each $a \in A$, $\varphi_j(t, \cdot)$ is continuous for each $t \in T$, and such that for some integrable function $\rho_j: T \to \mathbb{R}_+$, $|\varphi_j(\cdot, a)| \le \rho_j$ for all $a \in A$. Then given any measurable mapping $f: T \to M_P(A)$, there is a $\mathcal{T}$–$\mathcal{B}(A)$-measurable mapping $g: T \to A$ such that $g(t) \in \operatorname{supp} f(t)$ for almost all $t \in T$ (where $\operatorname{supp} f(t)$ denotes the support of the measure $f(t)$) and such that for all $k \in K$, $j \in J$, and all Borel subsets B of A,*

(1) $\int_T \int_A \varphi_j(t,a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t) = \int_T \varphi_j(t, g(t))\mu(\mathrm{d}t)$,
(2) $\int_T f(t)(B)\mu_k(\mathrm{d}t) = \mu_k(g^{-1}(B))$.

Proof The arguments of the proof of Corollary 2.4 in Loeb and Sun (2006) apply, with the invocation of Theorem 2.2 of Loeb and Sun (2006) there replaced by an appeal to our Theorem 2. We note for this that the assumption in Corollary 2.4 of Loeb and Sun (2006) that the probability space domain be an atomless Loeb measure space matters in the proof of that corollary only in connection with the invocation of Theorem 2.2 of Loeb and Sun (2006). But our Theorem 2 does the same job as does Theorem 2.2 of Loeb and Sun (2006). □

The class of super-atomless probability spaces is exactly the class of probability spaces for which the above corollary holds when the target space A may be any compact metric space. In fact, the following theorem shows that if the domain space is not super-atomless, then this corollary may fail in several aspects.

Theorem 3 *Let A be an uncountable compact metric space and let $(T, \mathcal{T}, \mu)$ be any probability space which is not super-atomless. Then:*

(A) *There is a measurable function $f : T \to M_p(A)$ such that there is no $\mathcal{T}$–$\mathcal{B}(A)$-measurable function $g : T \to A$ for which both $g(t) \in \operatorname{supp} f(t)$ for almost all $t \in T$ and $\int_T f(t)(B)\mu(\mathrm{d}t) = \mu(g^{-1}(B))$ for each $B \in \mathcal{B}(A)$.*

(B) *There are a measurable function $f : T \to M_p(A)$ and a function $\varphi : T \times A \to \mathbb{R}$, with $\varphi(\cdot, a)$ measurable for each $a \in A$, $\varphi(t, \cdot)$ continuous for each $t \in T$, and $|\varphi(\cdot, a)| \le \rho$ for some μ-integrable function $\rho : T \to \mathbb{R}_+$ and all $a \in A$, such that there exists no $\mathcal{T}$–$\mathcal{B}(A)$-measurable function $g : T \to A$ for which both*

$$\int_T \int_A \varphi(t, a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t) = \int_T \varphi(t, g(t))\mu(\mathrm{d}t)$$

and $\int_T f(t)(B)\mu(\mathrm{d}t) = \mu(g^{-1}(B))$ for each $B \in \mathcal{B}(A)$.

(C) *There are a measurable function $f : T \to M_p(A)$ and a probability measure γ on $(T, \mathcal{T})$, with γ being absolutely continuous with respect to μ, such that there exists no $\mathcal{T}$–$\mathcal{B}(A)$-measurable function $g : T \to A$ for which both $\int_T f(t)(B)\gamma(\mathrm{d}t) = \gamma(g^{-1}(B))$ and $\int_T f(t)(B)\mu(\mathrm{d}t) = \mu(g^{-1}(B))$ for each $B \in \mathcal{B}(A)$.*

The proof of Theorem 3 requires two lemmata.

Lemma 4 *Let A be an uncountable compact metric space and $(T, \mathcal{T}, \mu)$ an atomless probability space with Maharam type $\aleph_0$. Then there is a $\mathcal{T}$–$\mathcal{B}(A)$-measurable mapping $g : T \to A$ such that given any $E \in \mathcal{T}$ there exists a $B \in \mathcal{B}(A)$ such that $\mu(E \triangle g^{-1}(B)) = 0$.*

Proof Let $Y = \{0, 1\}^{\mathbb{N}}$, endowed with its usual topology (i.e. the product topology when $\{0, 1\}$ has the discrete topology). Let ν be the restriction to $\mathcal{B}(Y)$ of the usual measure on $\{0, 1\}^{\mathbb{N}}$. Write $(\mathfrak{A}, \hat{\mu})$ for the measure algebra of μ, and $(\mathfrak{C}, \hat{\nu})$ for that of ν. Further, write $F^\bullet$ for the element of $\mathfrak{C}$ determined by $F \in \mathcal{B}(Y)$, and analogously for the elements of $\mathfrak{A}$. Note that since μ is atomless with Maharam type $\aleph_0$, μ is in particular Maharam type homogeneous. Also note that since Y is metrizable, $\mathcal{B}(Y)$

coincides with the Baire σ-algebra of Y. In view of these facts, it follows from Fremlin (2002, 341Y(c), p. 173) that there exists a $\mathcal{T}$–$\mathcal{B}(Y)$-measurable mapping $h\colon T \to Y$ such that the mapping $\psi\colon \mathfrak{C} \to \mathfrak{A}$ given by $\psi(F^\bullet) = (h^{-1}(F))^\bullet$ is a bijection. Thus, in particular, given any $E \in \mathcal{T}$ there is an $F \in \mathcal{B}(Y)$ such that $(h^{-1}(F))^\bullet = E^\bullet$. Note that $(h^{-1}(F))^\bullet = E^\bullet$ is equivalent to $\mu(E \bigtriangleup h^{-1}(F)) = 0$.

Now because A is an uncountable compact metric space, there is a Borel isomorphism from Y to A, say ζ. (That is, ζ is a bijection from Y to A such that both ζ and its inverse ζ^{-1} are measurable for $\mathcal{B}(A)$ and $\mathcal{B}(Y)$.) Let $g\colon T \to A$ be defined as the composition $g = \zeta \circ h$. Then g is $\mathcal{T}$–$\mathcal{B}(A)$-measurable. Pick any $E \in \mathcal{T}$. By construction, there is an $F \in \mathcal{B}(Y)$ such that $\mu(E \bigtriangleup h^{-1}(F)) = 0$. Let $B = \zeta(F)$. Then $B \in \mathcal{B}(A)$ and $g^{-1}(B) = h^{-1}(\zeta^{-1}(\zeta(F))) = h^{-1}(F)$, so $\mu(E \bigtriangleup g^{-1}(B)) = 0$. Thus g does the job required. □

Lemma 5 *Let $(T, \mathcal{T}, \mu)$ be a probability space and A an uncountable compact metric space. Suppose that the measure μ is not super-atomless. Then there exists a $\mathcal{T}$–$\mathcal{B}(A)$-measurable mapping $h\colon T \to A$ such that given any $E \in \mathcal{T}$ there is a $B \in \mathcal{B}(A)$ such that $\mu(E \cap h^{-1}(B)) \neq \frac{1}{2}\mu(h^{-1}(B))$.*[5]

Proof Suppose first that the probability space $(T, \mathcal{T}, \mu)$ has an atom, say F. Pick points $a_0, a_1 \in A$ with $a_0 \neq a_1$ and let $h\colon T \to A$ be the measurable mapping given by

$$h(t) = \begin{cases} a_0 & \text{if } t \in F \\ a_1 & \text{if } t \in T \setminus F. \end{cases}$$

Consider any $E \in \mathcal{T}$ with $\mu(E) > 0$. Since F is an atom of $(T, \mathcal{T}, \mu)$, either $\mu(E \cap F) = 0$ or $\mu(E \cap F) = \mu(F)$. Because $h^{-1}(\{a_0\}) = F$ and $\mu(F) > 0$, it follows that in both cases $\mu(E \cap h^{-1}(\{a_0\})) \neq \frac{1}{2}\mu(h^{-1}(\{a_0\}))$.

Now suppose $(T, \mathcal{T}, \mu)$ is atomless. Then as $(T, \mathcal{T}, \mu)$ is not super-atomless, there is an $F \in \mathcal{T}$ with $\mu(F) > 0$ such that the subspace $(F, \mathcal{T}_F, \mu_F)$ has Maharam type $\aleph_0$. Applying Lemma 4 to $(F, \mathcal{T}_F, \mu_F)$ (with μ_F temporarily normalized so that $\mu_F(F) = 1$), we can select a $\mathcal{T}_F$–$\mathcal{B}(A)$-measurable mapping $g\colon F \to A$ such that for any given $G \in \mathcal{T}_F$ there is a $B \in \mathcal{B}(Y)$ for which $\mu_F(G \bigtriangleup g^{-1}(B)) = 0$. Fix any point $a_0 \in A$ so that $\mu_F(g^{-1}(\{a_0\})) = 0$. (This is possible because A is uncountable). Let $h\colon T \to A$ be the mapping given by

$$h(t) = \begin{cases} g(t) & \text{if } t \in F \\ a_0 & \text{if } t \in T \setminus F. \end{cases}$$

Evidently h is $\mathcal{T}$–$\mathcal{B}(A)$-measurable. Consider any $E \in \mathcal{T}$. By construction, there is a $B \in \mathcal{B}(A)$ such that $\mu_F((F \cap E) \bigtriangleup g^{-1}(B)) = 0$. Thus

$$\mu_F(g^{-1}(B)) = \mu_F(F \cap E \cap g^{-1}(B)) = \mu_F(F \cap E).$$

[5] Of course, for $E \in \mathcal{T}$ with $\mu(E) \neq \frac{1}{2}$ the existence of a $B \in \mathcal{B}(A)$ with the desired property is trivially guaranteed. However, in the applications of Lemma 5, the set E which matters will be so that $\mu(E) = \frac{1}{2}$.

In addition, by the facts that $\mu_F(g^{-1}(\{a_0\})) = 0$ and $h(t) = a_0$ for all $t \in T \setminus F$, we have

$$\mu_F(g^{-1}(B)) = \mu_F(g^{-1}(B \setminus \{a_0\})) = \mu(h^{-1}(B \setminus \{a_0\}))$$

as well as

$$\mu_F(F \cap E \cap g^{-1}(B)) = \mu_F(F \cap E \cap g^{-1}(B \setminus \{a_0\})) = \mu(E \cap h^{-1}(B \setminus \{a_0\})).$$

Consequently,

$$\mu(E \cap h^{-1}(B \setminus \{a_0\})) = \mu(h^{-1}(B \setminus \{a_0\})) \quad \text{and} \quad \mu(h^{-1}(B \setminus \{a_0\})) = \mu_F(F \cap E).$$

Thus since $\mu_F(F \cap E) = \mu(F \cap E)$, if $\mu(F \cap E) > 0$ then we must have

$$\mu(E \cap h^{-1}(B \setminus \{a_0\})) \neq \frac{1}{2}\mu(h^{-1}(B \setminus \{a_0\}))$$

as desired. Now if $\mu(F \cap E) = 0$ but $\mu(E) \neq \frac{1}{2}\mu(T \setminus F)$, then

$$\mu(E \cap h^{-1}(\{a_0\})) = \mu(E \cap (T \setminus F)) = \mu(E) \neq \frac{1}{2}\mu(T \setminus F) = \frac{1}{2}\mu(h^{-1}(\{a_0\})).$$

Finally, if $\mu(E) = \frac{1}{2}\mu(T \setminus F)$ then $\mu(E \cap h^{-1}(A)) = \mu(E) = \frac{1}{2}\mu(T \setminus F) < \frac{1}{2}\mu(h^{-1}(A))$ because $\mu(F) > 0$. □

Proof of Theorem 3 (A) Let $h\colon T \to A$ be a function chosen according to Lemma 5. In particular, h is $\mathcal{T}$–$\mathcal{B}(A)$-measurable. Fix a point a_0 such that $\mu(h^{-1}(\{a_0\})) = 0$ (as is possible since A is uncountable). Let $f\colon T \to M_P(A)$ be the function given by

$$f(t) = \frac{1}{2}\delta_{h(t)} + \frac{1}{2}\delta_{a_0}, \quad t \in T.$$

Since h is $\mathcal{T}$–$\mathcal{B}(A)$-measurable, it is plain that for any $B \in \mathcal{B}(A)$ the function $f(\cdot)(B)$ is measurable. Note that

$$\int_T f(t)(\{a_0\})\mu(\mathrm{d}t) = \frac{1}{2}$$

because $\mu(h^{-1}(\{a_0\})) = 0$.

Suppose $g\colon T \to A$ is a $\mathcal{T}$–$\mathcal{B}(A)$-measurable mapping with $\mu(g^{-1}(\{a_0\})) = \frac{1}{2}$ and such that for almost every $t \in T$, $g(t) = h(t)$ or $g(t) = a_0$, i.e. such that $g(t) \in \operatorname{supp} f(t)$ for almost all $t \in T$. Set $E = T \setminus g^{-1}(\{a_0\})$ and note that $g(t) = h(t)$ for almost all $t \in E$.

Now by choice of h, there is a $B \in \mathcal{B}(A)$ with $\frac{1}{2}\mu(h^{-1}(B)) \neq \mu(E \cap h^{-1}(B))$. Since $\mu(h^{-1}(\{a_0\})) = 0$ it follows that

$$\frac{1}{2}\mu(h^{-1}(B \setminus \{a_0\})) \neq \mu(E \cap h^{-1}(B \setminus \{a_0\})).$$

As noted above, $g(t) = h(t)$ for almost all $t \in E$, and thus

$$\mu(E \cap h^{-1}(B \setminus \{a_0\})) = \mu(E \cap g^{-1}(B \setminus \{a_0\})).$$

By the definition of E, $\mu(E \cap g^{-1}(B \setminus \{a_0\})) = \mu(g^{-1}(B \setminus \{a_0\}))$. Consequently,

$$\mu(g^{-1}(B \setminus \{a_0\})) \neq \frac{1}{2}\mu(h^{-1}(B \setminus \{a_0\})).$$

But $\frac{1}{2}\mu(h^{-1}(B \setminus \{a_0\})) = \int_T f(t)(B \setminus \{a_0\})\mu(\mathrm{d}t)$ as may readily be seen, and thus we have $\int_T f(t)(B \setminus \{a_0\})\mu(\mathrm{d}t) \neq \mu(g^{-1}(B \setminus \{a_0\}))$. Thus we have shown that for no $\mathcal{T} - \mathcal{B}(A)$-measurable mapping $g\colon T \to A$ we can have both $g(t) \in \operatorname{supp} f(t)$ for almost all $t \in T$ and $\mu(g^{-1}(B)) = \int_T f(t)(B)\mu(\mathrm{d}t)$ for all $B \in \mathcal{B}(A)$. This proves (A).

(B) Let h, a_0 and f be as in the proof of part (A). Let d denote the metric of A, and let $\varphi\colon T \times A \to \mathbb{R}_+$ be given by

$$\varphi(t, a) = \min\{d(a_0, a), d(h(t), a)\}, \quad t \in T, \ a \in A.$$

Then $\varphi(t, \cdot)$ is continuous for each $t \in T$ and $\varphi(\cdot, a)$ is measurable for each $a \in A$. (To see this latter fact, note that $d(h(\cdot), a)$ is the composition of the $\mathcal{T}$–$\mathcal{B}(A)$-measurable mapping h with the continuous mapping $d(\cdot, a)$.) Furthermore, from the compactness of A, there is a number $k > 0$ such that $\varphi(t, a) \leq k$ for all $t \in T$ and $a \in A$, so there is an integrable function $\rho\colon T \to \mathbb{R}_+$ such that $\varphi(t, a) \leq \rho(t)$ for all $t \in T$ and $a \in A$.

Note that $\int_A \varphi(t, a) f(t)(\mathrm{d}a) = \frac{1}{2}\varphi(t, h(t)) + \frac{1}{2}\varphi(t, a_0) = 0$ for any $t \in T$, so $\int_T \int_A \varphi(t, a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t) = 0$. Hence if $g\colon T \to A$ is any $\mathcal{T}$–$\mathcal{B}(A)$-measurable function such that $\int_T \int_A \varphi(t, a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t) = \int_T \varphi(t, g(t))\mu(\mathrm{d}t)$, then, since φ is a non-negative function, $\varphi(t, g(t)) = 0$ for almost all $t \in T$, whence, by the definition of φ, $g(t) \in \operatorname{supp} f(t)$ for almost all $t \in T$. But according to part (A), this implies that we cannot have $\mu(g^{-1}(B)) = \int_T f(t)(B)\mu(\mathrm{d}t)$ for all $B \in \mathcal{B}(A)$. Thus (B) has been shown.

(C) Since A is an uncountable compact metric space, there is a continuous surjection from A onto the unit interval $[0, 1]$, which shows that we can find a compact subset A_1 of A such that both A_1 and its complement in A, called A_2 in the sequel, are uncountable. In particular, A_1 and A_2 both are uncountable Borel subsets of the compact metric space A and thus there is a Borel isomorphism between A_1 and A_2, say $\zeta\colon A_1 \to A_2$. (That is, ζ is a bijection such that both ζ and its inverse ζ^{-1} are measurable for the restrictions of $\mathcal{B}(A)$ to A_1 and A_2, or, equivalently, measurable for the Borel σ-algebras generated by the subspace topologies of A_1 and A_2, respectively).

Now let $h_1 : T \to A_1$ be a function chosen according to Lemma 5, and let $h_2 : T \to A_2$ be defined as the composition $h_2 = \zeta \circ h_1$. Let $f : T \to M_P(A)$ be given by

$$f(t) = \frac{1}{2}\delta_{h_1(t)} + \frac{1}{2}\delta_{h_2(t)}, \quad t \in T.$$

Finally, let $\widehat{h} : A_1 \to \mathbb{R}_+$ be a bounded and injective function that is measurable for the Borel sets of A_1 and $\mathbb{R}$, and then let γ be the (positive) measure on $(T, \mathcal{T})$ given by $\gamma(F) = \int_F (\widehat{h} \circ h_1)(t)\mu(\mathrm{d}t)$, $F \in \mathcal{T}$. By an appropriate scaling of $\widehat{h}$, we can ensure that γ is a probability measure.

Suppose there is a $\mathcal{T}$–$\mathcal{B}(A)$-measurable function $g : T \to A$ such that both

$$\int_T f(t)(B)\mu(\mathrm{d}t) = \mu(g^{-1}(B)) \quad \text{for all } B \in \mathcal{B}(A) \tag{1a}$$

and

$$\int_T f(t)(B)\gamma(\mathrm{d}t) = \gamma(g^{-1}(B)) \quad \text{for all } B \in \mathcal{B}(A). \tag{1b}$$

In particular, then, for any bounded and non-negative Borel function $h : A \to \mathbb{R}$ we must have, by the monotone convergence theorem,

$$\int_T \int_A h(a) f(t)(\mathrm{d}a)\mu(\mathrm{d}t) = \int_T h(g(t))\mu(\mathrm{d}t) \tag{2a}$$

as well as

$$\int_T \int_A h(a) f(t)(\mathrm{d}a)\gamma(\mathrm{d}t) = \int_T h(g(t))\gamma(\mathrm{d}t). \tag{2b}$$

Now let $u : A \to \mathbb{R}_+$ be given by

$$u(a) = \begin{cases} \widehat{h}(a) & \text{if } a \in A_1 \\ \widehat{h}(\zeta^{-1}(a)) & \text{if } a \in A_2, \end{cases}$$

and let $v : A \to \mathbb{R}_+$ be given by $v(a) = (u(a))^2$. Evidently u and hence v are Borel functions.[6] Observe that for each $t \in T$,

$$\begin{aligned} \int_A u(a) f(t)(\mathrm{d}a) &= \frac{1}{2}u(h_1(t)) + \frac{1}{2}u(h_2(t)) \\ &= \frac{1}{2}\widehat{h}(h_1(t)) + \frac{1}{2}\widehat{h}(\zeta^{-1}(\zeta(h_1(t)))) \\ &= \widehat{h}(h_1(t)) \\ &= u(h_1(t)) \end{aligned}$$

[6] The idea to involve the square of a function is taken from Example 2.7 in Loeb and Sun (2006).

and, similarly,

$$\begin{aligned}\int_A v(a)f(t)(\mathrm{d}a) &= \frac{1}{2}(u(h_1(t)))^2 + \frac{1}{2}(u(h_2(t)))^2\\ &= \frac{1}{2}(\widehat{h}(h_1(t)))^2 + \frac{1}{2}(\widehat{h}(\zeta^{-1}(\zeta(h_1(t)))))^2\\ &= (\widehat{h}(h_1(t)))^2\\ &= v(h_1(t)).\end{aligned}$$

Hence, from (2a) and (2b),

$$\int_T v(h_1(t))\mu(\mathrm{d}t) = \int_T v(g(t))\mu(\mathrm{d}t) \tag{3a}$$

and

$$\int_T u(h_1(t))\gamma(\mathrm{d}t) = \int_T u(g(t))\gamma(\mathrm{d}t). \tag{3b}$$

According to its definition, γ is absolutely continuous with respect to μ, with Radon-Nikodym derivative $\widehat{h} \circ h_1$. Hence, in view of the definitions of v and u (and since h_1 has range space A_1), (3b) can equivalently be written in the form

$$\int_T v(h_1(t))\mu(\mathrm{d}t) = \int_T u(g(t))u(h_1(t))\mu(\mathrm{d}t). \tag{4}$$

Using (3a) and (4), it is readily seen that

$$\int_T (u(g(t)) - u(h_1(t)))^2\mu(\mathrm{d}t) = 0$$

whence $u(h_1(t)) - u(g(t)) = 0$ for almost all $t \in T$. That is, by the definition of u, for almost every $t \in T$,

$$\begin{aligned}\widehat{h}(h_1(t)) - \widehat{h}(g(t)) = 0 \quad &\text{if } g(t) \in A_1, \text{ and}\\ \widehat{h}(h_1(t)) - \widehat{h}(\zeta^{-1}(g(t))) = 0 \quad &\text{if } g(t) \in A_2.\end{aligned}$$

Since $\widehat{h}$ was chosen to be injective, it follows that for almost all $t \in T$, either $h_1(t) = g(t)$ or $h_1(t) = \zeta^{-1}(g(t))$. Since $\zeta \circ h_1 = h_2$, this means that for almost every $t \in T$, either $g(t) = h_1(t)$ or $g(t) = h_2(t)$. Thus, setting $E = g^{-1}(A_1)$, we have $g(t) = h_1(t)$ for almost all $t \in E$, because $h_2(t) \in A_2 \equiv A \setminus A_1$ for all $t \in T$.

By choice of h_1, there is a $B \in \mathcal{B}(A_1)$ with $\frac{1}{2}\mu(h_1^{-1}(B)) \neq \mu(E \cap h_1^{-1}(B))$. Evidently $\mu(E \cap h_1^{-1}(B)) = \mu(E \cap g^{-1}(B))$ since $h_1(t) = g(t)$ for almost all $t \in E$, and since $B \subset A_1$ we have $\mu(E \cap g^{-1}(B)) = \mu(g^{-1}(B))$ by the definition of E. Consequently, $\mu(g^{-1}(B)) \neq \frac{1}{2}\mu(h_1^{-1}(B))$.

On the other hand, for the function $f: T \to M_P(A)$ defined above, we have $\int_T f(t)(B)\mu(\mathrm{d}t) = \frac{1}{2}\mu(h_1^{-1}(B))$ because $B \subset A_1$ and $h_2(t) \in A_2 \equiv A \setminus A_1$ for all $t \in T$. We conclude that $\int_T f(t)(B)\mu(\mathrm{d}t) \neq \mu(g^{-1}(B))$, thus obtaining a contradiction to (1a), and thus proving part (C) of the theorem. □

Appendix

In this Appendix, we show that any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure. Recall first that a measure space $(T, \mathcal{T}, \mu)$ is called countably separated if there is a countable subfamily $\mathcal{C} \subset \mathcal{T}$ such that given any two distinct points $t, t' \in T$ there is an $E \in \mathcal{C}$ such that $t \in E$ but $t' \notin E$. Recall also that a measure space $(T, \mathcal{T}, \mu)$ is countably separated if and only if there is a mapping from T to $\mathbb{R}$ that is both $\mathcal{T}$–$\mathcal{B}(\mathbb{R})$-measurable and injective.

Let $\mathfrak{c}$ denote the cardinality of the continuum and let $\kappa = 2^{\mathfrak{c}}$. According to Fremlin (2005, Proposition 521P(b)) there is a probability space $(T, \mathcal{T}, \mu)$ which is countably separated but Maharam type homogeneous with Maharam type κ.[7] By what has been noted in the previous paragraph, since $(T, \mathcal{T}, \mu)$ is countably separated, we can identify T with a subset of $\mathbb{R}$ such that $B \cap T \in \mathcal{T}$ for each $B \in \mathcal{B}(\mathbb{R})$. Let the σ-algebra $\widehat{\mathcal{T}}$ on $\mathbb{R}$ be defined by

$$\widehat{\mathcal{T}} = \{F \subset \mathbb{R}\colon F \cap T \in \mathcal{T}\}$$

and let $\widehat{\mu}$ be the probability measure with domain $\widehat{\mathcal{T}}$ given by $\widehat{\mu}(F) = \mu(F \cap T)$ for $F \in \widehat{\mathcal{T}}$. Evidently the measure algebra of $\widehat{\mu}$ can be identified with that of μ. Thus $\widehat{\mu}$ is Maharam type homogeneous with Maharam type κ. Also, $\mathcal{B}(\mathbb{R}) \subset \widehat{\mathcal{T}}$, and we have $\widehat{\mu}(\{r\}) = 0$ for each singleton subset $\{r\}$ of $\mathbb{R}$.

Let μ^B be the restriction of $\widehat{\mu}$ to $\mathcal{B}(\mathbb{R})$. In particular, then, $\mu^B(\{r\}) = 0$ for each $r \in \mathbb{R}$, and thus the Borel measure μ^B on $\mathbb{R}$ is atomless. Now let Z be any Polish space and let ν be any atomless Borel probability measure on Z. Since both Z and $\mathbb{R}$ are Polish spaces, and both μ^B and ν are atomless Borel probability measures, the probability spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu^B)$ and $(Z, \mathcal{B}(Z), \nu)$ are isomorphic as measure spaces (see Fremlin 2003, 433X(f), p. 183). That is, there is a bijection $\zeta: \mathbb{R} \to Z$, which is measurable for $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(Z)$ in both directions, such that $\nu(B) = \mu^B(\zeta^{-1}(B))$ for each $B \in \mathcal{B}(Z)$. Let the σ-algebra Λ on Z be defined by

$$\Lambda = \{F \subset Z\colon \zeta^{-1}(F) \in \widehat{\mathcal{T}}\}$$

and let $\widehat{\nu}$ be the probability measure with domain Λ given by $\widehat{\nu}(F) = \widehat{\mu}(\zeta^{-1}(F))$ for $F \in \Lambda$. Then since ζ is a bijection, $(\mathbb{R}, \widehat{\mathcal{T}}, \widehat{\mu})$ and $(Z, \Lambda, \widehat{\nu})$ are isomorphic as measure

[7] Actually, in the statement of Proposition 521P(b) in Fremlin (2005) it is not spoken of a probability measure that is Maharam type homogeneous. However, inspecting the proof of that proposition shows, by using Fremlin (2002, 334X(g), p. 161), that the measure constructed there is actually Maharam type homogeneous. We remark also that the proof of Proposition 521P(b) in Fremlin (2005) shows that this proposition continues to hold when κ is any cardinal with $\aleph_1 \leq \kappa \leq 2^{\mathfrak{c}}$.

spaces, so they have isomorphic measure algebras. Consequently, $\widehat{\nu}$ is Maharam type homogeneous with Maharam type κ, and thus super-atomless as $\kappa > \aleph_0$. Also, since $\widehat{\mathcal{T}} \supset \mathcal{B}(\mathbb{R})$ and $\widehat{\mu}$ coincides with μ^B on $\mathcal{B}(\mathbb{R})$, the choice of ζ implies that $\Lambda \supset \mathcal{B}(Z)$ and that $\widehat{\nu}$ coincides with ν on $\mathcal{B}(Z)$. We conclude that $\widehat{\nu}$ is an extension of ν to a super-atomless probability measure.

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