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RESEARCH ARTICLE

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Tax schemes in a class of differential games

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Abstract This paper evaluates tax schemes in a class of differential games. The results indicate that there are many tax schemes that support efficient resource usage, but each may fail to implement the targeted resource because of the multiplicity of equilibria. Since all of the equilibria are subgame perfect, it is difficult to predict which specific one arises. Care must then be taken in using a tax scheme as a remedy for the "tragedy of the commons." The advantages of other policy instruments (including command-and-control regulation and a tradable permit system) are also discussed.

Keywords Common property resource · Tax · Markov-perfect Nash equilibrium · Indeterminacy

JEL Classification Numbers C72 · H23 · Q58

1 Introduction

Consider a resource-based economy with a finite number of identical infinitely lived agents harvesting a single freely available natural resource. Assume there is also a benevolent government that enforces the law. The use of this natural resource is inefficient, so the government seeks policy to encourage more efficient usage.

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Since the problem is intertemporal, and given that strategic interaction among users can play an important role in resource usage, differential game theory is an effective tool for evaluating policy measures. Even though there are a number of studies that characterize Nash equilibria, including a Pareto efficient equilibrium (e.g., Benhabib and Radner 1992; Dutta and Sundaram 1993a,b; Levhari and Mirman 1980; Tolwinski et al. 1986), few address policy measures themselves. In fact, tax subsidies and other incentive schemes in a differential game framework are studied only by Clemhout and Wan (1985), Sorger (2006), and Krawczyk and Tidball (2006).

These particular studies employ parametric models and examine a specific form of tax scheme. In contrast, this paper provides conditions for a tax scheme to support efficient resource usage in a general class of differential games. Several tax schemes that support efficient resource usage are presented. We demonstrate how each tax scheme may fail to implement the targeted resource usage because of the multiplicity of equilibria. This is a phenomenon that Clemhout and Wan (1994b) term "the dual indeterminacy property of the Markovian–Nash equilibrium," that is, the nonuniqueness of a tax scheme and the multiplicity of equilibria under a tax scheme. This latter indeterminacy, i.e., the existence of multiple equilibria, can be a serious problem for the policymaker, since one cannot ensure efficient resource usage with a tax scheme.

We focus on the strategic interaction among the agents and its remedy. To this end, we assume that the government has perfect information in the sense that agents' characteristics and actions are observable, whereas the agents do not respond strategically to the government, taking governmental intervention as given. We also abstract from uncertainty.

The rest of this paper is organized as follows. Section 2 introduces the tax scheme proposed by Clemhout and Wan (1985). Section 3 generalizes the tax scheme and derives the necessary and sufficient conditions for a tax scheme to support efficient resource usage. Section 4 illustrates the "dual indeterminacy property of the Markovian–Nash equilibrium." In Sect. 5, other policy instruments are introduced and compared with the tax scheme. Section 6 presents some concluding remarks. The proof of selected propositions and some auxiliary results are dealt with in the Appendix.

2 CW Tax

Clemhout and Wan (1985) study a differential game for a single nonrenewable resource with $n(\geq 2)$ identical infinitely lived players. The noncooperative problem of player $i \in N = \{1, \dots, n\}$ is:

$$\max_{h_i(t)\geq 0} \int_0^\infty \ln[h_i(t)] e^{-\rho t} \mathrm{d}t, \quad \rho > 0$$

$$\text{subject to } \dot{x}(t) = -h_i(t) - \sum_{j \in N\setminus\{i\}} \sigma_j[x(t)],\ x(t) \geq 0,\ x(0) = x_0 > 0 \text{ given,}$$

where $x(t)$ is the resource stock at time t, $h_i(t)$ is the amount harvested by player i at t, and $\sigma_j(x)$ is the harvest strategy of the other player j. If all other players use strategy $\sigma^*(x) = \rho x$, the same strategy is optimal also for player i. Therefore, strategy $\sigma^*(x)$ constitutes a Nash equilibrium. On the other hand, all players gain more if they use $\sigma^c(x) = (\rho/n)\, x$, since $\sigma^c(x)$ is the unique optimal strategy for the associated cooperative problem:

$$\max_{h(t)\geq 0} \int_0^\infty \ln[h(t)]e^{-\rho t}\mathrm{d}t \quad \text{subject to } \dot{x}(t) = -nh(t),\ x(t) \geq 0,$$
$$x(0) = x_0 > 0 \text{ given.}$$

Thus, the equilibrium strategy is inefficient.[1]

To correct this, Clemhout and Wan (1985) propose a tax-subsidy scheme. Let function $T\colon \mathbb{R}^2_+ \to \mathbb{R}$ stand for a tax-subsidy scheme. If the resource stock is x and if player i extracts the resource by h_i, the player is taxed (or subsidized) so that the net harvest becomes $h_i - T(h_i, x)$. Their tax scheme takes the form:

$$T(h_i, x) = \left[\frac{h_i}{x} - \frac{\rho}{n}\exp\left(\frac{h_i}{\rho x} - \frac{1}{n}\right)\right] x. \tag{2.1}$$

We refer to the tax scheme in Clemhout and Wan (1985) as the CW Tax.

Suppose that the government implements the CW Tax and players other than i adopt the cooperative strategy $\sigma^c(x)$. Then, the noncooperative problem of player i becomes:

$$\max_{\beta_i(t)\geq 0} \int_0^\infty \ln\left[\frac{\rho}{n}x(t)\exp\left(\frac{\beta_i(t)}{\rho} - \frac{1}{n}\right)\right] e^{-\rho t}\mathrm{d}t$$
$$\text{subject to } \dot{x}(t) = -\left[(n-1)\left(\frac{\rho}{n}\right) + \beta_i(t)\right]x(t),\ x(0) = x_0 > 0 \text{ given,}$$

where $\beta_i(t) = h_i(t)/x(t)$. By Proposition A.1 in the Appendix, the optimal control is a positive constant if a solution exists.[2] With $\beta_i(t) = \beta$, the objective functional is calculated as:

$$\int_0^\infty \left(\ln\frac{\rho x(0)}{n} - 1\right) e^{-\rho t}\mathrm{d}t.$$

The value of this objective functional is independent of β. This implies that with the CW Tax, cooperative strategy $\sigma^c(x)$ constitutes a Nash equilibrium. Note, also, that the CW Tax is designed to satisfy $T(\sigma^c(x), x) = 0$ so that the government can satisfy a balanced budget condition when all players use the cooperative strategy. In the next section, we generalize this tax scheme.

[1] For the derivation of σ^* and σ^c, apply Propositions A.1 and A.2 in the Appendix. ($\eta = 1$. $\varphi(\beta) = \beta$ for σ^* and $\varphi(\beta) = \beta/n$ with $\beta = nh/x$ for σ^c.) The optimality follows from Proposition A.3 in the Appendix.

[2] Control $\beta(t) = e^{\rho t}$ makes the player better off than $\beta(t) = \beta$. This implies that, as seen later in Proposition 4.1, for a solution to exist we need restrictions on the set of feasible controls.

3 Generalization

To generalize the CW Tax, we consider a class of differential games including that introduced above. A game of this class is described by a quartet $\Gamma = (n, u, \rho, f)$, where $n \geq 2$ is the number of identical players, u: $\mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is the instantaneous utility function, $\rho > 0$ is the discount rate, and f: $\mathbb{R}_+ \to \mathbb{R}$ is the natural growth function of the resource stock. We make the following assumptions:

(U.1) u is strictly increasing, strictly concave, and smooth.
(T.1) f is concave and smooth. $f(0) = 0$.

Associated with Γ, a noncooperative problem of player $i \in N$ is written as:

$$\max_{h_i(t)\geq 0} \int_0^\infty u\left[h_i(t)\right] e^{-\rho t} \mathrm{d}t$$

$$\text{subject to } \dot{x}(t) = f\left[x(t)\right] - \sum_{j\in N\setminus\{i\}} \sigma_j\left[x(t), t\right] - h_i(t),\ x(t) \geq 0,$$

$$x(0) = x_0 > 0 \text{ given.} \tag{3.1}$$

Throughout this paper, the optimality is understood in the sense of the catching-up criterion. That is, if a feasible control path $h_i^*(\cdot)$ satisfies $\liminf_{T\to\infty} \int_0^T \{u\left[h_i^*(t)\right] e^{-\rho t} - u\left[h_i(t)\right] e^{-\rho t}\}\mathrm{d}t \geq 0$ for any feasible $h_i(\cdot)$, then $h_i^*(\cdot)$ is a solution of (3.1). If optimal control $h_i^*(\cdot)$ exists and the integral in (3.1) converges with $h_i^*(\cdot)$, then $h_i^*(\cdot)$ maximizes the objective functional in the usual sense. See, for example, Dockner et al. (2000, Chapter 3) for the relationship of catching-up optimality with other criteria for an infinite-horizon optimal control problem.

A strategy $\sigma_i(x, t)$ is called a Markovian strategy, because the induced actions are determined only by the current state. If a profile of Markovian strategies $(\sigma_1^*(x, t), \ldots, \sigma_n^*(x, t))$ is a subgame perfect equilibrium, the equilibrium is called a Markov perfect Nash equilibrium (Dockner et al. 2000, Chapter 4). Formally:

Definition A profile of Markovian strategies $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Markovian Nash equilibrium, if for each i an optimal control path $h_i^*(\cdot)$ exists and is given by $h_i^*(t) = \sigma_i^*(x^*(t; x_0), t)$, where $x^*(\cdot)$ is the path induced by strategy profile σ^* and initial state $x(0) = x_0$. Furthermore, if for each $\tau \geq 0$ the restriction of σ^* to $t \geq \tau$ is also a Markovian Nash equilibrium for any initial state $x(\tau)$ reachable from x_0 at time τ, then the equilibrium is subgame perfect and is called a Markov-perfect Nash equilibrium (in short, MPNE).

We focus only on stationary and symmetric MPNE, as constituted by a time-independent and identical Markovian strategy $\sigma_i^*(x, t) = \sigma^*(x)$ for all $i \in N$. Correspondingly, the cooperative problem is defined in the class of symmetric strategies:

$$V(x) = \max_{h(t)\geq 0} \int_0^\infty u(h(t))e^{-\rho t}\mathrm{d}t$$

$$\text{subject to } \dot{x}(t) = f(x(t)) - nh(t),\ x(t) \geq 0,\ x(0) = x > 0. \tag{3.2}$$

We assume that there exists a solution $(x^c(t; x), h^c(t; x))$ for the cooperative problem from any $x > 0$ so that the cooperative value function $V : \mathbb{R}_{++} \to \mathbb{R}$ is well defined. We refer to $(x^c(t; x), h^c(t; x))$ as a cooperative solution. A function $\sigma^c : \mathbb{R}_+ \to \mathbb{R}_+$ is a cooperative strategy if $\sigma^c(0) = 0$ and for each $x > 0$, $h^c(t; x) = \sigma^c[x^c(t; x)]$ for almost all $t \geq 0$. We assume that σ^c is interior in the sense that $(x^c(t; x), \sigma^c[x^c(t; x)]) > 0$ for all $t \geq 0$ and for all $x > 0$.[3] Then, by Benveniste and Scheinkman (1979), V is continuously differentiable and satisfies:

$$V'(x) = \frac{u'(\sigma^c(x))}{n}. \tag{3.3}$$

Let $T : \mathbb{R}^2_+ \to \mathbb{R}$ be a tax-subsidy scheme as before. For a tax scheme T to support a cooperative solution, the following conditions are necessary and sufficient:

1. For any initial stock $x > 0$, $h(t) = \sigma^c[x^c(t; x)]$ is the optimal control for:

$$V(x) = \max_{h(t)\geq 0} \int_0^\infty u[h(t) - T(h(t), x(t))]e^{-\rho t}\mathrm{d}t$$
$$\text{subject to } \dot{x}(t) = f(x(t)) - (n-1)\sigma^c[x(t)] - h(t),\ \ x(t) \geq 0,\ \ x(0) = x. \tag{3.4}$$

2. $T(\sigma^c(x), x) = 0$ for all $x \geq 0$.

Note that with these conditions, the value function V in (3.4) is the same as the cooperative value function in (3.2). The HJB equation turns out as (3.5) because the problem in hand is an infinite-horizon autonomous program:

$$\begin{aligned} \rho V(x) &= u[\sigma^c(x)] + V'(x)[f(x) - n\sigma^c(x)] \\ &\geq u[h - T(h, x)] + V'(x)[f(x) - (n-1)\sigma^c(x) - h], \\ &\quad \text{all } h \text{ such that } \min\{h, h - T(h, x)\} \geq 0. \end{aligned} \tag{3.5}$$

Now we may characterize a tax scheme T that supports the efficient use of the resource.

Lemma 3.1 *Consider a game* $\Gamma = (n, u, \rho, f)$ *satisfying* $(U.1)$ *and* $(T.1)$. *Let* σ^c *be a cooperative strategy. If a tax scheme* T *implements a cooperative solution, as an MPNE, then the following three conditions are satisfied*:

1. $T(\sigma^c(x), x) = 0,$ (3.6)
2. $T_h(\sigma^c(x), x) = 1 - 1/n,$ *and* (3.7)
3. *For each* $x > 0$, $u[h - T(h, x)]$ *is concave as a function of* h *in a neighborhood of* $h = \sigma^c(x)$.

Proof Condition 1 is a balanced budget condition. Condition 2 follows from (3.3) and (3.5). Condition 3 follows from (3.5). □

[3] In what follows, we study parametric models, for which it is easily verified that a cooperative solution exists and satisfies this interiority condition.

For the following discussion, define the Hamiltonian associated with the problem (3.4) as:

$$\tilde{H}(h, x, \lambda) = u(h - T(h, x)) + \lambda[f(x) - (n-1)\sigma^c(x) - h]. \tag{3.8}$$

Throughout this paper, the Hamiltonians and costate variables are expressed in terms of current value. By (3.3), the costate variable that supports a cooperative solution satisfies $\lambda(t; x) = V'(x^c(t; x)) > 0$. If the maximized Hamiltonian $\tilde{H}^* : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ is well defined, then:

$$\tilde{H}^*(x, \lambda) = \max_{h \geq 0} \tilde{H}(h, x, \lambda). \tag{3.9}$$

We now describe the sufficient conditions under which tax scheme T can implement a cooperative solution as an MPNE.

Lemma 3.2 *Consider a game $\Gamma = (n, u, \rho, f)$ satisfying $(U.1)$ and $(T.1)$. Tax scheme $T(h, x)$, satisfying (3.6) and (3.7) in Lemma 3.1, implements a cooperative solution $(x^c(t; x), h^c(t; x))$ as a symmetric MPNE, if $u[h - T(h, x)]$, as a function of h, is concave and if either of the following two conditions are satisfied*:

1. *$V(x)$ is bounded from below, i.e.,* $\inf\{V(x)|x > 0\} > -\infty$.
2. *$\tilde{H}^*(x, \lambda)$ is well defined and concave in x. For each initial stock $x > 0$, the transversality condition,* $\lim_{t\to\infty} e^{-\rho t} V'[x^c(t; x)]x^c(t; x) = 0$, *is satisfied.*

Proof See Theorem 3.3 and Lemma 3.1 in Dockner et al. (2000). □

Remark 3.1 Later in the paper, Conditions 1 and 2 in Lemma 3.2 are used. Other sufficiency conditions are also applicable. See, for example, Dockner et al. (2000, Chapter 3) and Sorger (1989).

We conclude this section by arguing that there may be a continuum of tax schemes that implement a cooperative solution as an MPNE. An example is offered in the next section.

Proposition 3.1 *Suppose that game $\Gamma = (n, u, \rho, f)$ and two different tax schemes T^1 and T^2 satisfy the conditions in Lemma 3.1. A tax scheme $T^\mu = \mu T^1 + (1 - \mu) T^2$ ($\mu \in [0, 1]$) implements a cooperative solution as an MPNE, if either*:

1. *T^1 and T^2 are convex in h and Condition 1 in Lemma 3.2 holds, or*
2. *T^1 and T^2 are jointly convex in h and x, $f(x) - (n-1)\sigma^c(x)$ is concave as a function of x, and Condition 2 in Lemma 3.2 holds with T^1 and T^2.*

Proof We show that T^μ satisfies the conditions in Lemma 3.2. Since the other conditions obviously hold, we only examine the well-definedness of the maximized Hamiltonian associated with T^μ in Case 2. Assume not. Since the Hamiltonian is strictly concave in h, this is the case in which there exist $\mu = \tilde{\mu} \in (0, 1)$ and a pair $(x, \lambda) = (\tilde{x}, \tilde{\lambda}) > 0$ such that $\lim_{h\to\infty} \partial u[h - T^{\tilde{\mu}}(h, \tilde{x})]/\partial h = \varepsilon \geq \tilde{\lambda}$. On the other hand, since $\tilde{H}^*(\tilde{x}, \tilde{\lambda})$ is well defined with T^k $(k = 1, 2)$, there are $\varepsilon' > 0$ and $h^k > 0$ such that $\partial u[h^k - T^k(h^k, x)]/\partial h = \varepsilon' < \varepsilon$. Let $h^* = \max\{h^1, h^2\}$.

Pick an arbitrary $\tilde{h} > 0$ and let $\tilde{k} = \arg\min\{T^k(\tilde{h}, \tilde{x}) | k = 1, 2\}$. Then, from the concavity of u in h and from $T^{\tilde{\mu}}(\tilde{h}, \tilde{x}) \geq T^{\tilde{k}}(\tilde{h}, \tilde{x})$,

$$\begin{aligned}\varepsilon'(\tilde{h} - h^*) &\geq u[\tilde{h} - T^{\tilde{k}}(\tilde{h}, \tilde{x})] - u[h^* - T^{\tilde{k}}(h^*, \tilde{x})] \\ &\geq u[\tilde{h} - T^{\tilde{\mu}}(\tilde{h}, \tilde{x})] - u[h^* - T^{\tilde{k}}(h^*, \tilde{x})] \\ &\geq \varepsilon(\tilde{h} - h^*) + u[h^* - T^{\tilde{\mu}}(h^*, \tilde{x})] - u[h^* - T^{\tilde{k}}(h^*, \tilde{x})].\end{aligned}$$

Thus, we have:

$$(\varepsilon' - \varepsilon)(\tilde{h} - h^*) \geq \min_{k=1,2} \{u[h^* - T^{\tilde{\mu}}(h^*, \tilde{x})] - u[h^* - T^k(h^*, \tilde{x})]\}.$$

However, a sufficiently large $\tilde{h}$ violates this inequality. We obtain a contradiction. □

4 Multiple equilibria

Using (3.6) and (3.7) in Lemma 3.1, we can construct tax schemes as the CW Tax. First, suppose that T is additive: $T(h, x) = \alpha h + \beta \sigma^c(x)$. Immediately we have $\alpha = -\beta = 1 - 1/n$ and, thus:

$$T(h, x) = h - \frac{1}{n}\left[h + (n-1)\sigma^c(x)\right]. \tag{4.1}$$

Second, suppose that T contains a power function: $T(h, x) = h - h^\alpha \left[\sigma^c(x)\right]^\beta$. We have by (3.6) $\alpha + \beta = 1$ and by (3.7) $\alpha = 1/n$, so that:

$$T(h, x) = h - h^{\frac{1}{n}} \left[\sigma^c(x)\right]^{1-\frac{1}{n}}. \tag{4.2}$$

Finally, the following is a generalization of the CW Tax (2.1):

$$T(h, x) = h - \sigma^c(x) \exp\left[\frac{1}{n}\left(\frac{h}{\sigma^c(x)} - 1\right)\right]. \tag{4.3}$$

We refer to (4.1), (4.2), and (4.3) as the Convex Combination (CC) Tax, the Cobb–Douglas (CD) Tax, and the generalized CW (GCW) Tax, respectively.

We demonstrate that these tax schemes may implement efficient resource usage as an MPNE, but they may also allow inefficient equilibria. This section also illustrates that efficient resource usage is supported by countless tax schemes given as convex combinations of the CC and CD Taxes.

For game $\Gamma = (n, u, \rho, f)$, assume that u is isoelastic and f is linear:

(U.2) For all $c > 0$, $-cu''(c)/u'(c) = \eta > 0$.
(T.2) $f(x) = ax$, $a \in \mathbb{R}$.

By Propositions A.1–3 in the Appendix, a cooperative strategy $\sigma^c(x)$ is linear and given by:

$$\sigma^c(x) = b^c x = \frac{\rho - (1-\eta)a}{\eta n} x. \tag{4.4}$$

To ensure $b^c > 0$, we need an assumption for parameter values:
(P.1) $\rho - (1-\eta)a > 0$.

We consider the strategy spaces of linear and piecewise continuous functions.

4.1 Linear strategy

A strategy $\sigma(x) = bx$ $(b > 0)$ is called a linear strategy. Suppose that one of the tax schemes (4.1), (4.2), or (4.3) is implemented and the opponents of player i use a linear strategy $\sigma_j^*(x) = b^*x$ $(j \in N\backslash\{i\})$. Then, the noncooperative problem for i is written as:

$$\max_{\beta_i(t)\geq 0} \int_0^\infty u\left(x\varphi[\beta_i(t)]\right) e^{-\rho t} \mathrm{d}t \tag{4.5}$$

$$\text{subject to } \dot{x}(t) = [a - (n-1)b^* - \beta_i(t)]x(t), \;\; x(0) > 0 \text{ given,}$$

where the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by:

$$\varphi(\beta) = \begin{cases} \frac{1}{n}\left[\beta + (n-1)b^c\right] & \text{for the CC Tax,} \\ \beta^{\frac{1}{n}}(b^c)^{1-\frac{1}{n}} & \text{for the CD Tax,} \\ b^c \exp\left[\frac{1}{n}\left(\frac{\beta}{b^c} - 1\right)\right] & \text{for the GCW Tax.} \end{cases} \tag{4.6}$$

Proposition A.1, in the Appendix, is applicable to this problem. That is, against the opponents' linear strategy, i's best reply is also a linear strategy. If $\beta_i(t) = b^*$ is an optimal control, then b^*x constitutes an MPNE. Furthermore, if $b^* = b^c$, the MPNE is efficient. The following proposition shows that tax schemes (4.1), (4.2), and (4.3) can implement an efficient MPNE, but in the cases of the CC and GCW Taxes there is a continuum of MPNE under a particular set of parameter values.

Proposition 4.1 *Consider a game* $\Gamma = (n, u, \rho, f)$ *satisfying* $(U.1)$, $(U.2)$, $(T.2)$, *and* $(P.1)$.

(a) *The cooperative strategy* $\sigma^c(x) = b^c x$ *is an MPNE strategy if any of the following cases apply*:
   1. *The CC Tax is implemented.*
   2. *The CD Tax is implemented.*
   3. *The GCW Tax is implemented with the following restrictions on harvest ratios*:[4]
      (i) *When* $\eta = 1$: *There exist* $\xi < \rho$ *and* $\zeta > 0$ *such that* $\beta(t) < \zeta e^{\xi t}$ *for almost all* $t \geq 0$.
      (ii) *When* $\eta > 1$: $\beta(t)$ *is absolutely continuous and there exists* $\tilde{Z} > 0$ *such that*

$$\left| \frac{d^2 \exp\left[\int_0^t a - (n-1)\, b^c - \beta(s) ds\right]}{\mathrm{d}t^2} \right| < \tilde{Z} \textit{ for almost all } t \geq 0.$$

(b) *Any linear strategy* $\sigma(x) = b^*x$ $(b^* > 0)$ *constitutes a symmetric MPNE if either*:
   1. $\eta + n^{-1} = 1$ *and the CC Tax is implemented*, *or*
   2. $\eta = 1$ *and the GCW Tax is implemented subject to the above restriction.*

[4] The growth conditions are necessary to ensure the existence of the MPNE.

Proof See the Appendix. □

Remark 4.1 Under the assumptions in Proposition 4.1, Condition 2 in Proposition 3.1 is satisfied for the CC and CD Taxes. Therefore, there is a continuum of tax schemes that support efficient resource usage:

$$T^{\mu}(h,x) = h - \frac{\mu}{n}\left[h + (n-1)\sigma^c(x)\right] - (1-\mu)\, h^{\frac{1}{n}} \left[\sigma^c(x)\right]^{1-\frac{1}{n}}, \quad \mu \in [0,1].$$

Remark 4.2 When $0 < \eta < 1$, the GCW Tax cannot implement a symmetric MPNE, as shown in Lemma B.3 in the Appendix.

4.2 Most rapid extinction strategy

The multiplicity of equilibria shown in Proposition 4.1 is not robust because it disappears with a slight perturbation of the parameters. By considering a piecewise continuous strategy, we have an example of robust multiple equilibria. Assume that harvest rates for each player are bounded from above:
(T.3) There is $\hat{h} > 0$ such that $h_i(t) \in [0, \hat{h}]$ for all $t \in [0, \infty)$ and all $i \in N$.
For analytical simplicity, let the resource be nonrenewable. So, Assumption (T.2) is specified:
(T.2') $f(x) = 0$ for all $x \geq 0$.

The most rapid extinction (in short, MRE) strategy $\sigma_E : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as:

$$\sigma_E(x) = \begin{cases} \hat{h} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We consider a situation where the MRE strategy constitutes a symmetric MPNE and examine how a tax scheme can/cannot exclude this strategy from the equilibria. Note that the MRE strategy never constitutes an MPNE unless the utility function u is bounded from below. Thus, we can specify assumption (U.2) as follows:
(U.2') $u(c) = c^{1-\eta}$, $\eta < 1$.
Notice that with this assumption, the GCW Tax does not work as stated in Remark 4.2 below Proposition 4.1. We will only consider the CC and CD Taxes.

Let $x_0 > 0$ be the initial stock of the resource. When all players adopt the MRE strategy, the associated trajectory of the state $\hat{x}(t; x_0)$ is written as:

$$\hat{x}(t; x_0) = \begin{cases} x_0 - n\hat{h}t & \text{if } t \in [0, t_E(x_0)], \\ 0 & \text{if } t \geq t_E(x_0), \end{cases} \tag{4.7}$$

where $t_E(\cdot)$ is the extinction time of the resource, defined by:

$$t_E(x) = \frac{x}{n\hat{h}}. \tag{4.8}$$

We first show a necessary and sufficient condition for the MRE strategy to constitute an MPNE.

Lemma 4.1 *Consider a game* $\Gamma = (n, u, \rho, f)$ *satisfying* (*U*.2'), (*T*.2'), *and* (*T*.3). *The MRE strategy* $\sigma_E(x)$ *constitutes a symmetric MPNE if and only if*:

$$1 - \eta - n^{-1} \geq 0. \tag{4.9}$$

Proof Apply the proof of Sorger (1998, Theorem 2(a)). □

We now show that a game Γ with a CC Tax has two equilibria: a cooperative equilibrium and the MRE strategy equilibrium. Notice that as a special case of Proposition 4.1 (or as an application of Lemma 3.2 with Condition 1), the cooperative strategy constitutes an MPNE with the CC or CD Tax.[5]

Proposition 4.2 *Consider a game* $\Gamma = (n, u, \rho, f)$ *satisfying* (*U*.2'), (*T*.2'), (*T*.3), *and* (4.9). (*a*) *If the initial stock* x_0 *satisfies*:

$$x_0 \leq \frac{(1 - \eta - n^{-1})n\eta\hat{h}}{(1 - n^{-1})\rho}, \tag{4.10}$$

then the MRE strategy still constitutes a symmetric MPNE after the CC Tax is implemented. On the other hand, (*b*) *the CD Tax always precludes the MRE strategy for equilibrium strategies.*

Proof See the Appendix. □

4.3 Piecewise continuous strategy

Finally, we show that the CD Tax is not immune to multiple equilibria either. In fact, we see below a continuum of equilibrium strategies in the space of piecewise continuous functions. For analytical simplicity, we still assume that the resource is nonrenewable, but ignore the upper limits of the harvest rates.

Proposition 4.3 *Consider a game* $\Gamma = (n, u, \rho, f)$ *satisfying* (*U*.2') *and* (*T*.2'), *and suppose that the CD Tax is implemented. Then, the following strategy constitutes a symmetric MPNE*:

$$\sigma^*(x) = \begin{cases} (x/n)\left[(\rho/\eta) + \gamma x^{-n}\right] & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad \gamma \geq 0.$$

Proof See the Appendix. □

The equilibrium path is described as:

$$x(t; x_0) = \begin{cases} \left\{\left[(x_0)^n + \frac{\gamma\eta}{\rho}\right]\exp\left(-\frac{n\rho}{\eta}t\right) - \frac{\gamma\eta}{\rho}\right\}^{\frac{1}{n}} & \text{for } 0 \leq t < \frac{\eta}{n\rho}\ln\left[\frac{\rho(x_0)^n}{\gamma\eta} + 1\right], \\ 0 & \text{for } t \geq \frac{\eta}{n\rho}\ln\left[\frac{\rho(x_0)^n}{\gamma\eta} + 1\right]. \end{cases}$$

In the case of an inefficient equilibrium ($\gamma > 0$), players harvest more as a resource becomes scarcer once the resource stock falls short of $\left[(n-1)\eta\gamma/\rho\right]^{1/n}$. This critical level is eventually reached, since the resource is nonrenewable and $\sigma^*(x) > 0$ as far as $x > 0$. As a result, the resource is exhausted within a finite time period.

[5] With Assumption (T.3), the cooperative strategy is modified as $\sigma^c(x) = \min\{b^c x, \hat{h}\}$.

5 Other policy instruments

A tax scheme can implement efficient resource usage as an MPNE in a differential game, but will not necessarily eradicate undesirable equilibria. A natural question arises: is there a fail-proof tax scheme that implements efficient resource usage targeted by the government? The answer is "yes" for the following tax scheme:

$$T(h, x) = \begin{cases} 0 & \text{if } h = \sigma^c(x), \\ h & \text{otherwise.} \end{cases}$$

This is, however, substantially the same as command-and-control regulation. This indicates that in the presence of strategic interaction among agents, a forcible regulation may be more reliable than a tax scheme.

Clemhout and Wan (1994a) argue that strategic interaction causes multiple equilibria. The malfunctioning of a tax scheme could be because of this, since a tax scheme, except for the extreme type discussed above, cannot preclude strategic interaction among players. We can show that, in contrast to a tax scheme, a tradable permit system prevents this strategic interaction.

Consider a tradable permit system in which harvesting is allowed by exchange with the same amount of harvest permits. Permits with amount M_i are distributed to each player i at $t = 0$. The permits are tradable and storable. They also bear interest, the rate of which $r(t)$ is determined in advance by the government. Suppose that the government sets $r(t)$ and M_i as:

$$r(t) = f'[x^c(t)], \quad \text{and} \quad M_i = \int_0^\infty h^c(t) \exp\left[-\int_0^t r(s)ds\right] \mathrm{d}t, \tag{5.1}$$

where $(x^c(t), h^c(t))$ is a cooperative solution to the cooperative problem (3.2). We assume $M_i < \infty$. This holds, for example, under Assumptions (U.1), (U.2), (T.2), and (P.1). Suppose also that the following availability condition on permits is satisfied:

Availability condition. The government provides substitutes for harvests and guarantees that, if the resource stock should be exhausted, a permit is still exchangeable for the same value of substitutes as one unit of harvest.

Under the availability condition, the price of a permit is always one in terms of harvests. Note that there is no strategic interaction with other players under this permit system. Player i solves the following noncooperative problem:

$$\max_{h(t)\geq 0} \int_0^\infty u[h(t)]e^{-\rho t} \mathrm{d}t \text{ subject to } \dot{M}_i(t) = r(t)M_i(t) - h(t), \; M_i(0) = M_i,$$

$$\liminf_{t\to\infty} M_i(t) \exp\left[-\int_0^t r(s)ds\right] \geq 0. \tag{5.2}$$

The inequality in the constraint is a no-Ponzi-game condition, without which the player could harvest the resource without limit by borrowing permits from other players. Notice that by substituting the state equation, the condition becomes:

$$\liminf_{T\to\infty} M_i - \int_0^T h(t)\exp\left[-\int_0^t r(s)ds\right]\mathrm{d}t \geq 0.$$

We claim that the cooperative solution $h^c(t)$ to the problem (3.2) is the unique solution to this problem:

Proposition 5.1 *Under Assumption* $(U.1)$, *if* $M_i < \infty$ *and the availability condition is satisfied, then the tradable permit system characterized by* (5.1) *yields the efficient use of resource as a unique solution to the noncooperative problem* (5.2).

Proof By construction, $h^c(t)$ satisfies the no-Ponzi-game condition and is thus feasible. The cooperative solution with costate $\lambda(t) \geq 0$ satisfies $h^c(t) = \arg\max \{u(h) - n\lambda(t)h | h \geq 0\}$ and $\dot{\lambda}(t)/\lambda(t) = \rho - f'[x^c(t)]$ for almost all $t \geq 0$. For the noncooperative problem (5.2), the associated Hamiltonian is defined as $\tilde{H}(h, M, \mu, t) = u(h) + \mu\,[r(t)M - h]$. When $\mu(t) = n\lambda(t)(\geq 0)$, the following hold with $h^c(t)$:

$$h^c(t) = \arg\max\left\{\tilde{H}(h, M_i(t), \mu(t), t) | h \geq 0\right\},$$

$$\dot{\mu}(t)/\mu(t) = \rho - f'[x^c(t)] \text{ for almost all } t \geq 0, \text{ and}$$

$$\lim_{T\to\infty} e^{-\rho T}\mu(T)M_i(T) = \lim_{T\to\infty}\mu(0)\left[M_i - \int_0^T h^c(t)e^{-\int_0^t r(s)ds}\mathrm{d}t\right] = 0.$$

These are sufficient for $h^c(t)$ to be optimal for (5.2), since the maximized Hamiltonian is well defined and concave in M_i because of the concavity of $\tilde{H}$ in (h, M_i) [see, for instance, Dockner et al. (2000, Theorem 3.3)]. The uniqueness of the solution follows from the strict concavity of u and the linearity of the state equation: if there is another optimal control $\tilde{h}(\cdot)$ such that the set $\{t \in [0,\infty) | h^c(t) \neq \tilde{h}(t)\}$ has a positive Lebesgue measure, then with any $\xi \in (0, 1)$, a new control $\tilde{h}^c(\cdot) = \xi h^c(\cdot) + (1-\xi)\tilde{h}(\cdot)$ is feasible starting from the same initial state. The strict concavity of u implies a contradiction to the optimality of $h^c(\cdot)$: $\liminf_{T\to\infty}\int_0^T [u(h^c(t)) - u(\tilde{h}^c(t))]e^{-\rho t}\mathrm{d}t < 0$. □

This proposition illustrates that a tradable permit system can implement efficient resource usage, precluding other inefficient outcomes. In this class of differential games, the advantage of a tradable permit scheme, compared with a tax scheme, is to break off the strategic interaction among agents.

Remark 5.1 The key to breaking off the strategic interaction is the availability condition on permits. Without this condition, each agent would still have to take into account other agents' actions. For further discussion, see Akao (2001).

6 Concluding remarks

Common property resources include forests, fisheries, reefs, waterways, pastures, agricultural lands, and mineral resources. In the rural areas of developing countries, these are essential sources of income and food, especially for the poor. Recent research in India has found that 15–25% of household income is from common property resources with the share of household income rising to 25% in poorer households (World Resource Institute 2005, p.39). In many cases, those suffering from the misuse of common property resources are the poor (Dasgupta and Mäler 1997). Dasgupta (1982, Chapter 2) points out the distributional consequences of a tax imposed on those who barely eke out an existence. He cautions that without compensation, these households may be worse off; this may occur under the tax schemes considered in this paper. To make matters worse, the implementation of the tax may also cause harsher resource extraction than the status quo. It is now known that, under certain conditions, there is an approximately efficient MPNE without government intervention (Sorger 1998), whereas a tax scheme may unbalance the equilibrium strategy and change it to one that incurs resource exhaustion as demonstrated in Propositions 4.2 and 4.3. We must then be careful in using a tax scheme as a remedy for the tragedy of the commons.

Appendix

A On a class of linear optimal control models

We show a variant of the results established by Long and Shimomura (1998, Proposition 1). The propositions below are cited in the main text. Consider an optimal control problem:

$$\sup_{\beta(t)\geq 0} \int_0^\infty v\left[x(t), \beta(t)\right] e^{-\rho t} \mathrm{d}t, \quad v\left(x, \beta\right) = \frac{\left[x\varphi(\beta)\right]^{1-\eta} - 1}{1-\eta},$$
$$\text{subject to } \dot{x}(t) = [A - \beta(t)]x(t),\ x(0) = x_0 > 0 \text{ given,} \tag{A.1}$$

where $\eta > 0$, $\rho > 0$, $A \in \mathbb{R}$, and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. As in the main text, we use the catching-up criterion for optimality. Control $\beta(t)$ is feasible if it is nonnegative and measurable. Denote by $x(t; x_0, \beta(\cdot))$ the trajectory of the state variable generated by an initial state $x_0 > 0$ and a feasible control $\beta(t)$.

Lemma A.1 *If $\beta^*(t)$ is an optimal control from $x_0 > 0$, then it is also an optimal control from any initial state $x \geq 0$.*

Proof We only prove the case of $\eta \neq 1$. The proof for the case of $\eta = 1$ is quite similar to the following, and thus omitted. Fix $x_0 > 0$ and let $\beta^*(t)$ be an optimal control from x_0. Pick an arbitrary feasible control $\tilde{\beta}(t)$ and an arbitrary initial state $\tilde{x}$ such that $\tilde{x} \neq x_0$ and $\tilde{x} > 0$. Choose θ so that $x_0\theta^{\frac{1}{1-\eta}} = \tilde{x}$. Then, $\theta = (\tilde{x}/x_0)^{1-\eta}$, and

$$\theta v[x(t; x_0, \tilde{\beta}(\cdot)), \tilde{\beta}(t)] = v[x(t; \tilde{x}, \tilde{\beta}(\cdot)), \tilde{\beta}(t)] + (1-\theta)\,/\,(1-\eta)\,.$$

Therefore, we have:

$$0 \leq \liminf_{T\to\infty} \theta \int_0^T \{v[x(t; x_0, \beta^*(\cdot)), \beta^*(t)] - v[x(t; x_0, \tilde{\beta}(\cdot)), \tilde{\beta}(t)]\} e^{-\rho t} \mathrm{d}t$$

$$= \liminf_{T\to\infty} \int_0^T \{v[x(t; \tilde{x}, \beta^*(\cdot)), \beta^*(t)] - v[x(t; \tilde{x}, \tilde{\beta}(\cdot)), \tilde{\beta}(t)]\} e^{-\rho t}.$$

The first inequality follows from the optimality of $\beta^*(t)$. Notice that if $x(0) = 0$, every feasible control is optimal and so is $\beta^*(t)$. □

Proposition A.1 *If the optimal control problem* (*A.1*) *has a solution for a certain initial state,* $x_0 > 0$*, then there exists a nonnegative constant* β^* *such that* $\beta(t) = \beta^*$ *is an optimal control from any initial state.*

Proof Since the problem is autonomous, we can denote by $B(x) : \mathbb{R}_{++} \twoheadrightarrow \mathbb{R}_+$ the associated optimal policy correspondence. That is, $\beta(t)$ is optimal if and only if it is feasible and satisfies $\beta(t) \in B(x\,[t; x_0, \beta(\cdot)])$ for almost all $t \in [0, \infty)$. By Lemma A.1, $B(x)$ does not depend on x, so we can rewrite it as B. $B \neq \emptyset$ because we have assumed that an optimal control exists when the initial state is $x_0 > 0$. Therefore, we have $\beta(t) = \beta^* \in B$, which is an optimal control from any initial state. □

Proposition A.2 *If* $\beta^* > 0$*,* $\varphi(\beta^*) > 0$*, and* $\varphi'(\beta^*)$ *exists, then* (*a*) β^* *satisfies the following Euler equation*:

$$\varphi(\beta^*) - [\rho - (1-\eta)(A-\beta^*)]\varphi'(\beta^*) = 0. \tag{A.2}$$

Furthermore, if $\varphi'(\beta^*) \geq 0$*,* (*b*) *the maximal value of* (*A.1*) *is given by*:

$$V(x_0) = \frac{1}{\rho - (1-\eta)(A-\beta^*)} \left\{ \frac{\left[x_0\varphi\left(\beta^*\right)\right]^{1-\eta} - 1}{1-\eta} + \frac{A-\beta^*}{\rho} \right\}, \tag{A.3}$$

and (*c*) *the transversality condition holds*:

$$\lim_{t\to\infty} e^{-\rho t} V'\left[x\left(t; x_0, \beta^*\right)\right] x\left(t; x_0, \beta^*\right) = 0. \tag{A.4}$$

Proof Simple calculations yield these proofs. For (a), the reduced form of the utility function is $\tilde{u}(x, \dot{x}) = \left\{[x\varphi(A - \dot{x}/x)]^{1-\eta} - 1\right\}/(1-\eta)$ and the basic Euler equation is written as $\dot{q} - \rho q = -p$, where $q = -\partial\tilde{u}(x, \beta^* x)/\partial\dot{x}$ and $p = \partial\tilde{u}(x, \beta^* x)/\partial x$. For (b) and (c), note $\rho - (A-\beta^*)(1-\eta) > 0$, which follows from $\varphi(\beta^*) > 0$, $\varphi'(\beta^*) \geq 0$, and (A.2). □

Proposition A.3 *Let* $\tilde{H}(\beta, x, \lambda) = v(x, \beta) + \lambda(A - \beta)x$*. Assume that* φ *is concave and the maximized Hamiltonian of* (*A.1*), $\tilde{H}^*(x, \lambda) = \max\{\tilde{H}(\beta, x, \lambda) | \beta \geq 0\}$*, is well defined on* $\mathbb{R}^2_{++}$*. Then,* (*a*) $\tilde{H}^*(x, \lambda)$ *is concave in* x*.* (*b*) *If* $b^* > 0$ *satisfies* $\varphi(b^*) > 0$*,* $\varphi'(b^*) \geq 0$*, and the Euler equation* (A.2)*, then* $\beta(t) = b^*$ *is an optimal control for the problem* (A.1).

Proof (a) Arbitrarily select $\xi \in [0, 1]$, $\lambda > 0$, and $x^1, x^2 > 0$ such that $x^1 \neq x^2$. Let $\tilde{H}^*(x^k, \lambda)$ be attained at $\beta = \beta^k$ $(k = 1, 2)$. Define x^3 and ζ by $x^3 = \xi x^1 + (1-\xi)x^2$ and $\zeta = \xi x^1/x^3$. Note that $\zeta \in [0, 1]$ and $1-\zeta = (1-\xi)x^2/x^3$. Finally, let $\tilde{\beta} = \zeta\beta^1 + (1-\zeta)\beta^2$. Then, $\tilde{H}^*(x^3, \lambda) \geq \tilde{H}(\tilde{\beta}, x^3, \lambda) \geq \xi\tilde{H}^*(x^1, \lambda) + (1-\xi)\tilde{H}^*(x^2, \lambda)$. (b) Set $\lambda(t) = V'(x(t))$, where V is given by (A.3) in Proposition A.2 with $\beta^* = b^*$ and $x(t)$ is the solution of $\dot{x} = (A - b^*)x$ with given $x(0) > 0$. Then, the statement follows from the Arrow sufficiency theorem (see, for instance, Dockner et al. 2000, Theorem 3.3). □

B Proof of propositions

B.1 Proof of Proposition 4.1

The proof consists of a series of lemmas. Lemma B.1 shows a necessary condition for a linear strategy to constitute a symmetric MPNE of game Γ with each of the CC, CD, and GCW Taxes. The condition is derived from the two Euler equations for the cooperative and noncooperative problems. Therefore, if φ in (4.6) satisfies certain nonnegativity and concavity conditions, the linear strategy is in fact an equilibrium strategy, by Proposition A.3. Lemma B.2 shows that the conditions are satisfied for the CC and CD Taxes, but the GCW Tax fails in concavity. The nonconcavity requires separate treatment. Lemmas B.3–5 deal with the GCW Tax when $\eta <, >, = 1$. We assume (U.1), (U.2), (T.2), and (P.1). Note that the utility function satisfies $u'(h) > 0$ and $-hu''(h)/u'(h) = \eta > 0$ by (U.1) and (U.2). The natural growth function has the form of $f(x) = ax$ by (T.2). Then by Propositions A.1–3, the cooperative strategy is linear and given by (4.4), i.e., $\sigma^c(x) = b^c x$ with $b^c = [\rho - (1-\eta)a] / (\eta n)$. (P.1) ensures $b^c > 0$.

Lemma B.1 *If linear strategy $\sigma(x) = bx$ $(b > 0)$ constitutes a symmetric MPNE, then b satisfies*:

$$\begin{array}{ll} n\left(\frac{n-1}{n} - \eta\right)(b - b^c) = 0 & \text{for the CC Tax,} \\ -\eta n(b - b^c) = 0 & \text{for the CD Tax,} \\ (1-\eta)n(b - b^c) = 0 & \text{for the GCW Tax.} \end{array} \tag{B.1}$$

Proof Suppose that $\sigma(x) = bx$ constitutes a symmetric MPNE. That is, when other players adopt the strategy, the same strategy is optimal for the noncooperative problem (4.5). Then, applying (A.2) in Proposition A.2, we have $\rho - (1-\eta)(a - nb) - \varphi(b)/\varphi'(b) = 0$. From (4.4), b^c satisfies $\rho - (1-\eta)a - \eta n b^c = 0$. By combining these, we obtain $0 = \eta n b^c + (1-\eta)nb - \varphi(b)/\varphi'(b)$. This and (4.6) yield (B.1). □

Let $b^* > 0$ represent b, which satisfies (B.1).

Lemma B.2 *φ in (4.6) satisfies $\varphi(b^*) > 0$ and $\varphi'(b^*) > 0$. φ is concave for the CC and CD Taxes, but strictly convex for the GCW Tax.*

Proof $\varphi'' = \varphi / (nb^c)^2 > 0$ for the GCW Tax. The other properties are obvious. □

As for the GCW Tax, we separately consider the three cases of $\eta <, =, > 1$.

Lemma B.3 *If $\eta < 1$, then $\sigma^*(x) = b^* x$ does not constitute an MPNE under the implementation of the GCW Tax.*

Proof The proof goes by means of contradiction. Assume that $\sigma^*(x) = b^* x$ constitutes an MPNE. Then, $b^* = b^c$ from Lemma B.1. However, since $\partial^2 u[h - T(h, x)]/\partial h^2 = (-\eta + 1)\,\varphi'' u'/x > 0$, Condition 3 in Lemma 3.1 is not satisfied. □

Lemma B.4 *Let $\eta = 1$. If harvest ratios $\beta(t)$ are restricted to satisfy $\beta(t) < \zeta e^{\xi t}$ for almost all $t \geq 0$, with $\xi < \rho$ and $\zeta > 0$, then $\sigma^*(x) = b^* x$ constitutes an MPNE under the implementation of the GCW Tax.*

Proof We can write the utility function as $u\,(x\varphi(\beta)) = \ln x + \beta/\rho$ [Recall $b^c = \rho/n$ by (4.4)]. The trajectory of the state variable is given as:

$$x(t) = x_0 \exp\left\{\left[a - (n-1)\,b^*\right]t - \int_0^t \beta_i(s)ds\right\}.$$

Then, the lifetime utility of player i is calculated as follows:

$$\begin{aligned}&\int_0^\infty \left[\ln x(t) + \frac{\beta_i(t)}{\rho}\right] e^{-\rho t} \mathrm{d}t \\ &= \int_0^\infty \left[\ln x_0 + \left(a - (n-1)\,b^*\right)t - \int_0^t \beta_i(s)ds + \frac{\beta_i(t)}{\rho}\right] e^{-\rho t}\mathrm{d}t \\ &= \frac{\ln x_0}{\rho} + \frac{a - (n-1)b^*}{\rho^2} + \lim_{t\to\infty} \frac{e^{-\rho t}}{\rho}\int_0^t \beta_i(s)\mathrm{d}s.\end{aligned}$$

By the growth condition of $\beta_i(t)$, there are $\xi < \rho$ and $\zeta > 0$ and it holds that:

$$0 < e^{-\rho t}\int_0^t \beta_i(s)ds < \frac{\zeta}{\xi} e^{-\rho t}\left(e^{\xi t} - 1\right) \to 0 \quad \text{as } t \to \infty.$$

Therefore, we obtain:

$$\int_0^\infty \left[\ln x(t) + \frac{\beta_i(t)}{\rho}\right] e^{-\rho t}\mathrm{d}t = \frac{\ln x_0}{\rho} + \frac{a - (n-1)b^*}{\rho^2} \text{ (constant).}$$

This implies that all feasible controls are optimal, and so is $\beta_i(t) = b^*$. □

Lemma B.5 *Let $\eta > 1$. $\sigma^*(x) = b^* x = b^c x$ constitutes an MPNE under the implementation of the GCW Tax, if the harvest ratios are restricted to satisfy that $\beta(t)$ is absolutely continuous and there exists $\tilde{Z} > 0$ such that*:

$$\left|\frac{d^2 \exp\left[\int_0^t a - (n-1)\,b^c - \beta(s)ds\right]}{\mathrm{d}t^2}\right| < \tilde{Z} \quad \text{for almost all } t \geq 0. \qquad \text{(B.2)}$$

Proof If $\eta > 1$ and if there exists a solution to the noncooperative problem (4.5) with the GCW Tax, the solution is uniquely given by $b^* = b^c$ by (B.1) in Lemma B.1. Thus, $\sigma^c(x) = b^c x$ constitutes an MPNE. To ensure the existence, we apply the existence theorem in Romer (1986, p.899). Let $Z = \tilde{Z}x(0)$. By imposing a growth condition $|\ddot{x}(t)| < Z$, which is equivalent to (B.2), we can redefine utility function $u(c)$ as:

$$\tilde{u}\left[x\varphi\left(\beta\right), \ddot{x}\right] = \begin{cases} u\left[x\varphi\left(\beta\right), \ddot{x}\right] & \text{if } |\ddot{x}| < Z, \\ -\infty & \text{otherwise.} \end{cases}$$

By Assumption (U.2), $u(h) = h^{1-\eta}/(1-\eta)$, which is unique up to a positive affine transformation. Since $\eta > 1$, the utility is bounded from above, and we have for any feasible $(\beta(t), x(t))$,

$$\tilde{u}(x(t)\varphi\left[\beta(t)\right], z) \leq \sup_{c \geq 0} u(c) + Z^P - |z|^P \quad \text{all } z \in \mathbb{R},$$

where $P > 1$. Therefore, condition (ii) in Romer's theorem is satisfied. The other sufficient conditions for the theorem to apply obviously hold. □

Proof of Proposition 4.1 For the CC and CD Taxes, Lemmas B.1, B.2, and Proposition A.3(b) establish (a) 1, 2 and (b) 1. For the GCW Tax, (a) 3(i) and (b) 2 follow from Lemmas B.1 and B.4, and (a) 3(ii) follows from Lemmas B.1 and B.5. □

B.2 Proof of Proposition 4.2

Using φ in (4.6), we express the utility function as $u[x\varphi(h/x)]$. Let $\hat{V}_E(x_0)$ be the lifetime utility when either the CC or the CD Tax is implemented and all players use the MRE strategy:

$$\hat{V}_E(x_0) = \int_0^{t_E(x_0)} u\left[\hat{x}(t; x_0)\varphi\left(\hat{h}/\hat{x}(t; x_0)\right)\right] e^{-\rho t} \mathrm{d}t. \tag{B.3}$$

Since $\hat{x}(t; x_0)$ and $t_E(x_0)$, defined in (4.7) and (4.8), respectively, are continuously differentiable in $x_0 > 0$, $\hat{V}'_E(x_0)$ exists for all $x_0 > 0$. Therefore, we have:

$$\rho \hat{V}_E(x) = u[x\varphi(\hat{h}/x)] - \hat{V}'_E(x)n\hat{h} \quad \text{all } x > 0. \tag{B.4}$$

By (U.2'), $u(\cdot)$ is bounded below, and so is $\hat{V}_E(x_0)$. Then

$$u'[x\varphi(\hat{h}/x)]\varphi'(\hat{h}/x) - \hat{V}'_E(x) \geq 0 \quad \text{all } x \in (0, x_0]. \tag{B.5}$$

is a necessary and sufficient condition for the MRE strategy to constitute a MPNE (Dockner et al. 2000, Lemma 3.1). Using (U.2') and (B.4), arrange this inequality as follows:

$$0 \leq \frac{\rho}{n\hat{h}} \left\{ \hat{V}_E(x) - \left(1 - \frac{(1-\eta)n\hat{h}\varphi'(\hat{h}/x)}{x\varphi(\hat{h}/x)}\right) \frac{u[x\varphi(\hat{h}/x)]}{\rho} \right\}. \tag{B.6}$$

Recall (4.4). $\sigma^c(x) = [\rho/(n\eta)]x$ holds true because $a = 0$. (a) For the CC Tax, (B.6) holds if

$$\frac{(1-\eta)n\hat{h}\varphi'(\hat{h}/x)}{x\varphi(\hat{h}/x)} \geq 1, \tag{B.7}$$

since $\hat{V}_E(x)$ and $u[x\varphi(\hat{h}/x)]$ are positive for all $x > 0$. A straightforward calculation shows that (B.7) is equivalent to (4.10). (b) For the CD Tax, (B.6) is violated at a sufficiently small stock level of the resource. Notice that the condition in (B.6) has to hold for any $x \in (0, x_0]$.

B.3 Proof of Proposition 4.3

To find an equilibrium strategy, we use the method developed in Tsutsui and Mino (1990). This exploits the HJB equation as well as its derivative to derive a first-order ordinary differential equation for a Markovian equilibrium strategy. As another method, the sole use of the HJB equation brings a first-order ordinary differential equation for the value function. For the purpose of the present problem, the former is easier to obtain the closed form solution of an equilibrium strategy.

Let $y : \mathbb{R}_+ \to \mathbb{R}$ be a C^1 function that satisfies $y(x) > 0$ for all $x > 0$. Suppose that all players use $y(x)$ as a strategy under the CD Tax. Let $w(h, x) = u[h - T(h, x)]$, i.e.,

$$w(h, x) = h^{\frac{1-\eta}{n}}[\sigma^c(x)]^{\left(1-\frac{1}{n}\right)(1-\eta)}. \tag{B.8}$$

Let $W(x)$ be the associated lifetime utility. Assume that $W(x)$ is differentiable. (We will verify that this assumption holds true at the end.) Then, the following equation holds:

$$\rho W(x) = w\left[y(x), x\right] - nW'(x)y(x) \quad \text{all } x > 0. \tag{B.9}$$

Differentiate both sides of (B.9) with respect to x:

$$\rho W'(x) = w_h\left[y(x), x\right]y'(x) + w_x\left[y(x), x\right] - n\left[W''(x)y(x) + W'(x)y'(x)\right]. \tag{B.10}$$

w is concave in h and the lifetime utility W is bounded below by Assumption (U.2'). Therefore, if

$$w_h\left[y(x), x\right] - W'(x) = 0 \quad \text{all } x > 0, \tag{B.11}$$

then $y(x)$ constitutes an MPNE (Dockner et al. 2000, Lemma 3.1). By (B.10), (B.11) implies:

$$\rho w_h = w_h y' + w_x - n\left[\left(w_{hh}y' + w_{hx}\right)y + w_h y'\right] \quad \text{all } x > 0, \tag{B.12}$$

where all functions are evaluated at $(y(x), x)$. Using (B.8), (B.12) is calculated as follows:

$$\rho = \eta y' + \frac{\eta(n-1)y}{x}. \tag{B.13}$$

Let $z(x) = y(x)/x$ and rewrite the differential equation (B.13) as $\rho = \eta\left(z + xz'\right) + \eta(n-1)z$. Rearrange this to $dz/(\rho/\eta - nz) = dx/x$, and we have $z = (1/n)$

$\left(\rho/\eta + \gamma x^{-n}\right)$, where γ is an integration constant. Therefore, the differential equation (B.13) is solved as:

$$y(x) = \left(\frac{x}{n}\right)\left(\frac{\rho}{\eta} + \gamma x^{-n}\right), \quad \gamma \geq 0. \tag{B.14}$$

Notice that $\gamma < 0$ is ruled out by the nonnegativity of $y(x)$. The associated state equation $\dot{x}(t) = -ny(x)$ is a Bernoulli equation and is solved as:

$$x(t; x_0) = \left\{\left[(x_0)^n + \frac{\gamma\eta}{\rho}\right]\exp\left(-\frac{n\rho}{\eta}t\right) - \frac{\gamma\eta}{\rho}\right\}^{\frac{1}{n}}. \tag{B.15}$$

Thus, the resource is exhausted at time:

$$\tilde{T}(x_0) = \frac{\eta}{n\rho}\ln\left(\frac{\rho\,(x_0)^n}{\gamma\eta} + 1\right).$$

The lifetime utility W is expressed as:

$$W(x_0) = \int_0^{\tilde{T}(x_0)} w[y(x(t; x_0)), x(t; x_0)]e^{-\rho t}\mathrm{d}t.$$

Since $x(t; x_0)$ and $\tilde{T}(x_0)$ are continuously differentiable at any $x_0 > 0$, W is differentiable, completing the proof.

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