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RESEARCH ARTICLE

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On the characterization of efficient production vectors

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Abstract In this paper we study the efficient points of a closed production set with free disposal. We first provide a condition on the boundary of the production set, which is equivalent to the fact that all boundary points are efficient. When the production set is convex, we also give an alternative characterization of efficiency around a given production vector in terms of the profit maximization rule. In the non-convex case, this condition expressed with the marginal pricing rule is sufficient for efficiency. Then we study the Luenberger's shortage function. We first provide basic properties on it. Then, we prove that the above necessary condition at a production vector implies that the shortage function is locally Lipschitz continuous and the efficient points in a neighborhood are the zeros of it and conversely.

Keywords Production efficiency · Shortage function

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1 Introduction[1]

A concept of major interest is that of efficiency. We are often interested in knowing whether one can produce more with less. In this respect, it would be useful to

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characterize the efficient production plans. It would be more useful if the set of efficient points could be described as the set of zeros of a function.

It is known that points of the boundary of a closed production set satisfying the free-disposal assumption are always weakly efficient (see e.g. Bonnisseau and Cornet 1988, p. 120, or Villar 2000, p. 50). Of course, points in the interior of the production set are never efficient. It would be nice if one could state when points of the boundary of a production set are efficient. This paper addresses these issues.

We first provide a necessary and sufficient condition for (all) points of the boundary of a production set to be efficient. It works even if the production set is not convex. The condition involves the production plans on the boundary and it is clearly satisfied when the production set is strictly convex. We show at the end of the paper that this condition is incompatible with a standard assumption on the production set, which states that there is a differentiation of the inputs and the outputs. Together with the free-disposal assumption, this implies that some production plans on the boundary are not efficient.

Thus, we focus our attention on a local point of view. When the production set happens to be convex, we choose a dual approach by considering the supply set or the profit maximization rule. This is equivalent to considering the normal cone to the production set since the producer maximizes its profit at a production plan y with respect to a price vector p if and only if p belongs to the normal cone at y. It is known that if a strictly positive vector belongs to the normal cone at a production plan y, then y is efficient (see e.g. Villar 2000, p. 50). The converse is true when the production set is polyhedral, which is the case when the technology is defined by a finite number of activities (see Koopmans 1951). Nevertheless, the converse is false in the general case (see section 2 for a simple numerical example). But, we remark that there are non-efficient production plans on the boundary of the production set in every neighborhood of the production vector. We show that what happens in the example is always true, that is the normal cone at a production plan y on the boundary contains only strictly positive vectors if and only if the production plans on the boundary in a neighborhood of y are all efficient. This condition is equivalent to the fact that the production plan y is not in the supply set for a price vector, which is not strictly positive. Of course, this condition is equivalent to the previous one if and only if all points of the boundary are efficient.

In the non-convex case, things, unsurprisingly, are harder. The profit maximization rule is then generalized by considering the marginal pricing rule (see, Guesnerie 1975). A price vector p at a production plan y satisfies the marginal pricing rule if p satisfies a first order necessary condition for profit maximization. When the production set is smooth, this means that the relative prices must be equal to the marginal rates of transformation or of substitution. In the general case, to define the marginal pricing rule, the Clarke's normal cone is a convenient concept (see Cornet 1990; Quinzii 1992 and Villar 2000) even if it is possible to use other notions coming from non-smooth analysis (see Khan 1999). Thus the condition on the normal cone can be stated in the same way by replacing the normal cone of convex analysis by the Clarke's normal cone (see Cornet 1990). Nevertheless, the condition is only sufficient and we provide an example, which illustrates this point. Finally we study when efficient points are the zeros of well-defined class of functions. We address the question in term of the shortage function of Luenberger, which is a generalization of a function used in Bonnisseau and Cornet (1988).

In a nutshell, the shortage function measures the amount, in units of a fixed netput vector, by which any netput vector is short of reaching the boundary of the production set. Under the assumptions that the production set is closed and satisfies free disposal, Luenberger has shown that the shortage function describes the production set[2]. More precisely, the shortage function is a transformation function: a netput vector may be produced if and only if its image under the shortage function is negative. One can show, that if our sufficient condition expressed in terms of the marginal pricing rule holds true at a boundary point y, then, whatever is the fixed netput vector, the shortage function is locally Lipschitz continuous and the boundary points around y are the zeros of the shortage function. Thus, our condition implies that the boundary points are efficient and one can characterize them as the zeros of the shortage function.

The paper is organized as follows. We present our condition under which all boundary points are efficient in section 2. There, we also study the characterization of efficiency in the convex case in terms of the normal cone of boundary points and we provide the sufficient condition for the non-convex case. In the third section, we present some properties of the Luenberger function and, especially, when the zeros of this function are efficient points. The conclusion is in section four.

2 Characterization of efficient points

We consider a production set Y, which is a subset of $\mathbb{R}^m$. We posit the following basic assumption[3]:

Assumption P Y is closed, satisfies the free-disposal assumption $Y - \mathbb{R}^m_+ = Y$ and $Y \neq \mathbb{R}^m$.

Let us recall the definition of an efficient production plan. A production plan y in Y is efficient if there does not exist another production plan y' in Y such that: $y' \neq y$ and $y' \geq y$.

2.1 Characterization by a boundary assumption

When are points on the boundary efficient? It is known that, under Assumption P, they are always weakly efficient (Bonnisseau and Cornet 1988, p. 120; Villar 2000, p. 50). Let us consider the boundary condition:

Assumption B For all $(y, y') \in (\partial Y)^2$ such that $y' \neq y$ and $y' \geq y$, then for all $t \in]0, 1[$, $ty + (1 - t)y' \in \text{int}Y$.

[2] This is also the case of the distance and the benefit function, see Chamber et al. (1995) and the references quoted therein.

[3] Notations : if x and y are vectors of $\mathbb{R}^m$, $x \cdot y$ denotes the canonical inner product and $\|x\|$ the associated euclidean norm. If Y is a subset of $\mathbb{R}^m$, then int Y denotes the interior and ∂Y the boundary. $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m \mid x_h \geq 0, \forall h = 1, \ldots, m\}$, $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m \mid x_h > 0, \forall h = 1, \ldots, m\}$. 1_m denotes the vector of $\mathbb{R}^m$, whose coordinates are all equal to 1.

Remark If Y is strictly convex, then Y satisfies Assumption B[4]. Note that Assumption B is not compatible with an input-output differentiation as it is shown in the last section.

Theorem 1 *Let Y be a production set satisfying Assumption P. Then every point on the boundary of Y is efficient if and only if Y satisfies Assumption B.*

Proof Let us first assume that every point on the boundary of Y is efficient. Then, for all $y \in \partial Y$, one has $(y + \mathbb{R}^m_+) \cap Y = \{y\}$. Consequently, it does not exist $y' \in \partial Y$ such that $y' \neq y$ and $y' \geq y$. Hence Assumption B is satisfied.

Conversely, let $y \in \partial Y$. If y is not efficient, there exists $y' \in Y$ such that $y' \neq y$ and $y' \geq y$. If $y' \in \text{int}Y$, one gets a contradiction since $y \in \partial Y$ and $y \in y' - \mathbb{R}^m_+ \subset \text{int}Y - \mathbb{R}^m_+ = \text{int}Y$. The last equality comes from the free-disposal assumption.

If $y' \in \partial Y$, from Assumption B, $\tilde{y} = \frac{1}{2}(y + y')$ belongs to the interior of Y and $y \in \tilde{y} - \mathbb{R}^m_+$ and one gets a contradiction as above. □

As already said, the property that *all* boundary points are efficient is not compatible with input–output differentiation. Hence, it could be more relevant to follow a local approach, i.e. to see if a particular point is efficient or the stronger property that all points on the boundary in a neighborhood are efficient. To put it differently, one is not especially interested in the global property of the production set, but perhaps more in the local properties around one production plan. We follow a dual approach based on prices or on the supply set of the producer. This allows us to obtain a necessary and sufficient condition in the convex case.

2.2 Characterization of efficient points of a convex production set by the profit maximization rule

If a producer is a price taker and maximizes the profit, then a production plan $y \in Y$ belongs to the supply set for the price p if $p \cdot y \geq p \cdot y'$ for all $y' \in Y$. From the definition of the normal cone of the convex analysis, this is equivalent to say that the producer follows the profit maximization rule, that is, the producer chooses a production plan-price pair such that $p \in N_Y(y) = \{q \in \mathbb{R}^m \mid q \cdot (y - y') \geq 0,\ \forall y' \in Y\}$.

We now state the characterization of efficiency using the profit maximization rule.

Proposition 1 *Let Y be a convex production set satisfying Assumption P. Let y be a production plan on the boundary of Y. Then, there exists a neighborhood U of y such that for all $y' \in U \cap \partial Y$, y' is efficient if and only if $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$.*

The condition $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$ precisely means that the prices, which satisfy the profit maximization rule at y are strictly positive. Note that y can be efficient with $N_Y(y) \setminus \{0\} \cap \partial\mathbb{R}^m_+ \neq \emptyset$. Indeed, it is the case with $y = 0$ for $Y = \{(y_1, y_2) \subset \mathbb{R}^2 \mid y_1 \leq 0,\ y_2 \leq -y_1\}$. But, for all $\varepsilon > 0$, the point $(0, -\varepsilon)$ belongs to the boundary of Y and is not efficient since it is dominated by $(0, 0)$.

[4] Note that under the assumption of free-disposal, the usual definition of a strictly convex set (see, e.g. Aliprantis et al. 1989, p. 72) is exactly : a convex subset Y of $\mathbb{R}^m$ is strictly convex if $y, y' \in Y$, $y \neq y'$ and $\lambda \in (0, 1) \Rightarrow (1 - \lambda)y + \lambda y' \in \text{int}\, Y$.

Proof We first prove by contradiction that there exists a neighborhood U of y such that for all $y' \in U \cap \partial Y$, y' is efficient implies that $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$. If it is not true, then there exists $p \in N_Y(y) \setminus \{0\} \cap \partial \mathbb{R}^m_+$. Let $v \in \mathbb{R}^m_+ \setminus \{0\}$ such that $p \cdot v = 0$. For all $\varepsilon > 0$, we prove that $y^\varepsilon = y - \varepsilon v \in \partial Y$. Indeed, the free-disposal assumption implies that $y^\varepsilon \in Y$. If it is not in ∂Y, then $y^\varepsilon \in \text{int} Y$. Then, there exists $\rho > 0$ such that $y^\varepsilon + \rho 1_m \in Y$, where 1_m is the vector of $\mathbb{R}^m$ with all its coordinates equal to 1. But, $p \cdot (y^\varepsilon + \rho 1_m) = p \cdot y - \varepsilon p \cdot v + \rho p \cdot 1_m$. Since $p \cdot v = 0$ and $p \cdot 1_m > 0$, one deduces that $p \cdot (y^\varepsilon + \rho 1_m) > p \cdot y$, which contradicts $p \in N_Y(y)$. Consequently, $y^\varepsilon \in \partial Y$ and y^ε is dominated by y, which contradicts the assumption.

Conversely, if $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$, then there exists a neighborhood U of y such that for all $y' \in U \cap \partial Y$, $N_Y(y') \setminus \{0\} \subset \mathbb{R}^m_{++}$ since the graph of N_Y is closed. Thus, for all $y' \in U \cap \partial Y$, there exists $p \in N_Y(y') \setminus \{0\} \cap \mathbb{R}^m_{++}$ since $N_Y(y') \neq \{0\}$. Hence, for all $\tilde{y} \in Y$, $p \cdot \tilde{y} \leq p \cdot y'$. This implies that $\tilde{y} - y' \notin \mathbb{R}^m_+ \setminus \{0\}$ since $p \in \mathbb{R}^m_{++}$. Consequently, y' is efficient. □

One immediately deduces the following corollary, from which one gets that condition B is equivalent to the condition $y \in \partial Y$, $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$.

Corollary 1 *Let Y be a convex production set satisfying Assumption P. Then every point on the boundary of Y is efficient if and only if for all* $y \in \partial Y$, $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$.

Note also that the condition for all $y \in \partial Y$, $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$ can be equivalently formulated by saying that the supply set of the producer is empty for all nonzero price vectors on the boundary of the positive orthant. To put it differently, consider the smooth case. Y is defined by a smooth transformation function γ, that is $Y = \{y \in \mathbb{R}^m \mid \gamma(y) \leq 0\}$. We also assume that the gradient vector of γ, $\nabla\gamma$, does not vanish at every production plan y such that $\gamma(y) = 0$. The condition $y \in \partial Y$, $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$ means that $\nabla\gamma(y) \gg 0$ since $N_Y(y) = \{t\nabla\gamma(y) \mid t \geq 0\}$. The marginal rates of substitution or transformation are finite since they are equal to

$$\frac{\frac{\partial \gamma}{\partial y_i}(y)}{\frac{\partial \gamma}{\partial y_j}(y)}.$$

If $p \in \partial \mathbb{R}^m_+$, some prices are nil, it is then impossible to get the equality between relative prices and marginal rate of transformation or marginal rate of substitution. Then, the production plan y is not in the supply set for the price p.

2.3 Efficient points in the non-convex case

When the production set is not convex, we consider the marginal pricing rule instead of the profit maximization rule. This means that the producer chooses a price-production pair (p, y) such that p satisfies a first order necessary condition for profit maximization. To state this condition formally, we consider the Clarke's normal cone[5] to the production set at the production plan (see Clarke 1983; Cornet 1990; Quinzii 1992; Villar 2000), which is denoted $N_Y(y)$. Then, the necessary condition is $p \in N_Y(y)$. Since the Clarke's normal cone coincides with the normal cone of the convex analysis when the production set is convex, this generalizes the work of the previous section. If the production set is smooth, then the normal cone is the half-line generated by the gradient of the transformation function and the marginal pricing rule can be interpreted as the equality between the relative prices and the marginal rate of substitution or transformation.

If Y is not convex, Proposition 1 does not hold. Indeed, let us consider the following counter-example.

$$Y = \left\{ (y_1, y_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} y_1 \leq 0, \\ y_2 \leq -y_1, \text{ if } y_1 \geq -1, \\ y_2 \leq \sqrt{-y_1 - 1} + 1, \text{ if } y_1 \leq -1 \end{array} \right\}.$$

For this production set, every production plan on the boundary such that $y_2 \geq 0$ is efficient. But, at $y = (1, 1)$, $N_Y(1, 1) = \{p \in \mathbb{R}^2_+ \mid p_2 \leq p_1\}$, which contains $(1, 0)$ on the boundary of the positive orthant. Without convexity, we loose the necessary part but the condition on the marginal pricing rule remains sufficient.

Proposition 2 *Let Y be a production set satisfying Assumption P. Let y be a production plan on the boundary of Y such that $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$. Then, there exists a neighborhood U of y such that for all $y' \in U \cap \partial Y$, y' is efficient.*

Proof Since the normal cone has a closed graph [6], there exists a neighborhood U of y such that for all $y' \in U \cap \partial Y$, $N_Y(y') \setminus \{0\} \subset \mathbb{R}^m_{++}$. We now prove by contradiction that for all $y' \in U \cap \partial Y$, y' is efficient. If it is not true, there exists $\bar{y} \in U \cap \partial Y$, and $\tilde{y} \in \partial Y$ such that $\tilde{y} \geq \bar{y}$ and $\tilde{y} \neq \bar{y}$. For $t > 0$ close enough to 0, one has $y^t = t\tilde{y} + (1-t)\bar{y} \in U$. Since $y^t \geq \bar{y}$ and $\bar{y} \in \partial Y$, one has $y^t \in \partial Y$. Since $N_Y(y^t) \setminus \{0\} \subset \mathbb{R}^m_{++}$ and $\bar{y} - \tilde{y} \leq 0$, one has $(\bar{y} - \tilde{y}) \cdot p < 0$ for all $p \in N_Y(y^t)$, which implies $\bar{y} - \tilde{y} \in \text{int} T_Y(y^t)$. Consequently, from the characterization of the element of the interior of the Clarke's tangent cone (see, Clarke 1983, p.68), $\bar{y} - \tilde{y}$ is hypertangent[7] at y^t and one has $y^t + \varepsilon(\bar{y} - \tilde{y}) \in \text{int} Y$ for all ε small enough. But, $y^t + \varepsilon(\bar{y} - \tilde{y}) = (t - \varepsilon)\tilde{y} + (1 - t + \varepsilon)\bar{y}$. Consequently, for $\varepsilon < t$, one has $y^t + \varepsilon(\bar{y} - \tilde{y}) \geq \bar{y}$, hence $\bar{y} \in \text{int} Y$, which contradicts $\bar{y} \in \partial Y$. □

One immediately deduces the following corollary, which provides a sufficient condition for each point on the boundary of the production set to be efficient. Clearly, it is stronger than Assumption B.

Corollary 2 *Let Y be a production set satisfying Assumption P. If for all $y \in \partial Y$, $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$ then every point on the boundary of Y is efficient.*

[5] The Clarke's tangent cone to Y at $y \in Y$, denoted $T_Y(y)$, is the set of vectors $v \in \mathbb{R}^m$ such that for all sequences $(t^\nu, y^\nu) \in \mathbb{R}_+ \times Y$, which converges to $(0, y)$, there exists a sequence (v^ν), which converges to v, and such that $y^\nu + t^\nu v^\nu \in Y$. The Clarke's normal cone is the negative polar cone of $T_Y(y)$, that is the vectors $p \in \mathbb{R}^m$ such that $p \cdot v \leq 0$ for all $v \in T_Y(y)$.

[6] From Clarke (1983) Corollary 2 of Theorem 2.5.8, this is true when the interior of the tangent cone is nonempty. Under Assumption P, one easily checks that $-\mathbb{R}^m_+$ is included in the tangent cone and, consequently, $-\mathbb{R}^m_{++}$ is included in the interior of the tangent cone.

[7] A vector v is hypertangent at y if there exists $r \in]0, 1[$ such that for all $t \in [0, r]$, for all $u \in B(0, r)$, for all $y' \in B(y, r) \cap Y$, $y' - t(u + v) \in Y$.

3 Efficiency and the luenberger shortage function

It is well known that if Y is defined by a continuous transformation function f, i.e. $Y = \{y \in \mathbb{R}^m \mid f(y) \leq 0\}$, then an efficient point y_0 satisfies $f(y_0) = 0$ (see, e.g., Luenberger 1995, p. 22).

We would like to know when the converse is true. In order to do this, following Luenberger, we introduce a special transformation function, i.e. the shortage function. This function is defined as follows.

For every $g \in \mathbb{R}^m_+ \setminus \{0\}$, one considers the mapping $\sigma(., g) : \mathbb{R}^m \to \overline{\mathbb{R}}$:

$$\sigma(y, g) = \inf\{t \mid y - tg \in Y\}.$$

The special case where $g = 1_m$ was studied by Bonnisseau and Cornet (1988) (see their function $\lambda_j(.)$ page 139). The general case where introduced in economics by Luenberger (1992) and (1995) who names it the shortage function. Hence $\sigma(y, g)$ is the distance of y from the boundary of Y in the g direction. It measures the amount, in g units, by which any netput vector is short of reaching ∂Y. Suppose that, $\sigma(y, g) > 0$, then trying to produce y will lead to shortages, which means that it is not possible to produce the quantities of outputs with the quantities of inputs given by y. In the following proposition, we summarize the basic properties of the shortage function when the production set satisfies Assumption P.

Proposition 3 *If Y satisfies Assumption P, then, for all $g \in \mathbb{R}^m_+ \setminus \{0\}$, $\sigma(., g)$ is a transformation function. For all $y \in \mathbb{R}^m$, the set $\Sigma(y) = \{t \mid y - tg \in Y\}$ is a closed interval. If it is not empty, $\Sigma(y) + \mathbb{R}_+ = \Sigma(y)$ and if $\sigma(y, g)$ is finite, $y - \sigma(y, g)g \in \partial Y$.*

The proof showing that $\sigma(., g)$ is a transformation function is in Luenberger (1995) and the remaining is a direct consequence of the definition. Note also, that the shortage function needs not to be continuous but we get the following result.

Proposition 4 *If y is an efficient point then $\sigma(y, g) = 0$.*

Proof Suppose $y \in Y$ is an efficient point. Then, from the definition of $\sigma(., g)$, one has $\sigma(y, g) \leq 0$. If $\sigma(y, g) < 0$, then, again from the definition of $\sigma(., g)$, there exists $t < 0$ such that $y - tg \in Y$. Since $y - tg \geq y$ and $y - tg \neq y$, this contradicts the assumption y is efficient. □

This result is interesting since in general, as shown in the next proposition the shortage function is only lower semi-continuous.

Proposition 5 *If Y satisfies Assumption P, then $\sigma(., g)$ is lower semicontinuous.*

Proof $\sigma(., g)$ is lower semicontinuous if its epigraph is closed (see, Rockafellar 1970, Theorem 7.1). Let $(y_\alpha, t_\alpha)_{\alpha \geq 0}$ be a sequence of the epigraph of $\sigma(., g)$, which converges to (y, t). (y, t) belongs to the epigraph of $\sigma(., g)$ if $t \geq \sigma(y, g)$. From the above properties of $\Sigma(y)$, this is equivalent to $y - tg \in Y$. But, $y - tg$ is the limit of $y_\alpha - t_\alpha g$ and these points are in Y since (y_α, t_α) is in the epigraph of $\sigma(., g)$. Consequently, since Y is closed, one gets the result. □

We now explore in more depth the property of the shortage function. We remark that for all $y \in \mathbb{R}^m$, $z = y - (y \cdot g)/(\|g\|^2)g$ is the orthogonal projection of y on $g^\perp$, the orthogonal space to g. Thus, $\sigma(y, g) - (y \cdot g)/(\|g\|^2) = \sigma(z, g)$. Consequently, in the following, we will only consider the restriction of $\sigma(., g)$ to $g^\perp$ since all the properties of the restriction can be immediately extended to the whole space $\mathbb{R}^m$. We start with a sufficient condition for the local Lispschitzianity of the shortage function.

Proposition 6 *Consider a production set Y satisfying Assumption P. Let $y \in \partial Y$ such that $-g \in intT_Y(y)$. Then z, the orthogonal projection of y on $g^\perp$ is in the interior of the domain of $\sigma(., g)$, $y = z - \sigma(z, g)g$ and $\sigma(., g)$ is locally Lipschitz continuous at z.*

Before proving this result a few comments are in order. First, the condition $-g \in \text{int}T_Y(y)$ holds true if $g \in \mathbb{R}^m_{++}$. Indeed, from Assumption P, $-\mathbb{R}^m_+ \subset T_Y(y)$ for all $y \in Y$, so $-g \in \text{int}(-\mathbb{R}^m_+) \subset \text{int}T_Y(y)$. We now consider the case where $g \in \partial\mathbb{R}^m_+$. It is especially interesting for it allows special cases which have nice economic interpretations. Indeed, we could set all coordinates corresponding to outputs equal to zero. Then the shortage would be expressed in terms of inputs. Alternatively one could set all coordinates corresponding to input equal to zero, the shortage function would correspond to a production function. For all $g \in \mathbb{R}^m_+$, the condition $-g \in \text{int}T_Y(y)$ holds true if $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$. Indeed, for all $v \in -\mathbb{R}^m_+ \setminus \{0\}$, for all $p \in N_Y(y) \setminus \{0\}$, $p \cdot v < 0$, which implies that $-\mathbb{R}^m_+ \setminus \{0\} \subset \text{int}T_Y(y)$, and so $-g \in \text{int}T_Y(y)$. The consequences of these remarks are given in the propositions after the proof.

Proof From the characterization of the interior of the Clarke's tangent cone (see, Clarke 1983, Corollary 1 of Theorem 2.5.8) , $-g$ is hypertangent at y. Hence, there exists $r \in]0, 1[$ such that for all $t \in [0, r]$, for all $u \in B(0, r)$, for all $y' \in B(y, r) \cap Y$, $y' - t(g + u) \in Y$. This implies that $y' - t(g + u) \in \text{int}Y$ for all $t \in]0, r]$.

We first prove that $y = z - \sigma(z, g)g$. We first remark that $y = z + (y \cdot g)/(\|g\|^2)g \in Y$. Thus, from the definition of σ, one deduces that $\sigma(z, g) \leq (-y \cdot g)/(\|g\|^2)$. If $y \neq z - \sigma(z, g)g$, $\sigma(z, g) < (-y \cdot g)/(\|g\|^2)$. From the free-disposal assumption, for all $t \geq \sigma(z, g)$, $z - tg \in Y$. For $t < (-y \cdot g)/(\|g\|^2)$ and close enough to $(-y \cdot g)/(\|g\|^2)$, $y' = z - tg \in B(y, r) \cap Y$. Let τ be small enough, such that $t + \tau < (-y \cdot g)/(\|g\|^2)$ and $\tau < r$. One has $y' - \tau g \in \text{int}Y$ from the fact that $-g \in \text{int}T_Y(y)$. Furthermore $y' - \tau g = z - (t + \tau)g \geq z - (-y \cdot g)/(\|g\|^2) = y$. Hence, $y \leq y' - \tau g \in \text{int}Y$ implies $y \in \text{int}Y$ and this contradicts $y \in \partial Y$.

We now prove that $\sigma(., g)$ is bounded above on $B\left(z, \frac{r^2}{2}\right)$. For all $z' \in B\left(z, \frac{r^2}{2}\right)$, one has $y' = z' - \left(\sigma(z, g) + \frac{3\|z' - z\|}{2r}\right) g \in Y$. Indeed,

$$\begin{aligned} y' &= z - \sigma(z, g)g + z' - z - \frac{3\|z' - z\|}{2r} g \\ &= y - \frac{3\|z' - z\|}{2r} \left(g - \frac{2r}{3\|z' - z\|}(z' - z)\right). \end{aligned}$$

Since $\|z' - z\| < \frac{r^2}{2}$, $\frac{3\|z'-z\|}{2r} < r$ and $\frac{2r}{3\|z'-z\|}\|z' - z\| < r$. Thus, $y - \frac{3\|z'-z\|}{2r}(g - \frac{2r}{3\|z'-z\|}(z' - z)) = y - t(g + u)$ with $t \in [0, r[$ and $\|u\| < r$, which implies that it belongs to Y. Since $z' - \left(\sigma(z,g) + \frac{3\|z'-z\|}{2r}\right) g \in Y$, one deduces that $\sigma(z',g) \leq \sigma(z,g) + \frac{3\|z'-z\|}{2r}$. Hence $\sigma(.,g)$ is bounded above on $B\left(z, \frac{r^2}{2}\right)$.

We already know that $\sigma(.,g)$ is lower semicontinuous and it is finite at z. One then deduces that $\sigma(.,g)$ is bounded below on an open neighborhood V of z. Thus, $B\left(z, \frac{r^2}{2}\right) \cap V$ is included in the domain of $\sigma(.,g)$.

We now prove that $\sigma(.,g)$ is Lipschitz continuous on $B\left(z, \frac{r^2}{4}\right) \cap V$. Indeed, let $(z^1, z^2) \in B\left(z, \frac{r^2}{4}\right)$. Without any loss of generality, one can assume that $\sigma(z^1,g) \geq \sigma(z^2,g)$. Since $\|z^1 - z^2\| \leq \frac{r^2}{2}$, one proves with the same argument as above that $z^1 - \left(\sigma(z^2,g) + \frac{3\|z^1-z^2\|}{2r}\right) g \in Y$. Thus, $0 \leq \sigma(z^1,g) - \sigma(z^2,g) \leq \frac{3\|z^1-z^2\|}{2r}$. Hence $\sigma(.,g)$ is Lipschitz continuous on $B\left(z, \frac{r^2}{4}\right) \cap V$ with the constant $3/2r$. □

The next proposition considers the case where g is in the interior of the positive orthant.

Proposition 7 *Consider a production set Y satisfying Assumption P. Let $g \in \mathbb{R}^m_{++}$. Then the domain of definition of $\sigma(.,g)$ is $\mathbb{R}^m$ and it is locally Lipschitzian.*

Proof We have already remarked that $g \in \mathbb{R}^m_{++}$ implies $-g \in \operatorname{int} T_Y(y)$ at every $y \in Y$. From the previous proposition, it remains to prove that $\sigma(.,g)$ is finite everywhere or equivalently on $g^\perp$. Let $\bar{y} \in \partial Y$ and let $z \in g^\perp$. Let $\bar{t} = \max\left\{\frac{z_h - \bar{y}_h}{g_h} \mid h = 1, \ldots m\right\}$ and $\underline{t} = \min\left\{\frac{z_h - \bar{y}_h}{g_h} \mid h = 1, \ldots m\right\}$. One easily checks that $z - \bar{t}g \leq \bar{y} \leq z - \underline{t}g$. Consequently, since $\bar{y} \in \partial Y$ and $g \in \mathbb{R}^m_{++}$, one has $\bar{t} \in \Sigma(z) = \{t \mid z - tg \in Y\}$ and for all $t < \underline{t}$, $t \notin \Sigma(z)$. Thus, $\Sigma(z)$ is nonempty and bounded below, which implies that $\sigma(z,g)$ is finite. □

We now give a synthesis of the results, which gives a sufficient condition in order to characterize the efficient points as the zeros of the shortage function in a neighborhood of a given production vector.

Proposition 8 *Let Y be a production set satisfying Assumption P. Let $y \in \partial Y$ such that $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$. For every $g \in \mathbb{R}^m_+ \setminus \{0\}$, the mapping $\sigma(.,g)$ is defined on an open neighborhood U of y and it is locally Lipschitz continuous. Furthermore, $y' \in U$ is efficient if and only if $\sigma(y',g) = 0$.*

Note that if Y is convex, the result holds true around each point on the boundary if every production plan on the boundary is efficient.

Proof As already noticed, $N_Y(y) \setminus \{0\} \subset \mathbb{R}^m_{++}$ and $g \in \mathbb{R}^m_+ \setminus \{0\}$ imply $-g \in \operatorname{int} T_Y(y)$. Thus, proposition 6 implies that $\sigma(.,g)$ is defined on an open neighborhood of y and it is locally Lipschitz continuous. Proposition 2 implies that every production plan on the boundary of Y in a neighborhood of y is efficient. Thus, there exists a neighborhood U of y on which the two properties hold true. From Proposition 4, if y is efficient, $\sigma(y,g) = 0$. Finally, from Proposition 3, for all $y' \in U$, if $\sigma(y',g) = 0$, one has $y' = y' - \sigma(y',g)g \in \partial Y$ since obviously $\sigma(y',g)$ is finite. Thus, y' is efficient. □

4 Comments and conclusion

We first come back to the incompatibility between the input-output differentiation and the efficiency of every point on the boundary. As we assume that the production set satisfies the free-disposal assumption, the definition of input–output differentiation must be carefully stated since the output component of a production vector may be negative. That is why we posit the following definition.

Definition 1 *A production set Y of $\mathbb{R}^m$ satisfies the input–output differentiation if the set of commodities $\{1, \ldots, m\}$ is shared in a nonempty set of inputs $I = \{1, \ldots k\}$ and a nonempty set of outputs $O = \{k+1, \ldots, m\}$ and for all $y \in Y$, there exists $y' \in Y$ such that $y' \geq y$, $y'_h \geq 0$ for all $h \in O$ and $y'_h \leq 0$ for all $h \in I$.*

If Y satisfies Assumption P and the input–output differentiation, for all $y \in Y$, the production vector y' defined by $y'_h = \max\{y_h, 0\}$ for all $h \in O$ and $y'_h = y_h$ for all $h \in I$ is in Y. We now formally state the incompatibility.

Proposition 9 *Let Y be a production set satisfying Assumption P and the input–output differentiation. Then, there exists an element $y \in \partial Y$ such that y is not efficient.*

Proof Let $s \in 1_m^{\perp}$ be such that $s_h > 0$ for all $h \in I$ and $s_h < 0$ for all $h \in O$. One has $y = s - \sigma(s, 1_m)1_m \in \partial Y$. Since $y_h \leq 0$ for all $h \in I$, one deduces that $\sigma(s, 1_m) > 0$ and consequently, $y_h < 0$ for all $h \in O$. Thus, there exists $y' \in Y$ such that $y' \geq y$, $y'_h \geq 0$ for all $h \in O$ and $y'_h \leq 0$ for all $h \in I$. y is not efficient since $y' \geq y$ and $y' \neq y$. □

We also remark that the important feature of the production set satisfying Assumption P is the fact that the interior of the tangent cone is nonempty at every production vector. Thus, it should be possible to extend most of the previous results to the framework of an infinite dimensional space of commodities with a positive cone having a nonempty interior.

References

Aliprantis, C.D., Brown, D.J., Burkinshaw, O.: Existence and optimality of competitive equilibria. Berlin Heidelberg New-York: Springer-Verlag 1989

Bonnisseau, J.M., Cornet B.: Existence of equilibria when firms follows bounded losses pricing rules. J Math Econ **17**, 119–147 (1988)

Chamber R.G., Chung Y., Färe R.F.: Benefit and distance functions. J Econ Theory **70**(2), 407–419 (1995)

Clarke, F.H.: Optimization and nonsmooth analysis. New-York: Wiley (1983)

Cornet, B.: Existence of equilibria in economies with increasing returns. In: Cornet, B., Tulkens, H. (eds.) Contributions to operations reasearch and economics. Cambridge: MIT Press (1990)

Guesnerie, R.: Pareto optimaliy in nonconvex economies. Econometrica, **43**, 1–29 (1975)

Khan, M.A.: The Mordukhovich normal cone and the foundations of welfare economics. J Pub Econ Theory **1**, 309–338 (1999)

Koopmans, T.C.: Analysis of production as an efficient combination of activities. In: Koopmans T.C. (ed.) Activity analysis of production and allocation. New-York: Wiley (1951)

Luenberger, D.G.: New optimality principles for economic efficiency and equilibrium. J Opt Theory Appl **75**, 221–264 (1992)

Luenberger, D.G.: Microeconomic theory. New York: McGraw-Hill, International Edition 1995
Quinzii, M.: Increasing returns and efficiency. New-York: Oxford University Press 1992
Rockafellar, R.T.: Convex analysis. Princeton: Princeton University Press 1970
Villar, A.: Equilibrium and efficiency in production economies, second Edition. Berlin Heidelberg New-York: Spinger 2000