Economic Theory (2007) 31: 327–342
DOI 10.1007/s00199-006-0094-6

RESEARCH ARTICLE

Yaron Azrieli · Ehud Lehrer

Market games in large economies with a finite number of types

Received: 11 September 2005 / Accepted: 27 February 2006 / Published online: 5 April 2006

Abstract We study market games derived from an exchange economy with a continuum of agents, each having one of finitely many possible types. The type of agent determines his initial endowment and utility function. It is shown that, unlike the well-known Shapley–Shubik theorem on market games (Shapley and Shubik in J Econ Theory 1:9–25, 1969), there might be a (fuzzy) game in which each of its sub-games has a non-empty core and, nevertheless, it is not a market game. It turns out that, in order to be a market game, a game needs also to be homogeneous.

We also study investment games – which are fuzzy games obtained from an economy with a finite number of agents cooperating in one or more joint projects. It is argued that the usual definition of the core is inappropriate for such a model. We therefore introduce and analyze the new notion of *comprehensive core*. This solution concept seems to be more suitable for such a scenario. We finally refer to the notion of feasibility of an allocation in games with a large number of players.

Keywords Market games · Cooperative fuzzy games · Investment games · Core

JEL Classification Numbers C71 · D51

1 Introduction

We study two different economic models of cooperation. The first is an exchange economy that consists of a finite number of types of agents, each consisting of a

Some of the results in this paper appear in a previous draft distributed by the name "Cooperative investment games or Population games".

An anonymous referee of *Economic Theory* is acknowledged for his/her comments.

Y. Azrieli (✉) · E. Lehrer
School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
E-mail: azrieliy@post.tau.ac.il
E-mail: lehrer@post.tau.ac.il

large number of agents. The second model deals with a finite number of agents, each endowed with his own resource, who cooperate in one or more joint projects. The reason for combining these models into one paper is that the mathematical tool used for the analysis is the same.

In their seminal paper Shapley and Shubik (1969) characterize the set of those n-person TU cooperative games that can arise from an exchange economy where the agents have concave monetary utility functions. They write

> "... In reaching these conclusions, however, we were led to a positive result: a surprisingly simple mathematical criterion that tells precisely which games can arise from economic models of exchange (with money)."

Our first result is in this spirit. However, our setup differs from theirs in the cardinality of the set of agents involved. Here, we deal with an economy with a large number of agents divided into a finite number of *types*. It is assumed that agents of the same type are identical both in terms of their utility functions and their initial endowments. This assumption enables us to simplify the mathematics significantly. The reason is that, in order to describe a coalition, we only need to specify how many agents of each type are participating. Thus, we can identify every coalition with a point $c \in \mathbb{R}^n$ such that[1] $0 \leq c \leq Q$, where $Q \in \mathbb{R}^n_+$ specifies the total number of agents from each type (n is the number of types).

Every exchange economy will be defined by the vector $Q \in \mathbb{R}^n_+$ of the total number of agents from each type, and by the initial endowment and utility function of the agents of every type i $(1 \leq i \leq n)$. The worth of a coalition will be the maximal total utility that its members can achieve by reallocating their endowments. Thus, every such economy generates a characteristic function defined over the (compact polyhedral) set $F(Q) = \{c \in \mathbb{R}^n; 0 \leq c \leq Q\}$. We call it the *market game* generated by the economy. The following question is addressed: What are the necessary and sufficient conditions that a characteristic function over $F(Q)$ should satisfy in order for it to be a market game?

There is quite a vast literature on such characteristic functions. In cooperative games theory such a model is usually referred to as a *fuzzy game* (Aubin 1979, 1981). The idea is that the set of n players can choose their "level of participation" in a coalition. Thus, fuzzy coalitions composed of fractions of players can be formed. In this case, the domain of the characteristic function is extended from the vertices of the n-dimensional unit cube to the entire cube.

Another area where a similar model is studied is that of the pricing of a multiproduct monopoly (Sharkey and Telser 1978; Moulin 1988). Here, a monopoly produces n perfectly divisible goods. The cost of producing the bundle $c = (c^1, \ldots, c^n)$, where c^i is the amount of the i-th good being produced $(1 \leq i \leq n)$, is given by the number $v(c)$.

The first step towards answering the question just posed is to show that every market game has a non-empty *core*. Aubin (1979, 1981) defined the core of a fuzzy game to be the set of all vectors $x = \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n$ which are feasible in the sense that $\sum_{i=1}^n x^i = v(1, \ldots, 1)$ and satisfy[2] $xc \geq v(c)$ for all fuzzy coalition $c = \left(c^1, \ldots, c^n\right)$. The logic behind this definition is that when x^i is perceived as

[1] For two vectors z, w in $\mathbb{R}^n$, $z \leq w$ means that $z^i \leq w^i$ for $i = 1, \ldots, n$.
[2] for two vectors $z, w \in \mathbb{R}^n$, zw denotes the inner product $zw = \sum_{i=1}^n z^i w^i$.

a per-unit reward of agent i, then a core element x is resistant to claims of any (fuzzy) coalition[3].

Telser (1978, Chap. 4) makes a thorough study of the core of such characteristic functions. He considers a situation where, as in our case, a large population is composed of a finite number of types. His definition of the core does not enforce agents of the same type to be treated as equals and receive the same allocation. However, an important result he obtains is that, if the core is not empty, it must contain an allocation which gives identical shares to players of the same type.

The core is also important in the context of multi-products pricing. For a given production level $Q \in \mathbb{R}^n_+$, the question being asked is whether there is a prices vector $p \in \mathbb{R}^n$ such that the monopoly exactly covers its cost ($pQ = v(Q)$) and, moreover, $pc \le v(c)$ for any $c \in F(Q)$. This last requirement means that no subset of consumers subsidizes the rest of the population. Indeed, if $pc > v(c)$ for some $c \in F(Q)$, then a set of consumers who buy exactly the bundle c is subsidizing the rest of the economy. If p satisfies these conditions, then $-p$ is in the core of the fuzzy game defined by the function $-v$. Thus, existence of such a prices vector is equivalent to non-emptiness of the core of the fuzzy game defined by $-v$. Sharkey and Telser (1978) call the function v *supportable* if there is such a prices vector at every level of production $Q \in \mathbb{R}^n_+$.

Unlike the Shapley-Shubik theorem (Shapley and Shubik 1969), non-emptiness of the core of every sub-game of a given fuzzy game doesn't imply that this is a market game. It is shown that a necessary condition for market games is homogeneity of the function v. It turns out that homogeneity together with total balancedness is what characterizes market games in our setup.

In the second model we discuss, n agents can cooperate in a joint project. Each one of the agents is endowed with an individual resource or ability. An agent may choose to invest any part of it in the joint project. If agent i's ($1 \le i \le n$) level of investment (effort) is $c^i \ge 0$, then the worth of the investment profile $c = \left(c^1, \ldots, c^n\right)$ is the maximum total utility that the agents can derive from it by (optimally) splitting it among them. If $Q^i > 0$ is the total amount of agent i's resource, and $Q = \left(Q^1, \ldots, Q^n\right)$, then for any investment profile $c \in F(Q)$, $v(c)$ is the worth of c. Thus, every such economy defines a fuzzy game over $F(Q)$, called an *investment game*.

We argue that the standard definition of the core is inappropriate for investment games. Indeed, assume that the allocation x is blocked by the investment profile $c = \left(c^1, \ldots, c^n\right)$. That is, $xc < v(c)$. In this case, each agent i is left with amount $Q^i - c^i$ of his resource uninvested. It might be that, no matter how the agents invest this remainder, the total worth of their investment is smaller than that guaranteed by x. Thus, from a comprehensive perspective on the entire investment, x is a stable allocation.

The above argument suggests that in investment games a core allocation should take into account the total value of investing the entire resources, and not only a fraction of it. The notion of the *comprehensive core* we introduce captures this idea. An allocation x is in the comprehensive core if no subset of agents can do better than what is obtained by x, no matter how they choose to invest their entire

[3] Other papers discussing cores of fuzzy games include (Butnariu 1980) where a different definition of the core appears, and (Branzei et al. 2003) where cores of convex fuzzy games are studied.

resources. We prove that the comprehensive core always contains the standard core but not vise versa. We obtain a characterization of games whose comprehensive core is non-empty.

We also consider the case where the agents have "equal tastes" (identical utility functions) and the case of super-additive utility functions (not necessarily identical). It is shown that in both these cases the characteristic function obtained is super-additive on $F(Q)$. The converse is also true. That is, if v is a super-additive[4] characteristic function on $F(Q)$ then it is an investment game of agents with equal tastes and of agents with super-additive utility functions.

Finally, we comment on the feasibility of an allocation in the context of cooperative games with a large set of players. It is assumed in many economic models that the characteristic function is super-additive. However, if the population is very large, then it sometimes seems reasonable to assume that by splitting into several smaller coalitions the total payoff to the entire society will increase. Thus, an allocation that is not feasible by the usual definition of feasibility may become feasible if we allow the grand coalition to split.

We therefore introduce another core-like solution concept: the *split core*. The split core of a fuzzy game consists of those allocations which are feasible in this broader sense and, as in the standard core, no coalition can improve upon. It is obvious that the split core of a game contains its core and that the opposite direction need not be true. Conditions for non-emptiness of the split core are obtained from the conditions for the standard core.

In section 2 we formally define fuzzy games, market games and investment games. Section 3 contains the main result of the paper – characterization of market games. In section 4 we discuss investment games and in section 5 the split core is introduced. Some of the proofs appear in the Appendix.

2 The models

2.1 Fuzzy games

For every vector $Q \in I\!R^n$ with $Q \geq 0$, let $F(Q)$ be the box $F(Q) = \{c \in I\!R^n; 0 \leq c \leq Q\}$. The point Q is interpreted as the "grand coalition". The characteristic function is defined on the entire box $F(Q)$. Formally,

Definition 1 *A* fuzzy game *is a pair* (v, Q) *such that*

(i) $Q \in \mathbb{R}^n$ *and* $Q \geq 0$*;*
(ii) $v : F(Q) \to \mathbb{R}$ *is bounded and satisfies* $v(0) = 0$.

We will also be interested in sub-games of a given fuzzy game. These are naturally defined as in the theory of classic cooperative games.

Definition 2 *Let* (v, Q) *be a fuzzy game and fix some* $c \in F(Q)$*. The* sub-game *of* (v, Q) *with respect to* c *is* (v_c, c)*, where for every* $d \in F(c)$, $v_c(d) = v(d)$.

[4] That is, the worth of a coalition is not smaller than the sum of the worths of the coalitions in any partition of it.

2.2 Market games

Let $N = \{1, 2, \ldots, n\}$ be the set of *types* of agents in a large economy. The set of type i agents ($1 \leq i \leq n$) is identified with the closed interval $\left[0, Q^i\right]$, where $Q^i \geq 0$ is some measurement of the number of type i agents. A *coalition* is any product of measurable sets $C_1 \times \cdots \times C_n$ where $C_i \subseteq \left[0, Q^i\right], i = 1, \ldots, n$. if $C = C_1 \times \cdots \times C_n \subseteq F(Q)$ is a coalition and c^i is the measure of C_i then C can be identified (for our purposes) with the vector $c = \left(c^1, \ldots, c^n\right) \in F(Q)$. That is, we are only concerned with the number of agents of each type in the coalition and not with their identity. Therefore, from now on, a coalition is a point $c \in F(Q)$, and the set of all possible coalitions is $F(Q)$.

Each type of agent in the economy is characterized by an initial endowment and a utility function. The initial endowment of agents of type i is a bundle of goods which is represented by the point $w_i \in I\!R^{\ell}_{+}$. The utility function of type i agents is $u_i : I\!R^{\ell}_{+} \to I\!R$. We assume that, for every $1 \leq i \leq n$, u_i satisfies: (1) $u_i(0) = 0$; and (2) there is a constant M_i such that for every $c \in F(Q)$ either $u_i(c) \leq M_i$ or $u_i(c) \leq |c| M_i$, where $|c| = \sum_{i=1}^{n} c^i$ is the l_1 norm of c. Nothing like monotonicity or concavity is required.

Assume that the coalition $c \in F(Q)$ is formed. The entire endowment of c is $w_c = \sum_{i=1}^{n} c^i w_i$. The members in c can split w_c into bundles $y^i_k \in I\!R^l_{+} (1 \leq i \leq n, 1 \leq k \leq K_i)$, where $\sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k y^i_k = w_c$ and $\sum_{k=1}^{K_i} \gamma^i_k = c^i$ for every $i = 1, \ldots, n$. This means that type-i agents are split into K_i groups, such that group k is of size γ^i_k, and each agent in group k receives the bundle y^i_k. All together, the utility of the entire coalition with such allocation is $\sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k u_i\left(y^i_k\right)$.

Every economy $\left(Q, \{w_i, u_i\}_{i=1}^{n}\right)$ naturally defines a fuzzy game. If $c \in F(Q)$ is a coalition then $v(c)$ is the maximum achievable utility of the coalition c on its own. Formally,

Definition 3 *The* market game *induced by* $\left(Q, \{w_i, u_i\}_{i=1}^{n}\right)$ *is the fuzzy game* (v, Q) *where for every* $c \in F(Q)$,

$$v(c) = \sup \left\{ \sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k u_i(y^i_k); \sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k y^i_k \leq w_c, \sum_{k=1}^{K_i} \gamma^i_k = c^i, i = 1, \ldots, n \right\}.$$

(The obvious constraints $\gamma^i_k \geq 0$ *and* $y^i_k \in I\!R^{\ell}_{+}$ *are omitted.)*

Remark 1 (1) v is bounded over $F(Q)$. Indeed, for $c \in F(Q)$ if $\sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k y^i_k \leq w_c$ and $\sum_{k=1}^{K_i} \gamma^i_k = c^i, i = 1, \ldots, n$, then $\sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k u_i(y^i_k) \leq \sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k \left(M_i + \left|y^i_k\right| M_i\right) \leq (|c| + |w_c|) M$, where $M = \sum_{i=1}^{n} M_i$. Thus, $v(c) \leq (|c| + |w_c|) M \leq (|Q| + |w_Q|) M$ for every $c \in F(Q)$.
(2) If u_i is continuous and concave, there is a constant M_i such that for every c either $u_i(c) \leq M_i$ or $u_i(c) \leq |c| M_i$. Indeed, when u_i is continuous, there is M^1_i such that $u_i(c) \leq M^1_i$ when $|c| \leq 1$. Let M^2_i be the maximum of u_i over Δ. Then, by concavity, if $|c| \geq 1, u_i(c) \leq |c| M^2_i - u_i(0)$. Set, $M_i = \max\left(M^1_i, M^2_i + |u_i(0)|\right)$.

Note that, in the definition of a market game (Definition 3), $\sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k y^i_k$ is required to be not larger than, but not necessarily equal to w_c. Thus, we allow for free disposal of excessive quantity. It is therefore clear that $v(0) = 0$ and that v is monotonically non-decreasing.

This construction of a cooperative game from a market is in the spirit of Shapley and Shubik (1969). In their paper, they deal with an exchange economy that consists of a finite number of agents and, therefore, the resulting game is a (TU) cooperative game in its classical form. Here, in contrast, the resulting game is a fuzzy game.

2.3 Investment games

Consider a situation where a finite number of agents are involved in a joint project. Assume that each one of the agents contributes his own special ability or resource to the project. Moreover, assume that for every agent $i \in N$ (where N is the set of agents, $|N| = n$) $Q^i \geq 0$ is a measurement of the amount of the special resource that agent i has. However, an agent is not restricted to invest his entire quantity in the project, and can choose any participation level c^i $\left(0 \leq c^i \leq Q^i\right)$.

Each agent $i \in N$ is endowed with a utility function $u_i : F(Q) \to I\!R$. That is, if $c = \left(c^1, \ldots, c^n\right)$ is the *investments profile* of the various agents then the utility of agent i from the joint project is $u_i(c)$. Such a model naturally gives rise to a fuzzy game.

Definition 4 *The* investment game *induced by* $\left(Q, \{u_i\}_{i=1}^n\right)$ *is the fuzzy game* (v, Q) *where, for every* $c \in F(Q)$,

$$v(c) = \sup\left\{\sum_{i=1}^n u_i(d_i); \quad \sum_{i=1}^n d^i = c, d_i \in F(c), \ i = 1, \ldots, n\right\}.$$

3 Characterizing market games

This section contains the main result of the paper – a characterization of market games derived from large economies with a finite number of types of agents. In subsections 3.1 and 3.2 we introduce two necessary conditions for market games, namely, totally balancedness and homogeneity. Subsection 3.3 shows that these conditions (together) are also sufficient.

3.1 Totally balanced fuzzy games

Definition 5 *The* core *of the fuzzy game* (v, Q), *denoted* core(v, Q), *is the set of vectors* $x = \left(x^1, \ldots, x^n\right)$ *such that*

(i) $xQ = v(Q)$*; and*
(ii) $xd \geq v(d)$ *for any coalition* $d \in F(Q)$.

To remain consistent with the terminology of classical cooperative games, we say that the fuzzy game (v, Q) is *balanced*, if core$(v, Q) \neq \emptyset$. If each one of the sub-games of (v, Q) is balanced we say that (v, Q) is *totally balanced*.

Our first task is to find conditions on the function v itself which are equivalent to non-emptiness of core(v, Q). Such conditions were given by Aubin (1979,

1981) for the case of a homogeneous v. Telser (1978) called a totally balanced characteristic function *kind* and provided necessary and sufficient conditions even for the case of a continuum number of different types. Sharkey and Telser (1978) deal with a similar problem in the context of natural monopolies. For completeness, and since we think that our proof is interesting on its own, we bring here the proof in its full. We first need the following definition.

Definition 6 *Fix* $Q \in I\!R^n$, $Q \geq 0$ *and let* $v : F(Q) \to I\!R$.

(i) The Strong Super-Additive cover *of* v *is the function* $\mathbf{SSa}v : F(Q) \to I\!R$ *defined by*

$$\mathbf{SSa}v(d) = \sup\left\{\sum_{j=1}^{L} \lambda_j v(c_j);\, L \in \mathbb{N}, \sum_{j=1}^{L} \lambda_j c_j = d, \lambda_j \geq 0, c_j \in F(d), j = 1, \ldots, L\right\}.$$

(ii) v *is called* Strongly Super-Additive (SSA) *if* $v = \mathbf{SSa}v$ *on* $F(Q)$.

For some properties of the **SSa** operator the reader is referred to section 6.2 in the Appendix.

Proposition 1 *(i) The fuzzy game* (v, Q) *is balanced iff* $\mathbf{SSa}v(Q) = v(Q)$.
(ii) The fuzzy game (v, Q) *is totally balanced iff* v *is SSA.*

The proof of Proposition 1 appears in the Appendix. The next step is to use it in order to show that every market game is balanced. This is done in the following proposition.

Proposition 2 *Every market game has a non-empty core.*

Proof Let (v, Q) be the market game induced by $\left(Q, \{w_i, u_i\}_{i=1}^{n}\right)$. By Proposition 1 it is sufficient to show that $\mathbf{SSa}v(Q) = v(Q)$. Fix $\varepsilon > 0$ and suppose that $\mathbf{SSa}v(Q) \leq \sum_{j=1}^{L} \lambda_j v(c_j) + \varepsilon$ for some integer L, $\sum_{j=1}^{L} \lambda_j c_j = Q, \lambda_j > 0, c_j \in F(Q), j = 1, \ldots, L$. Furthermore, suppose that, for every $j = 1, \ldots, L$, the bundles $\left(y_{j,k}^i\right)_{i=1}^{n}$ in $I\!R_+^l$ satisfy $\sum_{i=1}^{n} \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i y_{j,k}^i \leq w_{c_j}, \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i = c_j^i$, $\gamma_{j,k}^i \geq 0$ and $v(c_j) \leq \sum_{i=1}^{n} \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i u_i \left(y_{j,k}^i\right) + \frac{\varepsilon}{L\lambda_j}$. Then,

$$\begin{aligned}\mathbf{SSa}v(Q) &\leq \sum_{j=1}^{L} \lambda_j \left(\sum_{i=1}^{n} \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i u_i \left(y_{j,k}^i\right) + \frac{\varepsilon}{L\lambda_j}\right) + \varepsilon \\ &= \sum_{i=1}^{n} \sum_{j=1}^{L} \sum_{k=1}^{K_{ij}} \lambda_j \gamma_{j,k}^i u_i \left(y_{j,k}^i\right) + 2\varepsilon \leq v(Q) + 2\varepsilon.\end{aligned}$$

The last inequality is due to $Q^i = \sum_{j=1}^{L} \lambda_j c_j^i = \sum_{j=1}^{L} \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i$ $(i = 1, \ldots, n)$ and to $w_Q = \sum_{j=1}^{L} \lambda_j w_{c_j} \geq \sum_{j=1}^{L} \lambda_j \sum_{i=1}^{n} \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i y_{j,k}^i = \sum_{j=1}^{L} \sum_{i=1}^{n} \sum_{k=1}^{K_{ij}} \lambda_j \gamma_{j,k}^i y_{j,k}^i$.

It follows that $\mathbf{SSa}v(Q) \leq v(Q) + 2\varepsilon$ for any $\varepsilon > 0$, and therefore, $\mathbf{SSa}v(Q) = v(Q)$. This shows that the core of (v, Q) is not empty. □

Corollary 1 *If (v, Q) is a market game then it is totally balanced.*

Proof It is clear that any sub-game of a market game is itself a market game. Therefore, (v, Q) is totally balanced. □

3.2 Homogeneous fuzzy games

In contrast of Shapley and Shubik (1969), not every totally balanced fuzzy game is a market game. To see this, consider the following example.

Example 1 Let $n = 1$, $Q = 1$ and $v(t) = t^2$ for every $0 \le t \le Q$. Clearly, v is SSA. Therefore, the core of any sub-game of v is not empty. If v is a market game, the coalition Q can reallocate the initial endowments of its members so as to derive a total utility of 1. The coalition $Q/2$, which has half of the resources that Q has, can reallocate these resources in precisely the same proportions as Q did. By so doing, coalition $Q/2$ may derive half of the utility of Q, which is $1/2 > v(Q/2) = 1/4$. Thus, v cannot be a market game.

It follows that more conditions should be added in order to characterize market games. The above example suggests that a homogeneity condition is missing.

Definition 7 *The fuzzy game (v, Q) is* homogeneous *if v is a homogeneous (of degree 1) function on $F(Q)$. That is $v(\lambda c) = \lambda v(c)$ whenever $c, \lambda c \in F(Q)$.*

When the game is homogeneous, the worth of any coalition c is $|c|v(c/|c|)$, which is the worth of a coalition whose size is 1 and its internal distribution is $(c/|c|)$ multiplied by the size of c.

Proposition 3 *Every market game is homogeneous.*

Proof Since (v, Q) is a market game, by Proposition 2 and by Corollary 1, v is SSA on $F(Q)$. Therefore, if $c \in F(Q)$ and $0 \le \lambda \le 1$, then $v(\lambda c) \le \lambda v(c)$. To obtain the inverse inequality, fix $\varepsilon > 0$ and let $\left\{\gamma_k^i, y_k^i\right\}_{1\le i\le n, 1\le k\le K_i}$ be such that $\sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i y_k^i \le w_c$, $\sum_{k=1}^{K_i} \gamma_k^i = c^i$, $i = 1, \ldots, n$ and $\sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i u_i\left(y_k^i\right) \ge v(c) - (\varepsilon/\lambda)$. The collection $\left\{\lambda\gamma_k^i, y_k^i\right\}_{1\le i\le n, 1\le k\le K_i}$ is possible in the definition of $v(\lambda c)$, and therefore $v(\lambda c) \ge \sum_{i=1}^n \sum_{k=1}^{K_i} \lambda\gamma_k^i u_i(y_k^i) \ge \lambda v(c) - \varepsilon$. Since this is true for every $\varepsilon > 0$, we get that $v(\lambda c) \ge \lambda v(c)$ and therefore v is homogeneous. □

3.3 Constructing a market from a game

We now show that if a fuzzy game is non-decreasing, homogeneous and totally balanced then it is a market game. The idea of the proof is similar to that of Shapley and Shubik (1969). That is, given a fuzzy game (v, Q) we identify the set of (l) goods with the set of (n) types of agents and the initial endowment of type i agents (w_i) with the i-th standard vector of $I\!R^n$. The utility functions of all the agents are identified with v itself. In the terminology of Shapley-Shubik (1969) this is the "direct market" generated by the fuzzy game (v, Q).

Theorem 1 *A fuzzy game (v, Q) is a market game if and only if:*

(i) v is monotonically non-decreasing on $F(Q)$;
(ii) (v, Q) is homogeneous; and
(iii) (v, Q) is totally balanced.

Proof The claim, that if (v, Q) is a market game then it is totally balanced is due to Corollary 1. The homogeneity of v is due to Proposition 3. It is also clear by the definition of a market game that v is non-decreasing on $F(Q)$.

Now suppose that (v, Q) is totally balanced and that v is a homogeneous non-decreasing function on $F(Q)$. We show that (v, Q) is a market game. Let w_i be the i-th standard basis vector of $I\!R^n$, and $u_i = v, i = 1, \ldots, n$ (notice that we take $\ell = n$).

Denote by (r, Q) the market game induced by $\left(Q, \{w_i, v\}_{i=1}^n\right)$. It remains to prove that $r = v$ on $F(Q)$. By the monotonicity of v and since $w_c = c$, we have,

$$r(c)=\sup\left\{\sum_{i=1}^{n}\sum_{k=1}^{K_i}\gamma_k^i v\left(y_k^i\right);\quad \sum_{i=1}^{n}\sum_{k=1}^{K_i}\gamma_k^i y_k^i = c,\quad \sum_{k=1}^{K_i}\gamma_k^i = c^i,\ i = 1, \ldots, n\right\}$$

(notice that the monotonicity of v enables us to replace the inequality in the definition of a market game with equality). Since every such partition of c is possible also in the definition of $\mathbf{SSa}v(c)$, we have that $r \leq \mathbf{SSa}v$. However, by Proposition 1, and since (v, Q) is totally balanced, v is SSA. Therefore, $r \leq \mathbf{SSa}v = v$.

To show that $r \geq v$, recall that r is homogeneous because (r, Q) is a market game and that v is homogeneous by assumption. Therefore we can assume w.l.o.g. that $\Delta \subseteq F(Q)$, where Δ is the unit simplex of $I\!R^n$. Fix $c \in \Delta$ and for $i = 1, \ldots, n$ take $K_i = 1, \gamma^i = c^i$ and $y^i = c$. Then by the definition of r, $r(c) \geq \sum_{i=1}^n c^i v(c) = v(c)$. It follows that $r = v$ on Δ and by homogeneity $r = v$ on $F(Q)$. Therefore, $(v, Q) = (r, Q)$ is a market game. □

As a last remark in this section, we note that if (v, Q) satisfies conditions (i), (ii) and (iii) of Theorem 1 then we can say more than that (v, Q) is a market game. Indeed, the proof of Theorem 1 implies that (v, Q) is a market game where the utility functions of all the agents in the economy are homogeneous and SSA. In the following lemma we show that a homogeneous SSA function is also concave. Thus, the utility functions in the economy can always be taken to be concave. Before proving the lemma we define the super-additive cover of a fuzzy game.

Definition 8 *Fix $Q \in I\!R^n$, $Q \geq 0$ and let $v : F(Q) \to I\!R$.*

(i) *The* Super-Additive cover *of v is the function* $\mathbf{Sa}v : F(Q) \to I\!R$ *defined by*

$$\mathbf{Sa}v(d) = \sup\left\{\sum_{j=1}^{L} v(c_j);\, L \in \mathbb{N}, \sum_{j=1}^{L} c_j = d, c_j \in F(d),\, j = 1, \ldots, L\right\}.$$

(ii) *v is called* Super-Additive (SA) *if $v = \mathbf{Sa}v$ on $F(Q)$.*

Some properties of the **Sa** operator appear in section 6.2 in the Appendix.

Lemma 1 *Let (v, Q) be a homogeneous fuzzy game. The following are equivalent:*

(i) v *is SSA on* $F(Q)$.
(ii) v *is SA on* $F(Q)$.
(iii) v *is concave on* $F(Q)$.

Proof It is straightforward that under homogeneity (i) and (ii) are equivalent.
$(ii) \Rightarrow (iii)$ If $c, d \in F(Q)$ and $\alpha \in (0,1)$ then $v(\alpha c + (1-\alpha)d) \geq v(\alpha c) + v((1-\alpha)d) = \alpha v(c) + (1-\alpha)v(d)$ so v is concave.
$(iii) \Rightarrow (ii)$ For $c, d \in F(Q)$ with $c + d \in F(Q)$, by concavity and homogeneity of v we have, $\frac{1}{2}v(c+d) = v\left(\frac{c+d}{2}\right) \geq \frac{1}{2}v(c) + \frac{1}{2}v(d)$ so v is SA. □

4 Cooperative investment games

4.1 The comprehensive core

The first issue we wish to address is the relevance of the core as a notion of stability in investment games. Recall Definition 5. An allocation x might not be in the core if there is a blocking investment profile $c \in F(Q)$ such that $v(c) > xc$. This means that the investment profile c yields a higher total utility than the share guaranteed by x. However, there is no reference to what remains after investing c. That is, every agent $i \in N$ is left with an excess amount of $Q^i - c^i$ unused resources. It might be that the yield of this remainder is so low that the total yield is less than the share guaranteed by x.

For every $S \subseteq N$, let Q_S be the n-dimensional vector which coincides with Q on the coordinates that belong to S, and is equal to zero otherwise. If a coalition S is not satisfied with the allocation x, it means that it has a comprehensive investment (rather than a partial one, as suggested by the core) of its entire resources, Q_S, that yields a higher total utility than xQ_S. By a comprehensive investment we mean investments $c_i \in F(Q_S)$, $i = 1, \ldots, L$, that satisfy $\sum_{i=1}^{L} c_i = Q_S$. The coalition S of players has a justified claim against x using such a comprehensive investment if $xQ_S < \sum_{i=1}^{L} v(c_i)$. A stable allocation in the context of investment games is, therefore, an allocation where none of the coalitions has a justified claim against using any comprehensive investment.

By Definition 8, for any coalition $S \subseteq N$, $\mathbf{Sa}v(Q_S)$ is the maximal achievable payoff of the coalition S using a comprehensive investment. The *comprehensive core* consists of all those allocations which there is no justified claim against using a comprehensive investment. Formally,

Definition 9 *The* comprehensive core *of a fuzzy game* (v, Q), *denoted* $\mathrm{C}-\mathrm{core}(v, Q)$, *is the set of vectors* $x = \left(x^1, \ldots, x^n\right)$ *such that*

(i) $xQ = \mathbf{Sa}v(Q)$;
(ii) $xQ_S \geq \mathbf{Sa}v(Q_S)$ *for any coalition* $S \subseteq N$.

The first observation we make is that any core allocation of a fuzzy game is also in the comprehensive core, while the converse is wrong.

Proposition 4 *(i) For any fuzzy game* (v, Q), $\mathrm{core}(v, Q) \subseteq \mathrm{C}-\mathrm{core}(v, Q)$.
(ii) There might be $x \in \mathrm{C}-\mathrm{core}(v, Q)$ *such that* $x \notin \mathrm{core}(v, Q)$.

Proof (i) Let $x \in \text{core}(v, Q)$. Since obviously $\mathbf{SSa}v(Q) \geq \mathbf{Sa}v(Q) \geq v(Q)$ and by Proposition 1 (i), we have $v(Q) = \mathbf{SSa}v(Q) \geq \mathbf{Sa}v(Q) \geq v(Q)$. This implies $xQ = \mathbf{Sa}v(Q)$. Next, fix $\epsilon > 0$, $S \subseteq N$ and let $c_i \in F(Q_S)$, $1 \leq i \leq L$ be such that $\mathbf{Sa}v(Q_S) - \epsilon \leq \sum_{i=1}^{L} v(C_i)$. Then $xQ_S = \sum_{i=1}^{L} xc_i \geq \sum_{i=1}^{L} v(c_i) \geq \mathbf{Sa}v(Q_S) - \epsilon$. This implies $xQ_S \geq \mathbf{Sa}v(Q_S)$ so $x \in \text{C} - \text{core}(v, Q)$.
(ii) The following is an example of a fuzzy game with empty core and non-empty comprehensive core. Let $n = 2$, $Q = (1, 1)$, $v(Q) = 2$, $v((1, 1/2)) = 3$, $v((0, t)) = -2t$ for $0 \leq t \leq 1/2$ and $v(c) = 0$ otherwise. We have that $Q = (1, 1/2) + (1/2)(0, 1)$ but $v(Q) = 2 < 3 = v((1, 1/2)) + (1/2)v((0, 1))$, so by Proposition 1 (i) the core of (v, Q) is empty. On the other hand, $x = (1, 1)$ is in $\text{C} - \text{core}(v, Q)$. When players 1 and 2 consider forming the mixed coalition $(1, 1/2)$, player 2 is left with an excess amount of $1/2$. Any way he might cut this amount into pieces yields -1. Thus, the net value is 2, which is what is given to this coalition by x. □

A condition characterizing fuzzy games with a non-empty comprehensive core can easily be obtained using the Bondareva–Shapley theorem (Bondareva 1962; Shapley 1967).

Proposition 5 $\text{C} - \text{core}(v, Q)$ *is not empty if and only if* $\sum_{S\subseteq N} \delta_S Q_S = Q$ *where* $\delta_S \geq 0$, $S \subseteq N$ *implies* $\mathbf{Sa}v(Q) \geq \sum_{S\subseteq N} \delta_S \mathbf{Sa}v(Q_S)$.

Proof Define the auxiliary (classical) cooperative game (N, v_N) by $v_N(S) = \mathbf{Sa}v(Q_S)$ for any $S \subseteq N$. This game has a non-empty core iff $\sum_{S\subseteq N} \delta_S \mathbb{I}_S = \mathbb{I}_N$ where $\delta_S \geq 0$ for every $S \subseteq N$ implies $\sum_{S\subseteq N} \delta_S v_N(S) \leq v_N(N)$. Since $\sum_{S\subseteq N} \delta_S \mathbb{I}_S = \mathbb{I}_N$ is equivalent to $\sum_{S\subseteq N} \delta_S Q_S = Q$, and $x = \left(x^1, \ldots, x^n\right) \in \text{C} - \text{core}(v, Q)$ iff $y = \left(Q^1 x^1, \ldots, Q^n x^n\right)$ is in the core of (N, v_N), the proof is complete. □

4.2 Investment games with "equal tastes"

In this subsection we consider the special case where all the agents have the same utility functions. That is, $u_i = u$, $1 \leq i \leq n$. In this case we say that the agents have *equal tastes*. We are interested in a characterization of investment games in this simple case. It turns out that super additivity is all what needed.

Proposition 6 *The fuzzy game* (v, Q) *is an investment game of agents with equal tastes if and only if* v *is SA on* $F(Q)$.

Proof Assume first that (v, Q) is the investment game generated by $\left(Q, \{u_i = u\}_{i=1}^{n}\right)$. Then, by definition (recall Definition 4), $v = \mathbf{Sa}u$. By Lemma 4 (3) in the Appendix v is SA.

Conversely, assume that v is SA on $F(Q)$. Consider the investment game generated by the utility functions $u_i = v$, $1 \leq i \leq n$. By definition, this investment game is $(\mathbf{Sa}v, Q)$. Since v is SA, $v = \mathbf{Sa}v$ and we are done. □

4.3 Investment games with super-additive utility functions

Another case which can be easily analyzed is when the utility functions are all SA. This case reflects a scenario where, for every agent, the utility from any two separated investments is smaller than the utility of their union. If this is the situation then the resulting investment game must also be SA. Formally,

Proposition 7 *The fuzzy game (v, Q) is an investment game of agents with super-additive utility functions if and only if v is SA on $F(Q)$.*

Proof Assume that (v, Q) is the investment game generated by $\left(Q, \{u_i\}_{i=1}^n\right)$ where u_i is SA for every $1 \leq i \leq n$. Fix $c \in F(Q)$ and assume that $\sum_{j=1}^L d_j = c$, where $d_j \in F(c)$ for $1 \leq j \leq L$. Let $\epsilon > 0$. For every $1 \leq j \leq L$ there are vectors $\left\{d_j^i\right\}_{i=1}^n$ such that $\sum_{i=1}^n d_j^i = d_j$ and $v(d_j) - (\epsilon/L) \leq \sum_{i=1}^n u_i\left(d_j^i\right)$. Since $\sum_{i=1}^n \sum_{j=1}^L d_j^i = c$ and by the super additivity of the utility functions, we have $v(c) \geq \sum_{i=1}^n u_i\left(\sum_{j=1}^L d_j^i\right) \geq \sum_{i=1}^n \sum_{j=1}^L u_i\left(d_j^i\right) \geq \sum_{j=1}^L (v(d_j) - (\epsilon/L)) = \sum_{j=1}^L v(d_j) - \epsilon$. This implies that v is SA.

Conversely, if v is SA then, similarly to the proof of Proposition 6, one can construct an investment game which equals v on $F(Q)$. □

5 On the feasibility of an allocation

What does it mean that the allocation $x \in I\!R^n$ is feasible for the grand coalition Q in a fuzzy game (v, Q)? The standard answer is, as in Definition 5, that $xQ = v(Q)$. An assumption underlying this definition is that the characteristic function v is super-additive and, thus, if the grand coalition will split into several smaller coalitions, the total worth will decrease. However, if the set of agents is very large, it might be the case that splitting is beneficial. That is, the standard assumption of v being super-additive seems doubtful.

If a fuzzy game (v, Q) is interpreted as a cooperative game with a continuum of players from a finite number of types (as in Telser 1978, for instance), then it seems reasonable that $\mathbf{Sa}v(Q) > v(Q)$. That is, there is a collection of coalitions $c_1, \ldots, c_L$ with $\sum_{i=1}^L c_i = Q$ such that $\sum_{i=1}^L v(c_i) > v(Q)$. A notion of feasibility of an allocation $x \in I\!R^n$ that better fits this scenario is, therefore, $xQ = \mathbf{Sa}v(Q)$.

Definition 10 *The* split core *of a fuzzy game (v, Q), denoted* $\mathrm{S-core}(v, Q)$*, is the set of vectors $x = \left(x^1, \ldots, x^n\right)$ such that*

(i) $xQ = \mathbf{Sa}v(Q)$;
(ii) $xc \geq v(c)$ for any coalition $c \in F(Q)$.

Here, the split-core allocation x is resistent to any blocking coalition, as in the core. However, it might be that $xQ > v(Q)$, meaning that x is not available if the grand coalition is formed. When the entire population splits, the total worth is increased and the allocation x becomes available.

It is important to note that, if $x \in \mathrm{S-core}(v, Q)$ then, for any $c \in F(Q)$, $xc \geq \mathbf{Sa}v(c)$. It implies that the allocation x is resistant to claims of any coalition c, even if c is allowed to split into smaller coalitions.

It is easy to see that $\mathrm{core}(v, Q) \subseteq \mathrm{S-core}(v, Q)$. The following proposition characterizes fuzzy games with non-empty split core.

Proposition 8 S − core(v, Q) *is non-empty if and only if* $\mathbf{SSa}v(Q) = \mathbf{Sa}v(Q)$.

Proof Consider the auxiliary fuzzy game (v', Q) defined by $v'(c) = v(c)$ if $c \neq Q$ and $v'(Q) = \mathbf{Sa}v(Q)$. By the definition of the split core we have that S − core(v, Q) = core(v', Q). By Proposition 1, core(v', Q) is not empty if and only if $v'(Q) = \mathbf{SSa}v'(Q)$. However, $v'(Q) = \mathbf{Sa}v(Q)$ by definition, and since v and v' coincide on $F(Q)\setminus\{Q\}$, it is obvious that $\mathbf{SSa}v(Q) = \mathbf{SSa}v'(Q)$. Therefore, $v'(Q) = \mathbf{SSa}v'(Q)$ is equivalent to $\mathbf{Sa}v(Q) = \mathbf{SSa}v(Q)$. □

In the following example (v, Q) has an empty core but a non-empty split core.

Example 2 Let $n = 2$, $Q = (1, 1)$ and $v(c) = v(c^1, c^2) = ((c^1)^2/(c^1 + c^2))$. Notice that v is homogeneous on $F(Q)$. This implies that $\boldsymbol{SSa}v = \mathbf{Sa}v$ so, by Proposition 8, S − core(v, Q) is not empty. On the other hand, $\boldsymbol{SSa}v(Q) \geq v(1, 0) + v(0, 1) = 1 > 1/2 = v(Q)$. Thus, by Proposition 1, (v, Q) has an empty core.

6 Appendix

6.1 Proof of Proposition 1

In order to prove the proposition, we define an auxiliary function on Δ – the unit simplex of $\mathbb{R}^n$.

Definition 11 *Let* (v, Q) *be a fuzzy game. Define* $u_{v,Q} : \Delta \to \mathbb{R} \cup \{+\infty\}$ *by* $u_{v,Q}(q) = \sup\{v(c)/|c|; c \in F(Q), q = (c/|c|)\}$.

To illustrate the definition consider the following example.

Example 3 Let $n = 2$, $Q = (2, 2)$, and for any $c \in F(Q)$ let $v(c) = \left(c^1 + c^2\right)^2$. Fix $q = \left(q^1, 1 - q^1\right) \in \Delta$. In order to compute $u_{v,Q}(q)$, we need to maximize $\left(\left(c^1 + c^2\right)^2 / \left(c^1 + c^2\right)\right) = c^1 + c^2$ with the constrains that $c \in F(Q)$ and $\left(c^1 / \left(c^1 + c^2\right)\right) = q^1$. A simple computation yields $u_{v,Q}(q) = \left(2/ (1 - \min \left\{q^1, 1 - q^1\right\}\right))$. □

Definition 12 *Assume that* $u_{v,Q}$ *is bounded. The* concavification *of* $u_{v,Q}$*, denoted* $\boldsymbol{Cav}u_{v,Q}$*, is defined as the minimum of all concave functions* $g : \Delta \to \mathbb{R}$ *such that* $g(q) \geq u_{v,Q}(q)$ *for every* $q \in \Delta$.

$\mathbf{Cav}u_{v,Q}$ is a concave function as a minimum of concave functions. Moreover, if $u_{v,Q}$ is continuous, then since Δ is a convex polygon, $\mathbf{Cav}u_{v,Q}$ is also continuous (see Laraki 2004).

Lemma 2 *Assume that* $u_{v,Q}$ *is bounded. Then for every* $q \in \Delta$,

$$\mathbf{Cav}u_{v,Q}(q) = \max\left\{\sum_{i=1}^{n+1}\alpha_i u_{v,Q}(q_i);\quad (i)\;\; q = \sum_{i=1}^{n+1}\alpha_i q_i; \quad (ii)\;\; \alpha_i \geq 0, i = 1, \ldots, n+1; \quad (iii)\; \sum_{i=1}^{n+1}\alpha_i = 1; \text{ and } \quad (iv)\;\; q_i \in \Delta, i = 1, \ldots, n+1\right\}.$$

Proof Denote by $w(q)$ the right-hand side of the equality without the restriction that the number of elements in the convex combination is $n+1$. First, it is clear that $w \geq u_{v,Q}$ on Δ. In addition, one can show that w is concave. Therefore, $w \geq \mathbf{Cav}u_{v,Q}$. On the other hand, if $q = \sum_{i=1}^{L} \alpha_i q_i$ where $\sum_{i=1}^{L} \alpha_i = 1$, $q_i \in \Delta, \alpha_i \geq 0, i = 1, \ldots, L$, then by concavity of $\mathbf{Cav}u_{v,Q}$ we have $\mathbf{Cav}u_{v,Q}(q) \geq \sum_{i=1}^{L} \alpha_i \mathbf{Cav}u_{v,Q}(q_i) \geq \sum_{i=1}^{L} \alpha_i u_{v,Q}(q_i)$. It follows that $w \leq \mathbf{Cav}u_{v,Q}$. Finally, it is enough to consider convex combinations of no more than $n+1$ elements by the Caratheodory theorem. □

The following lemma implies Proposition 1(i).

Lemma 3 *The following three conditions are equivalent:*

1. core(v, Q) *is non-empty.*
2. $\mathbf{SSa}v(Q) = v(Q)$.
3. $\mathbf{Cav}u_{v,Q}\left(\frac{Q}{|Q|}\right) = \frac{v(Q)}{|Q|}$.

Proof (1) ⇒ (2) Let $x \in \text{core}(v, Q)$ and assume that the equation $\sum_{j=1}^{L} \lambda_j c_j = Q$ holds, where $c_j \in F(Q)$, and $\lambda_j \geq 0$, $j = 1, \ldots, L$. Then $v(Q) = xQ = \sum_{j=1}^{L} \lambda_j x c_j \geq \sum_{j=1}^{L} \lambda_j v(c_j)$. Therefore, $v(Q) = \mathbf{SSa}v(Q)$.

(2) ⇒ (3) This is a consequence of Lemma 2. Indeed, since $v(Q) = \mathbf{SSa}v(Q)$, it follows that if $\sum_{j=1}^{L} \alpha_j (c_j/|c_j|) = (Q/|Q|)$, where $c_j \in F(Q), \alpha_j \geq 0$, $j = 1, \ldots, L$ and $\sum_{j=1}^{L} \alpha_j = 1$, then $\sum_{j=1}^{L} \alpha_j (v(c_j)/|c_j|) \leq (v(Q)/|Q|)$. This implies that $\sum_{i=1}^{n+1} \alpha_i u_{v,Q}(q_i) \leq (v(Q)/|Q|)$ whenever $\sum_{i=1}^{n+1} \alpha_i q_i = (Q/|Q|)$ and $\sum_{i=1}^{n+1} \alpha_i = 1, \alpha_i \geq 0, q_i \in \Delta, i = 1, \ldots, n+1$. By Lemma 2 we get that $\mathbf{Cav}u_{v,Q}(Q/|Q|) = (v(Q)/|Q|)$.

(3) ⇒ (1)$\mathbf{Cav}u_{v,Q}$ is concave over Δ. Let $x \in I\!R^n$ be a supporting hyperplane for $\mathbf{Cav}u_{v,Q}$ at the point $(Q/|Q|)$. Then $x(Q/|Q|) = \mathbf{Cav}u_{v,Q}(Q/|Q|) = (v(Q)/|Q|)$, so $xQ = v(Q)$. Also, for every coalition d, $xd = |d|x(d/|d|) \geq |d|\mathbf{Cav}u_{v,Q}(d/|d|) \geq |d|u_{v,Q}(d/|d|) \geq |d|(v(d)/|d|) = v(d)$. Therefore, $x \in \text{core}(v, Q)$. □

Part (ii) of Proposition 1 follows immediately from part (i). □

6.2 Some properties of the **Sa** and **SSa** operators

Lemma 4 *Let v, v' be two bounded functions on $F(Q)$ with $v(0) = v'(0) = 0$. Then*

(1) $v \leq \mathbf{Sa}v$.
(2) If $v \leq v'$ *then* $\mathbf{Sa}v \leq \mathbf{Sa}v'$.
(3) $\mathbf{Sa}v$ *is SA on* $F(Q)$.
(4) $\mathbf{Sa}v = \mathbf{SaSa}v$.
(5) The infimum of any family of SA functions is SA.
(6) $\mathbf{Sa}v = \inf\{g;\ g \geq v \text{ and } g \text{ is SA}\}$.

Lemma 5 *Let v, v' be two bounded functions on $F(Q)$ with $v(0) = v'(0) = 0$. Then*

(1) $v \leq \mathbf{SSa}v$.
(2) If $v \leq v'$ *then* $\mathbf{SSa}v \leq \mathbf{SSa}v'$.
(3) $\mathbf{SSa}v$ *is SSA on* $F(Q)$.
(4) $\mathbf{SSa}v = \mathbf{SSaSSa}v$.
(5) The infimum of any family of SSA functions is SSA.
(6) $\mathbf{SSa}v = \inf\{g;\, g \geq v$ *and* g *is SSA*$\}$.

The proof of Lemma 4 is similar to that of Lemma 5 and is therefore omitted.

Proof of Lemma 5 (1) and (2) are clear. As for (3), fix $c \in F(Q)$ and assume that the equation $\sum_{j=1}^{L} \lambda_j c_j = c$ holds where $\lambda_j \geq 0$ and $c_j \in F(c)$, $j = 1, \ldots, L$. Let $\varepsilon > 0$. By the definition of $\mathbf{SSa}v$, for any $j = 1, \ldots, L$, there exist $c_{j_1}, \ldots, c_{j_{K_j}} \in F(c_j)$ and non-negative numbers $\alpha_{j_1}, \ldots, \alpha_{j_{K_j}}$ such that $\sum_{i=1}^{K_j} \alpha_{j_i} c_{j_i} = c_j$ and $\sum_{i=1}^{K_j} \alpha_{j_i} v(c_{j_i}) \geq \mathbf{SSa}v(c_j) - \left(\varepsilon / \sum_{j=1}^{L} \lambda_j\right)$. It follows that,

$$\sum_{j=1}^{L} \lambda_j \mathbf{SSa}v(c_j) \leq \sum_{j=1}^{L} \lambda_j \left(\sum_{i=1}^{K_j} \alpha_{j_i} v(c_{j_i}) + \frac{\varepsilon}{\sum_{j=1}^{L} \lambda_j} \right)$$
$$= \varepsilon + \sum_{j=1}^{L} \sum_{i=1}^{K_j} \lambda_j \alpha_{j_i} v(c_{j_i}) \leq \varepsilon + \mathbf{SSa}v(c).$$

The last inequality is due to the fact that $\sum_{j=1}^{L} \sum_{i=1}^{K_j} \lambda_j \alpha_{j_i} c_{j_i} = c$. Since $\varepsilon > 0$ is arbitrary we have (Bondareva 1962).

(4) follows from (3). To prove (5), let $\{g_\alpha\}_{\alpha \in I}$ be a family of SSA functions and define $w = \inf_{\alpha \in I} g_\alpha$. Assume that the equation $\sum_{j=1}^{L} \lambda_j c_j = c$ holds where $\lambda_j \geq 0$ and $c_j \in F(c)$, $j = 1, \ldots, L$. Then for some fixed $\tilde{\alpha} \in I$, $g_{\tilde{\alpha}}(c) \geq \inf_{\alpha \in I} \sum_{j=1}^{L} \lambda_j g_\alpha(c_j) \geq \sum_{j=1}^{L} \lambda_j w(c_j)$. Since this is true for every $\tilde{\alpha}$, we get that $w(c) \geq \sum_{j=1}^{L} \lambda_j w(c_j)$, so w is SSA.

(6) follows from the previous claims. Indeed, denote $w = \inf\{g;\, g \geq v$ and g is SSA$\}$. By (3), $\mathbf{SSa}v$ is SSA on $F(Q)$ and by (1) it is above v. Therefore, $w \leq \mathbf{SSa}v$. On the other hand, if g is above v then by (2) $\mathbf{SSa}g \geq \mathbf{SSa}v$. If g is also SSA, then $g = \mathbf{SSa}g \geq \mathbf{SSa}v$. Since this is true for every such g, it follows that $w \geq \mathbf{SSa}v$ so $w = \mathbf{SSa}v$. □

References

Aubin, J.P.: Mathematical methods of games and economic theory. Amsterdam: North-Holland, 1979
Aubin, J.P.: Cooperative fuzzy games. Math Operat Res **6**, 1–13 (1981)
Bondareva, O.: The theory of the core in an n-person game (in Russian). Vestn Leningr Univ **13**, 141–142 (1962)
Branzei, R., Dimitrov, D., Tijs, S.: Convex fuzzy games and participation monotonic allocation schemes. Fuzzy Sets Syst **139**, 267–281 (2003)

Butnariu, D.: Stability and Shapley value for an n-persons fuzzy game. Fuzzy Sets Syst **4**, 63–72 (1980)

Laraki, R.: On the regularity of the convexification operator on a compact set. J Convex Anal **11**, 209–234 (2004)

Moulin, H.: Axioms of cooperative decision making, pp 98–102. Cambridge: Cambridge university press, 1988

Shapley, L.S.: On balanced sets and cores. Naval Res Logist Q **14**, 453–460 (1967)

Shapley, L.S., Shubik, M.: On market games. J Econ Theory **1**, 9–25 (1969)

Sharkey, W.W., Telser, L.G.: Supportable cost functions for the multiproduct firm. J Econ Theory **18**, 23–37 (1978)

Telser, L.G.: Economic theory and the core. Chicago: The university of Chicago press, 1978