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Non-commitment and savings in dynamic risk-sharing contracts*

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Summary. We characterize the solution to a dynamic model of risk sharing under non-commitment when saving is possible. Savings can play two important roles. First savings can be used to smooth aggregate consumption across different periods. Second, when savings are observable, they can act as a collateral that can be seized in the case of default. This relaxes the non-commitment constraint. When the aggregate income is fixed or when one of the agent is risk neutral, the allocation tends to first-best consumption. When one of the agent is risk neutral, this convergence occurs in an expected finite number of periods.

Keywords and Phrases: Dynamic contracts, Risk sharing, Non-commitment, Savings.

JEL Classification Numbers: C7, D9.

1 Introduction

The analysis of consumption dynamics under market incompleteness has been traditionally approached under one of two different environments. First, models of consumption and savings have studied how liquidity constraints can limit consumption smoothing. Second, models of bilateral risk sharing have presented the limits to insurance in long-term contracts due to imperfect information or imperfect commitment. In this paper, we bridge the gap between these two environments in a two-agent framework. Each agent is risk averse and receives a periodic random income that can be partly saved. Agents are liquidity constrained in the sense

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that they cannot borrow at the savings interest rate. They can, however, mitigate risk with bilateral contingent transfers. We describe the constrained efficient risk-sharing contract with savings when agents cannot fully commit to the contract and when they cannot borrow. We first give general properties of the optimal contract under general assumptions. We then study the cases where 1) there is no aggregate risk, and 2) one of the two agents is risk neutral. In these two particular cases, we show that the optimal second-best allocation converges to a first-best allocation. Furthermore, such convergence occurs in an expected finite time when one of the agents is risk neutral.

In dynamic models of consumption and savings with a representative agent, liquidity constraints have often been used as a reduced form for market incompleteness. They limit an agent's access to credit resulting in an imperfect smoothing of consumption over time. For example, Ljungqvist and Sargent (2004) show that, when income is independently and identically distributed over time, consumption is not perfectly smoothed, and savings and consumption converge to infinity in the limit.

In dynamic models of risk sharing, market incompleteness can endogenously arise from imperfections due to the lack of commitment by the parties engaged in a risk-sharing relationship.[1] Kocherlakota (1996) analyzes efficient risk sharing under non-commitment in a two-agent context with bilateral risk aversion, i.i.d. incomes, and no savings. He shows that (ex ante) expected intertemporal utilities converge monotonically towards first-best levels and that the distributions of individual consumptions converge towards limit distributions. An agent's consumption is always positively correlated with current and lagged incomes unless first best is feasible from the first period on. If first best is not feasible, perfect insurance is impossible. In a similar model with one-sided risk aversion, Thomas and Worrall (1988) have characterized an optimal risk-sharing labor contract with non-commitment. They show that the risk-neutral firm cannot generally fully smooth the worker's wage. In all periods, the wage varies depending on the last-period wage and the worker's current productivity. In this context, Gauthier et al. (1997) show that, if the agents can make a transfer before the realization of the state of nature, the commitment problem is alleviated. In these models, non-commitment is the source of imperfect insurance and smoothing.

The assumption of no-borrowing in the first class of models with liquidity constraints and that of no-savings in the second class of models of contracting under non-commitment may seem unreasonable in many circumstances. It therefore seems natural to bridge the gap between these two strands of literature by introducing savings in a model of bilateral risk sharing with non-commitment, or alternatively by allowing agents to sign contracts (even if plagued by non-commitment) in a model of savings with liquidity constraints.

Ligon et al. (2000) analyze a mutual insurance relationship with imperfect commitment and savings. They show that the possibility to save may not always increase welfare in a risk-sharing relationship because high levels of savings may

[1] Another source of market imperfections can be informational asymmetries. See Green (1987), Thomas and Worrall (1990), or Wang (2005).

encourage agents to breach the contract and live in autarky with their savings. Savings cannot be seized upon contractual breach. This assumption introduces non-convexities in the optimization and they cannot fully characterize the solution analytically. They provide simulated path for individual savings and welfare under different specifications. They conclude that savings have two beneficial effects. First, they help smooth aggregate consumption when aggregate income is risky. Second, they are a way for agents to make ex-ante transfers à la Gauthier et al. (1997) that relax future self-enforcing constraints. Those two effects are counterbalanced, however, by the outside opportunities that savings create, hence limiting the extent of risk sharing.

In this paper, we develop a two-agent risk-sharing contract with savings and non-commitment. We suppose that savings are observable and can be posted as a collateral to enhance commitment in the contractual relationship. Under this assumption, we avoid the non-convexity problem encountered by Ligon et al. (2000) and we are able to find analytic solutions. Under full commitment, savings and contractual risk sharing have complementary roles: bilateral transfers guarantee perfect risk sharing between agents, that is, the ratio of marginal utilities is maintained constant through time and states of nature, and savings are used to smooth the aggregate income over time. When perfect risk sharing is made impossible by imperfect commitment, savings act as a collateral relaxing the non-commitment problem. With non-commitment, savings become beneficial even when they have no use under full commitment. Savings can even lead to perfect insurance in some circumstances.

Huggett and Krasa (1996) study the role of fiat money in dynamic economies. They show that limited commitment can provide a role for fiat money since it facilitates intertemporal exchanges. The main difference with our model is that they do not consider dynamic relationships since agents meet with a different partner every period. Furthermore, money becomes useless when a savings is introduced.

In the next section, we present the model and describe the first-best solution. Section 3 presents results on the dynamics of financing and consumption with non commitment in the general case. In Section 4 we derive some more results under the restriction that there is no aggregate risk in the economy. In Section 5 we suppose there is aggregate risk and find the dynamics of consumption when one agent is risk neutral. The conclusion follows. All proofs are relegated to the Appendix.

2 The model

Consider two agents, 1 and 2, having an infinite-horizon life span. Each period, agent i receives an exogenous income endowment y_i which is a random variable on the space of events $\mathcal{S} = \{1, 2, \cdots, S\}$. Incomes y_i are i.i.d. on the time-independent discrete set of possible states of nature $\mathcal{S}$. The probability of state s is p^s, with $\sum_{s \in \mathcal{S}} p^s = 1$. The income of agent i in state s is y_i^s. Denote $y^s = \sum_{i=1}^{2} y_i^s$ the aggregate income and by S_i the state which maximizes agent i's income.

We define a particular case where individual endowments can be correlated. Outside this assumption, individual endowments are independently distributed.

Assumption E. *Individual endowments are perfectly negatively correlated, and there is no aggregate risk so that $y^s = \bar{y}$ for all $s \in \mathcal{S}$.*

Agent i has preferences represented by a state- and time-independent concave, and twice continuously differentiable utility function u_i. Formally, we assume that $u_i' > 0$, $u_i'' \leq 0$, and $u_i'(0) = \infty$. Agent 1 is strictly risk averse, that is, $u_1'' < 0$. At times we may assume that agent 2 is risk neutral.

Assumption RN. *Agent 2 is risk neutral with utility function $u_2(c) = c$.*

Both agents discount the future with a common factor $\beta = 1/(1+r)$.

Each agent has access to a savings account. Agent i's savings at the end of period $t-1$ are denoted by A_t^i. Initial savings are denoted by $A_0^i \geq 0$. Savings earn a time-independent interest rate r per period. Agents are, however, liquidity constrained in the sense that they cannot borrow at rate r. This amounts to assuming that savings must be non-negative at all time. The savings account can be interpreted as an investment in short-term riskless government bonds which earn r per dollar invested. The liquidity constraint simply means that an agent cannot issue riskless bonds.

In autarky, a risk-averse agent maximizes its lifetime utility by choosing consumption and savings in each period. We denote by $g_i(A_t^i, y_i^s)$ the value function that represents agent i's maximized expected lifetime utility at the beginning of period t given savings of A_t^i and income y_i^s. Under our assumptions, $g_i(\cdot, y_i^s)$ is continuous, strictly increasing, strictly concave, and continuously differentiable. Optimal savings A_{t+1}^i and consumption c_t^i are both increasing in current financial resources $A_t^i(1+r)+y_i^s$. In general, consumption remains risky and is imperfectly smoothed across periods.[2] Therefore, there are gains from trade to be realized in underwriting a risk-sharing contract with another agent.

The purpose of this paper is to study the risk-sharing agreement that the two agents can sign. The general model that we specify, although very stylized, can represent various economic environments. It can represent the relationship between a borrower and its bank. In this case, the borrower is large enough that the bank can be modelled as being risk averse (although this is not necessary since we also study the model under Assumption RN). The borrower could be a large firm or a sovereign country. The savings account represents investment in government bonds, a foreign government in the second case. The model could also represent the relationship between two large banks that seek to diversify their portfolio risk. The savings account could be government bonds or international financial markets. Finally, the model could represent the relationship between two insurance companies that exchange claims on their portfolios of policies. The savings account could be interpreted as the reinsurance market. For the remainder of the paper, we do not focus on any particular interpretation.

The two agents get together to share risks, and their relationship is governed by a contract signed at date 0. This risk-sharing contract specifies, for all dates,

[2] Deaton (1991) suggests that, if the discount rate is greater than r, that is, when savings are relatively costly, the agent is reluctant to save, even in good states of nature.

contingent transfers from agent 1 to agent 2, and contingent savings for both agents.[3] Transfers and savings depend on the state s_t realized at that date, and more generally on the whole history of states to date t, $h_t = (s_0, \cdots, s_t)$. Transfers and savings are denoted respectively by $b_t(h_t)$ and $A^i_{t+1}(h_t)$, $t = 0, \cdots, \infty$. Transfers can be either positive or negative, while savings must remain non-negative. Consumption is given by:

$$c^1_t(h_{t-1}, s) = y^s_1 + (1+r)A^1_t - A^1_{t+1}(h_{t-1}, s) - b_t(h_{t-1}, s),$$
$$c^2_t(h_{t-1}, s) = y^s_2 + (1+r)A^2_t - A^2_{t+1}(h_{t-1}, s) + b_t(h_{t-1}, s).$$

Let $C_i(h_t)$ denote a consumption plan starting in period t and for the remaining infinite horizon given that history h_t has occurred. Agent i's expected utility from period t on is:

$$\mathcal{U}_{i,t}(C_i, h_t) \equiv u_i(c^i_t(h_t)) + \mathrm{E}_t \sum_{\tau=1}^{\infty} \beta^\tau u_i(c^i_{t+\tau}(h_{t+\tau})), \qquad i = 1, 2.$$

Suppose the two agents can fully commit to a long-term risk-sharing contract. An optimal consumption plan shares the gains from trade between the two agents. The full commitment solution prescribes perfect insurance between the two agents, characterized by a constant ratio of marginal utilities. Savings are governed by two individual Euler conditions. The two individual savings accounts are not independent, however. Since an agent's consumption is adjusted through the contingent transfer $b_t(h_t)$, any change in individual savings can be offset by a compensating change in the transfer. Only the sum of savings $A_t = A^1_t + A^2_t$ is important in determining the agents' welfare.

In a first-best environment, savings play the same role in this model with two agents as they would in a standard model of consumption and savings with one agent having the aggregate endowment. Savings are used to smooth aggregate income as much as possible, and bilateral transfers offer protection against individual risks. Therefore, as long as there exists aggregate risk, aggregate savings evolve with aggregate income $y_1 + y_2$, so that savings increase when aggregate income is high, and decrease when aggregate income is low.

If there is no aggregate risk (Assumption E), however, the first-best solution is such that consumption and savings are constant across time and states of nature. In each period, the two agents consume the aggregate income endowment plus the return on initial savings. Any time-independent sharing of total assets is optimal. Since aggregate income is constant, there is no incentive to save. On the other hand, if agent 2 is assumed risk neutral (Assumption RN), optimal risk sharing trivially imposes that agent 2 fully insures agent 1 in every period and there is no savings.

The first-best model rests on the assumption that the two agents can fully commit to the transfers prescribed by the contract. When enforcement costs are high,

[3] We assume here that savings are verifiable and thus contractible. If savings were not observable, this would introduce moral hazard since agents could secretly select savings. Furthermore, once selected, savings become an attribute of an agent, that is, a type in the next period. Hence, there would be adverse selection as well. Although it would be interesting to study these (very hard) cases, it is still relevant to understand the case of contractible savings.

however, it is not possible to bind the agents to the contract in all circumstances. It is then relevant to study the optimal contract in an environment where agents cannot commit.

3 The risk-sharing contract with non-commitment

Each agent can decide to renege on the contract if the current payment to be made is greater than the expected future surplus of staying in the relationship. To prevent reneging, self-enforcing constraints are introduced into the contracting problem. These constraints impose that, in each period, following any history, agents have no incentives to renege on the contract.

When reneging on the contract, an agent is prevented from future trading and gets its autarkic utility level forever.[4] We assume that an agent loses its financial assets when breaching the contract. It is as if savings were put up as a collateral against each agent's borrowing. The reneging agent's autarkic life therefore starts with no savings.[5] The autarkic allocation is represented by the solution to the savings model with liquidity constraints characterized by the value function $g_i(0, y_i^s)$. Under these assumptions, self-enforcing constraints can be written as:

$$\mathcal{U}_{1,t}(C_1, h_{t-1}, s) \geq g_1(0, y_1^s) \;\; \forall\, (h_{t-1}, s),\; \forall\; t, \tag{1}$$

$$\mathcal{U}_{2,t}(C_2, h_{t-1}, s) \geq g_2(0, y_2^s) \;\; \forall\, (h_{t-1}, s),\; \forall\, t, \tag{2}$$

where g_i is the expected discounted utility of agent i under autarky.

Given initial savings (A_0^1, A_0^2), the optimal contract solves an optimization problem in $t = 0$ with the instruments being a sequence of savings and transfers for each date and possible history. The set $\Gamma(s_t)$ of instruments that satisfy self-enforcing constraints (1) and (2) and liquidity constraints $A_t^i \geq 0, \forall i, \forall t$, represents the set of feasible continuation contracts. If there are gains from trade, this set is nonempty because at least one agent can have more than autarky in each period. Furthermore, it is easy to show that it is compact and convex.[6]

As stated by Thomas and Worrall (1988), an optimal contract must be efficient starting in any period following any history. Consequently, an optimal contract maximizes at each date t the expected utility of agent 1 subject to a participation constraint for the agent 2 and subject to self-enforcing constraints (1) and (2) and liquidity constraints $A_t^i \geq 0, \forall i, \forall t$. However, only aggregate savings matter since income is transferred from an agent to the other through the contractual transfer $b(h_t)$. In the remainder of the paper, we denote total savings by $A = A^1 + A^2$ and aggregate financial resources by $X^s = y^s + (1 + r)A, \forall s$.

The optimal contract is characterized by a Pareto frontier $f(X^s, V)$ that gives the optimal utility agent 1 can get when agent 2 is granted utility level V and

[4] Asheim and Strand (1991) show that this punishment is renegotiation-proof in the repeated-game formulation of a closely related model.

[5] Ligon et al. (2000) consider the case where agents keep their savings following a breach of contract. This introduces non-convexities, so that they cannot find an analytical solution for the path of savings and consumptions.

[6] See Thomas and Worrall (1988) for a formal proof in a related model without savings.

aggregate available income is equal to X^s, given that self-enforcing constraints (1) and (2) hold in state s and individual savings are non-negative. Following Spear and Srivastava (1987), solving the problem amounts to picking current consumption c_t^2 of agent 2, future utility levels to be conceded to agent 2 next period contingent on the then realized state z, $\{V_{t+1}^z\}_{z\in\mathcal{S}}$, and a level of savings A_{t+1} for next period, and all this to maximize agent 1's expected future utility. Therefore, the optimal contract solves the following Bellman equation:

$$f(y^s+(1+r)A_t,V_t)=$$
$$\max_{A_{t+1},c_t^2,\{V_{t+1}^z\}_{z=1}^S} \quad u_1(y^s+(1+r)A_t-A_{t+1}-c_t^2)$$
$$+\beta \mathrm{E}_{z\in\mathcal{S}} f(y^z+(1+r)A_{t+1},V_{t+1}^z) \tag{3}$$
$$\text{s.t.}\quad f(y^z+(1+r)A_{t+1},V_{t+1}^z)\ge g_1(0,y_1^z)\ \ \forall\, z\in\mathcal{S}, \tag{4}$$
$$V_{t+1}^z \ \ge\ g_2(0,y_2^z) \qquad \forall\, z\in\mathcal{S}, \tag{5}$$
$$u_2(c_t^2)+\beta \mathrm{E}_z V_{t+1}^z\ge V_t, \tag{6}$$
$$A_{t+1}\ge 0. \tag{7}$$

Constraints (4)–(5) represent the self-enforcing constraints of agents 1 and 2 respectively, while constraint (6) ensures the intertemporal consistency of the optimal solution. Note that these constraints depend on individual incomes through the autarky levels of utility $g_i(0,y_i^s)$.

The value function f is state independent because states of nature are i.i.d., so that continuation utilities are independent of the current state. Hence, the period-t state matters only because it determines the level of aggregate income y^s and therefore, the available income at hand X^s that has to be shared between individual consumptions and aggregate savings.

Since agent 1's utility is increasing in X^s, so is function f. Moreover, an increase in agent 2's minimum utility level V shrinks the set of feasible contracts and, hence, the maximum utility agent 1 can get. Therefore, f is decreasing in V. The utility function u_1 is strictly concave and continuously differentiable so that f is concave and continuously differentiable in (X^s,V).

The variable V_{t+1}^z represents the expected utility promised to agent 2 in state z in period $t+1$. If there are gains from trade, the set of admissible values for V_{t+1}^z for a given A_{t+1} is non-empty. This set is bounded below by $\underline{V}^z=g_2(0,y_2^z)$, which is increasing in y_2^z. The upper bound $\overline{V}^z(X^z,y_1^z)$ depends on A_{t+1} as implicitly defined by condition (4): $f(X^z,\overline{V}^z(X^z,y_1^z))=g_1(0,y_1^z)$. $\overline{V}^z(X^z,y_1^z)$ increases with X^z and decreases with y_1^z. Hence, gains from trade increase with the level of savings A. Since agents lose access to the savings account if they breach the contract, it is as if savings act as a collateral that prevents them from leaving. Agent 1 can promise a higher expected utility to agent 2 tomorrow if a larger amount in the savings account is committed to today. This differs from the results of Thomas and Worrall (1988) and Kocherlakota (1996) who show that agent 2's surplus belongs to a time-independent interval.

The maximization problem on the right-hand side of the Bellman equation is a concave program. First-order conditions are therefore sufficient to characterize the optimal solution. Moreover, the policy functions $A_{t+1}(X_t^s, V_t)$, $c_t^2(X_t^s, V_t)$ and $\{V_{t+1}^z(X_t^s, V_t)\}_{z\in\mathcal{S}}$ are continuous.[7]

Let the variables $\beta p^z \theta^z$ and $\beta p^z \lambda^z$ represent the Lagrange multipliers of self-enforcing constraints (4) and (5) respectively, for all $z \in \mathcal{S}$. We denote by ψ the multiplier of constraint (6), and by μ, the multiplier of the liquidity constraint (7). First-order conditions with respect to A_{t+1}, c_t^2 and V_{t+1}^z for all z, and the envelope conditions can be rearranged to give the following set of conditions:

$$u_1'(c_t^1) = \mathrm{E}_z(1+\theta^z)u_1'(c_{t+1}^{1,z}) + \mu, \tag{8}$$

$$u_2'(c_t^2) = \mathrm{E}_z\left(1+\frac{\lambda^z}{\psi}\right)u_2'(c_{t+1}^{2,z}) + \frac{\mu}{\psi}, \tag{9}$$

$$\frac{u_1'(c_t^1)}{u_2'(c_t^2)} = (1+\theta^z)\frac{u_1'(c_{t+1}^{1,z})}{u_2'(c_{t+1}^{2,z})} - \lambda^z \ \ \forall\, z \in \mathcal{S}. \tag{10}$$

Equations (8)–(9) are the Euler equations that determine the optimal dynamics of consumption. Equation (10) characterizes the optimal risk sharing for period $t+1$, constrained by the presence of non-commitment. Note that if an agent's self-enforcing constraint binds in a state, the contract must give this agent a greater consumption in that state than what a standard Euler equation would otherwise suggest. The ratio of marginal utilities is modified accordingly.

If the policy function $c_t^2(X_t^s, V_t)$ is continuously differentiable, then the value function f is twice continuously differentiable. In that case, we can derive basic properties for the optimal consumption of each agent.

Proposition 1.

i) Agent 1's consumption is decreasing in V_t and agent 2's consumption is increasing in V_t, that is, $dc_t^1/dV_t < 0$ and $dc_t^2/dV_t > 0$.

ii) The consumption of agent 1 is increasing in X^s, that is, $dc_t^1/dX^s > 0$.

Agent 2's current consumption must increase with the expected utility V_t promised to agent 2. Since the contract solution lies on the Pareto frontier in intertemporal utility space, any increase in the utility promised to agent 2 entails a decrease in the utility of agent 1. This implies that agent 1's current consumption decreases with this promise. Finally, the consumption of agent 1 is increasing in available aggregate financial resources. The effect of X^s on the consumption of agent 2 is ambiguous. It all depends on its effect on risk sharing and smoothing. When increasing aggregate financial resources, the intertemporal utility of agent 2 remains at V_t, so that current consumption can either increase or decrease when X^s increases depending on how this increase affects future self-enforcing constraints. For example, it is easy to show that agent 2's consumption is unaffected by a change in X^s in the first best under Assumption E. In this case, its consumption is constant and determined solely by its participation constraint. All increases in financial resources accrue to agent 1.

A special case can be characterized when agent 2 is risk neutral (RN).

[7] See Stockey and Lucas (1989) and the proof of Lemma 1 in Thomas and Worrall (1988).

Proposition 2. *Under Assumption RN, the value function f can be written as*

$$f(y^s + (1+r)A, V) = h(y^s + (1+r)A - V),$$

where h is increasing and concave.

This proposition states that the value function f can be rewritten as a function of a single variable $y^s + (1+r)A - V$. The variable V can be interpreted as the value of the debt agent 1 has contracted with agent 2. The solution then only depends on the net asset value of agent 1 following the realization of the current state of nature s, $Y^s = y^s + (1+r)A - V$. Finally, the function h inherits the properties of the function f, namely monotonicity and concavity.

Before characterizing further the solution, it is instructive to see whether there is a level of savings such that the first best becomes self-enforcing. Under general conditions, it appears that this is not necessary the case. Recall that savings play a smoothing role in the first best and that they can vary with the history of income realizations. Following a long enough sequence of low incomes, savings in the first best tend towards zero. If the first best was feasible, this would imply that savings should tend to zero following such history. Since the first best is not necessarily self-enforcing with low (near zero) savings,[8] no level of savings could make the first best feasible.

As in the first-best case, it is difficult to provide a more detailed characterization of savings and consumption for arbitrary preferences and income distributions. We can, however, do so under two special but relevant cases, that of no aggregate risk and that of risk neutrality of agent 2. This is the object of the next two sections.

4 No aggregate risk

Assume there is no aggregate risk (E). The next two propositions characterize the dynamics of savings.

Proposition 3. *Fix the value of $\beta > 0$.*

(i) There exists a non-empty convex set $\Delta^*(\beta)$ such that if $(A_t, V_t) \in \Delta^*(\beta)$, the first best is feasible and hence optimal.
(ii) For any value of V_t, there is a finite level of savings such that $(A_t, V_t) \in \Delta^*(\beta)$.

Gains from trade relative to autarky increase with the level of savings since savings are confiscated if an agent reneges on the contract. When the level of savings is large enough, gains from trade are so high that it is always possible to find a time-independent sharing rule $\{\alpha_i\}$ that offers the optimal insurance and doesn't give incentives for the agents to cheat. It is the perspective of losing a high-value collateral (savings) that induces the agents to maintain the contractual relationship.

[8] It depends on the discount factor β. It must be high enough for self-enforcing constraints not to bind.

More specifically, the set $\Delta^*(\beta)$ is characterized by the following inequalities.

$$f(\bar{y}+(1+r)A_t, V_t) = u_1(\alpha_1(\bar{y}+rA_t))/(1-\beta) \geq g_1(0, y_1^{S_1}), \quad (11)$$

$$V_t = u_2(\alpha_2(\bar{y}+rA_t))/(1-\beta) \geq g_2(0, y_2^{S_2}), \quad (12)$$

where α_i is the constant share of aggregate resources consumed by agent i at first best, $\alpha_1 + \alpha_2 = 1$. Consider the first-best allocation. It provides utility equal to the term to the right of the equality sign. If this allocation is self-enforcing in the state for which autarky is mostly profitable, namely when income is $y_i^{S_i}$, then it is self-enforcing in all states. For a given V_t, which determines $\{\alpha_i\}$, there is a (finite) level of savings such that the inequalities (11–12) are satisfied. Note that the level of savings for which the first best can be implemented is not independent of agent 2's intertemporal utility. When this utility increases, the level of savings must increase to satisfy all of agent 1's self-enforcing constraints.

There is an important difference that savings bring to the feasibility of first best. In Thomas and Worrall's (1988) and Kocherlakota's (1996) models, there are no savings, and, in many cases, the first best is never feasible, unless β is close enough to 1. With savings, for any $\beta > 0$, there are values for (A, V) such that the first best is feasible. This is because savings act as a collateral or a bond that relaxes the agents' self-enforcing constraints. The future of the relationship is not the only means of binding agents. The perspective of losing a bond can be sufficient if it is large enough.

We can now show a stronger result. Even if $(A_0, V_0) \notin \Delta^*(\beta)$, agents still converge to a first-best allocation by accumulating savings.

Proposition 4. *Savings A_t increase until the first best is reached.*

With no aggregate risk, agents build up their collateral until first best can be implemented. With non-commitment, there are future benefits to save as long as the first best is not attained since savings can relax future self-enforcing constraints. In each period, agents trade off the benefit of relaxing future self-enforcing constraints to the immediate cost of reducing consumption. Savings increase as long as the first-best allocation is not self-enforcing, after which, savings and consumptions remain constant forever. This implies that the limit value of (A_t, V_t) is an element of $\Delta^*(\beta)$.

In the limit, our model is significantly different from the stationary distribution in autarky or that without savings. In autarky, Ljungqvist and Sargent (2004) have shown that the solution in the limit has infinite savings, while consumption also converges to infinity. In our model, convergence can occur for finite levels of savings, in which case aggregate consumption is mean income plus the net returns on savings, that is, $\bar{y} + rA$. This is certainly different from the autarkic situation.

In the model without savings, Kocherlakota (1996) has shown that there exists a limiting distribution of consumption. This limit is the first-best distribution only if the first best is self-enforcing. In our case, there is always a set $\Delta^*(\beta)$ of values for savings and individual utilities such that the first best is self-enforcing. The solution converges to an element of this set.

5 Agent 2 is risk neutral

If agent 2 is risk neutral, Proposition 2 states that the value function is $h(y^s + (1 + r)A - V)$. From first-order conditions (8–10), the first best implies a constant net asset value $Y = y^s + (1 + r)A_t - V_t^s$ for all t and all s, as well as a constant consumption $c^1 = u'^{-1}(h'(Y))$ for agent 1. The set of values for (A_t, Y) such that

$$Y \geq h^{-1}(g_1(0, y_1^z)),$$
$$(1 + r)A_t - Y \geq \max_z\{g_2(0, y_2^z) - y^z\},$$

makes the first best feasible and optimal. This set is convex and includes finite values.

Writing condition (9) with $u_2' = 1$, it is easy to show that when agent 2 is risk neutral, agent 2's self-enforcing constraints and the liquidity constraint are never binding. With the interest rate on savings exactly equal to the discount rate,[9] saving is as profitable as lending to agent 2. Since the savings account is not subject to self-enforcing constraints whereas saving through agent 2 is, savings effectively allow agent 1 to increase its future consumption in good states thus relaxing all agent 2's self-enforcing constraints.

The liquidity constraint is not binding because agent 1 can always save some negative amount by borrowing from agent 2. There is, however, an upper bound on the amount agent 1 can borrow. This amount is contingent on the amount of savings which act as a collateral in the case of default by agent 1. This means that V^s, the amount due to agent 2 in state s, can increase if the collateral A also increases sufficiently. This borrowing constraint has real consequences because the self-enforcing constraints of agent 1 can be binding, that is, in a high-income state, agent 1 may be tempted to not reimburse agent 2 and renege on the contract.

Without a binding liquidity constraint and self-enforcing constraints for agent 2, the model has some similarities with that of Harris and Holmström (1982), where agent 2 can fully commit to a long-term contract. We show below that the dynamics of consumption are similar to the dynamics they derive in their model.

Proposition 5. *Agent 1's net assets $y^s + (1 + r)A_t - V_t$ and consumption are non-decreasing in time.*

Agent 1's consumption in period t is equal to consumption in period $t + 1$ unless a self-enforcing constraint for agent 1 is binding in period $t + 1$, in which case consumption is equal to the minimum self-enforcing level $\underline{c}^s = u'^{-1}(h'(h^{-1}(g_1(0, y_1^s))))$. This minimum level of consumption $\underline{c}^s$ is increasing in agent 1's current income. Hence, a self-enforcing constraint can only be binding following a positive income shock for agent 1. This implies that consumption cannot decrease in time.

Proposition 6. *The first best is reached in an expected finite number of periods, when agent 1's net asset value and consumption reach a stationary state.*

[9] The result does not hold when the discount rate is not equal to the interest rate.

Suppose that income $y_1^{S_1}$ is realized for the first time in period t. The function h then attains a stationary value $h(y^{S_1} + (1+r)A_t - V_t^{S_1}) = h(Y_t)$ for all future states and periods since consumption and net assets are non-decreasing. No self-enforcing constraints can ever bind and the first-best becomes feasible. From then on, agent 2 bears all aggregate risk and agent 1's consumption remains constant.

There are two important characteristics of the stationary solution. First, since $Y_t = y^s + (1+r)A_t - V_t^s$ is constant, agent 2's utility is increasing in the states of nature, that is, $V^1 < \cdots < V^S$. Positive shocks to aggregate income are captured by agent 2 since agent 1's consumption is constant. Second, in any state, debt can increase to arbitrary large levels as long as the collateral A increases also. However, even though there exists an infinity of solutions in (A, V), an interesting characterization is that setting $A_t^s = A$ and $V_t^s = V^s$ (increasing in s) for all s and t. Savings do not need to tend to infinity to achieve perfect income smoothing as it does in models of savings with liquidity constraints.

6 Conclusion

We study the dynamic behavior of consumption when agents have access simultaneously to a risk-sharing contract and a savings account. We show that savings and contingent contractual transfers complement each other so that savings are used to smooth aggregate income whereas contingent transfers support risk sharing. When non-commitment makes transfers unable to provide perfect risk sharing, savings are accumulated (1) to act as a collateral that precommits both agents in the relationship, and (2) as a way to replace transfers when one agent would prefer to quit the relationship rather than make a transfer to the other agent.

In two special cases, we are able to characterize the path of savings. First, when individual incomes are perfectly correlated so that the aggregate income is constant, savings have no smoothing role, but still are accumulated to relax the non-commitment constraints. Second, when incomes are uncorrelated but one of the agent is risk-neutral, savings are necessary only as long as the best possible income for the risk-averse agent has not occurred. Agent 1 saves in order to avoid in the future resorting to the non-committed agent 2 for consuming more than its income.

The main characteristic of our solution is that the first best becomes self-enforcing when savings are introduced in the model. Thus, it appears that risk-mitigating instruments cannot be studied in isolation. Even if the imperfections proper to each instrument create market incompleteness, consumers may circumvent them by using a bundle of instruments, each reducing the imperfections created by the others.

A main concern of the literature on consumption is trying to explain co-movements in consumption and income. Our paper has characterized conditions under which such co-movements are limited, that is, when perfect risk sharing is achieved. It helps us, however, better understand when perfect risk sharing cannot be achieved. There are two issues that need to be addressed: that of intratemporal risk sharing, and that of intertemporal smoothing.

In the first best (with full commitment), when $\beta(1+r) = 1$, perfect risk sharing can be attained while some smoothing can be achieved with non-negative savings. Perfect smoothing can be achieved if there is no aggregate risk. In the second best (with non-commitment), perfect risk sharing and consumption smoothing can be achieved under either one of Assumptions RN or E. If these assumptions do not hold, namely, if there is widespread risk aversion and aggregate risk, it is unlikely that perfect risk sharing or consumption smoothing can be achieved. Consumption would vary with income, but co-movements would be reduced due to the presence of savings and risk sharing contracts. In the case where $\beta(1+r) < 1$, savings become costly and hence even less smoothing can be achieved in both the first best and the second best.

Appendix

The following proofs refer to the first-order conditions of the problem that write:

$$u_1'(c_t^1) = \mathrm{E}_z(1+\theta_t^z) f_X(X_{t+1}^z, V_{t+1}^z) + \mu, \tag{13}$$

$$u_1'(c_t^1) = \psi u_2'(c_t^2), \tag{14}$$

$$(1+\theta^z) f_V(X_{t+1}^z, V_{t+1}^z) = -\lambda^z - \psi \quad \forall\, z \in \mathcal{S}, \tag{15}$$

$$f_X(X_t^s, V_t) = u_1'(c_t^1) = u_1'(X_t^s - c_t^2 - A_{t+1}^s), \tag{16}$$

$$f_V(X_t^s, V_t) = -\psi. \tag{17}$$

Proof of Proposition 1

i) From (16) and (17), the derivatives of f are continuously differentiable if the policy function $c^2(X, V)$ is. Suppose this is the case. Differentiating the envelope condition (17) and using the first-order condition (14) to substitute for ψ, we obtain

$$f_{VV}(y^s + (1+r)A, V) = \frac{u_1'}{u_2'}\left[\frac{-u_1''}{u_1'}\frac{dc_t^1}{dV} - \frac{-u_2''}{u_2'}\frac{dc_t^2}{dV}\right] < 0,$$

since f is concave. We now show that dc_t^1/dV and dc_t^2/dV have opposite signs. Suppose they have the same sign and consider the dual problem:

$$F(y^s + (1+r)A, U) = \max\ \mathcal{U}_2\ s.t.\ \mathcal{U}_1 \geq U,$$

self-enforcing and liquidity constraints,

where $U = f(y^s + (1+r)A, V)$ and $F(y^s + (1+r)A, U) = V$. This problem gives the same solution as the primal does. Then, dc_t^1/dU and dc_t^2/dU should have the same sign given our supposition above. By symmetry of the primal and the dual problems, dc_t^2/dV and dc_t^1/dU must have the same sign. These relations imply that dc_t^1/dV and dc_t^1/dU have the same sign. This is not possible because of the negative relationship between U and V (the value functions are decreasing). Then, it must be that dc_t^1/dV and dc_t^2/dV have opposite signs. Finally, since $f_{VV}(y^s + (1+r)A, V) < 0$ above, we must have $dc_t^1/dV < 0$ and $dc_t^2/dV > 0$.

ii) Differentiating the envelope condition (16) with respect to X^s, we have:

$$f_{XX}(y^s + (1+r)A, V) = u_1''(c_t^1) dc_t^1/dX^s < 0,$$

with the sign coming from the concavity of the value function. This implies that $dc_t^1/dX^s > 0$. □

Proof of Proposition 2

When agent 2 is risk neutral, set w.l.o.g. $u_2' = 1$. Using condition (14) and the envelope conditions (16–17), we have

$$f_X(y^s + (1+r)A, V) + f_V(y^s + (1+r)A, V) = 0,$$

for all (X^s, V) and all s in $\mathcal{S}$. This is a homogeneous linear differential equation in f whose solution must be the functional form $h(X^s - V)$. Function h is increasing since $h'(X^s - V) = f_X(X^s, V) > 0$. Function h is concave since $h''(X^s - V) = f_{XX}(X^s, V) < 0$. □

Proof of Proposition 3

i) With no aggregate risk, the first best is characterized by a time-independent sharing rule $\{\alpha_i\}$ and constant savings. The sharing rule is such that agent 2 gets its reservation utility. For first best to be implemented, the solution to the non-commitment problem must have $V_{t+1}^z = V_t$ for all z, $A_{t+1} = A_t$. This solution is self-enforcing if and only if

$$f(\bar{y} + (1+r)A_t, V_t) = \frac{u_1(\alpha_1(\bar{y} + rA_t))}{1-\beta} \geq g_1(0, y_1^{S_1}), \tag{18}$$

$$V_t = \frac{u_2((1-\alpha_1)(\bar{y} + rA_t))}{(1-\beta)} \geq g_2(0, y_2^{S_2}). \tag{19}$$

Denote the set of (A_t, V_t) that satisfies these two equations by $\Delta^*(\beta)$.
We now show that it is convex. Use the l.h.s. equality of (19) to substitute for α_1 in (18). The set $\Delta^*(\beta)$ is then defined by

$$\frac{u_1\left(\bar{y} + rA_t - u_2^{-1}((1-\beta)V_t)\right)}{1-\beta} \geq g_1(0, y_1^{S_1}), \tag{20}$$

$$V_t \geq g_2(0, y_2^{S_2}).$$

Since u_1 and $-u_2^{-1}$ are concave functions, we can show that the l.h.s. of expression (20) is concave in both variables of interest A_t and V_t. This implies that $\Delta^*(\beta)$ is convex.
Using condition (20), we see that for any value of $\beta \in (0,1)$, the set $\Delta^*(\beta)$ is non-empty. For $\beta = 1$, we know from Thomas and Worrall (1988) that the first best is implementable even without savings. The set $\Delta^*(\beta)$ is therefore non-empty for all $\beta > 0$.

(ii) From condition (20), it is easy to find finite values of $(A, V) \in \Delta^*(\beta)$. □

Proof of Proposition 4

Consider a constrained version of the maximization problem on the r.h.s. of the Bellman equation (3–7) where $A_{t+\tau} = A_t$ for all $\tau \geq 1$ and denote by U_1^c agent 1's constrained maximized utility. This corresponds to the maximization problem in Kocherlakota (1996). In his solution agent 1's consumption is increasing in its individual income. Suppose that $y_1^{S_1}$ is realized in period t. We know that $c_t^1 \geq c_{t+1}^{1,z}$ for all $z \in \mathcal{S}$ with some strict inequality if some self-enforcing constraints for agent 2 are binding. This implies that the Euler equation (8) for savings is not satisfied, and hence that agent 1 would like to save more than A_t at this constrained solution.

Consider the set $\Gamma_t(S_1)$ of instruments (including savings) that satisfy all constraints of the maximization problem. We know that this set is convex. Take the intersection of this set and the set of instruments that yield strictly more utility to agent 1 than U_1^c. Since agent 1's intertemporal utility function is concave in instruments, this intersection set must be convex. Furthermore, the solution to (3–7) (where savings can be chosen optimally) must be in this set. From the previous paragraph we also know that this set includes instruments where $A_{t+1} > A_t$.

We want to argue that there can be in this set no instruments for which $A_{t+1} < A_t$. Suppose there were. By convexity, this would imply that there are also instruments where $A_{t+1} = A_t$. But this contradicts the fact that instruments in this set give strictly more utility than U_1^c. Hence, if $y_1^{S_1}$ is realized and if some self-enforcing constraints for agent 2 are binding, it must be that $A_{t+1} > A_t$. Since the solution to (3–7) is independent of the current state and is symmetric with respect to the two agents, it must be that $A_{t+1} > A_t$ as long as some self-enforcing constraint is binding. □

Proof of Proposition 5

Condition (8) with $\mu = 0$ implies that agent 1's consumption is non-decreasing in time. Conditions (16) and the concavity of h imply that net assets are also non-decreasing in time. □

The following lemmas help prove Proposition 6.

Lemma 1. *If $y_1^s > y_1^z$, then $\theta_t^s = 0$ implies $\theta_t^z = 0$.*

Proof of Lemma 1

Let $Y_t \equiv y + (1+r)A_t - V_t$ denote agent 1's net asset value. Suppose $\theta_t^z > 0$ and $\theta_t^s = 0$ with $y_1^s > y_1^z$. This gives: $h(Y_{t+1}^z) = g_1(0, y_1^z) < g_1(0, y_1^s) \leq h(Y_{t+1}^s)$. By first-order condition (15), $\theta_t^z > 0$ also implies $h'(Y_{t+1}^z) < h'(Y_{t+1}^s)$. But, since h is increasing and concave, these inequalities lead to $Y_{t+1}^z < Y_{t+1}^s$ and $Y_{t+1}^z > Y_{t+1}^s$, that is, a contradiction. □

Lemma 2.

(i) If state s is realized in period t, then agent 1's self-enforcing constraint in that state cannot be binding in subsequent periods. Formally, $s_t = s \Rightarrow \theta_{t+\tau}^s = 0 \ \forall \tau \geq 1$.

(ii) If state S_1 is realized in period t, then from $t+1$ on, no self-enforcing constraint for agent 1 is ever binding. Formally, $s_t{=}S_1 \Rightarrow \theta^s_{t+\tau} = 0 \; \forall \; s, \; \forall \tau \geq \; 1$.

Proof of Lemma 2

(i) Suppose state s realizes in period t. In period $t+\tau$, suppose that state s realizes and that the self-enforcing constraint for agent 1 is binding: $\theta^s_{t+\tau} > 0$. Then, from first-order conditions (15) and (16), we have $h'(Y^s_{t+\tau+1}) < h'(Y_{t+\tau}) \leq h'(Y^s_t)$. By the concavity of h, this implies $Y^s_{t+\tau+1} > Y^s_t$. However, a binding constraint in period $t+\tau$ implies: $h(Y^s_{t+\tau+1}) = g_1(0, y^s_1) \leq h(Y^s_t)$, that is, $Y^s_{t+\tau+1} \leq Y^s_t$ which is a contradiction.

(ii) Lemma 1 implies ($\theta^s_t = 0 \;\; \Rightarrow \;\; \theta^z_t = 0$) for all $y^s_1 > y^z_1$ and part (i) of this Lemma ($s_t = s \;\; \Rightarrow \;\; \theta^s_\tau = 0$) for all $\tau > t$. □

Proof of Proposition 6

From Lemma 2, no constraint can be binding after state S_1 has occurred. Consumption and net assets then become constant as soon as S_1 has realized, that is, first best is reached. The probability of state S_1 being reached in period n for the first time is $p^{S_1}(1-p^{S_1})^{n-1}$ where p^{S_1} is the probability of state S_1 occurring. The expected time for first best to be reached is then

$$p^{S_1} \sum_{n=1}^{\infty} n(1-p^{S_1})^{n-1}.$$

This converges to $1/p^{S_1}$ which is finite for any $p^{S_1} > 0$. □

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