Economic Theory (2006) 29: 163–179
DOI 10.1007/s00199-005-0015-0

RESEARCH ARTICLE

Andrea Galeotti

One-way flow networks: the role of heterogeneity

Received: 28 September 2004 / Accepted: 25 July 2005 / Published online: 11 October 2005

Abstract I study a one-way flow connections model in which players are heterogeneous with respect to values and the costs of establishing a link. I show that values and costs heterogeneity are equally important in determining the level of connectedness and the architecture of equilibrium networks. I also show that when asymmetries are independent of the potential partner there are distributions of costs and values for which centrality is a distincitive feature of equilibrium networks. This sharply contrasts with the homogeneous case.

Keywords Network formation · Heterogeneity

JEL Classification Numbers D85 · C72

1 Introduction

The role of social and economic networks in shaping individual behavior and aggregate phenomena has been widely documented in recent years.[1] This has lead scholars in different disciplines to investigate the structural properties that networks exhibit in reality. The most stable empirical finding is that networks have very asymmetric architectures. Specifically, they exhibit high level of centrality: there are few nodes having many links, while the majority of nodes maintain few links. The connections model is the primary model used to explain the strategic

I thank an anonymous referee for useful comments. The paper has benefited from discussion with Gabriella Conti, Jose Luis Moraga-Gonzalez, Fernando Vega-Redondo and Sanjeev Goyal.

A. Galeotti
Social and Information Science Laboratory, CALTECH, Pasadena, CA 91125, USA
Department of Economics, University of Essex, Colchester, UK
E-mail: agaleotti@ist.caltech.edu

[1] There is a large body of work on this subject. See e.g., Burt (1992) on careers of professional managers, Montgomery (1991) on wage inequality in labour markets, Granovetter (1974) on the flow of job information, and Coleman (1966) on diffusion of medical drugs.

formation of networks.[2] Variants of this model have been proposed in order to analyze different social and economic situations. Nevertheless, much of the work has explored the formation of *undirected* networks: a link induces benefits to both parties, i.e., two-way flow networks. In reality many networks are *directed*: the flow of benefits is directed only towards the investor of the link, i.e., one-way flow networks. For example, the World Wide Web is a directed network: nodes are agents maintaining a webpage and links are hyperlinks that point from one web page to the other.[3]

Bala and Goyal (2000) analyze a non-cooperative model of network formation where players are homogeneous and there is one-way information flow. They show that if a player's payoffs are increasing in the number of other players accessed and decreasing in the number of links formed, a strict Nash network is either a wheel (a connected network in which each player creates and receives one link) or the empty network (with no links). The intuition for this result is as follows. Consider a minimally connected network where player 1 initiates a link with players 2 and 3, and each of these players has a link with player 1. Under the assumption of homogeneous costs of linking and values this network is not a strict equilibrium: player 2 is indifferent between maintaining the link with 1 and switching to player 3, instead. A generalization of this argument implies that a connected strict equilibrium is symmetric and has a wheel architecture. It is worth emphasizing two aspects of this result. The first aspect is that while centrality appears to be a crucial property of directed networks, equilibrium networks are symmetric when players are homogeneous.[4] Secondly, the findings of Bala and Goyal (2000) depend on the assumption of homogenous players. To observe this, assume that player 1 is just slightly cheaper to be linked with than players 2 and 3, ceteris paribus. This small introduction of heterogeneity implies that the network described in the example above becomes a strict equilibrium.

In the current paper, I study the role played by heterogeneous players in shaping the equilibrium architecture of directed networks. Players are heterogeneous in terms of the costs of linking and the values of accessing other players. Ex-ante asymmetries across players arise quite naturally in reality. For instance, in the context of information networks it is often the case that some individuals are more interested in particular issues and therefore better informed, which makes them

[2] This model has been extensively studied in the literature; see e.g., Aumann and Myerson (1989), Bala and Goyal (2000), Dutta and Jackson (2000), Goyal (1993), Haller and Sarangi (2005), Jackson and Wolinsky (1996), Johnson and Gilles (2000), McBride (2004), Slikker and van den Nouweland (2001), and Watts (2001a,b). The terminology "connections model" has been introduced by Jackson and Wolinsky (1996).

[3] In general the one-way information flow technology is compelling for the study of communication networks where nodes are "traffic" providers and a link initiated by i towards j signifies that j allows the transit of the traffic to i. A specific example is the internet interconnection networks, where nodes are internet providers. Citations networks are another example, where nodes are published papers and links are reference to previously published papers. A final example are E-mail networks. Here, nodes are address books of individuals and a link from A to B signifies that B's E-mail address appears in A's address book.

[4] Albert and Barabási (2002) report that centrality is widely observed on the web: few nodes have a very high number of outgoing and incoming links. They also report similar findings for the internet interconnection networks. Newman, Forrest and Balthrop (2002) find similar properties for E-mail networks. Redner (1998) reports high centrality for citation networks of papers cataloged by the Institute of Scientific Information.

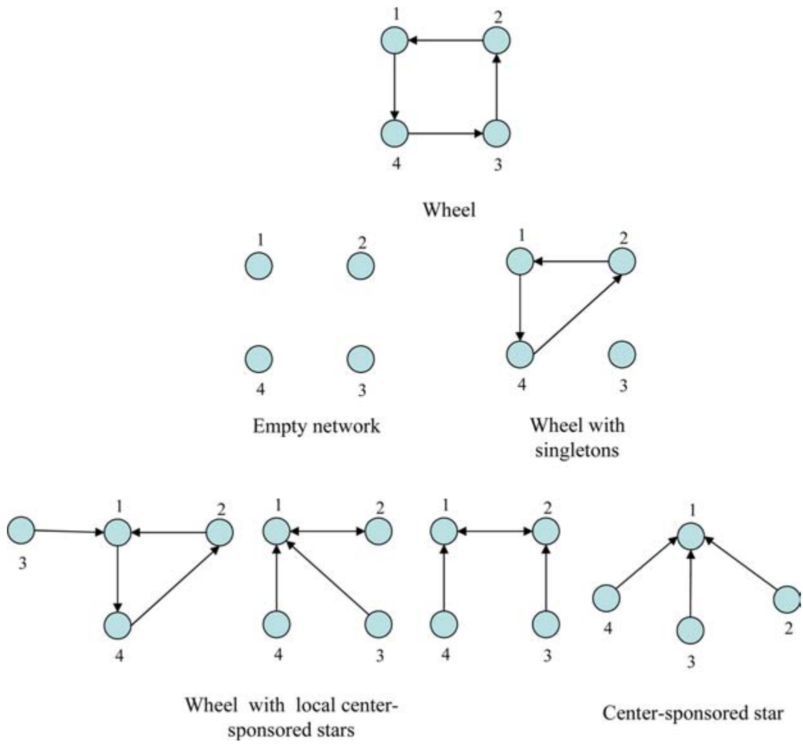

Fig. 1 Partner independent heterogeneity: equilibria

more valuable contacts. Similarly, individuals differ in communication and social skills and it seems natural that forming links is cheaper for some individuals as compared to others.[5]

I start with a setting where values and costs of linking are heterogeneous across players but the heterogeneity is not partner specific: the cost for player i to invest in a social tie is c_i, and the benefit to player i to access another player is V_i. In addition, I assume that the length of the path connecting player i to j does not matter in defining the benefits. Here, I provide a complete equilibrium characterization: a connected equilibrium is a wheel network and an unconnected equilibrium network is a center-sponsored star, a wheel with local center-sponsored stars, a wheel with singletons or empty (Proposition 3.1). Figure 1 illustrates all strict equilibria in a society with four players. This result provides three main insights. The first is that the wheel architecture is robust to asymmetries which are independent of the potential partner.

Secondly, players' heterogeneity alters the level of connectedness of equilibrium networks. Nonempty unconnected equilibria have well defined properties: there is a set of players sharing a maximum amount of information while the remaining players are socially isolated (they do not access any information). In

[5] In other settings players can be classified in term of their cost of being accessed. For example, on the web the terminology user-friendly web site is used to describe home pages which are easier to access as compared to others.

sharp contrast with the homogeneous setting these equilibria are asymmetric and central players may emerge: (1) the players maximally informed are connected in a wheel component; and the players socially isolated are either (2)a. singletons or b. spokes of center-sponsored stars. Third, I show that the property of centrality uniquely and only emerges when the distribution of costs of linking and values of accessing other players satisfies two conditions. On the one hand, there are players, the centers, who have a very low costs of linking as compared to the values of accessing other players. On the other hand, there are players, isolate players, who have a very high costs of forming links as compared to the values of accessing other agents.

I then turn to settings where heterogeneity also depends on the potential partner. I show that as far as the costs of linking are not partner-specific even if values vary freely, at equilibrium, every nonsingleton component has still a wheel architecture (Proposition 3.2). Differently, when the costs of linking are allowed to depend on the potential partner almost any minimal network is a strict equilibrium for some costs and values (Proposition 3.3). Thus, when information flow without frictions, it is the costs of forming links heterogeneity which shapes the architecture of equilibrium networks. Is this result an artifact of the frictionless information flow assumption? The answer to this question is positive. Indeed, I show that as far as a small amount of decay in the information flow is introduced, values' asymmetries are as important as costs' asymmetries in determining the architecture of equilibrium networks (Proposition 3.4).

This paper is a contribution to the theory of network formation. This is a very active area of research currently (see references in footnote 2). Most of the existing literature focuses on the two-way flow connections model and it assumes homogeneous players. I elaborate on the respective roles of values and costs of forming links heterogeneity in shaping equilibrium architectures in a one-way flow connections model. My findings indicate that values and costs heterogeneity are equally important in determining the level of connectedness and the architecture of equilibrium networks. Furthermore, under players' heterogeneity unconnected equilibrium networks are asymmetric and central players arise under well-defined conditions. The emergence of equilibrium networks with central players sharply contrasts with the homogeneous model.

The works that come closer to mine are Galeotti et al. (2005) and Kim and Wong (2003). Galeotti et al. (2005) study the role of heterogeneity in a two-way flow model, while I focus on the one-way flow network technology. It is worth noting that the two assumptions on the information flow provide individuals with very different incentives and this makes the two models belonging to two distinct class of games. Galeotti et al. (2005) confirm that equilibrium networks exhibit short distances and high centrality even in settings with substantial heterogeneity. My findings show that in directed networks, players' heterogeneity allows for the emergence of central players, a property which cannot be obtained in a homogeneous model.

Kim and Wong (2003) study a one-sided connections model with heterogeneous players where agents form two-flow connections but basic links are only one-flow. In other words, this implies that a player i accesses player j only if there exists a sequence of basic links connecting i to j and vice versa. They find that a nonempty equilibrium network is either the wheel or the wheel with singletons.

My work departs from Kim and Wong (2003) in two directions. First, I do not distinguish between basic links and flow connections, which implies that in my framework a player can access another individual, without the reverse being necessarily true.[6] Proposition 3.1 in the current paper shows that when the distinction of basic links and flow connections is not considered the property of centrality may emerge in equilibrium. Second, I analyze different form of players' asymmetries, while Kim and Wong (2003) focus exclusively on settings where asymmetries are not partner specific. This allows me to understand the role that different forms of players' asymmetries play on social interaction.

Finally, I relate my findings to a recent experimental paper by Falk and Kosfeld (2003). This paper shows that the predictions based on Nash and Strict Nash equilibria for the one-way flow model are consistent with the experimental results, while they generally fail in the two-way flow model.[7] The authors argue that the success of the one-way flow model relies, among other things, on the strategic symmetry (symmetric distribution of links) which characterizes equilibrium networks under the one-way flow assumption. The analysis developed in the current paper shows that the property of symmetric distribution of links depends on the assumption of homogeneous players. An experiment which takes into account ex-ante asymmetries in the costs of forming links may help to understand the role played by strategic symmetry in the formation process of a network.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the results on equilibrium networks under general cost and value heterogeneity. Section 4 concludes. Proofs are provided in the Appendix.

2 The model

Let $N = \{1, ..., n\}$ be a set of players and let i and j be typical members of this set. I shall assume throughout that the number of players is $n \geq 3$. Each player is assumed to possess some information of value to himself and to other players. He can augment his information by communicating with other people; this communication takes resources, time and effort and is made possible via *pair-wise* links. I start by considering a setting where information flow without friction in the network, i.e., there is no decay.[8]

A strategy of player $i \in N$ is a (row) vector $g_i = \left(g_{i,1}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n}\right)$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$. I say that there is a link from player i to j if $g_{i,j} = 1$.[9] I assume throughout the paper that a link from i to j allows player i to access j's information. The set of strategies of player i is denoted by $\mathcal{G}_i$. I shall restrict the attention to pure strategies. Since player i has the option of forming or not forming a link with each of the remaining $n-1$ players, the number of strategies of player i is clearly $|\mathcal{G}_i| = 2^{n-1}$. The set $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$ is the space of pure strategies of all the players.

6 For example, the web is characterized by one-way link and one-way flow connections.

7 Bala and Goyal (2000) show that with homogeneous players, when the information flow is bidirectional, a strict equilibrium is either a center-sponsored star (only one player, the center, promotes all the links) or empty (no links).

8 The assumption of no-decay is mainly used for tractability reasons. In Section 3.1 I will relax this assumption.

9 Note that $g_{i,j} = 1$ does not imply that $g_{j,i} = 1$.

A strategy profile $g = (g_1, ..., g_n)$ in $\mathcal{G}$ can be represented as a directed network. Let $g \in \mathcal{G}$, I say that in g there is a path directed from i to j if either $g_{i,j} = 1$ or there exist players $j_1, \cdots, j_m$ distinct from each other and i and j such that $\left\{g_{i,j_1} = \cdots = g_{j_m,j} = 1\right\}$. I write $i \overset{g}{\to} j$ to indicate a path directed from i to j in g. Given two players i and j in g, the geodesic distance, $d(i, j; g)$, is defined as the length of the shortest path directed from i to j. Furthermore, I define $N^d(i; g) = \left\{k \in N \,\middle|\, g_{i,k} = 1\right\}$ as the set of players with whom i maintains a link. This is the set of players that agent i accesses directly. The set $N(i; g) = \left\{k \in N \,\middle|\, i \overset{g}{\to} k\right\} \cup \{i\}$ represents the set of players that i accesses in g both directly and indirectly. Let $\mu_i^d : \mathcal{G} \to \{1, ..., n\}$ and $\mu_i : \mathcal{G} \to \{1, ..., n\}$ be defined as $\mu_i^d(g) = \left|N^d(i; g)\right|$ and $\mu_i(g) = |N(i; \overline{g})|$.

Given a network g, a nonsingleton component of g is a nonsingleton set $C(g) \subset N$ where $\forall i, j \in C(g)$ there exists a path directed from i to j and vice versa, and $\forall i \in C(g)$ there is not a path directed from i to k, where $k \in N \diagdown C(g)$. A component $C(g)$ of a network g is minimal if $C(g)$ is no longer a component upon replacement of a link $g_{i,j} = 1$ between two agents $i, j \in C(g)$ by $g_{i,j} = 0$, ceteris paribus. A network g is minimal if every component of g is minimal. A network g is connected if it has a unique component containing all players. If the unique component is minimal the network g is minimally connected. A network which is not connected is unconnected. Given a network g, a player i is a singleton player if $g_{i,j} = g_{j,i} = 0$ for any $j \in N$. Finally, the empty network, denoted as to g^{e}, is an unconnected network where no links are formed.

To complete the definition of a normal-form game of network formation, I specify the payoffs. Let $V_{i,j}$ denotes the benefits to player i from accessing player j. Similarly, let $c_{i,j}$ denotes the cost for player i of forming a link with player j. The payoff to player i in a network g can be written as follows:

$$\Pi_i(g) = \sum_{j \in N(i;g)} V_{i,j} - \sum_{j \in N^d(i;g)} c_{i,j} \tag{1}$$

I shall assume that $c_{i,j} > 0$ and $V_{i,j} > 0$ for all $i, j \in N$.[10]

Given a network $g \in \mathcal{G}$, let g_{-i} denote the network obtained when all of player i's links are removed. Note that the network g_{-i} can be regarded as the strategy profile where i chooses not to form a link with anyone. The network g can be written as $g = g_i \otimes g_{-i}$ where the '$\otimes$' indicates that g is formed as the union of the links in g_i and g_{-i}. The strategy g_i is said to be a *best response* of player i to g_{-i} if:

$$\Pi_i(g_i \otimes g_{-i}) \geqslant \Pi_i\left(g_i' \otimes g_{-i}\right) \qquad \text{for all } g_i' \in \mathcal{G}_i. \tag{2}$$

The set of all of player $i's$ best responses to g_{-i} is denoted by $\mathcal{BR}_i(g_{-i})$. Furthermore, a network $g = (g_1, ..., g_n)$ is said to be a *Nash network* if $g_i \in \mathcal{BR}_i(g_{-i})$ for each i, i.e., players are playing a Nash equilibrium. If a player has multiple best responses to the equilibrium strategies of the other players then this could make the network less stable as the player can switch to a payoff equivalent strategy. This switching possibility in nonstrict Nash networks has been exploited and has been

10 The results developed further qualitatively carry on when relaxing the linearity assumption of the payoffs functions.

shown to be important in refining the set of equilibrium networks in an earlier work (see e.g., Bala and Goyal 2000). So I will focus on strict Nash equilibria in the present paper. A *strict* Nash equilibrium is a Nash equilibrium where each player gets a strictly higher payoff from his current strategy than he would with any other alternative strategy.

3 Heterogeneity

In this section I investigate the effects of values and costs of linking heterogeneity on the level of connectedness and the architecture of strict equilibria. I start by considering a setting in which each player has a distinct cost of linking as well as a distinct benefit of accessing other agents. While these costs and values vary across players, they are independent of the identity of the partner, i.e., $V_{i,j} = V_i$ and $c_{i,j} = c_i$, for any $i, j \in N$. For example, some individuals are more expert in surfing the web as compared to others; this allows them to access other internet members to a lower cost, ceteris paribus.[11] I first introduce some architectures which will prove useful in the analysis.

A center-sponsored star architecture is an unconnected network where there exists a player i, the center, who forms links with every other player, i.e., $g_{i,j} = 1$ for any $j \in N \setminus \{i\}$, and no other links are formed. A nonsingleton component has a wheel architecture if players within the component are arranged as $\{i_1, ..., i_n\}$ with $g_{i_2,i_1} = \cdots = g_{i_n,i_{n-1}} = \cdots = g_{i_1,i_n} = 1$ and there are no other links between players within the component. I will refer to a wheel component of a network g as $C^W(g)$ and $\mu^W = \left|C^W(g)\right|$. A wheel architecture is a connected network with the unique component being a wheel. A wheel network with local center-sponsored stars is an unconnected network with a unique wheel component, say $C^w(g)$, and where $\forall j \notin C^w(g)$, $\exists! i \in C^w(g)$ such that $g_{i,j} = 1$. Finally, a wheel network with singleton players is an unconnected network with a unique wheel component composed of at least three players and where $g_{i,j} = g_{j,i} = 0$ for any $i \notin C^w(g)$ and for any $j \in N$.

The next result provides a full characterization of equilibrium networks in this model.

Proposition 3.1 *Let payoffs satisfy* (1) *and assume that* $c_{i,j} = c_i$ *and* $V_{i,j} = V_i$, $\forall j \in N \setminus \{i\}$. *The following network architectures are the only equilibria*:

1. *The wheel is a strict equilibrium if and only if* $c_i < (n-1)V_i$, $\forall i \in N$.
2. *The wheel with singletons is a strict equilibrium if and only if* $c_i \in (V_i, (\mu^w - 1)V_i)$, $\forall i \in C^w(g)$ *and* $c_j > \mu^w V_j$, $\forall j \notin C^w(g)$.
3. *A wheel with local center sponsored stars is an equilibrium if and only if each center i has* $c_i < V_i$, *while* $c_j < (n-1)V_j$, $\forall j \in C^w(g)$ *and* $c_j > (n-1)V_j$, $\forall j \notin C^w(g)$.
4. *A center sponsored star is an equilibrium if and only if the center i has* $c_i < V_i$, *while* $c_j > (n-1)V_j$, $\forall j \in N \setminus \{i\}$.
5. *The empty network is an equilibrium if and only if* $c_i > V_i$, $\forall i \in N$.

[11] Similarly, some individuals may value more information provided on the web as compared to others. In general, individuals differ in communication and social skills and it seems natural that the costs of establishing links as well as the values of accessing information vary across individuals.

Figure 1 illustrates all strict equilibria in a society composed of four players. I represent a link $g_{i,j} = 1$ as an edge starting at j with the arrowhead pointing at i.

The proof of Proposition 3.1 proceeds as a sequence of Lemmas. I sketch here the main steps. I first show that a strict Nash network is minimal. This follows from the no-decay assumption. Secondly, using a standard switching argument I show that each player receives at most one link (Lemma 3.1). Third, using this equilibrium property it follows that each nonsingleton component has a wheel architecture (Lemma 3.2). Therefore, a connected strict equilibrium is a wheel. Fourth, I take up the case of nonempty unconnected equilibria in which each component is composed of a single player. Using the finiteness of the set of players I show that an equilibrium is a center-sponsored star network (Lemma 3.3). Finally, an elaboration of the arguments used in the previous lemmas establishes the result for unconnected equilibria which have at least a nonsingleton component.

Proposition 3.1 provides some interesting insights. As in the homogeneous setting, the unique connected equilibrium is the wheel. Therefore, the wheel architecture is prominent also in settings where costs and values asymmetries are partner independent. Next, values and costs heterogeneity alters the level of connectedness of strict equilibria. In any unconnected (and nonempty) equilibrium there is a set of players accessing a maximum amount of information while all the other players are socially isolated (they do not access any information). Furthermore, the maximally informed players are connected in a wheel, while the isolated players are either singletons or spokes of center-sponsored stars. Thus, unconnected equilibria are generally asymmetric and central players may emerge. This contrasts with the findings of Kim and Wong (2003). In a similar setting, they obtain that an equilibrium is either the wheel or the wheel with singletons. The reason behind these differences is that Kim and Wong (2003) distinguish between basic links and flow connections so that a player i accesses player j only if there is a path directed from i to j and vice versa. This prevents the emergence of equilibria with central players. I finally note that the results presented in Proposition 3.1 carry on in settings with homogenous values (resp. costs of linking) and heterogeneous costs of linking (resp. values). This implies that as far as heterogeneity is independent of the partner, costs and values asymmetries have equivalent effects on strategic interaction.

It is interesting to describe more systematically the conditions on the distribution of costs and values which allow to sustain the different equilibrium architectures described in Proposition 3.1.

Corollary 3.1 *Let payoffs satisfy* (1) *and assume that* $c_{i,j} = c_i$ *and* $V_{i,j} = V_i$, $\forall j \in N \setminus \{i\}$. *The following holds*:

1. *If* $c_j < (n-1)\,V_j$, $\forall j \in N$ *an equilibrium is the wheel, the wheel with singletons or the empty network. Further, if for some* $i \in N$, $c_i < V_i$ *the wheel network is the unique equilibrium.*
2. *If for some* $j \in N$, $c_j > (n-1)\,V_j$ *and for some* $i \in N$, $c_i < V_i$ *then an equilibrium is either a center sponsored star or a wheel with local center sponsored stars. Every central player has* $c_i < V_i$.
3. *If for all* $j \in N$, $c_j > (n-1)\,V_j$ *the empty network is the unique equilibrium.*

This corollary clarifies two aspects of Proposition 3.1. First, equilibrium networks with central players emerge uniquely and only when the distribution of costs and values across players satisfies two conditions. One, some agents have low costs of linking as compared to the values of accessing other players, i.e. $c_i < V_i$ for some i. Only these players are candidates for a central position. Two, some players have high costs of linking as compared to the values of accessing others, $c_j > (n-1)\,V_j$ for some j. These players are isolated. Second, the distribution of costs and values which sustains the wheel architecture and its variants have the feature that no players has extremely high costs of linking as compared to the values of accessing other agents. In such a case, it is enough that a single player has $c_i < V_i$ for the wheel architecture to be the unique equilibrium.

I now turn to two more general models which allow for partner dependency heterogeneity in values and costs, respectively. The next result clarifies the role of values heterogeneity in shaping the architecture of equilibrium networks.

Proposition 3.2 *Let payoffs satisfy* (1). *Assume $c_{i,j} = c_i$ while values vary freely. A strict equilibrium is either the empty network or a minimal network where each nonsingleton component is a wheel and each player receives at most one link. Conversely, any such network is a strict equilibrium for some* $\{c_i, V_{i,j}\}_{i,j\in N}$.

I note that as far as costs asymmetries are partner independent, even if values vary freely each equilibrium component has (still) a wheel architecture. The main difference from the case where values are partner independent is that more that one nonsingleton component can be sustained in equilibrium. This result also holds with homogenous costs of linking. I now investigate the role of costs heterogeneity in shaping the architecture of equilibrium networks.

Proposition 3.3 *Let payoffs satisfy* (1). *Assume $V_{i,j} = V_i$ while costs vary freely. A strict equilibrium is either the empty network or a minimal network where, if* $g_{i,j} = 1$, *then there exists one and only one path from i to j. Conversely, any such network is a strict equilibrium for some* $\{c_{i,j}, V_i\}_{i,j\in N}$.

In sharp contrast with Proposition 3.2, when the costs of linking are allowed to vary freely across players almost any minimal network can be sustained as a strict equilibrium.[12] Figure 2 illustrates some architectures which are equilibria when costs vary freely, but that are not strategically viable otherwise. Table 1 summarizes the role that different forms of values and costs heterogeneity have in shaping equilibrium networks.

The findings in this section show that it is the costs' heterogeneity which is mainly responsible for shaping the architecture of equilibrium networks. Galeotti et al. (2005) find a similar result for the two-way information flow model. Thus, with frictionless information flow, regardless of the network technology, the heterogeneity of the costs plays a primary role in shaping the architecture of equilibrium networks. The reason for this result is that when costs are allowed to vary freely, almost any minimal network can be made strategically viable by setting the costs

[12] The property that for any $g_{i,j} = 1$ there exists one and only path connecting i to j (see second part of Proposition 3.3) is crucial because it rules out networks which are minimal but not (strict) equilibria. The following example illustrates this. Consider a network composed of four players. Suppose that player 1 promotes a link with 2 and vice versa. Suppose also that both players 1 and 2 promote a link with player 3. It is readily seen that this network is minimal; however it is not an equilibrium because player 1 (2) strictly gains by deleting the link with player 3.

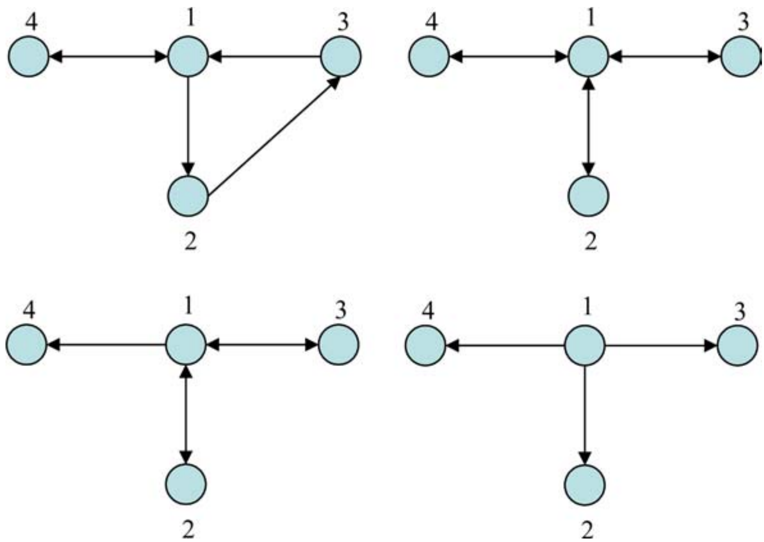

Fig. 2 Free costs heterogeneity: equilibria

Table 1 Equilibrium networks and players' heterogeneity

Costs/values	Homogeneous	Partner-independent	Partner-dependent
Homogeneous	Empty network and wheel network	Empty network, wheel, wheel with local center-sponsored stars, wheel with singletons, center-sponsored stars.	Empty network and minimal networks in which each non-singleton component has a wheel and each player receives at most one link
Partner-independent	Empty network, wheel, wheel with local center-sponsored stars, wheel with singletons, center-sponsored stars.	Empty network, wheel, wheel with local center-sponsored stars, wheel with singletons, center-sponsored stars.	
Partner-dependent	Empty network and minimal networks in which if $g_{i,j} = 1$ then there exists no addtional path from i to j		

each player pays for each actual link sufficiently lower than the costs of switching to another player. Differently, when values vary freely, a player is indifferent as between sponsoring a link with a low valued agent and switching to a high valued player, as far as there is a path directed from the former to the latter and vice versa. This property rules out many minimal network architectures. Are these findings an artifact of the assumption of frictionless information flow?

3.1 A small amount of decay

I measure the level of decay by a parameter $\delta \in (0, 1)$. For a network g it is assumed that if the shorter path directed between i and j has $q \geq 1$ links, then the value of

$j's$ information to i is $V_{i,j}\delta^q$. The payoff (1) to player $i \in N$ in a network g can be rewritten as:

$$\Pi_i(g) = \sum_{j \in N(i;g)} \delta^{d(i,j;g)} V_{i,j} - \sum_{j \in N^d(i;g)} c_{i,j} \tag{3}$$

The next proposition shows that in the presence of a small amount of decay values' heterogeneity is as important as the costs' heterogeneity in shaping the architecture of equilibrium networks.

Proposition 3.4 *Let payoffs satisfy* (3). *Assume* $c_{i,j} = c_i$ *while values vary freely. There exists a* $\tilde{\delta} < 1$ *such that for all* $\delta \in (\tilde{\delta}, 1)$ *a strict equilibrium is either the empty network or a minimal network where, if* $g_{i,j} = 1$, *then there exists one and only one path from* i *to* j. *Conversely, for all* $\delta \in (\tilde{\delta}, 1)$, *any such network is a strict equilibrium for some* $\{c_i, V_{i,j}\}_{i,j \in N}$.

It is worth noting that the result obtained in Proposition 3.3 carries on for small amount of decay. This fact, in combination with Proposition 3.4, shows that as soon as some decay is taken into account, values and costs' asymmetries play the same role in shaping the architecture of equilibrium networks. In particular, when the costs of linking and/or the values of accessing other players are allowed to be partner specific, social interaction leads to a '*everything is possible*' type of result.

4 Discussions

I have studied a connections model where heterogeneous players decide unilaterally to invest in social ties which leads a direct return only to the investor. I have shown that both values and costs asymmetries alter the level of connectedness of equilibrium networks as well as the shape of their architecture. Interestingly, in unconnected equilibria the property of centrality uniquely and only emerges under well defined conditions on the distribution of costs and values. An open question, which is left for further research, is whether we can say something systematic about the architectural properties of equilibrium networks in partner specific heterogeneous models.

Appendix

I first introduce some additional notation. Given a network g, I denote $g' = g - g_{i,j}$ the network obtained by deleting the link $g_{i,j} = 1$ in g. Similarly, $g' = g + g_{i,j}$ is the network obtained by adding a link from i to j in g.

Proof of Proposition 3.1 I start by claiming that if g is a strict equilibrium then g is one of the following: the empty network, the wheel, the wheel with singletons, the wheel with local center-sponsored stars or the center-sponsored star. I first note that an equilibrium network is minimal. This follows from the assumption of no-decay in the information flow. The proof of the claim now proceeds as a sequence of Lemmas. The next result shows that in equilibrium each player receives at most one link. □

Lemma 3.1 *Let g be a strict equilibrium. If $g_{i,j} = 1$ then $g_{k,j} = 0$ for any $k \in N \setminus \{i\}$.*

Proof Suppose, for a contradiction that $g_{i,j} = 1$ and $g_{k,j} = 1$. Since g is minimal, i does not access player k in g; however, in this case player i strictly prefers to delete the link with player j and linking up with player k, instead. This is a contradiction. □

Using this result, I show that each nonsingleton component part of a strict equilibrium is a wheel

Lemma 3.2 *Let $C(g)$ be a nonsingleton component part of a strict equilibrium g. Then $C(g)$ has a wheel architecture.*

Proof I note that if a player i belongs to a nonsingleton component, say $C(g)$, then $g_{i,j} = 1$ for at least one player $j \in C(g)$ and $g_{k,i} = 1$ for at least one player $k \in C(g)$. These two observations and Lemma 3.1 imply that each player $i \in C(g)$ receives one and only one link from the players belonging to $C(g)$. I now claim that for any player $i \in C(g)$, then $g_{i,j} = 1$ for only one player $j \in C(g)$. Suppose, for a contradiction, that for some $i \in C(g)$, $g_{i,j} = g_{i,k} = 1$ for some $j, k \in C(g)$ and $j \neq k$. Since $j, k \in C(g)$ then j and k must access player i; therefore, there exist two paths $\{g_{i,k} = g_{k,k_1} = ...g_{k_{n-1},k_n} = g_{k_n,i} = 1\}$ and $\{g_{i,j} = g_{j,j_1} = ...g_{j_{n-1},j_n} = g_{j_n,i} = 1\}$. Since i receives only one link it must be the case that $j_n = k_n$. However, the same argument applies for player $j_n (= k_n)$, and therefore it must be the case that $j_{n-1} = k_{n-1}$. By induction, it follows that $k_1 = j_1$; since $k \neq j$, it follows that $k_1 (= j_1)$ must receive more than one link. This constitutes a contradiction. These observations altogether implies that $C(g)$ is minimally connected and it has a symmetric architecture. It is readily seen that the unique directed graph which satisfies these properties is the wheel. This proves the Lemma. □

Lemmas 3.1 and 3.2 prove that a connected strict equilibrium network is a wheel. I now take up the case of non-empty unconnected strict equilibria in which each component is a singleton. The next Lemma proves the result.

Lemma 3.3 *A non-empty unconnected strict equilibrium where each component is a singleton has a center-sponsored star architecture.*

Proof I start by claiming that a nonempty unconnected network where each component is a singleton has a center-sponsored star architecture. Since g is nonempty there exists some $g_{i,j} = 1$. There are two cases. First, suppose $g_{j,j'} = 0$ for any $j' \in N$. Since g is strict Nash it must hold that $V_i - c_i > 0$; this implies that $\exists i \overset{g}{\rightarrow} j'$ for any $j' \in N$. Select player k which is at the maximum distance from i in g, i.e., $k = \arg\max_{j' \in N} d_{i,j'}(g)$. If $d_{i,k}(g) = 1$ player i accesses each other player directly and the proof trivially follows. If $d_{i,k}(g) > 1$, it must be the case that $\{g_{i,j_1} = g_{j_1,j_2} = ... = g_{j_m,k} = 1\}$ and $g_{k,s} = 0$ for any $s \in N$. Since g is strict Nash then $V_{j_m} - c_{j_m} > 0$ and therefore player j_m accesses any player in g. This implies that player i and j_m belongs to a nonsingleton component, which constitutes a contradiction. Second, suppose $g_{j,j'} = 1$ for some $j' \in N$. Since g has only singleton components it follows $j' \in N\backslash\{i\}$. Therefore, if $g_{j',k} = 0$ for any $k \in N$, the previous argument applies and we end-up with a contradiction. If $g_{j',k} = 1$ for some k, then it must be the case that $k \in N\backslash\{i, j\}$. Since the number of players is finite, there must exist a player h who is accessed by player i via the link $g_{i,j} = 1$ and such that $g_{h,h'} = 1$ and $g_{h',h''} = 0$ for any $h'' \in N$. However, also in this case the fact that g is strict Nash implies that $V_h - c_h > 0$ and therefore player h must access player i in g. This constitutes a contradiction. This completes the proof. □

I now turn to unconnected strict equilibria where at least a nonsingleton component exists. Let $C_1(g), C_2(g), ..., C_m(g)$ be the components of an unconnected strict equilibrium g. Lemmas 3.1 and 3.2 imply that: (a) $C_x(g)$ is a wheel $\forall x = 1, ..., m$; (b) $g_{j,i} = 0$, $\forall i \in C_x(g)$ and $\forall\ j \in N\backslash\{C_x(g)\}$, $\forall\ x \in \{1, ..., m\}$.

Lemma 3.4 *Let g be a strict equilibrium and let $i \in C_x(g)$. If $g_{i,j} = 1$ where $j \notin \cup_{y=1}^m C_y(g)$, then $g_{j,k} = 0$ for any $k \in N$.*

Proof Suppose, for a contradiction $g_{i,j} = g_{j,k} = 1$. Lemma 3.1 implies that $k \notin \cup_{y=1}^m C_y(g) \cup \{j\}$; moreover, it also implies that if $g_{k,h} = 1$ then $h \notin \cup_{y=1}^m C_y(g) \cup \{j, k\}$. Suppose that $g_{k,h} = 0$ for any $h \notin \cup_{y=1}^m C_y(g) \cup \{j, k\}$; since g is a strict Nash it follows that $V_j > c_j$. In this case player j strictly gains by forming a link with player i. This constitutes a contradiction. If $g_{k,h} = 1$ for some $h \notin \cup_{y=1}^m C_y(g) \cup \{j, k\}$ we can iterate the argument above and since the number of players is finite the proof follows. □

Lemmas 3.1, 3.3, and 3.4 imply that any pair of players, say i and j, belonging to two different components, say $C_x(g)$ and $C_y(g)$, access two distinct set of players, i.e. if $i \in C_x(g)$ and $j \in C_y(g)$, with $x \neq y$, then $N_i(g) \cap N_j(g) = \Phi$. The next Lemma uses this observation to prove that a strict equilibrium network has at most one nonsingleton component.

Lemma 3.5 *A strict equilibrium has at most one nonsingleton component.*

Proof Suppose not and, without loss of generality, let $|N_i(g)| \geq \left|N_j(g)\right|$, where $i \in C_x(g)$ and $j \in C_y(g)$, and $x \neq y$. Since g is strict Nash it follows that $\left|N_j(g)\right| V_j - c_j > 0$; however, if this is the case, player j is weakly better off by deleting his link in $C_y(g)$ and linking up with player i, i.e., $|N_i(g)|\ V_j - c_j \geq \left|N_j(g)\right| V_j - c_j > 0$. This contradiction proves the lemma. □

The next lemma completes the analyses of unconnected strict equilibria which have a nonsingleton component.

Lemma 3.6 *Let g be an unconnected strict equilibrium with a nonsingleton component. Then $g_{j,j'} = 0$ for any $j, j' \notin C(g)$.*

Proof Suppose, for a contradiction, that $g_{j,j'} = 1$. Lemma 3.1 implies that each player outside the component does not access players belonging to the component. Therefore, Lemma 3.5 applies to the set of players $N\backslash\{C(g)\}$, i.e., $g_{j,j'} = 1$ for some $j' \notin C(g)$. However, in this case player j strictly gains by creating a link with a player $i \in C(g)$. This constitutes a contradiction. Hence, Lemma 3.5 follows. □

The combination of Lemmas 3.2, 3.4, 3.5 and 3.6 implies that an unconnected strict equilibrium with some nonsingleton components is a wheel with local center-sponsored stars, a wheel with singleton players or a wheel with some local center-sponsored star and some singleton player. It is immediately seen that this last architecture cannot be sustained as a strict equilibrium. Hence, the claim follows.

I now provide the necessary and sufficient conditions for each of the aforementioned network architectures to be a strict equilibrium.

(1) Let g be a wheel network. It is immediately seen that if $c_i < (n-1) V_i$ for any $i \in N$ then g is a strict equilibrium. Suppose now that $c_i \geq (n-1) V_i$ for some $i \in N$ and let $g_{i,j} = 1$. Note that $\Pi_i (g) \leq \Pi_i \left(g - g_{i,j}\right) = 0$. Hence, g is not a strict equilibrium.

(2) Let g be a wheel with singletons. First, suppose that $c_i \in (V_i, (\mu^w - 1) V_i)$ $\forall i \in C^w (g)$ and $c_j > \mu^w V_j$ for any $\forall j \notin C^w (g)$. The fact that $c_i > V_i$ implies that every player $i \in C^w (g)$ does not want to form a link with any player $j \notin C^w (g)$. The fact that $c_i < (\mu^w - 1) V_i$ implies that every player i wants to keep the link he has with the other member of the wheel component. Since $c_j > \mu^w V_j$ $\forall j \notin C^w (g)$ the strategy of not forming links is optimal for every such player j. Thus g is a strict equilibrium. Suppose now that at least one of the conditions is not satisfied. If $c_i \leq V_i$ for some $i \in C^w (g)$ then player i weakly gains by forming a link with a player $j \notin C^w (g)$. If $c_i \geq (\mu^w - 1) V_i$ for some $i \in C^w (g)$, then player i weakly gains by deleting the link he has with a player $i' \in C^w (g)$. Suppose $c_j \leq \mu^w V_j$ for some $j \notin C^w (g)$, then player j weakly gains by forming a link with a player $i \in C^w (g)$.

(3) Let g be a wheel with x local center-sponsored stars and let $\{i_1, \ldots, i_x\} \in C^w (g)$ be the centers. Suppose $c_i < (n-1) V_i$ for any $i \in C^w (g)$, $c_{i_l} < V_{i_l}$ for any $l = 1.\ldots, x$ and $c_j > (n-1) V_j$ for any $j \notin C^w (g)$. It is immediately seen that g is a strict equilibrium. Suppose now that $c_i \geq (n-1) V_i$ for some $i \in C^w (g)$. Since $i \in C^w (g)$, there exists a player $h \in C^w (g)$ such that $g_{i,h} = 1$; note that $\Pi_i (g) \leq \Pi_i \left(g - g_{i,j}\right) = 0$. Suppose that $c_{i_l} \geq V_{i_l}$ for some $l = 1, \ldots, x$ and that $g_{i_l,j} = 1$ for some $j \notin C^w (g)$ then $\Pi_{i_l} (g) = (n-1) V_{i_l} - x c_i \leq \Pi_{i_l} \left(g - g_{i_l,j}\right) = (n-2) V_{i_l} - (x-1) c_i$. Suppose that $c_j \leq (n-1) V_j$ for some $j \notin C^w (g)$. Then $\Pi_j (g) = 0 \leq \Pi_j \left(g + g_{j,i}\right) = (n-1) V_j - c_j$, where $i \in C^w (g)$.

(4) Let g be a center-sponsored star and let i be the center. It is immediately seen that if $c_i < V_i$ for the center and $c_j > (n-1) V_j$ for any $j \neq i$, then g is a strict equilibrium. Suppose now that the $c_i \geq V_i$ for the center i, then $\Pi_i (g) \leq \Pi_i (g^{\mathrm{e}}) = 0$, where g^{e} is the empty network. Suppose $c_j \leq (n-1) V_j$ for some $j \neq i$, then $\Pi_j (g) = 0 \leq \Pi_j \left(g + g_{j,i}\right) = (n-1) V_j - c_j$.

(5) Let g be the empty network. It is clear that if $c_i > V_i$ for any i, then g is a strict equilibrium. Suppose $c_i \leq V_i$ for some i, then $\Pi_i (g) = 0 \leq \Pi_i \left(g + g_{i,j}\right) = V_i - c_i$.

The proof of Proposition 3.1 is completed. □

Proof of Corollary 3.1 The proof follows immediately from Proposition 3.1. □

Proof of Proposition 3.2 Suppose that $c_{i,j} = c_i$, $\forall j \in N$ and $V_{i,j}$ varies freely. I first note that an equilibrium network is minimal; this follows from the no decay assumption. Let g be a nonempty (strict) equilibrium network. The result of Lemmas 3.1 and 3.2 in Proposition 3.1 applies also when values vary freely. Hence, the proof follows. I now prove the converse. If g is the empty network, the proof is trivial and therefore omitted. Let g be a minimal network satisfying Lemma 3.1. I introduce some notations. For any $g_{i,j} = 1$ let $I_{i,j}(g)$ be the set of players including j whom player i exclusively accesses via the link $g_{i,j} = 1$. Select an arbitrary player $i \in N$. For any $g_{i,j} = 1$ set (1) $\sum_{j' \in I_{i,j}(g)} V_{i,j'} > c_i$, while set $\sum_{j' \notin N_i(g)} V_{i,j'} < c_i$. The optimality of maintaining the link of each player i follows by (1), the optimality of not forming any additional link follows by (2). For any $i \in N$ such that $g_{i,j} = 0\ \forall j \in N$ let $\sum_{j \in N} V_{i,j'} < c_i$. This condition implies that i is playing his unique best response. This completes the proof. □

Proof of Proposition 3.3 Suppose that $V_{i,j} = V_i$, $\forall j \in N$, and $c_{i,j}$ varies freely. The no-decay assumption implies that an equilibrium network is minimal. I now note that if g is an equilibrium and $g_{i,j} = 1$ then there cannot be any other additional path connecting i to j. For otherwise player i would strictly gain by deleting the link with j and still access the same set of players. I now prove the converse. Let g be a minimal network with the property that if $g_{i,j} = 1$ there is no additional path connecting i to j. For any $g_{i,j} = 1$ let $I_{i,j}(g)$ be the number of player whom player i accesses exclusively via the link $g_{i,j} = 1$ and set $c_{i,j} < I_{i,j}(g) V_i$. For any $g_{i,j} = 0$ let $c_{i,j} > (n-1) V_i$. These two conditions assure that each player is playing his unique best response. This completes the proof of the Proposition. □

Recall that given a network g, $N^d(i; g) = \left\{k \in N \,\middle|\, g_{i,k} = 1\right\}$ is the set of players that agent i accesses directly. Further, $N(i; g) = \left\{k \in N \,\middle|\, i \overset{g}{\to} k\right\} \cup \{i\}$ is the set of players that i accesses in g. Let $P(i; g) = N(i; g) \setminus \left\{N^d(i; g)\right\}$.

Proof of Proposition 3.4 Suppose that $c_{i,j} = c_i$, $\forall j \in N$, and $V_{i,j}$ varies freely. Note that when $\delta = 1$, Proposition 3.2 implies that a strict equilibrium network is minimal. By invoking continuity it then follows that there exists a $\tilde{\delta} < 1$ such that for any $\delta \in \left(\tilde{\delta}, 1\right)$ a strict equilibrium network is minimal. I now note that if g is an equilibrium and $g_{i,j} = 1$ then there cannot be any other additional path directed from i to j. Suppose that this is not the case. Let $g' = g - g_{i,j}$, it is seen immediately that $N(i; g) = N\left(i; g'\right)$ and that $N^d(i; g) = N^d\left(i; g'\right) \cup \{j\}$, which implies that $\Pi_i(g) - \Pi_i\left(g'\right) = \sum_{j \in N(i;g)} \left(\delta^{d(i,j;g)} - \delta^{d\left(i,j;g'\right)}\right) V_{i,j} - c_i < 0$ as $\delta \to 1$. This completes the proof of the first part of the proposition.

I now prove the converse. Let g be a minimal network with the property that if $g_{i,j} = 1$ there is not an additional path directed from i to j. Consider an arbitrary player i and fix $\delta \in \left(\tilde{\delta}, 1\right)$. First, for any $k \in N^d(i; g)$, set $c_i < \delta V_{i,k}$. Second, for any $j \in N \setminus \left\{N^d(i; g)\right\}$ set $c_i > \sum_{j \in N \setminus \left\{N^d(i;g)\right\}} V_{i,j}$. The latter condition implies that player i does not wish to form any additional link. The former condition implies that player i strictly prefers to maintain the link with k, instead of deleting the link with k and forming a new link with a player h who either (a) does not access k in g or (b) who accesses k in g via a path which includes i. For otherwise, after the switching, player i will not access player k. Thus, the only type of deviation to consider is one where: player i deletes the link with k and forms a new link with

h, given that in g there exists a path directed from h to k which does not include i. Let h be such a player and let $g' = g - g_{i,k} + g_{i,h}$. Note that

$$\Pi_i(g) = \sum_{j \in N^d(i;g)} \delta V_{i,j} + \sum_{j' \in P(i;g)} \delta^{d(i,j';g)} V_{i,j'} - \mu_i^d c_i$$

Note also that

$$\Pi_i\left(g'\right) = \sum_{j \in N^d(i;g)\setminus\{k\}} \delta V_{i,j} + \delta V_{i,h} + \delta^{d\left(i,k;g'\right)} V_{i,k} + \sum_{j' \in P(i;g')} \delta^{d\left(i,j';g'\right)} V_{i,j'} - \mu_i^d c_i$$

Thus,

$$\begin{aligned}\Pi_i(g) - \Pi_i\left(g'\right) = {} & \delta V_{i,k}\left(1 - \delta^{d\left(i,k;g'\right)-1}\right) - \delta V_{i,h} \\ & + \sum_{j' \in P(i;g)} \delta^{d\left(i,j';g\right)} V_{i,j'} - \sum_{j' \in P(i;g')} \delta^{d\left(i,j';g'\right)} V_{i,j'}\end{aligned}$$

For a given $\delta \in \left(\tilde{\delta}, 1\right)$, fix $V_{i,h}$ and $V_{i,j'}$ for any $j' \in P(i;g) \cup P\left(i;g'\right)$, and set $V_{i,k}$ sufficiently large so that $\Pi_i(g) - \Pi_i\left(g'\right) > 0$. This implies that player i strictly prefers g to g'. Since i is an arbitrary player the proof follows. □

References

Albert, R., Barabási, A-L.: Statistical mechanics of complex networks. Rev Mod Phys **74**, 47–97 (2002)

Aumann, R., Myerson, R.: Endogenous formation of links between players and coalitions: an application of the shapley value. In: Roth, A. (ed.) The shapley value. Cambridge: Cambridge University Press 1989

Bala, V., Goyal, S.: A non-cooperative model of network formation. Econometrica **68**, 1181–1229 (2000)

Burt, R.: Structural holes: the social structure of competition. Cambridge: Harvard University Press 1992

Coleman, J.: Medical innovation: a diffusion study. New York: Bobbs-Merrill 1966

Dutta, B., Jackson, M.: The stability and efficiency of directed communication networks. Rev Econ Des **5**, 251–272 (2000)

Falk, A., Kosfeld, M.: It's all about connections: evidence on network formation. IZA Discussion Paper 777 (2003)

Galeotti, A., Goyal, S., Kamphorst J.: Network formation with heterogeneous players. Games Econ Behav (forthcoming) (2005)

Goyal, S.: Sustainable communication networks. Tinbergen Institute, Discussion paper (1993)

Granovetter, M.: Getting a job: a study of contacts and careers. Chicago: University of Chicago Press 1974

Haller, H., Sarangi, S.: Nash networks with heterogeneous links. Math Soc Sci **50** (2), 181–201 (2005)

Jackson, M., Wolinsky, A.: A strategic model of social and economic networks. J Econ Theory **71**, 44–74 (1996)

Johnson, C., Gilles, R. P.: Spatial social networks. Rev Econ Design **5**, 273–301 (2000)

Kim C., Wong, K. C.: Network formation and stable equilibrium. Mimeo (2003)

McBride, M.: Imperfect monitoring in communication networks. J Econ Theory (forthcoming) (2004)

Montgomery, J.: Social networks and labor market outcomes: toward an economic analysis. Am Econ Rev **81**, 1408–1418 (1991)

Newman, M., Forrest, S., Balthrop, J.: Email networks and the spread of computer viruses. Phys Rev E **66**, 035101 (2002)
Redner, S.: How popular is your paper? An empirical study of the citation distribution. Eur Phys J B, **4**, 131–134 (1998)
Slikker M., van den Nouweland, A.: A one-stage model of link formation and payoff division. Games Econ Behav **34**, 153–175 (2001)
Watts, A.: A dynamic model of network formation. Games Econ Behav **34**, 331–341 (2001a)
Watts, A.: Non-mopic formation of circle networks. Econ Lett **74**, 272–282 (2001b)