Economic Theory (2006) 29: 111–121
DOI 10.1007/s00199-005-0014-1

RESEARCH ARTICLE

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Characterizations of strategy-proof and fair mechanisms for allocating indivisible goods

Received: 25 May 2005 / Accepted: 21 June 2005 / Published online: 29 September 2005

Abstract We study the problem of allocating indivisible goods when monetary compensations are possible. First, we characterize the set of strategy-proof and envy-free mechanisms. Second, we identify the Pareto undominated subset in the set of strategy-proof and envy-free mechanisms. These characterizations can be interpreted as envy-free selections of Groves mechanisms.

Keywords Indivisible goods · Strategy-proofness · Envy-freeness · Groves mechanisms

JEL Classification Numbers C72 · D63 · D71 · D82

1 Introduction

We study the problem of allocating indivisible goods when monetary compensations are possible. We consider economies with n agents, $s(1 \leq s \leq n-1)$ units of homogeneous indivisible goods, and money. Each agent consumes at most one indivisible good, and each indivisible good must be consumed by one agent. We allow each agent to have a negative valuation of indivisible goods, and thus our study includes an analysis of allocating indivisible "bads", such as dangerous missions and noxious facilities. For this allocation problem, we construct direct revelation mechanisms, which determine how to allocate indivisible goods and make monetary compensations.

I would like to thank Jun Matsuyama, Hervé Moulin, Satoko Okuyama, Takehiko Yamato, Naoki Yoshihara, an associate editor, two anonymous referees, and seminar participants at Rice University and Tokyo Institute of Technology for helpful suggestions and detailed comments. Research was partially supported by Grant-in-Aid for Scientific Research 17730126 of the Ministry of Education, Culture, Sports, Science and Technology in Japan.

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We want to realize a fair allocation of indivisible goods and money. One of the most important concepts of fairness is envy-freeness (Foley 1967). [1] Envy-freeness requires that no agent should prefer another agent's consumption bundle to his own. However, it is not sufficient for realizing a fair allocation to construct mechanisms which always choose an envy-free allocation. Since preferences are private information, selfish agents may misrepresent their preferences in order to realize an allocation in their favor. To prevent such strategic behavior, we should impose an incentive compatibility constraint on mechanisms. One of the most appealing constraint is strategy-proofness (Gibbard 1973; Satterthwaite 1975). [2] Strategy-proofness requires that truthfully reporting his preferences should be a weakly dominant strategy for each agent.

For the case of one indivisible good and money, it follows from Tadenuma and Thomson (1995) that there is no strategy-proof and envy-free mechanism under the "budget balance" condition. Ohseto (2000) focuses on the size of the set of preferences and provides the following negative result: there is no strategy-proof and envy-free mechanism under the budget balance condition when the set consists of more than three quasi-linear preferences. Without imposing envy-freeness, characterizations of strategy-proof and budget balanced mechanisms satisfying auxiliary axioms are well established. For the case of one indivisible good and money, Ohseto (1999) provides such a characterization. For the case of heterogeneous indivisible goods and money, Schummer (2000) investigates the properties of strategy-proof and "nonbossy"[3] mechanisms, and Miyagawa (2001) and Svensson and Larsson (2002) provide complete characterizations using the auxiliary axioms such as nonbossiness and "individual rationality". The mechanisms in these characterizations admit only a finite number of monetary compensations, and therefore they are not envy-free.

In this paper, without imposing budget balance, we present two characterizations for the case of homogeneous indivisible goods and money. First, we characterize the set of strategy-proof and envy-free mechanisms (Theorem 1). Second, we identify the Pareto undominated subset in the set of strategy-proof and envy-free mechanisms (Theorem 2). Our study is closely related to the literature on Groves mechanisms.[4] Holmström's (1979) general result shows that the set of Groves mechanisms is equivalent to the set of strategy-proof and "decision-efficient" mechanisms in our model. Since Groves mechanisms are rich, we face to solve a selection problem. Since envy-freeness implies decision-efficiency in our model (Svensson 1983), our study can be interpreted as finding the set of envy-free Groves mechanisms, and the Pareto undominated subset in that set.

For the case of heterogeneous indivisible goods and money, Svensson (2004) recently provides a characterization that is parallel to our Theorem 2. He focuses on the set of "optimal fair allocation mechanisms" introduced by Sun and Yang (2003), and characterizes this set by the axioms of strategy-proofness, envy-freeness,

[1] For the problem of allocating indivisible goods and money, the existence and selections of envy-free allocations are extensively studied by Svensson (1983), Alkan et al. (1991), and Tadenuma and Thomson (1991).

[2] Mas-Colell et al. (1995, Section 23.C), Sprumont (1995), and Barberà (2001) provide excellent surveys of the literature on strategy-proofness.

[3] This axiom is introduced by Satterthwaite and Sonnenschein (1981).

[4] See, among others, Vickrey (1961), Clarke (1971), Groves (1973), Green and Laffont (1979), and Mas-Colell et al. (1995, Section 23.C).

"budget balance at one preference profile v", and "efficiency at preference profile v". His axiom of "efficiency at preference profile v" requires that monetary compensations $(t_1, ..., t_i, ..., t_n)$ should never be worse than the monetary compensation $(t_1^*, ..., t_i^*, ..., t_n^*)$ at preference profile v in the sense that $t_i < t_i^*$ for all agents. One difference between Svensson's characterization and our Theorem 2 is that he uses the above-mentioned auxiliary axioms, whereas we use the "Pareto dominance relation" defined in Section 2. Another difference between two characterizations is that Svensson studies the heterogeneous case and we study the homogeneous case. Although the set of preferences over heterogeneous indivisible goods contains the set of preferences over homogeneous indivisible goods, a characterization result on the former set is logically independent of that on the latter set.[5]

The rest of the paper is organized as follows. Section 2 contains notation and definitions. Section 3 presents main characterizations. Section 4 offers some remarks. Proofs are presented in the Appendix.

2 Notation and definitions

Let $N = \{1, ..., n\}$ $(n \geq 2)$ be the set of agents. There are $s (1 \leq s \leq n-1)$ units of homogeneous indivisible goods and some amount $T \in \mathbb{R}$ of a transferable good (often regarded as money). We assume that each agent consumes one indivisible good at most and some amount of money. We prohibit each agent from disposing of the indivisible good even if it is "bad" for him. We allow negative consumptions of money. Agent i's consumption space is the set of consumption bundles $(s_i, t_i) \in \{0, 1\} \times \mathbb{R}$, where s_i denotes his consumption of indivisible goods and t_i denotes his consumption of money. The set of *feasible allocations* is $Z = \{z = (z_1, ..., z_n) = ((s_1, t_1), ..., (s_n, t_n)) \in [\{0, 1\} \times \mathbb{R}]^n \mid \sum_{i \in N} s_i = s$ and $\sum_{i \in N} t_i \leq T\}$.

Each agent i has a *valuation* $v_i \in \mathbb{R}$ of indivisible goods, and his preference can be represented by a quasi-linear utility function $U((s_i, t_i); v_i) = v_i s_i + t_i$. Given any $\alpha, \beta \in \mathbb{R}$, let $V_i = [\alpha, \beta] \subset \mathbb{R}$ be the set of agent i's possible valuations of indivisible goods. Let V be the Cartesian product of V_i, and an element $v = (v_1, ..., v_n) \in V$ is called a *valuation profile*. For any $v \in V$, let $\gamma_q(v) (q = 1, ..., n)$ denote the qth highest valuation in v. When there is a tie for the qth highest valuation, we may break it arbitrarily. Hence $\gamma_1(v) \geq \cdots \geq \gamma_s(v) \geq \gamma_{s+1}(v) \geq \cdots \geq \gamma_n(v)$ for any $v \in V$. Given any coalition $C \subset N$, let (v'_C, v_{-C}) denote the valuation profile whose ith component is v'_i if $i \in C$ and v_i if $i \notin C$. For simplicity of notation, we often use (v'_i, v_{-i}) instead of $(v'_{\{i\}}, v_{-\{i\}})$.

A *mechanism* is a function $f : V \to Z$, which associates a feasible allocation with each valuation profile. Given a mechanism f and $v \in V$, we write $f(v) = ((s_1(v), t_1(v)), ..., (s_n(v), t_n(v)))$, $f_i(v) = (s_i(v), t_i(v))$ for any $i \in N$, and $C(v) = \{i \in N \mid s_i(v) = 1\}$. Note that $C(v)$ denotes the consumers of indivisible goods at $v \in V$.

[5] This is not a universal statement. A characterization on a larger preference domain can imply a characterization on a smaller preference domain if no axioms are used that link different preference profiles with each other in a particular way (e.g., Pareto efficiency, envy-freeness, etc.). In this paper, this statement is true because the use of strategy-proofness crucially depends on the domain.

We introduce three standard axioms. Strategy-proofness requires that truth-telling should be a weakly dominant strategy for each agent. Envy-freeness requires that no agent should prefer another agent's consumption bundle to his own. Decision-efficiency requires that s agents with s highest valuations (when there is a tie for the sth highest valuation, we can break it arbitrarily) should consume indivisible goods.

Strategy-proofness. A mechanism f is strategy-proof if for any $v \in V$, any $i \in N$, and any $v_i' \in V_i$, $U(f_i(v); v_i) \geq U(f_i(v_i', v_{-i}); v_i)$.

Envy-freeness. A mechanism f is envy-free if for any $v \in V$ and any $i, j \in N$, $U(f_i(v); v_i) \geq U(f_j(v); v_i)$.

Decision-efficiency. A mechanism f is decision-efficient if for any $v \in V$ and any coalition $C \subset N$ consisting of s agents, $\sum_{i \in C(v)} v_i \geq \sum_{i \in C} v_i$.

It follows from Svensson (1983) that *envy-freeness implies decision-efficiency* in our model. Formally, assume that a mechanism f is envy-free, but not decision-efficient. Then there are some $v \in V$ and $i, j \in N$ such that $v_i > v_j$, $i \notin C(v)$, and $j \in C(v)$. Let $f_i(v) = (0, t_i(v))$ and $f_j(v) = (1, t_j(v))$. By envy-freeness, $t_i(v) \geq v_i + t_j(v)$ and $v_j + t_j(v) \geq t_i(v)$, which contradict $v_i > v_j$.

We introduce the definition of Groves mechanisms in our model. It follows from Holmström's (1979) that *the set of Groves mechanisms is equivalent to the set of strategy-proof and decision-efficient mechanisms* in our model.

Groves mechanisms. A mechanism f is a Groves mechanism if f is a decision-efficient mechanism such that for any $v \in V$ and any $i \in N$, $t_i(v) = \sum_{j \in C(v) \setminus \{i\}} v_j + h_i(v_{-i})$, where $h_i(v_{-i})$ is an arbitrary function of v_{-i}.

We use the following Pareto dominance relation for welfare comparisons between two mechanisms. A mechanism f *Pareto dominates* another mechanism f' if for any $v \in V$ and any $i \in N$, $U(f_i(v); v_i) \geq U(f_i'(v); v_i)$, and for some $v \in V$ and some $i \in N$, $U(f_i(v); v_i) > U(f_i'(v); v_i)$.

3 Main characterizations

In this section we characterize the set of strategy-proof and envy-free mechanisms (Theorem 1) and identify the Pareto undominated subset in the set of strategy-proof and envy-free mechanisms (Theorem 2).

We state a direct result of envy-freeness. If a mechanism f is envy-free, then agents who (or who do not) consume indivisible goods must have the same amount of money. For any $v \in V$, let $t_w(v)$ and $t_l(v)$ denote the amounts of money allocated to the " winners" and the "losers", respectively, of indivisible goods, i.e. $t_i(v) = t_w(v)$ for any $i \in C(v)$ and $t_j(v) = t_l(v)$ for any $j \notin C(v)$.

We introduce the following new class of mechanisms.

Definition 1 *Given any nonnegative-valued function* $\pi : \mathbb{R} \to \mathbb{R}_+$ *satisfying*

[Condition A] for any $x, y \in \mathbb{R}$ $(x < y)$, $-\dfrac{n-s}{n} \leq \dfrac{\pi(y) - \pi(x)}{y - x} \leq \dfrac{s}{n}$, *and*

[Condition B] for any $x, y \in \mathbb{R}$ $(x < y)$, $\frac{s(n-s)(y-x)}{n} \leq s\pi(x) + (n-s)\pi(y)$,
let f^{π} be a decision-efficient mechanism such that for any $v \in V$, $t_w^{\pi}(v) = \frac{T-(n-s)\gamma_{s+1}(v)}{n} - \pi(\gamma_{s+1}(v))$ and $t_l^{\pi}(v) = \frac{T+s\gamma_s(v)}{n} - \pi(\gamma_s(v))$.

Let F^1 be the set of all mechanisms $\{f^{\pi}\}$ introduced in Definition 1. First, we mention that Condition A is related to envy-freeness. Let $v \in V$ be such that $\gamma_s(v) = y$ and $\gamma_{s+1}(v) = x$ $(x < y)$. Then $t_w^{\pi}(v) = \frac{T-(n-s)x}{n} - \pi(x)$ and $t_l^{\pi}(v) = \frac{T+sy}{n} - \pi(y)$. Since π satisfies Condition A, $\gamma_s(v) + t_w^{\pi}(v) \geq t_l^{\pi}(v)$ and $t_l^{\pi}(v) \geq \gamma_{s+1}(v) + t_w^{\pi}(v)$. Since f^{π} is decision-efficient, this means that the agent with the sth highest valuation does not envy the agent with the $(s+1)$th highest valuation and vice versa. It is easy to check the other cases and thus each mechanism f^{π} is envy-free. The necessity of Condition A for envy-freeness is shown in Lemma 4. Second, we demonstrate that Condition B is related to feasibility, i.e., each mechanism f^{π} associates a feasible allocation with each valuation profile. For any $v \in V$, the total amount of money allocated to agents is $T(v) = st_w^{\pi}(v) + (n-s)t_l^{\pi}(v) = T + \frac{s(n-s)\{\gamma_s(v) - \gamma_{s+1}(v)\}}{n} - \{s\pi(\gamma_{s+1}(v)) + (n-s)\pi(\gamma_s(v))\}$. When $\gamma_s(v) = \gamma_{s+1}(v)$, since π is a non-negative-valued function, $T(v) \leq T$. When $\gamma_s(v) > \gamma_{s+1}(v)$, since π satisfies Condition B, $T(v) \leq T$. The necessity of Condition B for feasibility is proved in Lemma 4. Third, we check that each mechanism f^{π} is a Groves mechanism. It is easy to see that $t_w^{\pi}(v)$ can be written as $t_i(v) = \sum_{j \in C(v)\setminus\{i\}} v_j + h_i(v_{-i})$, where $h_i(v_{-i}) = -\sum_{j \in C(v)\setminus\{i\}} v_j + \frac{T-(n-s)\gamma_{s+1}(v)}{n} - \pi(\gamma_{s+1}(v))$ for any $i \in C(v)$, and $t_l^{\pi}(v)$ can be written as $t_i(v) = \sum_{j \in C(v)\setminus\{i\}} v_j + h_i(v_{-i})$, where $h_i(v_{-i}) = -\sum_{j \in C(v)\setminus\{i\}} v_j + \frac{T+s\gamma_s(v)}{n} - \pi(\gamma_s(v))$ for any $i \notin C(v)$.

We characterize the set of strategy-proof and envy-free mechanisms.

Theorem 1 *F^1 is the set of strategy-proof and envy-free mechanisms.*

Proof See the Appendix.

We define an interesting subset of mechanisms in F^1. For any $p \in [\alpha, \beta] \subset \mathbb{R}$, let $\pi^p : \mathbb{R} \to \mathbb{R}_+$ be the function such that $\pi^p(x) = ((n-s)/n)(p-x)$ if $x < p$ and $\pi^p(x) = (s/n)(x-p)$ if $x \geq p$. It is easy to check that each π^p satisfies Conditions A and B. Letting $\pi = \pi^p$ in Definition 1, with some computations, we have the following mechanisms.

Definition 2 *Given any $p \in [\alpha, \beta] \subset \mathbb{R}$, let f^p be a decision-efficient mechanism such that for any $v \in V$, $t_w^p(v) = (T-(n-s)p)/n - \max\{0, \gamma_{s+1}(v) - p\}$ and $t_l^p(v) = (T+sp)/n - \max\{0, p - \gamma_s(v)\}$.*

Let F^2 be the set of all mechanisms $\{f^p\}$ introduced in Definition 2. This set consists of the pivotal mechanism (the case of $p = 0$) and its variants.[6]

[6] The pivotal mechanism is prominent in the set of Groves mechanisms. Some characterization results are established in a public good context (Moulin 1986).

Each mechanism f^p has a natural interpretation. The allocation of indivisible goods is determined by decision-efficiency and some tie-breaking rule. The allocation of money is determined as follows. The total amount T of money is allocated to agents equally. Each winner must pay a *tentative price* p. The total amount sp of revenue is allocated to agents equally. Moreover, if the tentative price is " too low" (i.e., $p < \gamma_{s+1}(v)$), each winner must pay $\gamma_{s+1}(v) - p$ additionally, and if it is " too high" (i.e., $p > \gamma_s(v)$), each loser must pay back $p - \gamma_s(v)$. This amount of money is not reallocated to agents.

We compute the budget surplus of the mechanism f^p. For any $v \in V$, the budget surplus is (1) $\mathrm{BS}(v) = s\{\gamma_{s+1}(v) - p\} > 0$ if $\gamma_s(v) \geq \gamma_{s+1}(v) > p$, (2) $\mathrm{BS}(v) = 0$ if $\gamma_s(v) \geq p \geq \gamma_{s+1}(v)$, and (3) $\mathrm{BS}(v) = (n-s)\{p - \gamma_s(v)\} > 0$ if $p > \gamma_s(v) \geq \gamma_{s+1}(v)$. As we will see, this budget surplus is the minimum cost of requiring strategy-proofness and envy-freeness.

We characterize the set of Pareto undominated mechanisms in the set of strategy-proof and envy-free mechanisms.

Theorem 2 *F^2 is the set of Pareto undominated mechanisms in F^1. More precisely, (1) for any $f^\pi \in F^1 \backslash F^2$, there is some $f^p \in F^2$ that Pareto dominates f^π, and (2) for any $f^p \in F^2$, there is no $f^\pi \in F^1$ that Pareto dominates f^p.*

Proof See the Appendix.

We state three remarks on Theorem 2.

Remark 1 One may consider the restriction of the set of valuation profiles in order to construct a more desirable mechanism. Let V and V' be such that for any $i \in N$, $V_i = [\alpha, \beta]$ and $V'_i = [\alpha', \beta']$, where $\alpha < \alpha' < \beta' < \beta$. Theorem 2 suggests that we can find no new interesting strategy-proof and envy-free mechanism even if we change the set of valuation profiles from V to V'. Therefore, the planner should choose a sufficiently large set of valuation profiles in order that it always includes true valuations of agents.

Remark 2 For simplicity of discussion, we fix here some tie-breaking rule on the allocation of indivisible goods. Let g^β be the decision-efficient mechanism such that for any $v \in V$ and any $i \in N$, $t_i^\beta(v) = \sum_{j \in C(v)\backslash\{i\}} v_j + (T - (n-1)s\beta)/n$. Ohseto (2004) characterizes the mechanism g^β as the *best* strategy-proof, decision-efficient, and " 0-egalitarian-equivalent" mechanism in the sense that it Pareto dominates any other strategy-proof, decision-efficient, and 0-egalitarian-equivalent mechanism. It is easy to show that the mechanism $f^\beta \in F^2$ Pareto dominates g^β.

Remark 3 We characterize mechanisms in F^2 that always allocate all agents nonnegative consumptions of money. Let $f^p \in F^2$ and $v \in V$. Nonnegative consumptions of money require that

(1) $t_w^p(v) = \dfrac{T - (n-s)p}{n} - \{\gamma_{s+1}(v) - p\} \geq 0$ and $t_l^p(v) = \dfrac{T + sp}{n} \geq 0$ when $\gamma_s(v) \geq \gamma_{s+1}(v) > p$,

(2) $t_w^p(v) = \dfrac{T - (n-s)p}{n} \geq 0$ and $t_l^p(v) = \dfrac{T + sp}{n} \geq 0$ when $\gamma_s(v) \geq p \geq \gamma_{s+1}(v)$, and

(3) $t_w^p(v) = \dfrac{T-(n-s)p}{n} \geq 0$ and $t_l^p(v) = \dfrac{T+sp}{n} - \{p - \gamma_s(v)\} \geq 0$ when $p > \gamma_s(v) \geq \gamma_{s+1}(v)$.

With some computations on (1) and (3), we have (1) $(T+sp)/n \geq max\{0, \gamma_{s+1}(v)\}$ and (3) $(T-(n-s)p)/n \geq max\{0, -\gamma_s(v)\}$, and hence (2) is redundant. We set $\gamma_{s+1}(v) = \beta$ in (1) and $\gamma_s(v) = \alpha$ in (3), and we conclude that p should be such that $(n \cdot \max\{0, \beta\} - T)/s \leq p \leq (T - n \cdot \max\{0, -\alpha\})/(n-s)$. When $T \leq 0$, such p almost never exist.[7] When $T > 0$, such p may or may not exist. For example, let $[\alpha, \beta] = [-100, 100], T = 6000, n = 50, s = 25$. Then $-40 \leq p \leq 40$.

4 Concluding remarks

We characterized the set of Pareto undominated mechanisms in the set of strategy-proof and envy-free mechanisms. Each mechanism f^p is related to a tentative price p of indivisible goods. The planner should estimate the sth highest and the $(s+1)$th highest valuations of indivisible goods and choose a mechanism f^p whose tentative price is between them. If his choice is correct, the mechanism realizes a Pareto efficient allocation. If not, the mechanism induces a welfare loss in the form of a budget surplus, the amount of which is proportional to the minimum difference between his choice and correct ones.

We comment on further research. For the problem of allocating heterogeneous indivisible goods and money, Svensson (2004) characterizes the set of strategy-proof and envy-free mechanisms satisfying some auxiliary axioms. His model includes the case of indivisible "bads", but assumes that each agent consumes exactly one indivisible good. In a multi-object auction model, Pápai (2003) characterizes the set of strategy-proof and envy-free mechanisms when the valuations of indivisible goods are superadditive. Her model does not include the case of indivisible "bads", but allows each agent to consume any number of indivisible goods. Given the maximum and minimum numbers of indivisible goods each agent has to consume, we expect some general characterization result to be established for the problem of allocating homogeneous/heterogeneous indivisible "bads".

Appendix

We prepare four lemmas that describe the conditions on $t_w(v)$ and $t_l(v)$ implied by strategy-proofness and envy-freeness. In each lemma we assume (but do not state in the statement) that a mechanism f is strategy-proof and envy-free, and thus by Svensson (1983), f is decision-efficient. In each proof of the lemmas, without loss of generality, we permute the indexes of agents such that $v_1 \geq \cdots \geq v_s \geq v_{s+1} \geq \cdots \geq v_n$ for the valuation profile denoted by v.

First, we consider the following subset of valuation profiles. Let $V^* = \{v \in V \mid \gamma_s(v) = \gamma_{s+1}(v)\}$ be the set of valuation profiles where the sth highest valuation is equal to the $(s+1)$th highest valuation. Lemma 1 is a direct consequence of envy-freeness on V^*.

[7] The exception is the trivial case of $T = \alpha = \beta = p = 0$.

Lemma 1 *For any* $v \in V^*$, $\gamma_s(v) = t_l(v) - t_w(v)$.

Proof By decision-efficiency, without loss of generality, we can assume $C(v) = \{1, ..., s\}$. Hence by envy-freeness, $f_s(v) = (1, t_w(v))$ and $f_{s+1}(v) = (0, t_l(v))$. By envy-freeness, $\gamma_s(v) + t_w(v) \geq t_l(v)$ and $t_l(v) \geq \gamma_{s+1}(v) + t_w(v)$. Hence $\gamma_s(v) = t_l(v) - t_w(v)$. □

For any $x \in \mathbb{R}$, let $v^x \in V^*$ be the valuation profile where the valuation of every agent is x, i.e., $v_i^x = x$ for any $i \in N$. Lemma 2 shows that for any $v \in V^*$, if the sth (equivalently, the $(s+1)$th) highest valuation in v is x, then the amounts of money allocated to the winners and the losers at v are equal to those at v^x.

Lemma 2 *For any* $v \in V^*$, *if* $\gamma_s(v) = x$, *then* $t_w(v) = t_w(v^x)$ *and* $t_l(v) = t_l(v^x)$.

Proof First, we show $t_w(v_1, v_{-1}^x) = t_w(v^x)$ and $t_l(v_1, v_{-1}^x) = t_l(v^x)$. This is clear when $v_1 = x$. We consider the case of $v_1 > x$.[8] By decision-efficiency and envy-freeness, $f_1(v_1, v_{-1}^x) = (1, t_w(v_1, v_{-1}^x))$. By envy-freeness, either (1) $f_1(v^x) = (1, t_w(v^x))$ or (2) $f_1(v^x) = (0, t_l(v^x))$. Consider the case (1). By strategy-proofness, $t_w(v_1, v_{-1}^x) = t_w(v^x)$. Consider the case (2). By decision-efficiency and strategy-proofness, $f_1(v_1', v_{-1}^x) = f_1(v_1, v_{-1}^x)$ for any $v_1' \in V_1$ such that $v_1' > x$. By strategy-proofness, $v_1' + t_w(v_1, v_{-1}^x) \geq t_l(v^x)$ for any $v_1' \in V_1$ such that $v_1' > x$, and $t_l(v^x) \geq x + t_w(v_1, v_{-1}^x)$. Hence $x = t_l(v^x) - t_w(v_1, v_{-1}^x)$. Note that by Lemma 1, $x = t_l(v^x) - t_w(v^x)$ and $x = t_l(v_1, v_{-1}^x) - t_w(v_1, v_{-1}^x)$, and thus $t_w(v_1, v_{-1}^x) = t_w(v^x)$ and $t_l(v_1, v_{-1}^x) = t_l(v^x)$ in both cases. Applying the same arguments to agent 2, 3, ..., $s-1$ successively, we have $t_w(v_{\{1,...,s-1\}}, v_{-\{1,...,s-1\}}^x) = t_w(v^x)$ and $t_l(v_{\{1,...,s-1\}}, v_{-\{1,...,s-1\}}^x) = t_l(v^x)$.

Let $\overline{v} = (v_{\{1,...,s-1\}}, v_{-\{1,...,s-1\}}^x)$. Next, we show $t_w(v_n, \overline{v}_{-n}) = t_w(\overline{v})$ and $t_l(v_n, \overline{v}_{-n}) = t_l(\overline{v})$. This is clear when $v_n = x$. We consider the case of $v_n < x$. By decision-efficiency and envy-freeness, $f_n(v_n, \overline{v}_{-n}) = (0, t_l(v_n, \overline{v}_{-n}))$. By envy-freeness, either (1) $f_n(\overline{v}) = (0, t_l(\overline{v}))$ or (2) $f_n(\overline{v}) = (1, t_w(\overline{v}))$. Consider the case (1). By strategy-proofness, $t_l(v_n, \overline{v}_{-n}) = t_l(\overline{v})$. Consider the case (2). By decision-efficiency and strategy-proofness, $f_n(v_n', \overline{v}_{-n}) = f_n(v_n, \overline{v}_{-n})$ for any $v_n' \in V_n$ such that $v_n' < x$. By strategy-proofness, $t_l(v_n, \overline{v}_{-n}) \geq v_n' + t_w(\overline{v})$ for any $v_n' \in V_n$ such that $v_n' < x$, and $x + t_w(\overline{v}) \geq t_l(v_n, \overline{v}_{-n})$. Hence $x = t_l(v_n, \overline{v}_{-n}) - t_w(\overline{v})$. Note that by Lemma 1, $x = t_l(\overline{v}) - t_w(\overline{v})$ and $x = t_l(v_n, \overline{v}_{-n}) - t_w(v_n, \overline{v}_{-n})$, and thus $t_w(v_n, \overline{v}_{-n}) = t_w(\overline{v})$ and $t_l(v_n, \overline{v}_{-n}) = t_l(\overline{v})$ in both cases. Applying the same arguments to agent $n-1, n-2, ..., s+2$ successively, we have $t_w(v_{\{s+2,...,n\}}, \overline{v}_{-\{s+2,...,n\}}) = t_w(\overline{v})$ and $t_l(v_{\{s+2,...,n\}}, \overline{v}_{-\{s+2,...,n\}}) = t_l(\overline{v})$.

Hence $t_w(v_{\{s,s+1\}}^x, v_{-\{s,s+1\}}) = t_w(v^x)$ and $t_l(v_{\{s,s+1\}}^x, v_{-\{s,s+1\}}) = t_l(v^x)$. Note that $v_s = v_s^x$ and $v_{s+1} = v_{s+1}^x$. Therefore $t_w(v) = t_w(v^x)$ and $t_l(v) = t_l(v^x)$. □

[8] One referee suggests an alternate line of proof using the form of Groves mechanisms but refraining from exploiting strategy-proofness. We establish $t_w(v_1, v_{-1}^x) = t_w(v^x)$ and $t_l(v_1, v_{-1}^x) = t_l(v^x)$ when $v_1 > x$: By the definition of Groves mechanisms, $1 \in C(v_1, v_{-1}^x)$ and $t_1(v_1, v_{-1}^x) = t_w(v_1, v_{-1}^x) = (s-1)x + h_1(v_{-1}^x)$. Then we need to show $t_1(v_1, v_{-1}^x) = t_w(v^x)$. If $1 \in C(v^x)$, then by the definition of Groves mechanisms, $t_1(v^x) = t_w(v^x) = (s-1)x + h_1(v_{-1}^x)$. Otherwise, if $1 \notin C(v^x)$, then by the definition of Groves mechanisms, $t_1(v^x) = t_l(v^x) = sx + h_1(v_{-1}^x)$. By Lemma 1, $t_w(v^x) = t_l(v^x) - \gamma_s(v^x) = (s-1)x + h_1(v_{-1}^x)$. Hence $t_w(v_1, v_{-1}^x) = t_w(v^x)$. Now the claim on t_l follows from Lemma 1. One can obtain Lemmas 2 and 3 by repeated application of this argument.

It follows from Lemma 2 that for any $v \in V^*$, the amounts of money allocated to the winners and the losers depend only on the sth (equivalently, the $(s+1)$th) highest valuation in v, i.e., for any $v, v' \in V^*$ such that $\gamma_s(v) = \gamma_s(v') = x$, $t_w(v) = t_w(v') = t_w(v^x)$ and $t_l(v) = t_l(v') = t_l(v^x)$. Hence we can define new functions $t_w^* : \mathbb{R} \to \mathbb{R}$ and $t_l^* : \mathbb{R} \to \mathbb{R}$ in such a way that $t_w^*(x) = t_w(v)$ and $t_l^*(x) = t_l(v)$ for any $v \in V^*$ such that $\gamma_s(v) = x$. Lemma 3 shows that for any $v \in V$, the amount of money allocated to the winners at v depends only on the $(s+1)$th highest valuation in v and the amount of money allocated to the losers at v depends only on the sth highest valuation in v.

Lemma 3 *For any $v \in V$, $t_w(v) = t_w^*(\gamma_{s+1}(v))$ and $t_l(v) = t_l^*(\gamma_s(v))$.*

Proof Lemma 2 proves the case of $v_s = v_{s+1}$. We consider here the case of $v_s > v_{s+1}$.

First, we show $t_w(v) = t_w^*(\gamma_{s+1}(v))$. By decision-efficiency and envy-freeness, $f_s(v) = (1, t_w(v))$. Let $v_s' \in V_s$ be such that $v_s' = v_{s+1}$. By Lemma 2, either (1) $f_s(v_s', v_{-s}) = (1, t_w^*(\gamma_{s+1}(v)))$ or (2) $f_s(v_s', v_{-s}) = (0, t_l^*(\gamma_{s+1}(v)))$. Consider the case (1). By strategy-proofness, $t_w(v) = t_w^*(\gamma_{s+1}(v))$. Consider the case (2). By decision-efficiency and strategy-proofness, $f_s(v_s'', v_{-s}) = f_s(v)$ for any $v_s'' \in V_s$ such that $v_s'' > v_s'$. By strategy-proofness, $v_s'' + t_w(v) \geq t_l^*(\gamma_{s+1}(v))$ for any $v_s'' \in V_s$ such that $v_s'' > v_s'$, and $t_l^*(\gamma_{s+1}(v)) \geq v_s' + t_w(v)$. Hence $t_l^*(\gamma_{s+1}(v)) - t_w(v) = v_s'$. By Lemma 1, $\gamma_s(v_s', v_{-s}) = t_l(v_s', v_{-s}) - t_w(v_s', v_{-s})$. By Lemma 2, $t_w(v_s', v_{-s}) = t_w^*(\gamma_{s+1}(v))$ and $t_l(v_s', v_{-s}) = t_l^*(\gamma_{s+1}(v))$. Hence $v_{s+1} = t_l^*(\gamma_{s+1}(v)) - t_w^*(\gamma_{s+1}(v))$. Therefore $t_w(v) = t_w^*(\gamma_{s+1}(v))$.

Next, we show $t_l(v) = t_l^*(\gamma_s(v))$. By decision-efficiency and envy-freeness, $f_{s+1}(v) = (0, t_l(v))$. Let $v_{s+1}' \in V_{s+1}$ be such that $v_{s+1}' = v_s$. By Lemma 2, either (1) $f_{s+1}(v_{s+1}', v_{-\{s+1\}}) = (0, t_l^*(\gamma_s(v)))$ or (2) $f_{s+1}(v_{s+1}', v_{-\{s+1\}}) = (1, t_w^*(\gamma_s(v)))$. Consider the case (1). By strategy-proofness, $t_l(v) = t_l^*(\gamma_s(v))$. Consider the case (2). By decision-efficiency and strategy-proofness, $f_{s+1}(v_{s+1}'', v_{-\{s+1\}}) = f_{s+1}(v)$ for any $v_{s+1}'' \in V_{s+1}$ such that $v_{s+1}'' < v_{s+1}'$. By strategy-proofness, $t_l(v) \geq v_{s+1}'' + t_w^*(\gamma_s(v))$ for any $v_{s+1}'' \in V_{s+1}$ such that $v_{s+1}'' < v_{s+1}'$, and $v_{s+1}' + t_w^*(\gamma_s(v)) \geq t_l(v)$. Hence $t_l(v) - t_w^*(\gamma_s(v)) = v_{s+1}'$. By Lemma 1, $\gamma_s(v_{s+1}', v_{-\{s+1\}}) = t_l(v_{s+1}', v_{-\{s+1\}}) - t_w(v_{s+1}', v_{-\{s+1\}})$. By Lemma 2, $t_w(v_{s+1}', v_{-\{s+1\}}) = t_w^*(\gamma_s(v))$ and $t_l(v_{s+1}', v_{-\{s+1\}}) = t_l^*(\gamma_s(v))$. Hence $v_s = t_l^*(\gamma_s(v)) - t_w^*(\gamma_s(v))$. Therefore $t_l(v)$=$t_l^*(\gamma_s(v))$. □

By Lemma 3, the functions $t_w^* : \mathbb{R} \to \mathbb{R}$ and $t_l^* : \mathbb{R} \to \mathbb{R}$ completely define $t_w(v)$ and $t_l(v)$ for any $v \in V$. Lemma 4 characterizes the properties of such functions.

Lemma 4 *There is a nonnegative-valued function $\pi : \mathbb{R} \to \mathbb{R}_+$ satisfying Conditions A and B such that for any $x \in \mathbb{R}$, $t_w^*(x) = (T - (n-s)x)/n - \pi(x)$ and $t_l^*(x) = (T + sx)/n - \pi(x)$.*

Proof Given any $x \in \mathbb{R}$, let $v \in V^*$ be such that $\gamma_s(v) = \gamma_{s+1}(v) = x$. By Lemma 3, $t_w(v) = t_w^*(x)$ and $t_l(v) = t_l^*(x)$. By feasibility at v, $st_w^*(x) + (n-s)t_l^*(x) \leq T$. By Lemma 1, $x = t_l^*(x) - t_w^*(x)$. Hence $t_w^*(x) \leq (T - (n-s)x)/n$ and $t_l^*(x) \leq (T + sx)/n$ for any $x \in \mathbb{R}$. Note that $(T + sx)/n - (T - (n-s)x)/n = x$. By Lemma 1, there is a nonnegative-valued function $\pi : \mathbb{R} \to \mathbb{R}_+$ such that for any $x \in \mathbb{R}$, $t_w^*(x) = (T - (n-s)x)/n - \pi(x)$ and $t_l^*(x) = (T + sx)/n - \pi(x)$.

Given any $x, y \in \mathbb{R}$ $(x < y)$, let $v \in V$ be such that $\gamma_s(v) = y$ and $\gamma_{s+1}(v) = x$. By decision-efficiency, $C(v) = \{1, ..., s\}$. By Lemma 3, $f_s(v) = (1, t^*_w(x))$ and $f_{s+1}(v) = (0, t^*_l(y))$. By envy-freeness, $y + t^*_w(x) \geq t^*_l(y)$ and $t^*_l(y) \geq x + t^*_w(x)$. A simple computation proves that π satisfies Condition A. By feasibility at v, $st^*_w(x) + (n-s)t^*_l(y) \leq T$. A simple computation proves that π satisfies Condition B. □

We now prove the two main characterizations.

Proof of Theorem 1. First, we show that each $f^\pi \in F^1$ is strategy-proof and envy-free. Since f^π is a Groves mechanism, f^π is strategy-proof. It is easy to check that by Condition A, f^π is envy-free. Next, the fact that envy-freeness implies decision-efficiency (Svensson 1983) and Lemmas 1–4 prove that if a mechanism f is strategy-proof and envy-free, then $f \in F^1$. □

Proof of Theorem 2. (1) We must show that for any $f^\pi \in F^1 \backslash F^2$, there is some $f^p \in F^2$ such that $\pi^p(x) \leq \pi(x)$ for any $x \in [\alpha, \beta]$. First, let $p = \alpha + (n/(n-s))\pi(\alpha)$ if $\alpha + (n/(n-s))\pi(\alpha) < \beta$. Note that $\pi^p(\alpha) = \pi(\alpha)$. By Condition A, $-(n-s)/n \leq (\pi(x) - \pi(\alpha))/(x - \alpha)$ for any $x \in (\alpha, p]$. Hence $\pi^p(x) = \pi(\alpha) - ((n-s)/n)(x-\alpha) \leq \pi(x)$ for any $x \in [\alpha, p]$. Suppose that there is $x^* \in (p, \beta]$ such that $\pi^p(x^*) > \pi(x^*)$. Note that $s\pi^p(\alpha) + (n-s)\pi^p(x^*) = s((n-s)/n)(p-\alpha) + (n-s)(s/n)(x^* - p) = (s(n-s)(x^* - \alpha)/n$. By Condition B, $s(n-s)(x^* - \alpha)/n \leq s\pi(\alpha) + (n-s)\pi(x^*)$. This contradicts $\pi^p(\alpha) = \pi(\alpha)$ and $\pi^p(x^*) > \pi(x^*)$. Second, let $p = \beta$ if $\alpha + (n/(n-s))\pi(\alpha) \geq \beta$. Note that $\pi^p(\alpha) \leq \pi(\alpha)$. By Condition A, $-(n-s)/n \leq (\pi(x) - \pi(\alpha))/(x - \alpha)$ for any $x \in (\alpha, \beta]$. Hence $\pi^p(x) = ((n-s)/n)(\beta - x) \leq \pi(\alpha) - ((n-s)/n)(x - \alpha) \leq \pi(x)$ for any $x \in [\alpha, \beta]$. Since π^p is different from π, $\pi^p(x) < \pi(x)$ for some $x \in [\alpha, \beta]$.

(2) Note that for any two mechanisms $f^p, \overline{f}^p \in F^2$, f^p neither Pareto dominates, nor is Pareto dominated by $\overline{f}^p$. Suppose that for some $f^p \in F^2$, there is some $f^\pi \in F^1 \setminus F^2$ that Pareto dominates f^p. By (1), there is some $\overline{f}^p \in F^2$ that Pareto dominates f^π. This contradicts transitivity of the Pareto dominance relation. □

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