Economic Theory 27, 179–211 (2006)
DOI: 10.1007/s00199-004-0582-5

Economic Theory

Debt, liquidity and dynamics*

Céline Rochon[1] **and Herakles M. Polemarchakis**[2]

[1] THEMA, Université de Cergy-Pontoise, 95011 Cergy-Pontoise, FRANCE
(e-mail: celine.rochon@eco.u-cergy.fr)
[2] Department of Economics, Brown University, Providence, RI 02912, USA
(e-mail: herakles_polemarchakis@brown.edu)

Received: October 5, 2004; revised version: November 5, 2004

Summary. Money, which provides liquidity, is distinct from debt. The introduction of a bank that issues money in exchange for debt and pays out its profit as dividend to shareholders modifies the model of overlapping generations. The set of equilibrium paths, their dynamic properties, as well as the scope and effectiveness of monetary policy are significantly altered: though low rates of interest are associated with superior steady state allocations, stability of the steady state may require a nominal rate of interest above a certain minimum: without production, a decrease in the nominal rate of interest may result in explosive behavior or convergence to an endogenous cycle, while in an economy with production, an increase in the nominal rate of interest may lead to indeterminacy and fluctuations.

Keywords and Phrases: Overlapping generations, Money, Liquidity constraints, Debt, Dynamics.

JEL Classification Numbers: E30, E32, E50, E52.

1 Introduction

The model of overlapping generations of Allais [1], Diamond [11] and Samuelson [25] abstracts from the role of money in the provision of liquidity, and it does not distinguish between debt and money; it may be inadequate for the study of monetary equilibria and the analysis of monetary policy.

The introduction of a bank or monetary-fiscal authority that issues money in exchange for debt and pays out its profit as dividend to shareholders modifies the model of overlapping generations: it distinguishes money, which provides liquidity, from debt, which does not.

* We thank Jean-Michel Grandmont for helpful comments.

Correspondence to: C. Rochon

A monetary authority lends money in exchange for interest bearing bonds; also, it accepts deposits and pays depositors the same rate of interest which they charge on loans, which is not the case in [17]; equivalently, there are interest bearing nominal assets that dominate money as a store of value.

There exist alternative ways of modelling the cost and inconvenience of non-monetary transactions. Most simply, they are captured by a cash-in-advance constraint of Clower [10], modified by a parameter that reflects the exogenous velocity of circulation of money balances.

Economies that extend over an infinite horizon and are characterized by an operative cash-in-advance constraint have been extensively employed in theoretical macroeconomics and monetary theory, starting with Grandmont and Younès [19, 20].

The indeterminacy of equilibrium paths for a set path of real balances or, alternatively, of the rate of interest arises under certainty, as in Sargent and Wallace [26], but is most interesting under uncertainty, which, here, is absent. It depends, first, on the distinction introduced in Woodford [28]: at a Ricardian policy, in Bloise, Drèze and Polemarchakis [7], the monetary-fiscal authority is restricted to satisfy a budget constraint at all date-events, while at a non-Ricardian policy, in Dubey and Geanakoplos [13] or Woodford [28], the monetary-fiscal authority may violate the budget constraint and accumulate debt at terminal date-events, possibly at infinity. It also depends on the financial policy adopted by the monetary-fiscal authority, as in Nakajima and Polemarchakis [23] or the stationarity of monetary policy, as in Lucas and Stokey [22].

The distinction between money, which provides liquidity services and is issued by a bank that distributes its profit as dividend, and debt, a store of value, modifies the set of equilibrium paths, their dynamic properties, as well as the scope and effectiveness of monetary policy:

1. there is a continuum of distinct steady state paths, indexed by the nominal rate of interest, and low rates of interest are associated with superior steady state allocations, as in Friedman [14];
2. the stability of the steady state may require a nominal rate of interest above a certain minimum;
3. without production, a decrease in the nominal rate of interest may result in explosive behavior or convergence to an endogenous cycle, while with production, an increase in the nominal rate of interest may result in indeterminacy and fluctuations.

With a cash-in-advance constraint, the nominal rate of interest is an ad valorem tax on purchases, while the distribution of seignorage in the form of a dividend acts as a lump sum transfer: indeed, the reduced form of the equilibrium conditions coincides with those in an economy with debt and fiscal policy that taxes the purchase of commodities but distributes tax revenue as lump sum transfers to individuals. The indeterminacy of steady state paths is not surprising; only, it arises naturally from the distribution of seignorage necessary for a consistent specification of the monetary sector, and not from autonomous redistributive policy.

Economies of overlapping generations have provided an appropriate framework for the study of endogenous cycles, following Benhabib and Day [5] and Grandmont

[16]. Stationary economies may have cyclical equilibrium paths and, as a structural parameter, such as the elasticity of savings with respect to the real rate of interest varies, the economy may switch from an economy in which the steady state is stable in the equilibrium dynamics to one where a cycle of arbitrary order is attractive. With production, in Benhabib and Laroque [6], fluctuations are likely to occur either when savings are a decreasing function of the rate of interest or else the quantity of outside money is negative at the golden rule and there is enough complementarity in the production technology; in Reichlin [24], fluctuations appear under strong complementarity of inputs.

When money provides liquidity and the nominal rate of interest along an equilibrium path is indeterminate, it is possible to study the dynamic properties of equilibrium paths for different levels of real balances or, alternatively, of the nominal rate of interest, alone or in conjunction with parameters that describe the fundamentals of the economy: structural change is, at least in part, the outcome of a policy decision, such as the targeted nominal rate of interest or the money supply.

The paper is organized as follows. In Section 2, the specifics of the pure exchange economy model is presented. The equilibrium is defined and characterized, and the stationary states and various monetary policies are analyzed. In Section 3, the specifics of the model with production is presented. The linearized dynamical system at the golden rule is studied, as well as its stability properties as a function of the nominal rate of interest.

2 The exchange economy

Discrete time, t, extends into the infinite future as well as, possibly, the infinite past. One, perishable commodity is exchanged and consumed at each date. The price of the commodity is $p_t > 0$. The rate of inflation is

$$\pi_{t+1} = \frac{p_{t+1}}{p_t} - 1.$$

The economy is stationary: individuals differ only on the dates at which they are active, and the rate of growth of population, n, is a constant. An individual, t, has an economic life that spans dates t and $t + 1$. A generation, t, consists of

$$N_t = (1 + n)^t$$

identical individuals.

2.1 Individuals

The preferences of an individual are described by an intertemporally separable utility function

$$V(X_t) + U(C_{t+1}),$$

where X_t is the consumption of an individual at the first date in his life span and C_{t+1} the consumption at the second. The cardinal utility indices, U and

V, are smooth, strictly monotonically increasing and concave functions that satisfy the boundary conditions $\lim_{X\to 0} V'(X) = \infty$, $\lim_{C\to 0} U'(C) = \infty$, and $\lim_{C\to +\infty} U'(C) = 0$.

The endowment of an individual is $\overline{X} > 0$, at the first date that he is active, while it is $\overline{C} > 0$ at the second date that he is active.

Debt held by individuals at their first date of activity, B_t^1, is only used to finance the first period money holdings, M_t^1, and bears nominal interest $r_t \geq 0$. Debt held by individuals at their second date of activity, B_{t+1}^2, finances the second period money holdings, M_{t+1}^2, and serves as a store of value between dates t and $t+1$. It bears nominal interest $r_{t+1} \geq 0$. The real rate of interest is

$$\rho_{t+1} = \frac{1 + r_{t+1}}{1 + \pi_{t+1}} - 1,$$

and indexed debts are

$$\beta_t^1 = \frac{B_t^1}{p_t}, \quad \text{and} \quad \beta_{t+1}^2 = \frac{B_{t+1}^2}{p_{t+1}}$$

respectively.

Money provides liquidity services and serves as a store of value that bears no interest; balances, M_t^1 and M_{t+1}^2, are issued at dates $t-1$ and t and provide liquidity at dates t and $t+1$, respectively. Real balances are

$$\mu_t^1 = \frac{M_t^1}{p_t}, \quad \text{and} \quad \mu_{t+1}^2 = \frac{M_{t+1}^2}{p_{t+1}}.$$

The budget constraints of an individual, t, are

$$M_t^1 + B_t^1 = 0,$$

$$p_t X_t + M_{t+1}^2 + B_{t+1}^2 = p_t \overline{X} + (1 + r_t) B_t^1 + M_t^1,$$

$$p_{t+1} C_{t+1} = p_{t+1} \overline{C} + (1 + r_{t+1}) B_{t+1}^2 + M_{t+1}^2 + T_{t+1},$$

where T_{t+1} is the profit of the monetary authority redistributed as a lump sum transfer to the individual at his second date of activity; these constraints reduce to the intertemporal budget constraint

$$X_t + r_t \mu_t^1 + \frac{C_{t+1}}{1 + \rho_{t+1}} + \frac{r_{t+1}}{1 + \rho_{t+1}} \mu_{t+1}^2 = \overline{X} + \frac{\overline{C}}{1 + \rho_{t+1}} + \frac{\tau_{t+1}}{1 + \rho_{t+1}},$$

where $\tau_{t+1} = T_{t+1}/p_{t+1}$.

Liquidity constraints are operative in both dates in the life span of an individual; they are described by

$$\gamma p_t X_t \leq M_t^1, \quad \text{and} \quad \delta p_{t+1} C_{t+1} \leq M_{t+1}^2$$

or, equivalently,

$$\gamma X_t \leq \mu_t^1, \quad \text{and} \quad \delta C_{t+1} \leq \mu_{t+1}^2,$$

where $1 \geq \gamma, \delta \geq 0$, $\gamma \neq \delta$ are the reciprocals of the velocity of circulation of balances applicable, respectively, for individuals in the first and second dates of activity. Unless otherwise mentioned, $\gamma > 0$ and $\delta > 0$, since, for $\gamma = 0$ and $\delta = 0$ the liquidity constraints vanish. As long as the nominal rate of interest is positive, $r_t > 0$, the liquidity constraints bind.

Per-capita aggregate real balances at t are

$$\mu_t = \kappa \left(\mu_t^1 + \frac{1}{1+n} \mu_t^2 \right),$$

where

$$\mu_t^2 = \left(\overline{X} + \frac{\overline{C}}{1+n} - \frac{\mu_t^1}{\gamma} \right) \delta(1+n), \quad \text{and} \quad \kappa = \frac{1+n}{2+n}.$$

Whenever $\delta = \gamma$, the path of per-capita aggregate real balances is constant through time: at all dates, $\mu_t = \delta\kappa \left(\overline{X} + (\overline{C}/1+n) \right)$. Non-trivial dynamic properties of the money supply require that $\delta \neq \gamma$. Though this poses problems if the parameters γ and δ are to be interpreted as the reciprocal of the velocity of circulation of balances, it is not unnatural that individuals at different stages of their life cycle face different liquidity constraints.

Alternatively, the cash-in-advance constraint applies to the net trades as opposed to the consumption of individuals. One or the other formulation is appropriate according to the interpretation of the aggregate consumption commodity. The formulation here is analytically tractable. In the alternative formulation, $\gamma = \delta$ does not force constant per-capita aggregate real balances.

2.2 The monetary-fiscal authority

At dates $t-1$ and t, a monetary-fiscal authority issues per-capita balances, $\overline{M}_t^1 \geq 0$ and $\overline{M}_{t+1}^2 \geq 0$, that provide liquidity at dates t and $t+1$ respectively. The rate of growth of per capita balances is

$$\sigma_{t+1} = \frac{\overline{M}_{t+1}^1}{\overline{M}_t^1} - 1 = \frac{\overline{M}_{t+1}^2}{\overline{M}_t^2} - 1.$$

The monetary authority supplies balances in exchange for debt through open market operations; part of the debt issued by the old individuals, $\tilde{B}_{t+1}^2$, is the counterpart of second period balances:

$$\overline{M}_{t+1}^2 + \tilde{B}_{t+1}^2 = 0.$$

The total amount of debt issued by individuals at their first date of activity is the counterpart of first period balances: $\tilde{B}_t^1 = B_t^1$ and $\overline{M}_t^1 + B_t^1 = 0$. The profit of the monetary authority out of open market operations at date t that are concluded at date $t+1$ is seignorage,

$$r_{t+1}(\overline{M}_{t+1}^1(1+n) + \overline{M}_{t+1}^2),$$

which is distributed as dividend to individuals at their second date of activity at date $t+1$. Hence

$$T_{t+1} = r_{t+1}(\overline{M}^1_{t+1}(1+n) + \overline{M}^2_{t+1}).$$

The supplies of real balances per-capita are denoted by

$$\overline{\mu}^1_t = \frac{\overline{M}^1_t}{p_t}, \qquad \overline{\mu}^2_{t+1} = \frac{\overline{M}^2_{t+1}}{p_{t+1}},$$

and the real per-capita dividends by

$$r_t\overline{\mu}^1_t, \qquad r_{t+1}\overline{\mu}^2_{t+1}.$$

The budget constraint of the fiscal authority that manages the debt is

$$(\beta^2_{t+1} + \overline{\mu}^2_{t+1})(1+r_{t+1}) = \frac{1+\rho_{t+1}}{1+n}(\beta^2_t + \overline{\mu}^2_t)(1+r_t).$$

2.3 Equilibrium

A competitive equilibrium is a path,

$$((X_t, C_t, \mu^1_t, \mu^2_t, \beta^1_t, \beta^2_t, \overline{\mu}^1_t, \overline{\mu}^2_t, \rho_t, r_t) : t = \dots, 0, 1, \dots),$$

such that

i. $(X_t, C_{t+1}, \mu^1_t, \mu^2_{t+1}, \beta^1_t, \beta^2_{t+1})$ is a solution to the optimization problem of individual t at real rate of interest ρ_{t+1}, nominal rate of interest r_{t+1}, and real dividends $r_t\overline{\mu}^1_t$, and $r_{t+1}\overline{\mu}^2_{t+1}$,
ii. $X_t + \frac{C_t}{1+n} = \overline{X} + \frac{\overline{C}}{1+n}$, and
iii. $\mu^1_t = \overline{\mu}^1_t, \quad \text{and} \quad \mu^2_t = \overline{\mu}^2_t.$

The debt market clears as a residual.

A steady state is a competitive equilibrium that is stationary: at all dates,

$$X_t = X, \quad C_t = C.$$

Competitive equilibria are fully described by paths of nominal rates of interest and per-capita aggregate supplies of real balances. Equilibria are of two kinds: with debt, or autarkic.

Paths along which debt held by the old individuals exceeds bonds holdings that finance the provision of liquidity:

$$0 < X_t < \frac{\overline{X}}{1+\gamma r_t} \quad \text{or, else,} \quad 0 < C_t - X_t\gamma r_t(1+n) < \overline{C},$$

are equilibria with debt.

Paths along which all bonds holdings finance the provision of liquidity:

$$X_t(1+\gamma r_t) = \overline{X}, \quad \text{or else} \quad C_t - X_t\gamma r_t(1+n) = \overline{C},$$

are autarkic.

From the first order conditions for individual optimization, it follows that a path of nominal rates of interest and allocations determines a competitive equilibrium path with debt if and only if

$$\frac{V'(X_t)}{U'(C_{t+1})} = (1+\rho_{t+1})\frac{(1+\gamma r_t)}{(1+\delta r_{t+1})}.$$

From the market clearing conditions,

$$X_t = \frac{\mu_t^1}{\gamma} = \frac{\mu_t - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)},$$

$$C_{t+1} = \frac{\mu_{t+1}^2}{\delta},$$

$$\beta_t^1 = -\mu_t^1,$$

$$\beta_{t+1}^2 = \frac{\frac{\mu_{t+1}^2}{\delta}(1-\delta-\delta r_{t+1}) - \overline{C} - r_{t+1}\mu_{t+1}^1(1+n)}{1+r_{t+1}},$$

$$1+\rho_{t+1} = (1+n)\frac{\overline{X} - \frac{\mu_{t+1}^1}{\gamma}(1+\gamma r_{t+1})}{\overline{X} - \frac{\mu_t^1}{\gamma}(1+\gamma r_t)}$$

$$= (1+n)\frac{\frac{\mu_{t+1}^2}{\delta} - r_{t+1}\mu_{t+1}^1(1+n) - \overline{C}}{\frac{\mu_t^2}{\delta} - r_t\mu_t^1(1+n) - \overline{C}}.$$

As a consequence, an equilibrium path of rates of interest and consumption is described by the non-linear difference equation

$$\frac{V'(X_t)}{U'((1+n)(\overline{X} - X_{t+1}) + \overline{C})} = (1+n)\frac{\overline{X} - X_{t+1}(1+\gamma r_{t+1})}{\overline{X} - X_t(1+\gamma r_t)}\frac{(1+\gamma r_t)}{(1+\delta r_{t+1})};$$

equivalently, with real balances as opposed to consumption,

$$\frac{V'\left(\frac{\mu_t - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}\right)}{U'\left((\overline{X} + \frac{\overline{C}}{1+n} - \frac{\mu_{t+1} - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)})(1+n)\right)} =$$

$$\frac{(1+\gamma r_t)}{(1+\delta r_{t+1})}(1+n)\left(\frac{\overline{X} - \frac{\mu_{t+1} - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}(1+\gamma r_{t+1})}{\overline{X} - \frac{\mu_t - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}(1+\gamma r_t)}\right).$$

Along an equilibrium path with debt, the rate of growth of real balances is

$$1+\sigma_{t+1} = \frac{1+r_{t+1}}{1+n}\frac{\overline{X} - \frac{\mu_t^1}{\gamma}(1+\gamma r_t)}{\overline{X} - \frac{\mu_{t+1}^1}{\gamma}(1+\gamma r_{t+1})}\frac{\mu_{t+1}^1}{\mu_t^1},$$

and the rate of inflation is

$$1+\pi_{t+1}=\frac{1+r_{t+1}}{1+n}\frac{\overline{X}-\frac{\mu_t^1}{\gamma}(1+\gamma r_t)}{\overline{X}-\frac{\mu_{t+1}^1}{\gamma}(1+\gamma r_{t+1})}.$$

2.4 Stationary states

A constant path of nominal rates of interest and supplies of real balances,

$$(r,\mu),$$

determines a steady state with debt if and only if

$$V'\left(\frac{\mu-\delta\kappa(\overline{X}+\frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}\right)=$$
$$U'\left((\overline{X}+\frac{\overline{C}}{1+n}-\frac{\mu_{-}\delta\kappa(\overline{X}+\frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)})(1+n)\right)\frac{1+\gamma r}{1+\delta r}(1+n).$$

At a steady state with debt, the real rate of interest coincides with the rate of growth of population:

$$\rho=n;$$

the rate of inflation coincides with the rate of growth of per capita balances:

$$\sigma=\pi.$$

Alternatively, a constant path of nominal rates of interest determines an autarkic steady state with

$$X_t=\frac{\overline{X}}{1+\gamma r},\quad C_{t+1}=\overline{C}+\frac{\overline{X}\gamma r}{1+\gamma r}(1+n),\quad \mu_t=\kappa\left(\overline{X}\gamma\frac{1+\delta r}{1+\gamma r}+\frac{\overline{C}\delta}{1+n}\right)$$

if and only if

$$V'\left(\frac{\overline{X}}{1+\gamma r}\right)=U'\left(\overline{C}+\frac{\overline{X}\gamma r}{1+\gamma r}(1+n)\right)\frac{1+\gamma r}{1+\delta r}(1+\rho).$$

The real rate of interest indexes the one-dimensional continuum of autarkic steady states.

Proposition 1. *There exists a one-dimensional continuum of distinct steady states with debt indexed by the per-capita aggregate supply of real balances or the nominal rate of interest, and a one-dimensional continuum of autarkic steady states indexed by the nominal rate of interest. A steady state with debt is Pareto superior to a steady state with debt with a higher nominal rate of interest; in particular, the steady state with debt with nominal rate of interest* $r=0$ *is Pareto superior to all other steady states.*

Proof. At a steady state with debt, for any given aggregate supply of real balances, μ, the corresponding nominal rate of interest is

$$r = \frac{U'(\overline{C} + (1+n)(\overline{X} - X))(1+n) - V'(X)}{\delta V'(X) - \gamma U'(\overline{C} + (1+n)(\overline{X} - X))(1+n)},$$

where

$$X = \frac{\mu - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma - \delta)}.$$

From the boundary behavior and the concavity of the cardinal utility indices and since $\delta \neq \gamma$, it follows that there exists an interval of non-negative rates of interest, $[0, \overline{r})$, in which there is a one-to-one relation between the nominal rate of interest and the allocation of resources and money balances at the associated steady state.

Alternatively, for a given nominal rate of interest, r, the corresponding supply of real balances is given implicitly by

$$V'(X) = U'(\overline{C} + (1+n)(\overline{X} - X))\frac{1+\gamma r}{1+\delta r}(1+n).$$

At the autarkic steady state, the real rate of interest is given by

$$(1+\rho) = \frac{V'\left(\frac{\overline{X}}{1+\gamma r}\right)\frac{1+\delta r}{1+\gamma r}}{U'\left(\overline{C} + \frac{\overline{X}\gamma r}{1+\gamma r}(1+n)\right)}.$$

For the optimization problem

$$\max_r V(X(r)) + U(C(r)),$$

where $X(r)$ and $C(r)$ are the steady state values of consumption with debt, the derivative of the objective function is

$$V'(X(r))\left(\frac{\mu'(r)}{\kappa(\gamma - \delta)}\right) + U'(C(r))\left(\frac{-\mu'(r)}{\kappa(\gamma - \delta)}\right)(1+n),$$

which reduces to

$$\mu'(r)U'(C(r))\frac{(1+n)}{\kappa}\frac{r}{1+\delta r}.$$

Since $V'(X(r)) = (1+n)[(1+\gamma r)/(1+\delta r)]U'(C(r))$, one finds that

$$\mu'(r) = \frac{(1+n)\kappa\frac{(\gamma-\delta)^2}{(1+\gamma r)^2}U'(C(r))}{V''(X(r)) + \frac{(1+\gamma r)}{(1+\delta r)}(1+n)^2 U''(C(r))} < 0.$$

Consequently, for any positive rate of interest, the derivative of the objective function is negative, and the unique solution to the maximization problem is $r = 0$. As $\rho = n$ at any steady state with debt, the corresponding optimal level of inflation is $-n/(1+n)$. Moreover, since the derivative of the objective function is negative, the lower the nominal rate of interest the higher the welfare at the steady state. □

The allocation of resources at autarkic steady states with different nominal rates of interest are distinct as well.

The nominal rate of interest r for which $\rho = n$ at the associated autarkic steady state is implicitly given by

$$V'\left(\frac{\overline{X}}{1+\gamma r}\right)\frac{1+\delta r}{1+\gamma r} = U'\left(\overline{C} + \frac{\overline{X}\gamma r}{1+\gamma r}(1+n)\right)(1+n).$$

It corresponds to a limit point of the set of steady states with debt. Indeed, a steady state with debt is characterized by aggregate real balances and nominal rates of interest, (μ, r), for which debt is not only the counterpart of money: $\beta^2 \neq -\mu^2$, while, in an autarkic steady state, debt is exactly the counterpart of money: $\beta^2 = -\mu^2$.

Since, at a steady state with debt, the rate of inflation coincides with the rate of growth of the supply of balances while the real rate of interest coincides with the rate of growth of population,

$$1 + r = (1+\sigma)(1+n),$$

the rate of growth of per capita balances indexes the one-dimensional continuum of steady states with debt.

2.5 Transfers

The nominal rate of interest acts as a consumption tax, while the distribution of seignorage to individuals acts as a lump sum transfer that alleviates the burden of the tax.

Indeed, as shown below, an economy with debt and fiscal policy that taxes the purchase of commodities but distributes tax revenue as lump sum transfers to individuals generates dynamics that is analytically equivalent to that of a monetary economy with liquidity constraints.

Importantly, this equivalence does not hold for an economy with lump sum transfers but no commodity taxes.

Assume debt, D_{t+1}, serves as a store of value and bears nominal interest, r_{t+1}. The real rate of interest is ρ_{t+1} and indexed debt is β_{t+1}.

No liquidity constraints are operative and balances that provide liquidity are absent.

At dates t and $t+1$, a fiscal authority imposes tax rates, z_t^1 and z_{t+1}^2, on the purchases of the consumption good. It distributes the revenue as lump sum per capita transfers, w_t^1 and w_{t+1}^2, to individuals in the corresponding date of activity; the real per-capita transfers are τ_t^1 and τ_{t+1}^2.

The budget constraints of an individual, t, are

$$(1+z_t^1)p_t X_t + \tfrac{1}{1+r_{t+1}} D_{t+1} = p_t\overline{X} + w_t^1,$$

$$(1+z_{t+1}^2)p_{t+1}C_{t+1} = p_{t+1}\overline{C} + D_{t+1} + w_{t+1}^2,$$

which reduce to the intertemporal budget constraint

$$(1+z_t^1)X_t + \frac{1+z_{t+1}^2}{1+\rho_{t+1}}C_{t+1} = \overline{X} + \tau_t^1 + \frac{\overline{C}+\tau_{t+1}^2}{1+\rho_{t+1}}.$$

The budget constraints of the fiscal authority are

$$z_t^1 X_t = \tau_t^1, \qquad z_{t+1}^2 C_{t+1} = \tau_{t+1}^2.$$

A competitive equilibrium is a path

$$((X_t, C_t, \beta_t, \tau_t^1 \tau_t^2, \rho_t, r_t, z_t^1, z_t^2) : t = \ldots, 0, 1, \ldots),$$

such that

i. $(X_t, C_{t+1}, \beta_{t+1})$ is a solution to the optimization problem of individual t at real rate of interest ρ_{t+1}, nominal rate of interest r_{t+1}, consumption taxes z_t^1 and z_{t+1}^2 and transfers τ_t^1 and τ_{t+1}^2,
ii. $X_t + \frac{1}{1+n}C_t = \overline{X}$, and
iii. $\tau_t^1 = z_t^1 X_t$ and $\tau_{t+1}^2 = z_{t+1}^2 C_{t+1}$.

A path of tax rates and indexed per capita debt,

$$((z_t^1, z_t^2, \beta_t) : t = \ldots, 0, 1, \ldots),$$

such that

$$\beta_t < (1+n)\overline{X},$$

determines a non-autarkic competitive equilibrium path if and only if

$$V'(\overline{X} - \frac{1}{1+n}\beta_t) = U'(\overline{C} + \beta_{t+1})(1+\rho_{t+1})\left(\frac{1+z_t^1}{1+z_{t+1}^2}\right),$$

where

$$1+\rho_{t+1} = \frac{\beta_{t+1}}{\beta_t}(1+n).$$

The equilibrium paths for the monetary economy with operative liquidity constraints and for the economy with active fiscal policy coincide whenever

$$z_t^1 = \gamma r_t, \qquad z_{t+1}^2 = \delta r_{t+1}.$$

2.6 Monetary policy and dynamics

The monetary authority sets the rate of interest, $r > 0$, constant over time, and accommodates the demand for real balances; prices adjust and markets clear.

Equivalently, the monetary authority sets the rate of growth of the per capita supply of balances, σ, constant over time; this is the case, since, along an equilibrium path with debt, the rate of interest is determined by the rate of growth of the supply of balances and vice-versa.

The path of the aggregate supply of real balances $(\mu_t : t = \ldots, 0, 1, \ldots)$ along an equilibrium with debt is described by the initial supply, μ_0, and the first order, non-linear equation

$$\frac{V'\left(\frac{\mu_t - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}\right)}{U'\left((\overline{X} + \frac{\overline{C}}{1+n} - \frac{\mu_{t+1} - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)})(1+n)\right)} =$$

$$\frac{(1+\gamma r)}{(1+\delta r)}(1+n)\left(\frac{\overline{X} - \frac{\mu_{t+1} - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}(1+\gamma r)}{\overline{X} - \frac{\mu_t - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}(1+\gamma r)}\right).$$

A first order approximation at a steady state (μ, r) yields the path of linear deviations $(\tilde{\mu}_t)$ from the steady state aggregate supply of real balances; it is described by the equation

$$\tilde{\mu}_{t+1} = A\tilde{\mu}_t,$$

where

$$A = \frac{1 + \frac{\overline{V}}{X}\frac{\overline{X} - X(1+\gamma r)}{1+\gamma r}}{1 - \frac{\overline{U}}{C}\frac{(1+n)(\overline{X} - X(1+\gamma r))}{1+\gamma r}},$$

with

$$X = \frac{\mu(r) - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma - \delta)}, \quad \frac{C}{1+n} = \overline{X} + \frac{\overline{C}}{1+n} - X.$$

For computational reasons, one restricts attention to the case $\gamma = 0$ and $\overline{C} = 0$, under which

$$A = \frac{1 + \overline{V}\frac{\overline{X} - X}{X}}{1 - \overline{U}}, \quad \text{and} \quad X = \overline{X} - \frac{\mu}{\delta\kappa},$$

and a parametric specification of the preferences of the individuals.

Proposition 2. *If the intertemporal utility function is*

$$V(X_t) + U(C_{t+1}) = \frac{1}{\alpha}X_t^\alpha + \frac{\nu}{\alpha}C_{t+1}^\alpha, \quad \alpha < 1, 0 < \nu,$$

then the dynamic behavior of real balances along an equilibrium path with debt is described as follows:

1. *for $-1 < \alpha < 1$, the associated path of linear deviations from the steady state supply of real balances is unstable,*
2. *the associated path of linear deviations from the steady state supply of real balances is stable if and only if $\alpha < -1$ and $r > \overline{r}$ where*

$$\overline{r} = \frac{1}{\delta}\left(\nu(1+n)^\alpha\left(-\frac{1-\alpha}{1+\alpha}\right)^{1-\alpha} - 1\right), \quad \alpha < -1$$

3. *if $\alpha < -1$, there is no equilibrium path of supplies of real balances with $\mu_0 \neq \mu(r)$ whenever $r < \overline{r}$.*

Proof. A first order approximation at a steady state, (r, μ), yields the path of linear deviations $(\tilde{\mu}_t : t = \dots, 0, 1, \dots)$ from the steady state supply of real balances; it is described by the initial deviation, $\tilde{\mu}_0$, and the first order, linear equation

$$\tilde{\mu}_{t+1} = A\tilde{\mu}_t,$$

where

$$A = \frac{1 + (1-\alpha)\frac{\mu}{\overline{X}\delta\kappa - \mu}}{\alpha},$$

and the steady state value of real balances is

$$\mu = \mu(r) = \delta \frac{\left(\frac{\nu(1+n)^\alpha}{1+\delta r}\right)^{\frac{1}{1-\alpha}} \overline{X}\kappa}{1 + \left(\frac{\nu(1+n)^\alpha}{1+\delta r}\right)^{\frac{1}{1-\alpha}}}.$$

The sign and magnitude of the coefficient A determine the dynamic behavior of deviations or, equivalently, the local dynamic behavior of the supply of real balances.

The path of linear deviations from the steady state supply of real balances is stable if and only if $|A| < 1$; by direct computation, this is the case if and only if $\alpha < -1$ and $r > \overline{r}$. □

Proposition 2 reports only the linear case as the nonlinear dynamics associated with this specification is not well defined over an unrestricted domain.

The minimal nominal rate of interest that prevents instability of the steady state increases with the rate of time preference, ν; and it decreases with the velocity of circulation of δ. In particular, any nominal rate of interest allows for stability if

$$\left(-\frac{1-\alpha}{1+\alpha}\right)^{1-\alpha} \nu(1+n)^\alpha \leq 1, \quad \text{with } \alpha < -1$$

while, if the inequality is reversed, rates of interest sufficiently close to the first best rate lead to instability.

In the case of no growth, $n = 0$, and for $\alpha \in (-100, -1)$, rates of interest sufficiently close to 0 lead to instability as long as $\nu > .13$.

Figure 1 shows the relationship between α and $\overline{r}$, the minimal rate of interest that eliminates instability, for $\delta = 0.5$, $n = 0.02$ and $\nu = 0.98$.

With a policy of low nominal rates, any supply of real balances other than the one corresponding to the steady state may eventually lead to a failure of feasibility.

More generally, the elasticity of the marginal utility of consumption differs in the two dates of activity of the individual and the elasticity of intertemporal substitution is not constant.

The intertemporal utility function is

$$V(X_t) + U(C_{t+1}) = \frac{1}{\alpha}X_t^\alpha + \frac{\nu}{\theta}C_{t+1}^\theta, \quad \alpha < 1, \theta < 1, 0 < \nu.$$

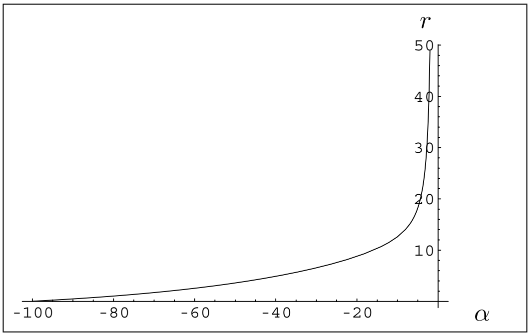

Figure 1. For any value of the nominal rate of interest below the curve or for $\alpha > -1$, the steady state is unstable

It is necessary to assign values to the parameters. Let $\alpha = -1$ and $\theta = -3$. If the intertemporal utility function is

$$V(X_t) + U(C_{t+1}) = -X_t^{-1} - \frac{\nu}{3} C_{t+1}^{-3}, \quad \nu > 0,$$

then the steady state value of real balances is

$$\mu(r) = \frac{\delta\kappa}{1+n}\left(\overline{X}\left(\frac{\nu(1+n)}{1+\delta r}\right)^{\frac{1}{2}} + \frac{\nu}{4(1+\delta r)(1+n)}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{\nu}{(1+\delta r)(1+n)}\right)^{\frac{1}{2}}.$$

The associated path of linear deviations from the steady state supply of real balances is unstable if and only if $r < \overline{r}$, where

$$\overline{r} = \frac{1}{\delta}\left(\frac{\nu}{(1+n)^3}\frac{4}{\overline{X}^2} - 1\right).$$

If $\overline{X} > 2(\nu/(1+n)^3)^{\frac{1}{2}}$, no choice of the nominal rate of interest leads to instability.

Low endowments $\overline{X}$ are compatible with asymptotic stability in the deviations of the supply of real balances for high values of the nominal rate of interest only.

Alternatively, preferences display constant elasticity of the marginal utility of consumption at the first day an individual is active but constant partial elasticity of the marginal utility of consumption at the second date.

Proposition 3. *If the intertemporal utility function is*

$$V(X_t) + U(C_{t+1}) = -\frac{1}{\alpha}\exp(-\alpha X_t) + \frac{\nu}{\theta} C_{t+1}^{\theta}, \quad 0 < \alpha, \theta < 1, 0 < \nu,$$

the dynamic behavior of real balances along an equilibrium path with debt is described as follows:

1. *the associated path of linear deviations from the steady state supply of real balances is unstable if and only if $\theta > -1$ or $r < \overline{r}$, where*

$$\overline{r} = \frac{1}{\delta}\left(\nu(1+n)^{\theta}\left(-\frac{1+\theta}{\alpha}\right)^{\theta-1}\exp(\alpha\overline{X} + 1 + \theta) - 1\right), \quad \textit{with } \theta < -1$$

2. *according to simulations for $\theta \ll -1$, in the non-linear dynamical system in the supply of real balances, if $r < \overline{r}$ the steady state is unstable: if $\mu_0 \neq \mu(r)$, then the equilibrium path with debt of the supply of real balances converges to a cycle of period two.*

Proof. By direct computation, the dynamical system $(\mu_t : t = \ldots, 0, 1, \ldots)$ is described by the equation

$$\mu_{t+1} = \psi(\mu_t; r),$$

where

$$\psi(\mu_t; r) = \left(\frac{\nu(1+n)}{1+\delta r} \left(\frac{\delta\kappa}{1+n} \right)^{1-\theta} \frac{\exp\left(\alpha\left(\overline{X} - \frac{\mu_t}{\delta\kappa}\right)\right)}{\mu_t} \right)^{-\frac{1}{\theta}}.$$

Similarly, the steady state value of real balances is implicitly given by the equation

$$\frac{\alpha}{\delta\kappa}\mu(r) - \alpha\overline{X} = \ln\left(\frac{\nu(1+n)}{1+\delta r} \left(\frac{\delta\kappa}{1+n} \right)^{1-\theta} \mu(r)^{\theta-1} \right).$$

As the steady state value of μ cannot be computed explicitly, it is convenient to translate ψ by the steady state value of μ for which a bifurcation occurs, as this value of μ is independent of r.

The path of linear deviations from the steady state supply of real balances is unstable if and only if

$$|\psi'(\mu(r))| > 1;$$

by direct computation, this is the case if and only if $\theta > -1$ or $r < \overline{r}$ (with $\theta < -1$).

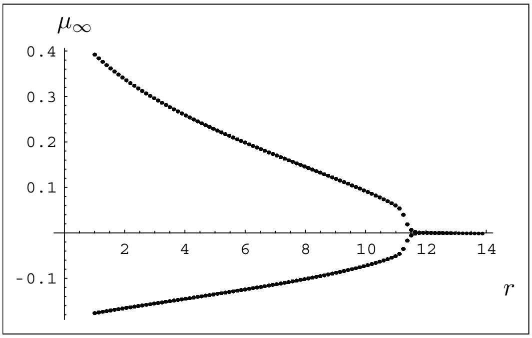

Figure 2. The bifurcation diagram of ψ. The μ_∞ are the stable fixed points

Figure 2 shows the bifurcation diagram of ψ, for $n = 0.02$, $\nu = 0.98$, $\alpha = -\theta = 10$ and $\delta = 0.5$, after 2000 iterations.

For values of the nominal rate of interest below the bifurcation point $\overline{r}$, there is a stable cycle of period two. □

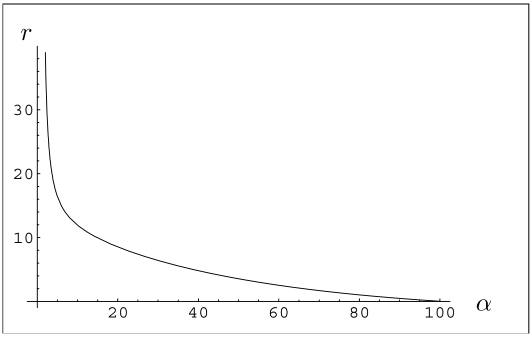

Figure 3. Below the curve or for values of $\theta > -1$, the steady state is unstable

In simulations, values of the nominal rate of interest that do not allow for instability and, possibly cycles, are moderate as long as $\alpha = -\theta$ and α is high. Variations in $\overline{X}$ do matter.

Figure 3 shows the relationship between α and $\overline{r}$, the minimal rate of interest that eliminates instability, for $\theta = -\alpha$, $\delta = 0.5$, $\overline{X} = 1$, $n = 0.02$, and $\nu = 0.98$.

The speed of convergence to the steady state of the non-linear path increases with the nominal rate of interest.

With a policy of low nominal rates, any supply of real balances other than the one corresponding to the steady state may lead to endogenous equilibrium fluctuations.

2.7 Remarks

Aspects of the specification of the model play an important role, which should be brought forward.

Controlling the circulation of real balances. The monetary authority could, alternatively, set the aggregate supply of real balances, $\mu > 0$, constant over time, and allow the nominal rate of interest to adjust to maintain equilibrium.

The path of the nominal rate of interest $(r_t : t = \ldots, 0, 1, \ldots)$ along an equilibrium with debt is described by the initial rate, r_0, and the first order, non-linear equation

$$\frac{V'\left(\frac{\mu - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}\right)}{U'\left((\overline{X} + \frac{\overline{C}}{1+n} - \frac{\mu - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)})(1+n)\right)} =$$

$$\frac{(1+\gamma r_t)}{(1+\delta r_{t+1})}(1+n)\left(\frac{\overline{X} - \frac{\mu - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}(1+\gamma r_{t+1})}{\overline{X} - \frac{\mu - \delta\kappa(\overline{X} + \frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}(1+\gamma r_t)}\right).$$

A first order approximation at a steady state (μ, r) yields the path of linear deviations $(\tilde{r}_t)$ from the steady state nominal rate of interest; it is described by the equation

$$\tilde{r}_{t+1} = A\tilde{r}_t,$$

where

$$A = \left(\frac{1+\delta r}{1+\gamma r}\right)\frac{\gamma\overline{X}}{\delta\overline{X}+X(\gamma-\delta)}, \quad \text{and} \quad X = \frac{\mu - \delta\kappa(\overline{X}+\frac{\overline{C}}{1+n})}{\kappa(\gamma-\delta)}.$$

For $\gamma = 0$, and with real balances constant over time, the nominal rate of interest is necessarily constant as well; for $\gamma \neq 0$, the dynamics in the nominal rate of interest is not trivial. The coefficient A is always positive, since, with debt, $\overline{X} > X$. The system is stable for $A < 1$, but unstable for $A > 1$; cyclical behavior cannot arise.

If one considers fixed nominal balances M rather than fixed real balances μ, then $\mu_t = \frac{M}{p_t} = f(r_t)$, for some decreasing function f of the nominal rate of interest r. At a steady state (M, r), prices are fully determined and fixed and one recovers a fixed supply of real balances μ.

Monetary policy that sets the supply of nominal balances is not tractable.

The liquidity constraint. With the consumption of individuals different from their net trades at the date the transactions constraint is operative, it is possible to impose the cash-in-advance requirement on transactions:

$$\delta p_{t+1} \max\{0, (C_{t+1} - \overline{C})\} \leq M^2_{t+1},$$

or on consumption:

$$\delta p_{t+1} C_{t+1} \leq M^2_{t+1}.$$

The constraint on transactions is more convincing; it introduces computational complications that may not be without interest. The constraint on consumption can be rationalized by arguing that the endowment of individuals is labor employed to produce the consumption good with a linear technology.

The monetary authority. In the spirit of Wilson [27], one could imagine that at the beginning of time, a bank, whose value represents the discounted stream of liquidity services, is owned by an individual with continuous preferences over streams of consumption and a finite or infinite horizon of economic activity.

Shares to the bank constitute stores of value and absence of arbitrage between shares and debt is a necessary condition for equilibrium.

In [8], the optimization of the initial shareholders of the bank suffices to guarantee the near optimality of competitive allocations, as long as the nominal rate of interest does not vanish.

3 The economy with production

We consider now an economy with production.

Individuals work during their first period of life and consume in both periods. They can save their labor wage in the form of three assets, namely, capital, bonds and money.

The good is the numeraire. The price of money in terms of the good at date t is Q_t and the real wage is W_t. Investments in capital in period t, K_{t+1}, become productive in period $t+1$. The gross return on capital is denoted R_{t+1}. Debt (bonds),

B_{t+1}, serves as a store of value between dates t and $t+1$. In period t, the price of a unit of a bond that will give one unit of money in period $t+1$ is $S_t = \frac{Q_t}{1+r_{t+1}}$, where r_{t+1} is the nominal rate of interest. In period $t+1$, the value of that unit of bond is Q_{t+1}. Capital and bonds are substitute assets; their returns satisfy the arbitrage condition

$$R_{t+1} = \frac{Q_{t+1}}{S_t}.$$

As before, money provides liquidity services and serves as a store of value that yields no interest. It is issued by a bank through open market operations at date t and provides liquidity at date $t+1$.

3.1 The agents

Under perfect foresight, an individual now chooses consumption levels, X_t and C_{t+1}, savings in the form of capital, K_{t+1}, money, M_{t+1}, and bonds, B_{t+1}, to maximize his utility, $V(X_t) + U(C_{t+1})$, subject to the following constraints:

$$\begin{aligned} &X_t = W_t - [K_{t+1} + Q_t M_{t+1} + S_t B_{t+1}], \\ &C_{t+1} = R_{t+1} K_{t+1} + Q_{t+1} M_{t+1} + Q_{t+1} B_{t+1} + T_{t+1}, \\ &Q_{t+1} M_{t+1} \geq \gamma C_{t+1}, \end{aligned}$$

where T_{t+1} is the profit of the bank redistributed as a lump sum transfer to the individual at his second date of activity (its expression is given in equilibrium below), and $\gamma \in (0,1)$ is the reciprocal of the velocity of circulation of money.

The first two constraints are the current and expected budget constraints. The third constraint is a liquidity constraint, applicable in the second period of activity. It can be rewritten as $C_{t+1} \leq \Gamma Q_{t+1} M_{t+1}$, with $\Gamma = 1/\gamma > 1$. The constraint indicates that the individual finances his consumption not only from his *initial* money holdings, but also by borrowing from the bank against his *end of period* non monetary incomes,

$$R_{t+1} K_{t+1} + Q_{t+1} B_{t+1} + T_{t+1}, \quad \text{up to a limit of} \quad Q_{t+1} M_{t+1} \left(\frac{1}{\gamma} - 1 \right).$$

Hence the constraint involves an intra-period overdraft that is proportional to the initial money holdings, with a coefficient $1/\gamma - 1 > 0$. The usual Clower cash-in-advance constraint has $\Gamma = 1$, and involves no such intra-period borrowing.

Whenever the transfer T_{t+1} is low, most of the agent's income accrues when young. The agent thus saves a part of his labor wage in capital or bonds in order to consume in the next period. For $\gamma > 0$, part of the wage is also saved in the form of money $M_{t+1} > 0$ to finance C_{t+1}, as requested by the constraint. Whenever the transfer T_{t+1} is large, then savings can be negative ($B_{t+1} < 0$), partly to smooth lifetime income, partly to acquire $M_{t+1} > 0$ as required by the liquidity constraint to finance C_{t+1} when $\gamma > 0$.

As bonds and capital are substitutes, the real demand for bonds $S_t B_{t+1}$ (which can be positive or negative) and for capital K_{t+1} (which is positive) is indeterminate.

The actual real demand for bonds will be determined in equilibrium by the supply of the bank, which will itself be determined by the monetary policy. The budget constraint of the bank is presented below.

Note that B_{t+1} can be negative even for low T_{t+1}, in which case savings of the young are positive. Indeed, as bonds and capital are substitutes, the agent can decide to hold B_{t+1} negative and K_{t+1} positive as long as these choices are consistent with positive savings, namely $K_{t+1} + S_t B_{t+1} \geq 0$.

The nominal rate of interest, r_{t+1}, will be assumed to be strictly positive. Whenever $r_{t+1} = 0$, then the three assets, capital, bonds and money are substitutes.

For any $\gamma \geq 0$, money is dominated as a store of value by the other assets (capital and bonds) when the nominal rate of interest is positive:

$$\frac{Q_{t+1}}{Q_t} < \frac{Q_{t+1}}{S_t}.$$

In an economy with no liquidity constraint ($\gamma = 0$ or equivalently $\Gamma = +\infty$), the individual never holds any money when $r_{t+1} > 0$. The optimum is obtained in that case by maximizing the utility under the sole intertemporal budget constraint

$$X_t + \frac{C_{t+1}}{R_{t+1}} = A_{t+1}, \tag{3.1}$$

where $A_{t+1} = W_t + \frac{T_{t+1}}{R_{t+1}}$ represents the income and $M_{t+1} = 0$.

Under a liquidity constraint with $\gamma > 0$, the optimum with $\gamma = 0$ cannot be reached because the liquidity constraint requires $M_{t+1} > 0$.

Hence, the liquidity constraint is binding and the optimum is obtained in that case by maximizing the utility under the intertemporal budget constraint

$$X_t + C_{t+1}\frac{1 + \gamma r_{t+1}}{R_{t+1}} = A_{t+1}, \tag{3.2}$$

for the same A_{t+1} as above.

Comparing (3.1) and (3.2), one observes that for a fixed liquidity constraint $\gamma > 0$, the nominal rate of interest acts as a tax on second period consumption. The last section shows that the equilibrium path of this monetary economy is equivalent to that of a non-monetary economy in which second period consumption is taxed and these taxes are redistributed as lump-sum transfers to the individuals in their second period of activity.

The first order condition of the maximization problem of the individual reads

$$U'(C_{t+1})\frac{(1 + r_{t+1})}{(1 + \pi_{t+1})} = V'(X_t)(1 + \gamma r_{t+1}).$$

Under a fixed rate of inflation π, an increase in r has a larger effect on the marginal utility of second period consumption, $U'(C)$, than the marginal utility of first period consumption, $V'(X)$, when $\gamma > 0$. Figure 4 shows how the marginal rate of substitution, $\frac{V'}{U'}$, varies with respect to r, for $0 < \gamma < 1$ and $\gamma = 0$ and a fixed rate of inflation π.

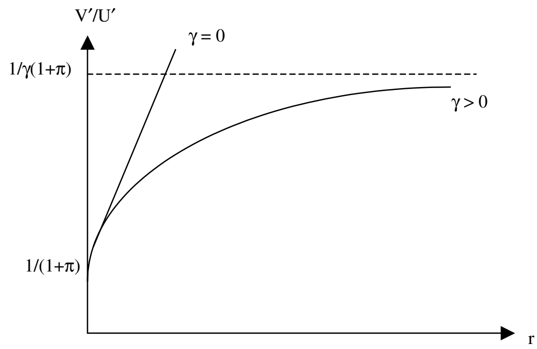

Figure 4. The marginal rate of substitution, $\frac{V'}{U'}$, with respect to r, for $0 < \gamma < 1$ and $\gamma = 0$ and a fixed rate of inflation π

An increase in the nominal rate of interest r has two effects: an intertemporal substitution effect and a tax effect. As r increases, the real rate of interest, $R = \frac{1+r}{1+\pi}$, increases as well. The ratio of the marginal utilities $\frac{V'}{U'}$ increases, and the intertemporal substitution effect leads to an increase in C and a decrease in X. This observation holds both when $\gamma = 0$ and when $\gamma > 0$. However, in the latter case, the extent of the intertemporal substitution effect is reduced, due to the tax effect. Indeed, with $\gamma > 0$, the ratio $\frac{R}{1+\gamma r}$ increases but less than R. Overall, whenever $\gamma > 0$, the intertemporal substitution effect results in an increase in C and a decrease in X as r increases, but these variations are smaller than under no constraint ($\gamma = 0$).

3.2 Firms

Firms act competitively. The *production technology* is described by a production function $G(N_t K_t, N_t)$ homogeneous of degree one. The supply of labor in period t, L_t, is assumed equal to the number of agents for which period t is the first period of activity: N_t. By assumption, $G(N_t K_t, N_t) = N_t G(K_t, 1) = N_t \overline{G}(K_t)$.

The function $\overline{G} : I\!R_+ \to I\!R_+$ is positive, strictly increasing, strictly concave, and smooth; $\lim_{K\to\infty} \overline{G}(K)/K < 1$, $\lim_{K\to 0} \overline{G}'(K) = +\infty$, $\lim_{K\to+\infty} \overline{G}'(K) = 0$.

Profit maximization implies that the wage is equal to the marginal productivity of labor while the price of capital is equal to the marginal productivity of capital:

$$W_t = G'_N = \overline{G}(K_t) - K_t \overline{G}'(K_t),$$
$$R_t = G'_{NK} = \overline{G}'(K_t).$$

3.3 The monetary authority

A bank issues per capita balances, $\overline{M}_{t+1}$, at date t, that provide liquidity at date $t+1$. The bank supplies balances in exchange for debt through open market operations.

The per capita budget constraint of the bank is

$$Q_t((1+n)\overline{M}_{t+1} - \overline{M}_t) = Q_t B_t - (1+n) S_t B_{t+1} + T_t.$$

Profits of the bank from money creation, $r_t Q_t \overline{M}_t$, are redistributed to the old agents. Therefore, $T_t \equiv r_t Q_t \overline{M}_t$.

3.4 Equilibrium

An *intertemporal competitive equilibrium with perfect foresight* is a sequence

$$\{(X_t, C_{t+1}, K_{t+1}, M_{t+1}, B_{t+1}, \overline{M}_{t+1}, R_t, W_t, Q_t, S_t) : t \in I\!N\},$$

satisfying

i. $(X_t, C_{t+1}, K_{t+1}, M_{t+1}, B_{t+1})$ is an optimal action of an individual born in period t, given the price system;
ii. Firms maximize profits under the technological constraint, given the prices (R_t, W_t);
iii. (good market equilibrium) $\overline{G}(K_t) = X_t + \frac{C_t}{1+n} + (1+n)K_{t+1}$, for all t;
iv. (money market equilibrium) $\overline{M}_{t+1} = M_{t+1}$, for all t;
v. (bank's budget constraint)

$$Q_t((1+n)\overline{M}_{t+1} - \overline{M}_t) = Q_t B_t - (1+n) S_t B_{t+1} + r_t Q_t \overline{M}_t.$$

An equilibrium of this economy will be characterized by means of three variables: the second period consumption, the stock of capital and the nominal rate of interest.

Proposition 4. *A sequence*

$$\{Y_t = (C_t, K_t, r_t) : t \in I\!N\}$$

describes a competitive equilibrium of an economy characterized by $(U, V, \overline{G}, \gamma)$ *if and only if for all* t *the following conditions are satisfied:*

$$\begin{aligned} V'(X_t)(1+\gamma r_{t+1}) &= U'(C_{t+1})\overline{G}'(K_{t+1}), \\ \frac{C_{t+1}}{1+n} - \overline{G}'(K_{t+1})K_{t+1} &= \frac{\overline{G}'(K_{t+1})}{1+n}\left[\frac{C_t}{1+n} - \overline{G}'(K_t)K_t\right], \end{aligned} \tag{3.3}$$

where

$$X_t = \overline{G}(K_t) - (1+n)K_{t+1} - \frac{C_t}{1+n}.$$

The proposition states that to completely characterize the equilibria of the economy, it is enough that the first order condition of the maximization problem of the household and the firms be satisfied, along with the constraint of the bank and the good market clearing condition.

Let η_t denote outside money. By definition, $\eta_t = \mu_t(1+r_t) + \beta_t$, where $\mu_t = Q_t M_t$ and $\beta_t = Q_t B_t$ are the real money supply and the real debt, respectively.

Debt is the counterpart of money whenever $\eta_t = 0$. It corresponds to a situation where all money is inside money. Observe that η_t is expressed in real terms and can equivalently be written as $\eta_t = \frac{C_t}{1+n} - \overline{G}'(K_t)K_t$.

A value of η_t larger than 0 refers to a situation where either debt β_t is positive, or it is negative but smaller in absolute value than the stock of money $\mu_t(1 + r_t)$. A value of η_t less than 0 corresponds to a situation where debt β_t is negative (as determined by the bank) and greater in absolute value than $\mu_t(1 + r_t)$.

In terms of the value of outside money, the first period per capita budget constraint is

$$X_t = W_t - \frac{1+n}{R_{t+1}}(R_{t+1}K_{t+1} + \eta_{t+1}).$$

In equilibrium, the second period per capita budget constraint is precisely

$$C_{t+1} = (1 + n)[R_{t+1}K_{t+1} + \eta_{t+1}].$$

Hence $R_{t+1}K_{t+1} + \eta_{t+1} \geq 0$ and whenever η_{t+1} is negative, then it is, in absolute value, less than $R_{t+1}K_{t+1}$. Note that whenever η_{t+1} is negative, then $C_{t+1} < (1 + n)R_{t+1}K_{t+1}$.

Proposition 4 presents a two-dimensional dynamical system in the three variables C_t, K_t, r_t. There is one degree of freedom in the determination of the equilibrium path. In the sequel, the monetary policy fixes the nominal interest rate r_t to a constant level, r, over time.

3.5 Stationary states

A first approach to the problem of determining the equilibria of the economy is to focus on stationary equilibria. Given a nominal rate of interest r, one studies the golden rule steady state, where the real rate of interest is equal to the rate of growth of population and outside money is typically different from 0.

As is well known in monetary OLG models, there is another type of stationary state involving no outside money, where debt is the exact counterpart of money. This type of steady states shall not be considered.

The golden rule. Assume $\eta_t = \frac{C_t}{1+n} - \overline{G}'(K_t)K_t \neq 0$ for all t. A sequence $\{(C_t, K_t) : t \in I\!N\}$ with $C_t = C$ and $K_t = K$ for all t defines a stationary equilibrium if and only if it satisfies (3.3), in which case

$$V'(X)(1 + \gamma r) = U'(C)\overline{G}'(K), \qquad \overline{G}'(K) = (1 + n),$$

where

$$X = \overline{G}(K) - (1 + n)K - \frac{C}{1+n}.$$

Under the assumptions on preferences and production, and with

$$\eta = \frac{C}{1+n} - \overline{G}'(K)K \neq 0,$$

there exists a unique golden rule stationary state for any given nominal rate of interest $r > 0$. Indeed, under the assumptions on production, there exists a unique capital stock, $K = K^*_\gamma$ such that $\overline{G}'(K) = 1 + n$. Given such a K, there exists a unique consumption level, $C = C^*_\gamma$ such that

$$V'\left(\overline{G}(K) - (1+n)K - \frac{C}{1+n}\right)(1+\gamma r) = U'(C)\overline{G}'(K), \qquad (3.4)$$

by the assumptions on preferences. Indeed, the left-hand side (LHS) of (3.4) is positive and increases to $+\infty$ as $C \to \overline{C} = (1+n)[\overline{G}(K) - (1+n)K] > 0$, while the right-hand side (RHS) of (3.4) is positive and decreasing from $+\infty$ to 0 as C increases from 0 to $+\infty$. The above discussion about the existence and uniqueness of an equilibrium is summarized in Figure 5.

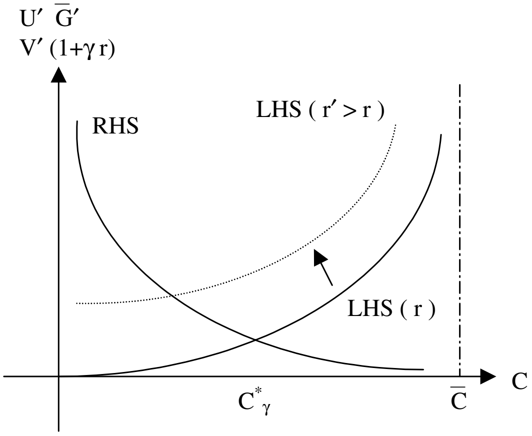

Figure 5. Existence and uniqueness of the stationary equilibrium with outside money

In real terms, all magnitudes are constant at the golden rule stationary state. As the real rate of interest R is fixed, the rate of inflation π is determined by the monetary policy r since $1 + \pi = (1 + r)/R$. Nominal magnitudes are not constant:

$$B_{t+1} = (1+\pi)B_t, \quad M_{t+1} = (1+\pi)M_t, \quad Q_{t+1} = \frac{1}{1+\pi}Q_t.$$

Figure 5 allows to analyze easily the effect of changing the nominal rate of interest r on the golden rule stationary state, both when the liquidity constraint is absent ($\gamma = 0$) and when a liquidity constraint is operative ($\gamma > 0$).

If no liquidity constraint is present, the monetary policy is superneutral; the nominal rate of interest has no real effect. Indeed, at the golden rule steady state, the real rate of interest, $R = \overline{G}'(K^*_\gamma)$, which is the only determinant of the consumption levels, X^*_γ and C^*_γ, is fixed. If there is no liquidity constraint, a permanent increase in the nominal rate of interest r is simply compensated by a permanent increase in the rate of inflation, with no real consequences.

The introduction of a constraint allows the nominal rate of interest to play a role in the determination of the equilibrium path. Whenever $\gamma > 0$, an increase in the nominal rate of interest r is still matched by an increase of the rate of inflation so that the real rate of interest $R = \overline{G}'(K_\gamma^*)$ is left unchanged. However, the increase in r here acts as a tax by modifying the relative price of current and of future consumption. The first period consumption increases in r, while the second period consumption decreases in r: $X_\gamma^{*\prime}(r) > 0$ and $C_\gamma^{*\prime}(r) < 0$. The curve corresponding to the left-hand side of equation (3.4) in Figure 5 is shifted upward following an increase in the nominal rate of interest from r to r'.

The next lemma summarizes the results and states their implications for the value of outside money at the golden rule.

Lemma 1.

- *Whenever $\gamma = 0$, the golden rule consumption levels, X_0^* and C_0^*, are independent of r.*
- *Whenever $\gamma > 0$, then as functions of r, $X_\gamma^*(r)$ is increasing, while $C_\gamma^*(r)$ is decreasing and converges to 0 as r goes to $+\infty$.*
- *The value of outside money at the golden rule, $\eta_\gamma^*(r) = \frac{C_\gamma^*(r)}{1+n} - \overline{G}'(K_\gamma^*)K_\gamma^*$ is a decreasing function of r, which converges to $-\overline{G}'(K_\gamma^*)K_\gamma^* < 0$ as r goes to $+\infty$. In particular, if $\eta_\gamma^*(0) > 0$, then there exists a unique value of r, namely $\overline{r}$, such that $\eta_\gamma^*(\overline{r}) = 0$ and $\eta_\gamma^*(r)$ is negative for $r > \overline{r}$. If $\eta_\gamma^*(0) < 0$, then $\eta_\gamma^*(r)$ is negative for all values of r.*

One important piece of information is the sign of $\eta_\gamma^*(0)$. It actually depends in a complex manner on the fundamentals of the economy. This issue is analyzed through an example with CRRA utilities and CES technology at the end of the paper.

3.6 Equilibria near a stationary state with outside money and bifurcation analysis

The effect of changing the nominal rate of interest r on the local dynamics near the golden rule stationary state is now studied. One observes that increasing r increases the possibility of indeterminacy and bifurcations.

Consider an economy whose dynamics is described by Proposition 4 and concentrate on the golden rule equilibrium with $\eta_\gamma^* = \frac{C_\gamma^*}{1+n} - K_\gamma^*(1+n) \neq 0$.

Proposition 5. *Let*

$$\epsilon = -\frac{K_\gamma^* \overline{G}''(K_\gamma^*)}{\overline{G}'(K_\gamma^*)}$$

be the elasticity of the marginal productivity of capital. The linearized dynamics at the golden rule steady state is given by the system

$$A\begin{pmatrix} \tilde{C}_{t+1} \\ \tilde{K}_{t+1} \end{pmatrix} = B\begin{pmatrix} \tilde{C}_t \\ \tilde{K}_t \end{pmatrix}$$

where $\{(\tilde{C}_t, \tilde{K}_t) : t \in I\!N\}$ *denotes a sequence of deviations from the steady state path* $\{(C^*_\gamma, K^*_\gamma)\}$.

The characteristic polynomial of the matrix $A^{-1}B$ *is* $Q(Z) = Z^2 - TZ + D$, *where* T *is the trace of* $A^{-1}B$ *and* D *its determinant, respectively defined by*

$$D = \frac{1}{|A|}\,\epsilon\,(1+n)\frac{\overline{V}}{X^*_\gamma},$$

$$T = 1 + D + \frac{1}{|A|}\frac{\epsilon}{K^*_\gamma}\eta^*_\gamma\left(\frac{\overline{V}}{X^*_\gamma} + \overline{U}\frac{(1+n)}{C^*_\gamma}\right)$$

where

$$\overline{V} \;=\; \frac{-X^*_\gamma V''(X^*_\gamma)}{V'(X^*_\gamma)}, \qquad \overline{U} \;=\; \frac{-C^*_\gamma U''(C^*_\gamma)}{U'(C^*_\gamma)}$$

are the coefficients of relative risk aversion at the steady state and

$$|A| = \frac{\epsilon}{K^*_\gamma}(1-\overline{U}) + (1+n)\left(\frac{\overline{V}}{X^*_\gamma} + \frac{\overline{U}}{C^*_\gamma}(1+n)\right)$$

is the determinant of the matrix A.

Proof. The matrices A and B are

$$A = \begin{pmatrix} -\frac{\overline{U}}{C^*_\gamma} & -\frac{\overline{V}}{X^*_\gamma}(1+n) - \frac{\epsilon}{K^*_\gamma} \\ \\ 1 & -(1+n)^2(1-\epsilon) + \frac{\epsilon}{K^*_\gamma}(C^*_\gamma - (1+n)^2 K^*_\gamma) \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\overline{V}}{X^*_\gamma(1+n)} & -\frac{\overline{V}}{X^*_\gamma}(1+n) \\ \\ 1 & -(1+n)^2(1-\epsilon) \end{pmatrix}.$$

The matrix $A^{-1}B$ summarizes the dynamics, and is given by

$$(A^{-1}B)_{11} = \frac{1}{|A|}\frac{\epsilon}{K^*_\gamma}\left(\frac{\overline{V}}{X^*_\gamma}\frac{C^*_\gamma}{1+n} + 1\right);$$

$$(A^{-1}B)_{12} = \frac{1}{|A|}\frac{\epsilon}{K^*_\gamma}(1+n)^2\left[\frac{\overline{V}}{X^*_\gamma}\left(K^*_\gamma(1+n) - \frac{C^*_\gamma}{1+n}\right) - (1-\epsilon)\right];$$

$$(A^{-1}B)_{21} = -\frac{1}{|A|}\left[\frac{\overline{V}}{X^*_\gamma(1+n)} + \frac{\overline{U}}{C^*_\gamma}\right];$$

$$(A^{-1}B)_{22} = \frac{1+n}{|A|}\left(\frac{\overline{V}}{X^*_\gamma} + \frac{\overline{U}}{C^*_\gamma}(1+n)(1-\epsilon)\right).$$

□

Let λ_1 and λ_2 denote the eigenvalues of $A^{-1}B$. Then the trace T and the determinant D of $A^{-1}B$ are $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$. Following Azariadis [2] and Grandmont et al. [18], the stability properties of the system can be determined by finding the position of the pair (T, D) in the plane as a function of the economic parameters.

Figure 6 gives a graphical representation of the stability regions in the (T, D) plane parameterized by ϵ, the elasticity of the marginal productivity of capital.

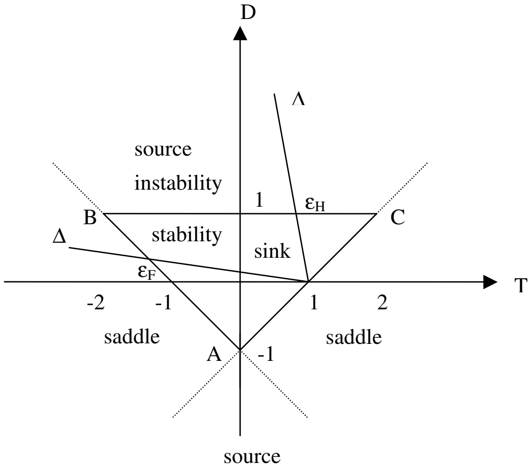

Figure 6. Stability analysis

The linear dynamics described in Proposition 5 will be stable whenever $|\lambda_i| < 1$, which equivalently means that the pair (T, D) lies inside the triangle ABC in Figure 6. Whenever one eigenvalue is of modulus equal to +1, then (T, D) belongs to the line AC in Figure 6; when one eigenvalue is of modulus equal to -1, then (T, D) belongs to the line AB; when the two eigenvalues are complex conjugate of modulus equal to 1, then (T, D) belongs to the segment BC.

For any (T, D) in the interior of the triangle ABC, the stationary solution is a sink. The interior of the triangle ABC in the (T, D) plane corresponds to the stable and indeterminate region, where for any neighborhood of the steady state and any initial capital stock K_0 close to the steady state level of the capital stock, there are infinitely many equilibria corresponding to K_0 that remain in that neighborhood for all times.

The exterior of the triangle ABC corresponds to the unstable and determinate region. For any (T, D) outside the triangle ABC, the stationary solution is either a saddle or a source. For any neighborhood W of the steady state and for any initial capital stock K_0 close to the steady state level of the capital stock, there is a unique equilibrium corresponding to K_0 that remains in W and eventually converges to the golden rule whenever the characteristics of the economy are such that it lies outside the triangle ABC.

One observes that T and D are ratios of polynomials of degree one in ϵ, the elasticity of the marginal productivity of capital. Keeping every other parameters fixed and letting ϵ vary from 0 to $+\infty$, then the pair $(T(\epsilon), D(\epsilon))$ moves along a half-line Δ in the plane. The method is to locate this half-line in the (T, D) plane, as it contains all the information pertaining to the dynamics of the economy. Figure 6 gives a graphical representation of some of the likely positions of the half line Δ in the (T, D) plane parameterized by ϵ.

The half line Δ has as origin the coordinates $(1, 0)$ in the (T, D) plane, from Proposition 5. Under the assumption of gross substitutability ($\overline{U} < 1$), and for any $\epsilon > 0$, the determinant of the matrix A, $|A|$, is strictly positive. In this case, $D(\epsilon) > 0$ and the half line Δ lies above the horizontal axis.

The same method allows an easy visualization of the changes of stability as the parameter ϵ moves, while the other parameters are held fixed. Indeed, whenever the half line Δ, whose slope depends on η_γ^* and the nominal rate of interest r, intersects one of the sides of the triangle, that is, when varying ϵ while keeping the other parameters fixed the modulus of an eigenvalue takes the value one, a change in stability occurs.

When the half-line Δ intersects the line AB ($\epsilon = \epsilon_F$), one eigenvalue is -1 and one observes a flip bifurcation; when Δ intersects the segment BC ($\epsilon = \epsilon_H$), the eigenvalues are complex conjugate of modulus one and one observes a Hopf bifurcation.

In most studies (e.g. Grandmont et al. [18]), stability changes are driven by changes in technological parameters only. Here, in contrast, the graphical representation does depend on a policy variable. Indeed, the slope of the half line Δ depends on the nominal rate of interest r.

The slope of the parameterized line Δ is given by

$$\Delta' = \frac{D'(\epsilon)}{T'(\epsilon)} = \frac{K_\gamma^*(1+n)\frac{\overline{V}}{X_\gamma^*(r)}}{K_\gamma^*(1+n)\frac{\overline{V}}{X_\gamma^*(r)} + \eta_\gamma^*(r)\left(\frac{\overline{V}}{X_\gamma^*(r)} + \frac{\overline{U}}{C_\gamma^*(r)}(1+n)\right)},$$

where

$$\eta_\gamma^*(r) = \frac{C_\gamma^*(r)}{1+n} - K_\gamma^*(1+n).$$

Alternatively,

$$\Delta' = \frac{1}{1+f(r)}$$

where

$$f(r) = \eta_\gamma^*(r)\left(\frac{\overline{V}C_\gamma^*(r) + \overline{U}X_\gamma^*(r)(1+n)}{C_\gamma^*(r)K_\gamma^*(1+n)\overline{V}}\right) \qquad (3.5)$$

and

$$X_\gamma^*(r) = \overline{G}(K_\gamma^*) - (1+n)K_\gamma^* - \frac{C_\gamma^*(r)}{1+n}.$$

The objective is to find the position of the half line Δ (or equivalently evaluate its slope Δ') as a function of the nominal rate of interest r. The analysis can be divided in two steps. First, one looks at how Δ varies as a function of η_γ^* seen as a parameter independent of r. Assume r is fixed, in which case X_γ^* and C_γ^* are fixed as well, and $\overline{U}$ and $\overline{V}$ are constant.

Proposition 6. *Assume* $\overline{U} < 1$ *(gross substitutability), so that the half line* Δ *described by* $(T(\epsilon), D(\epsilon))$ *when* ϵ *increases from 0 to* $+\infty$*, given the other parameters of the model, has an origin at* $T = 1$*,* $D = 0$ *and lies above the horizontal axis.*

- *Whenever* η_γ^* *goes to* $+\infty$*, the slope of* Δ *tends to 0.*
- *Whenever* η_γ^* *is positive, the slope of* Δ *lies between 0 and 1 and the golden rule is a saddle point; hence it is determinate.*
- *Whenever* $\eta_\gamma^* = 0$*, then the slope of* Δ *is equal to 1:* Δ *coincides with the line* AC*.*
- *Whenever* η_γ^* *is negative, the half line* Δ *lies above the line* AC*. The golden rule is indeterminate for low values of* ϵ*. For negative but small values of* η_γ^* *in absolute value, there is a change of stability when* ϵ *increases through a Hopf bifurcation. When* η_γ^* *is negative but large in absolute value, the change of stability occurs through a flip bifurcation.*
- *When* $\eta_\gamma^* \to -\infty$*, then the half line* Δ *becomes almost horizontal.*

Figure 7 gives a graphical representation of the results presented in the above proposition. The results follow immediately from the expression $\Delta' = \frac{1}{1+f(r)}$, where $f(r)$ is of the form $P\eta_\gamma^*$, with P constant.

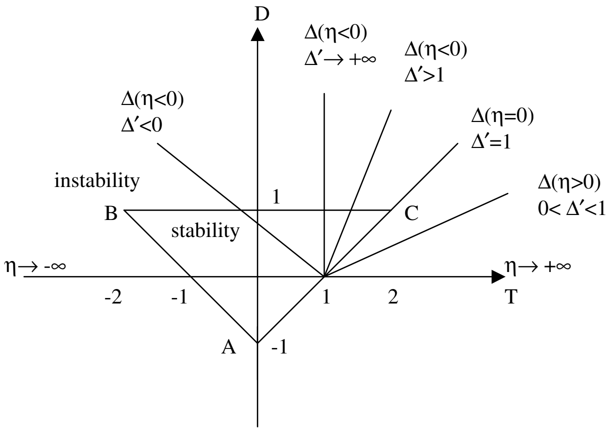

Figure 7. The half line Δ as a function of $\eta = \eta_\gamma^*$

As shown in Figure 7, whenever η_γ^* is negative, the slope of the half line Δ is either positive or negative. In either case, an increase in ϵ eventually leads to a change of stability.

One proceeds to the second step of the analysis of the position of the half line Δ by taking into account the variations of the nominal rate of interest r and the fact that the value of outside money η_γ^* is indeed a function of r.

It is not clear in general how the slope of the half line Δ varies with the nominal rate of interest r. Indeed, as r increases, $f(r)$ can either increase or decrease, as it is the product of a decreasing function of r, $\eta_\gamma^*(r)$, and an increasing function of r, the second term in (3.5). The latter term depends essentially on the ratio X_γ^*/C_γ^* which is increasing in r given $\overline{U}$ and $\overline{V}$.

The next result states that the slope of the half line $\varDelta$ increases with r when $\overline{V}$ and $\overline{U}$ are constant and $\overline{V} \geq \overline{U}$.

Proposition 7. *Assume $\overline{U} < 1$, and $\overline{U}$ and $\overline{V}$ are independent parameters. The slope of the half line $\varDelta$ increases with r and rotates counterclockwise around its origin as r increases whenever $\overline{V} \geq \overline{U}$.*

Proof. The slope of the line $\varDelta$ is

$$\varDelta' = \frac{1}{1+f(r)}, \qquad \text{where} \qquad f(r) = \eta^*_\gamma(r)\left(\frac{\overline{V}C^*_\gamma(r) + \overline{U}X^*_\gamma(r)(1+n)}{C^*_\gamma(r)K^*_\gamma(1+n)\overline{V}}\right)$$

and $X^*_\gamma(r) = \overline{G}(K^*_\gamma) - (1+n)K^*_\gamma - \frac{C^*_\gamma(r)}{1+n}$. As

$$\frac{d}{dr}\varDelta' = \frac{-f'(r)}{(1+f(r))^2},$$

one must study the sign of $-f'(r)$. Computations give

$$-f'(r) = \frac{-C^{*\prime}_\gamma(r)}{C^*_\gamma K^*_\gamma(1+n)\overline{V}}\left\{\frac{C^*_\gamma(\overline{V}-\overline{U})}{1+n} + \frac{K^*_\gamma}{C^*_\gamma}(1+n)^2\overline{U}(\overline{G}(K^*_\gamma) - (1+n)K^*_\gamma)\right\}$$

where $C^{*\prime}_\gamma(r) < 0$, and $\frac{K^*_\gamma}{C^*_\gamma}(1+n)^2\overline{U}(\overline{G}(K^*_\gamma) - (1+n)K^*_\gamma) > 0$. □

In general, one cannot say much more. However from Proposition 6 and Lemma 1, if $\eta^*_\gamma(0) > 0$, then for low values of r, $\eta^*_\gamma(r) > 0$ and the slope of $\varDelta$ is positive and less than 1. Consequently, the golden rule is determinate. As r increases, $\eta^*_\gamma(r)$ becomes eventually negative and indeterminacy and fluctuations occur, as a result of the tax effect.

If $\eta^*_\gamma(r)$ is negative for all $r > 0$, then for low ϵ one observes indeterminacy while for large values of ϵ one observes bifurcations.

The next proposition gives the explicit expressions for ϵ_F and ϵ_H, the values of ϵ for which a flip or a Hopf bifurcation occurs.

Proposition 8.

- *A flip bifurcation will generically occur for*

$$\epsilon_F = \frac{2K^*_\gamma(1+n)\left(\frac{\overline{V}}{X^*_\gamma} + \frac{\overline{U}}{C^*_\gamma}(1+n)\right)}{-2 + \left(\frac{\overline{U}}{C^*_\gamma}(1+n) - \frac{\overline{V}}{X^*_\gamma}\right)(2K^*_\gamma(1+n) + \eta^*_\gamma)}.$$

- *A Hopf bifurcation will generically occur for*

$$\epsilon_H = \frac{(1+n)\left(\frac{\overline{V}}{X^*_\gamma} + \frac{\overline{U}}{C^*_\gamma}(1+n)\right)}{(1+n)\frac{\overline{V}}{X^*_\gamma} - \frac{1}{K^*_\gamma}(1-\overline{U})}.$$

Proof. One observes a flip bifurcation if $Q(-1) = 0$ or $D = -(1+T)$. In this case,

$$\epsilon_F = \frac{2K_\gamma^*(1+n)\left(\frac{\overline{V}}{X_\gamma^*} + \frac{\overline{U}}{C_\gamma^*}(1+n)\right)}{-2 + \left(\frac{\overline{U}}{C_\gamma^*}(1+n) - \frac{\overline{V}}{X_\gamma^*}\right)(2K_\gamma^*(1+n) + \eta_\gamma^*)}.$$

One observes a Hopf bifurcation whenever $D = 1$ and $|T| < 2$. In this case

$$\epsilon_H = \frac{(1+n)\left(\frac{\overline{V}}{X_\gamma^*} + \frac{\overline{U}}{C_\gamma^*}(1+n)\right)}{(1+n)\frac{\overline{V}}{X_\gamma^*} - \frac{1}{K_\gamma^*}(1-\overline{U})}.$$

□

3.7 An example

Consider an example involving utilities with constant relative risk aversion and a production technology with constant elasticity of substitution. Under these specifications, one is able to determine the way in which the slope of the line Δ varies with the nominal rate of interest r. One is also able to determine the sign of the value of outside money when $r = 0$ in terms of the technological and preference parameters of the economy.

Let preferences be of the CRRA class:

$$V(X) + U(C) = \frac{1}{\beta}X^\beta + \frac{\delta}{\beta}C^\beta,$$

with $\beta < 1$, and the production be of the CES class:

$$\overline{G}(K) = (1+n)a(K^\alpha/a + 1 - 1/a)^{\frac{1}{\alpha}},$$

where $a > 1$ and $\alpha < 1$.

The coefficients of relative risk aversion $\overline{U}$ and $\overline{V}$ are equal and constant: $\overline{U} = \overline{V} = 1 - \beta$. Let $\overline{r}(r) = \left(\frac{\delta(1+n)}{1+\gamma r}\right)^{\frac{1}{1-\beta}}$. The golden rule values of the capital stock and consumption levels are

$$K_\gamma^* = 1, \qquad X_\gamma^*(r) = \frac{(1+n)^2(a-1)}{1+n+\overline{r}(r)}, \qquad C_\gamma^*(r) = \frac{(1+n)^2(a-1)\overline{r}(r)}{1+n+\overline{r}(r)}.$$

The elasticity of the marginal productivity of capital, the elasticity of the marginal productivity of labor and the elasticity of substitution between capital and labor at the golden rule are respectively

$$\epsilon = \frac{(1-\alpha)(a-1)}{a}, \qquad \omega = \frac{1-\alpha}{a}, \qquad \text{and} \qquad \sigma = \frac{1}{1-\alpha}.$$

Let $s = \frac{K\overline{G}'(K)}{\overline{G}(K)}$. Given the above specifications, the golden rule value of s is $s = \frac{1}{a}$. Hence

$$\epsilon = \frac{1-s}{\sigma}, \qquad \omega = \frac{s}{\sigma} \qquad \text{and} \qquad \frac{\epsilon}{\omega} = \frac{1-s}{s}.$$

The golden rule value of outside money is

$$\eta^*_\gamma(r) = (1+n)\left(\frac{1-s}{s}\frac{\overline{r}(r)}{1+n+\overline{r}(r)} - 1\right),$$

and the slope of the line Δ is

$$\Delta' = \frac{1}{\frac{1-s}{s} - \frac{1+n}{\overline{r}(r)}}.$$

The next lemma summarizes the additional information one can extract from this example, and follows Propositions 6 and 7 which are of course valid here.

Lemma 2. *Let* $\tau = \frac{1-s}{s}$.

- *The slope of the line* Δ *increases with the nominal rate of interest* r *and rotates counterclockwise around its origin as* r *increases.*
- *The value of outside money* $\eta^*_\gamma(r)$ *is a constant function of* τ, *equal to* $-(1+n)$ *when* $\tau = 0$. *There exists a value of* τ, $\tau^* = \frac{1}{(\delta(1+n)^\beta)^{\frac{1}{1-\beta}}} + 1$, *such that whenever* $0 < \tau < \tau^*$, *then* $\eta^*_\gamma(0)$ *is negative. For* $\tau > \tau^*$, *then* $\eta^*_\gamma(0)$ *is positive.*

Note that whenever τ converges to 0, then ϵ converges to 0 while ω converges to $1/\sigma$.

For low values of $\tau = \epsilon/\omega$, the half line Δ intersects the interior of the triangle ABC at $r = 0$ and hence the golden rule is indeterminate. As r increases, the slope of the half line Δ increases as well and for low enough ϵ, the steady state remains indeterminate.

For large values of τ ($\tau > \tau^*$) and low values of r, the golden rule is determinate. As r increases, the steady state becomes indeterminate and bifurcations may occur.

3.8 Taxes and transfers

Consider a variation of the model where no balances are present, only debt and capital as defined previously and using the same notations. A fiscal authority imposes a tax, z_{t+1}, on the second period consumption and redistributes the revenue as lump sum per capita transfer to the individuals at their second date of activity. Let τ_{t+1} denote the real per capita transfer.

The budget constraints of an individual at date t are

$$X_t = W_t - [K_{t+1} + S_t B_{t+1}],$$
$$C_{t+1}(1 + z_{t+1}) = R_{t+1}K_{t+1} + S_{t+1}B_{t+1} + \tau_{t+1}.$$

The budget constraint of the fiscal authority is

$$z_{t+1}C_{t+1} = \tau_{t+1}.$$

An *intertemporal competitive equilibrium with perfect foresight* is a sequence

$$\{(X_t, C_{t+1}, K_{t+1}, B_{t+1}, z_t, R_t, W_t, S_t) : t \in I\!N\},$$

satisfying

i. $(X_t, C_{t+1}, K_{t+1}, B_{t+1})$ is an optimal action of an individual born in period t, given the price system;
ii. Firms maximize profits under the technological constraint, given prices (R_t, W_t);
iii. (good market equilibrium) $\overline{G}(K_t) = X_t + \frac{C_t}{1+n} + (1+n)K_{t+1}$, for all t;
iv. $\tau_{t+1} = z_{t+1}C_{t+1}$.

Proposition 9. *A sequence*

$$\{Y_t = (C_t, K_t, z_t) : t \in I\!N\}$$

describes a competitive equilibrium of an economy characterized by $(U, V, \overline{G})$ *if and only if for all* t *the following conditions are satisfied:*

$$V'(X_t)(1 + z_{t+1}) = U'(C_{t+1})\overline{G}'(K_{t+1}),$$
$$\frac{C_{t+1}}{1+n} - \overline{G}'(K_{t+1})K_{t+1} = \frac{\overline{G}'(K_{t+1})}{1+n}\left[\frac{C_t}{1+n} - \overline{G}'(K_t)K_t\right],$$

where

$$X_t = \overline{G}(K_t) - (1+n)K_{t+1} - \frac{C_t}{1+n}.$$

The equilibrium paths for the monetary economy with a liquidity constraint and the economy with an active fiscal policy coincide for

$$z_t = \gamma r_t.$$

Hence, the nominal rate of interest is nothing but a consumption tax.

4 Extensions

Arguments for the effectiveness of monetary policy have focused, following Lucas [21], on the information revealed by prices.

When the asset market is incomplete, the role of money as a unit of account renders monetary policy effective through its impact on the attainable reallocations of revenue across date-events; Balasko and Cass [3], Cass [9], Geanakoplos and Mas-Colell [15].

Wealth effects and price rigidities or structural constraints on price adjustments, as in Benassy [4] and Drèze [12] or, more recently, Woodford [29] also make for effective monetary policy.

The considerations above are absent from the argument here, but well within the scope of extensions.

References

1. Allais, M.: Economie et intérêt. Imprimerie Nationale, France (1947)
2. Azariadis, C.: Intertemporal macroeconomics. Oxford: Blackwell 1995
3. Balasko, Y., Cass, D.: The structure of financial equilibrium, I. Exogenous yields and unrestricted participation. Econometrica **57**, 135–162 (1989)
4. Bénassy, J.-P.: Neo-keynesian disequilibrium theory in a monetary economy. Review of Economic Studies **42**, 503–523 (1975)
5. Benhabib, J., Day, R.: A characterization of erratic dynamics in the overlapping generations model. Journal of Economic Dynamics and Control **4**, 37–55 (1982)
6. Benhabib J., Laroque, G.: On competitive cycles in productive economies. Journal of Economic Theory **45**, 145–170 (1988)
7. Bloise, G., Drèze, J.H., Polemarchakis, H.M.: Monetary equilibria over an infinite horizon. Economic Theory (forthcoming) (2005)
8. Bloise, G., Polemarchakis, H.M.: An argument for positive nominal interest rates. Mimeo (2004)
9. Cass, D.: On the number of equilibrium allocations with financial markets. Working Paper No. 8500, CARESS, University of Pennsylvania (1985)
10. Clower, R.: A reconsideration of the microfoundations of monetary theory. Western Economic Journal **6**, 1–9 (1967)
11. Diamond, P.: National debt in a neoclassical growth model. The American Economic Review **55**, 1125–1150 (1965)
12. Drèze, J. H.: Existence of an exchange equilibrium under price rigidities. International Economic Review **16**, 301–320 (1975)
13. Dubey, P., Geanakoplos, J.D.: Inside-outside money, gains to trade and IS-LM. Economic Theory (forthcoming)
14. Friedman, M.: The quantity theory of money: a restatement. In: Friedman. M. (ed.) Studies in the quantity theory of money, pp. 3–21. Chicago: University of Chicago Press 1956
15. Geanakoplos, J. D., Mas-Colell, A.: Real indeterminacy with financial assets. Journal of Economic Theory **47**, 22–38 (1989)
16. Grandmont, J.-M.: On endogenous competitive business cycles. Econometrica **53**, 995–1045 (1985)
17. Grandmont, J.-M., Laroque, G.: On money and banking. Review of Economic Studies **42**, 207–236 (1975)
18. Grandmont J.-M., Pintus, P., de Vilder, R.: Capital-labor substitution and competitive nonlinear endogenous business cycles. CREST 9728 (1997)
19. Grandmont, J.-M., Younès, Y.: On the role of money and the existence of a monetary equilibrium. Review of Economic Studies **39**, 355–372 (1972)
20. Grandmont, J.-M., Younès, Y.: On the efficiency of monetary equilibrium. Review of Economic Studies **40**, 149–165 (1973)
21. Lucas, R. E.: Expectations and the neutrality of money. Journal of Economic Theory **4**, 103–124 (1972)
22. Lucas, R. E., Stokey, N.L.: Money and rates of interest in a cash-in-advance economy. Econometrica **55**, 491–513 (1987)
23. Nakajima, T., Polemarchakis, H.: Money and prices under uncertainty. Review of Economic Studies (forthcoming)
24. Reichlin P.: Equilibrium cycles in an overlapping generations economy with production. Journal of Economic Theory **40**, 89–102 (1986)
25. Samuelson, P.: An exact consumption-loan model of interest with or without the social contrivance of money. Journal of Political Economy **66**, 467–482 (1958)
26. Sargent, T., Wallace, N.: Rational expectations, the optimal monetary instrument and the optimal money supply rule. Journal of Political Economy **83**, 241–254 (1975)
27. Wilson, C.: Equilibrium in dynamic models with an infinity of agents. Journal of Economic Theory **24**, 95–111 (1981).
28. Woodford, M.: Monetary policy and price level determinacy in a cash-in-advance economy. Economic Theory **4**, 345–380 (1994)
29. Woodford, M.: Interest and prices. Mimeo (2002)