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Type space on a purely measurable parameter space*

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Summary. Several game theoretical topics require the analysis of hierarchical beliefs, particularly in incomplete information situations. For the problem of incomplete information, Harsányi suggested the concept of the type space. Later Mertens and Zamir gave a construction of such a type space under topological assumptions imposed on the parameter space. The topological assumptions were weakened by Heifetz, and by Brandenburger & Dekel. In this paper we show that at very natural assumptions upon the structure of the beliefs, the universal type space does exist. We construct a universal type space, which employs purely a measurable parameter space structure.

Keywords and Phrases: Coherent beliefs, Incomplete information, Type space.

JEL Classification Numbers: C70, C79, D80, D82.

1 Introduction

Modeling rationally behaving actors in a multi-person decision problem involves the analysis of players' information about all aspects, which have influence on the decision making. During the decision making process the rational players use all available information, so its analysis is necessary for modeling the actors' behavior. Aumann[1] introduced a formal definition for the idea of common knowledge. The distinction between common knowledge and knowledge leads to, among others, the research of hierarchies of beliefs.

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The problem of incomplete information is related to the problem of hierarchical beliefs. In an incomplete information situation, some parameters of the model are not common knowledge. If something is not common knowledge, we must deal with hierarchies of beliefs, that is, we have to consider arguments like what every agent believes about what every agent believes about what every agent believes and so on, which makes the model very complicated.

Harsányi [3] assumed a ready-made type space, which includes all possible types of players, and hence, their knowledges, beliefs as well. Simultaneously he assumed a probability measure, defined on the product of the parameter space and the type spaces. This probability measure induces hierarchies of beliefs, so we can consider this probability measure as a "summary of hierarchies of beliefs". However, the opposite question remains: how can we build a type space from hierarchies of beliefs?

A very important step in this direction was made by Mertens and Zamir [10] who built a universal type space based on a compact parameter space. Later, Heifetz [4] relaxed the compactness, but other topological assumptions were retained. Almost parallel Brandenburger and Dekel [2] proved the existence of a universal type space in presence of a complete, separable metric (Polish) parameter space. More recently, Mertens et al. [9] gave an elegant proof for the existence of a universal type space in cases of parameter spaces with various structures. Ultimately, all of the above proofs are based on the Kolmogorov's Existence Theorem and its generalizations.

In 1998 Heifetz and Samet [5] proved the existence of a universal type space, which possesses a purely measurable structure. In contrast to our paper, the authors make a distinction between universal type space, and space of coherent hierarchies of beliefs. They also gave an illuminating discussion on the problem of type spaces, beliefs spaces. The same authors gave a counterexample showing that in general circumstances, coherent beliefs are not always types (see Heifetz and Samet [6]).

Quite recently, Meier [8] investigated the problem of the existence of a universal type spaces, his model is based on finitely additive measures. By regarding the opinions as finitely additive measures, the problem of existence of σ-additive measures on type spaces can be eliminated. On the other hand, the author discusses how "rich" the structure of a universal type space can be. This work brings to the surface that, the problem of existence of σ-additive measures on type spaces is not only the problem of σ-additivity.

Mertens and Zamir [10], Heifetz [4], Brandenburger and Dekel [2], and Mertens et al. [9] use the concept of projective limit for proving the existence of a universal type space. In all four papers the structure of beliefs is inherited from the topology of lower ranked beliefs spaces or the parameter space, moreover beliefs are modeled by compact regular probability measures.

Our main goal is to build a universal type space, that is apparently "purely measurable", and in which every coherent hierarchy of beliefs is a type. The structure on the beliefs is naturally generated by the Baire sets of the pointwise convergence topology. For metric spaces Baire sets and Borel sets coincide. However, in non-metrizable cases (for instance when the cardinality of the players is greater than countable), our approach results in a weaker then Borel structure, but this structure allows the players to distinguish between any pair of beliefs (i.e., regular probability measures) yet.

An other new idea in this paper is that we cut the parameter space off the beliefs space. This truncated space has a sufficiently good topological structure (i.e., a projective system of completely regular topological spaces), so the measure projective limit exists. After this, we re-fit the parameter space to the measure projective limit, and we construct the universal type space. It is clear that the existence of a measure projective limit crucially depends on topological assumptions. However, if we remove finitely many elements of the projective system of measure spaces, it does not influence the existence of the measure projective limit.

In the next section we build up our model. In Section 3, we prove the main result of our paper, finally, in Section 4 an illustrative example is provided.

2 The model

If something is common knowledge, then everybody knows that, everybody knows that everybody knows that, and so on. So, common knowledge is more than knowledge, it is some kind of knowledge that is the strongest knowledge in the situation. If something is common knowledge, then somebody's knowledge of this fact does not influence the situation. If something is not common knowledge, then the rational players must concern with the beliefs of other players, beliefs about beliefs of other players and so on.

Therefore, if we have a parameter space S, and this includes all parameters of the game, then we are about to construct a space generated by S, that includes all reasonable beliefs, beliefs about beliefs and so on. This space is called the beliefs space.

Definition 1 *The parameter space is a measurable space $(S, \mathcal{A}_S)$, where $\mathcal{A}_S$ is a σ-algebra defined on S.*

This space S contains all parameters, which have impact on the game. We assume only measurability on this space. The players think in ideas like probability, events, thus a purely measure theoretic model seems to be adequate. However, as is well known from Heifetz and Samet [6], a purely measure theoretic universal type space does not exist in our context.

Definition 2 *Let $\Delta(S, \mathcal{A}_S)$ denote the space of the probability measures on $(S, \mathcal{A}_S)$, and put $d(\mu_1, \mu_2) = \sup_{A \in \mathcal{A}_S} |\mu_1(A) - \mu_2(A)|$. Then $(\Delta(S, \mathcal{A}_S), d)$ or briefly (Δ, d) is a metric space. The collection of all Baire sets of (Δ, d) is denoted by $B(\Delta, d)$*

If it will not lead to misunderstanding, instead of $\Delta(S, \mathcal{A}_S)$ we use the shorter notation $\Delta(S)$ or simply Δ. Analogously, $B(\Delta(S), d)$ is replaced by $B(\Delta(S))$.

Definition 3 *Let us define a sequence of spaces recursively, where M stands for set of the players:*

$$\begin{aligned}
T_0 &= (S, \mathcal{A}_S) \\
T_1 &= T_0 \otimes (\Delta(T_0)^M, B(\Delta(T_0)^M)) \\
T_2 &= T_1 \otimes (\Delta(T_1)^M, B(\Delta(T_1)^M)) \\
&= T_0 \otimes (\Delta(T_0)^M, B(\Delta(T_0)^M)) \otimes (\Delta(T_1)^M, B(\Delta(T_1)^M)) \\
&\vdots \\
T_n &= T_{n-1} \otimes (\Delta(T_{n-1})^M, B(\Delta(T_{n-1})^M)) \\
&= T_0 \otimes \otimes_{j=0}^{n-1} (\Delta(T_j)^M, B(\Delta(T_j)^M)) \\
&\vdots
\end{aligned}$$

where $\otimes$ denotes the product measurable structure.

A point in T_0 is called parameter value, simply a parameter of the game. A point in T_1 is a combination of a parameter value and a 1-st order beliefs (the players' beliefs on the parameter values), and so on.

Consider the infinite product $T_\infty = S \times \times_{j=0}^{\infty} \Delta(T_j)^M$. If $t \in T_\infty$ then it has the form $t = (s, \mu_1^1, \mu_1^2, \dots, \mu_2^1, \mu_2^2, \dots)$, where μ_j^i means the "i" player's j-th order belief. So, every element of T_∞ describes an *hierarchy of beliefs*, i.e., $(\mu_1^i, \mu_i^2, \dots)$ for all players and a possible parameter, therefore it is a possible state of the world. We call beliefs space the spaces of type of T_∞.

Remark 1 The elements of $(s, \mu_1^1, \mu_1^2, \dots, \mu_2^1, \mu_2^2, \dots)$ can be regarded as members of a generalized sequence, where the ordering is: the least element is s, and $\mu_j^i < \mu_k^l$ iff $j < k$.

Definition 4 *Fix an $i \in M$. A hierarchy of beliefs $(\mu_1^i, \mu_2^i, \dots)$ is coherent if* $n \geq 2$

- $marg_{T_{n-2}} \mu_n^i = \mu_{n-1}^i$
- $marg_{[\Delta(T_{n-2})]^i} \mu_n^i = \mu_{\mu_{n-1}^i}^i$,

where μ_n^i is taken from $[\Delta(T_{n-1})]^i$ (which is the i-th copy of $\Delta(T_{n-1})$), furthermore, $marg_{T_n}$ denotes the marginal distribution on T_n, and $\mu_{\mu_{n-1}^i}^i$ stands the Dirac measure concentrated on the "point" μ_{n-1}^i.

The first condition declares the fact that the beliefs over some aspects of the game do not change in the hierarchy. The second condition states that the players know exactly their own beliefs (cf. Harsányi [3]). These two conditions describe the "logic" of the players, we assume this logic to be *common knowledge*.

Remark 2 The measurable structure on $[\Delta(T_{n-1})]^i$ $\forall i, n$ is defined by the Baire sets, which coincide with Borel sets in the case of metric spaces, hence any singleton is measurable.

Consider an element $(s, \mu_1^1, \mu_1^2, \ldots, \mu_2^1, \mu_2^2, \ldots)$ from T_∞ such that the hierarchies of beliefs $(\mu_1^i, \mu_i^2, \ldots)$ are coherent for every $i \in M$. The set all those elements is denoted by T_∞^c and called the *coherent subspace* of T_∞. (The superscript ^{c} will be used in the same context throughout the paper.)

Definition 5 *Fix an $i \in M$ and set*

$$T^i = (\times_{k=0}^{\infty}[\Delta(T_k^c)]^i)^c.$$

T^i is called the type space for player i. A point in T^i is a possible type of player i.

The type space of player i consists of all coherent hierarchies of beliefs. In particular, if $t \in T^i$, then $t = (\mu_1^i, \mu_2^i, \mu_3^i, \ldots)$, and t is coherent.

Corollary 1 *T^i is metrizable since it is a subspace of a countable product of metric spaces. This metric is given by $d_p(\mu, \mu') = \sum_n \frac{1}{2^n} d(\mu_n, \mu'_n)$ where $\mu, \mu' \in T^i$, and $\mu_n, \mu'_n \in [\Delta(T_{n-1}^c)]^i$ (d is given in Definition 2).*

Remark 3 If the cardinality of M is more than countable, then the Baire structure of $\Delta(T_n)^M$ is weaker than the Borel structure. On the other hand, this structure (Baire sets) coincides with the product measurable structure $\otimes_{m \in M} B(\Delta(T_n))^m$. It is worth noting that our construction very similar to a purely measurable type space, because no topology is used to make a stronger measurable structure for product spaces.

Corollary 2 *For a given $i \in M$,*

$$(((T_n^c, B(T_n^c), \mu_{n+1}^i), pr_{mn})_{m<n}) \tag{1}$$

is a projective system of measure spaces, where pr_{mn} is the coordinate projection from T_n^c to T_m^c, and $(\mu_1^i, \ldots, \mu_{n+1}^i, \ldots) \in T^i$.

Proof. For the definition of projective systems we refer to M. M. Rao ([11], p. 117).

- $pr_{mn} = pr_{mk} \circ pr_{kn}$ $\forall m < k < n$, by the definition of coordinate projections.
- $pr_{nn} = id_{T_n^c}$ $\forall n$ follows from the definition of coordinate projections.
- pr_{mn} is measurable $\forall m < n$, because of the definition of product measurable spaces.
- $\mu_{n+1}^i(pr_{mn}^{-1}(A)) = \mu_{m+1}^i(A)$ $\forall m < n$ and $\forall A \in B(T_m^c)$ is a consequence of the coherency of beliefs.

The above Corollary establishes the connection between the idea of projective system and beliefs space. The main question is that, whether or not a proper projective limit of the above defined system exists.

3 The main result

Before we take the next step, we clarify the role of Baire sets in our model. In Mertens and Zamir [10], the opinions were modeled by regular probability measures on Borel sets of a compact space. However, if there is a compact regular probability measure on the Baire sets of a topological space, then it can uniquely be extended to the Borel sets as a compact regular measure. So, there is one-to-one correspondence between compact regular probability measures on Baire sets and on Borel sets. In conclusion, regular probability measures are compact regular measures on a compact topological space hence, there is a bijection between opinions in Mertens and Zamir [10] and opinions in our model.

In Brandenburger and Dekel [2], the opinions are compact regular probability measures on the Borel sets of a Polish (separable, complete, metric) space. As is well known, Borel sets and Baire sets coincide in the case of metric spaces, and all regular probability measures on Borel sets of a Polish space are compact regular. Therefore, the opinions in Brandenburger and Dekel [2] and the opinions in our model are related the same way as Mertens and Zamir [10] and our model, respectively.

In Heifetz [4], and Mertens et al. [9] the opinions are compact regular probability measures on different kinds of spaces. According to our previous discussion, all compact regular probability measures on Borel sets are regular probability measures on Baire sets, but there may be regular probability measures on Baire sets, which are not necessarily compact regular. In an informal way we may say that the set of opinions in our model is, in a certain context broader than that in Heifetz [4], or Mertens et al. [9].

As we have seen, the collection of Baire sets is essentially smaller than the collection of Borel sets if the cardinality of M is more than countable. In this case, a point is not measurable in T^c_n $n > 0$ space. We can interpret this phenomenon as the players' inability of knowing what the others' beliefs exactly are. The players can concentrate on countably many players' beliefs only. We often meet the following argument: "I don't know who, but I'm sure somebody believes that ....!". In the language of probability theory: "Mr. X believes that ...." is the outcome, "somebody believes that ..." is the event. In this example, we mean that the players cannot make an argument like "Mr. i believes that ..., Mr. j believes that ..., " for all players, but our players can argue that "Mr. 1 believes that ..., Mr. 2 believes that ..., ..., somebody believes that ...". This feature of our model is a typical pure measure theoretic feature.

In the next proposition we show that, the central question in our model is the σ-additivity of μ^i in the projective limit (definition is given in the Appendix).

Proposition 1 *Let $i \in M$ be fixed. The projective limit $(T, \mathcal{A}_T, \mu^i) = \varprojlim(((T^c_n, B(T^c_n), \mu^i_{n+1}), pr_{mn})_{m<n})$ of the projective system (1) exists. Further, $T = T^c_\infty$, $\mathcal{A}_T$ is a field and μ^i is an additive set function on $\mathcal{A}_T$.*

Proof. The proof essentially follows the ideas of Rao ([11], p. 118).

Since every pr_{mn} is a coordinate projection we deduce that T is not empty and $T = T^c_\infty$. Pick an $A \in \mathcal{A}_T$, then there is an index n, and $B \in B(T^c_n)$, $A = p_n^{-1}(B)$. Moreover, if $B \in B(T^c_n)$, then also $\complement B \in B(T^c_n)$, so $\complement A = p_n^{-1}(\complement B) \in \mathcal{A}_T$. If $A_1, ..., A_m \in \mathcal{A}_T$, then for every $1 \leq j \leq m$ there exists an index n_j such that $A_j = p_{n_j}^{-1}(B_j)$. Let k be the maximal element of $\{n_1, ..., n_m\}$, and let

$K_j = p_{n_j k}^{-1}(B_j)$, we know $K_j \in B(T_k^c)\ \forall j$, so $\cup_j K_j \in B(T_k^c)$. Making use of $A_j = p_k^{-1}(K_j)$ we obtain $\cup_j A_j \in \mathcal{A}_T$. Thus, $\mathcal{A}_T$ is an algebra.

Since every p_{nm} is a coordinate projection, we conclude that p_n is onto. This implies that p_n^{-1} is one-to-one. Therefore, the set function μ^i defined by the equality $\mu^i \circ p_n^{-1} = \mu_n^i$ is uniquely defined.

Take $A_1, ..., A_m \in \mathcal{A}_T$ disjoint sets, then $\cup_j A_j \in \mathcal{A}_T$. For each $1 \le j \le m$ select B_j and K_j as above. We know K_js are disjoint, and therefore, $\sum_j \mu_{k+1}^i(K_j) = \mu_{k+1}^i(\cup_j K_j)$, and $\sum_j \mu^i(A_j) = \sum_j \mu^i(p_k^{-1}(K_j)) = \sum_j \mu_{k+1}^i(K_j) = \mu_{k+1}^i(\cup_j K_j) = \mu^i(\cup_j p_k^{-1}(K_j))$, hence μ^i is finitely additive $\forall i$.

Proposition 1 concentrates on the additivity of μ^i. Generally, the problem of existence of a proper measure projective limit is twofold: the first problem is the "richness" of the projective limit set (Heifetz and Samet [6] address this problem), the second is the problem of σ-additivity of μ^i. We use the idea of coordinate projections in the projective system, which ensures that the projective limit set is "rich" enough. The second problem demands regularity (but not compact regularity).

In the next proposition, we take preliminary steps for proving our main result.

Proposition 2 *Let us define the following sequence of truncated spaces (c.f. Definition 3):*

$$\begin{aligned}
C_0 &= (\Delta(T_0)^M, B(\Delta(T_0)^M)) \\
C_1 &= C_0 \otimes (\Delta(T_1)^M, B(\Delta(T_1)^M)) \\
&= (\Delta(T_0)^M, B(\Delta(T_0)^M)) \otimes (\Delta(T_1)^M, B(\Delta(T_1)^M)) \\
&\vdots \\
C_n &= C_{n-1} \otimes (\Delta(T_{n-1})^M, B(\Delta(T_{n-1})^M)) \\
&= \otimes_{j=0}^{n-1} (\Delta(T_j)^M, B(\Delta(T_j)^M)) \\
&\vdots
\end{aligned}$$

Consider the projective limit

$$(C, \mathcal{A}_C, \nu^i) = \varprojlim(((C_n^c, B(C_n^c), \nu_n^i), pr_{mn})_{m<n}),$$

where $\nu_n^i = marg_{C_n^c} \mu_{n+2}^i$. *Then* ν^i *is* σ*-additive for every* $i \in M$.

Proof. The proof based on M. M. Rao ([12], pp. 357–358).

Let $i \in M$ be fixed and arbitrary.

The preceding proposition tells us that $\mathcal{A}_C$ is a field, and ν^i is an additive set function on it for each i. Furthermore, $\mathcal{A}_C \subset B(C)$ because all p_n are continuous with respect to the product topology on C (which is the weakest topology for which all p_n are continuous).

Since the topological product of completely regular spaces is completely regular, it follows that C enjoys complete regularity. It is not hard to verify that ν^i is inner regular set function.

The completely regular topological spaces are characterized by the fact, that they can be embedded into a compact space as a dense set ($\hat{C}$ech-Stone compactification). Let I be the one-to-one function, which embeds C into a K compact space, and let $\nu_K^i = \nu^i \circ I^{-1}$ be a set function on $\mathcal{A}_K$, the subsets of K, which are defined by $\mathcal{A}_K = \{X \subseteq K | I^{-1}(X) \in \mathcal{A}_C\}$. The direct corollary of this definition that, ν_K^i is inner regular, therefore (inner) compact regular as well.

As is well known, if an additive set function is compact regular, then it is σ-additive as well. Hence, ν_K^i is σ-additive. On the other hand, C contains the support of ν_K^i, and ν^i is the restriction of ν_K^i on C, hence ν^i is σ-additive as well.

Consequently, ν^i is σ-additive on $\mathcal{A}_C$ $\forall i$.

Remark 4 The role of compact regularity in the proofs of existence theorems of measure projective limit is twofold. First, compact regularity ensures σ-additivity. On the other hand, every compact regular measure can uniquely be extended from Baire sets to Borel sets. This later proves to be very important in the case of stochastic processes (the measurability of the sample function), but it is not relevant in our problem. We do not want to introduce events into our model that cannot be deduced directly by probabilistic logic.

The next theorem is our main result.

Theorem 1 *T^i is universal type space, so there exists a homeomorphism $f : T^i \to (\Delta(\mathcal{A}_T), \tau_p)$, where $(\Delta(), \tau_p)$ means the pointwise topology on $\Delta()$.*

The proof of the theorem is basically divided into two parts.

Definition 6 *Let $g : \Delta(\mathcal{A}_T) \to T^i$ that associates with every measure μ a point $t = (\mu_1^i, \mu_2^i, \ldots, \mu_n^i, \ldots)$ in T^i, where*

$$\mu_n^i = marg_{T_{n-1}} \mu$$

for every integer n.

Lemma 1 *Let $(M, \mathcal{A}_M, \mu_M), (N, \mathcal{A}_N, \mu_N)$ be probability measure spaces, and let μ be an additive set function on $\mathcal{A}_M \otimes \mathcal{A}_N$, and let p_M and p_N denote the coordinate projections. If $\mu \circ p_M^{-1} = \mu_M$ and $\mu \circ p_N^{-1} = \mu_N$, then μ is σ-additive on the field $\mathcal{A}$ generated by the cylinder sets.*

Proof. It is easy verify that every element of $\mathcal{A}$ has the form $\cup_j M_j \times N_j$, where $j < \infty$, $M_j \in \mathcal{A}_M$, $N_j \in \mathcal{A}_N$. It is well known ([7]) that, μ is σ-additive on $\mathcal{A}$ iff for a sequence $A_{n+1} \subseteq A_n$, $\cap_n A_n = \emptyset \Longrightarrow \lim_{n\to\infty} \mu(A_n) = 0$. For every finite intersection $\cap_n A_n = \cup_j (M_j \times N_j)$, for a finite set of indices j. Therefore, if the countable intersection $\cap_n A_n = \emptyset$, then the corresponding $M_j \times N_j = \emptyset$. Let us divide the sets $M_j \times N_j$ into two groups. Let the first group contain those products $M_j \times N_j$ where $M_j = \emptyset$, and let the second contain the others. Let us take the union of the members of the first group, it has the form $\emptyset \times (\cup_j N_j)$. Similarly, the union of the elements of the second group can be expressed as $(\cup_j M_j) \times \emptyset$. We have $\mu(\emptyset \times (\cup_j N_j)) = \mu((\cup_j M_j) \times \emptyset) = 0$, from the additivity of μ, $\mu(\emptyset \times (\cup_j N_j)) + \mu((\cup_j M_j) \times \emptyset) = \mu(\emptyset)$, which implies $\lim_{n\to\infty} \mu(A_n) = 0$, hence μ is σ-additive on $\mathcal{A}$.

Lemma 2 *g is a bijection.*

Proof. First we show that g is injective. If $\mu \in \Delta(\mathcal{A}_T)$ is given, then μ determines its marginals, in other words, it determines a unique point in T^i.

Now we verify that g is onto. Let a point $t \in T^i$ be given. From Proposition 1 and 2 we have that $\mathcal{A}_S \times \mathcal{A}_C \subset \mathcal{A}_T$. Let us define $q_1 : (T, \mathcal{A}_T) \to (S, \mathcal{A}_S)$, and $q_2 : (T, \mathcal{A}_T) \to (C, \mathcal{A}_C)$ as coordinate projections. Define μ on the cylinder sets by the equalities:

$$\mu = \mu_1^i \circ q_1, \quad \text{and} \quad \mu = \nu^i \circ q_2$$

(see Definition 3 and Proposition 2). On the cylinder sets, μ and μ^i coincide (μ^i is taken from the projective limit, see Proposition 1) and μ^i is an additive set function, hence we can extend μ to the field generated by the cylinder sets, in the way that, μ and μ^i coincide on this field. From Lemma 1 μ is σ-additive set function on this field, so it can be extended uniquely onto $\mathcal{A}_T$. We prove that $\mu = \mu^i$ on $\mathcal{A}_T$. Indeed, if there were an $A \in \mathcal{A}_T$ with $\mu(A) \neq \mu^i(A)$, then there would exist a k, and $B \in B(T_k^c)$ such that $A = p_k^{-1}(B)$. We know μ_{k+1}^i is σ-additive, hence $\mu = \mu^i$ on T_k^c, which is a contradiction. Thus, g is a bijection.

Definition 7 *Set $f = g^{-1}$.*

Lemma 3 *f is a homeomorphism.*

Proof. f is continuous ($t_k \xrightarrow{d_p} t \Longrightarrow f(t_k) \xrightarrow{p} f(t)$): $t_k \xrightarrow{d_p} t$ means $\forall l$, $\forall A_l \in B(T_l^c)$ $t_k^l(A_l) \to t^l(A_l)$, moreover $p_l^{-1}(A_l) \in \mathcal{A}_T$, and $f(t_k) \circ p_l^{-1}(A_l) = t_k^l(A_l)$, hence $f(t_k) \xrightarrow{p} f(t)$ on $\mathcal{A}_T$.

f^{-1} is continuous ($\mu_k \xrightarrow{p} \mu \Longrightarrow f^{-1}(\mu_k) \xrightarrow{d_p} f^{-1}(\mu)$): $\mu_k \xrightarrow{p} \mu$ on $\mathcal{A}_T$, which means the marginals of μ_k converge to μ pointwise, so $f^{-1}(\mu_k) \xrightarrow{d_p} f^{-1}(\mu)$.

Proof of the Theorem. Let f be defined by Definition 7.

From Lemma 2, f is a bijection.

From Lemma 3 f is a homeomorphism.

Remark 5 We proved the homeomorphism for $\mathcal{A}_T$, but not for $\sigma(\mathcal{A}_T)$, because the homeomorphism is not valid in the latter case. Our theorem can be extended to the $\sigma(\mathcal{A}_T)$, if the structure of $\sigma(\mathcal{A}_T)$ is induced by the pointwise convergence topology on $\mathcal{A}_T$.

Remark 6 This Theorem shows the importance of pointwise convergence topology. If T is a topological space, then the weak or weak* topology is less then our structure on $\Delta(\sigma(\mathcal{A}_T))$.

4 Conclusion

The main advantage of this model comes from the pointwise convergence topology on beliefs, that is independent of the topology of the original space. This space is a completely regular topological space, so we can use Kolmogorov's Existence Theorem in a general form (Proposition 2, Theorem 1).

Let us see an example for the usage of this model.

Example 1 Let there be two players, every player has two strategies. This game in normal form is a point in $\mathbb{R}^8$. There are two random variables, which determine the payoffs of the players. Therefore, the parameter space: $S = \mathbb{R}^{8^{\mathbb{R}^2}}$ (the parameters are functions from $\mathbb{R}^2$ to $\mathbb{R}^8$). S is not compact, nor Polish, so Mertens and Zamir's and Brandenburger and Dekel's construction do not work in this case. Let the measurable structure of S be the Borel sets of S. In our model, the opinions are the probability measures on S, but these are not necessarily compact regular, so Heifetz's, Mertens, Sorin and Zamir's models are less general, than ours.

It seems that, our model performs better, than the previous ones. On the other hand, recently, Simon [13] showed that, there may be problem with the existence of measurable equilibrium of the games with incomplete information. Hence, a model, in which , the beliefs of the players are modeled by probability measures, is not necessarily appropriate for some problems.

We think the existence of measurable equilibrium is out of the scope of our paper, hence we refer to this problem as an open problem in general, so in the case of our model as well.

Appendix: Definition of measure projective limit

We define the idea of projective limit of measure spaces for completeness. Rao's [11] definition is a little bite different from ours.

Definition 8 *Let* $(((M_n, \mathcal{M}_n, \mu_n), p_{mn})_{m<n}, I)$ *be a projective system, where* $(M_n, \mathcal{M}_n, \mu_n)$*s are measure spaces,* p_{mn}*s are the measurable projections, and I is a directed set. The projective limit of* $(((M_n, \mathcal{M}_n, \mu_n), p_{mn})_{m<n}, I)$ *is* $(M, \mathcal{M}, \mu) = \varprojlim(((M_n, \mathcal{M}_n, \mu_n), p_{mn})_{m<n}, I)$*, where*

- $pr_n : \times_n M_n \to M_n$ *coordinate projection,*
- $M = \{\omega \in \times_n M_n | pr_m(\omega) = p_{mn} \circ pr_n(\omega), \forall m < n \in I\}$,
- $p_n = pr_n|_M$,
- $\mathcal{M} = \cup_n \Sigma_n$*, where* $\Sigma_n = \{p_n^{-1}(A) | A \in \mathcal{M}_n\}$,
- μ *is on* $\mathcal{M}$*, defined by the equality* $\mu \circ p_n^{-1} = \mu_n$ $\forall n$*, and it is unique.*

The main difference between our and Rao's definition is in the properties of μ. Rao recommends μ to be σ-additive, we do not. Our definition makes the discussion more clear.

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