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Economic Theory

Comparative statics of properness in two-moment decision models

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Summary. Properness has been introduced in the expected utility framework and it recently has been transfered to mean-variance utility functions. Here, we show that properness implies the slope of the mean-standard deviation indifference curve being convex in the standard deviation. This indifference curve property allows us to characterize the comparative static effects of changing the background risk and dependency structure in a simple portfolio choice model.

Keywords and Phrases: Properness, Mean-standard deviation, Comparative statics.

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1 Introduction

The mean-standard deviation (μ, σ) and mean-variance (μ, v) approach have received increased attention in recent years (Lajeri and Nielsen [10], Lajeri-Chaherli [8], Wagener [16, 17], and Eichner and Wagener [3, 4]). The above mentioned authors translate concepts developed in expected utility models such as prudence, temperance, risk vulnerability, properness or standardness into the two-moment framework.

Mean-variance approach and expected utility approach are, in general, two different models under risks. There is a bulk of literature providing pros and cons for both models or studying the conditions under which both frameworks are compatible.[1] We do not contribute to this discussion but rather consider the mean-variance apprach as a model which stands on its own and bear in mind that our comparative

[1] For a survey of this literature we refer to Lajeri-Chaherli [9].

static results also hold in the expected utility approach if and only if the multivariate risks are assumed to be jointly elliptically symmetric distributed (Chamberlain [1]).

Starting point of this note is Lajeri-Chaherli [8] who demonstrates that Pratt's and Zeckhauser's [12] properness is equivalent to the mean-variance utility function being quasi-concave and displaying decreasing absolute risk aversion. Here, we show that properness is sufficient for the (σ, μ)-indifference curve slope being convex in σ. We undertake a detour via the convexity of the (σ, μ)-indifference curve slope in σ since this property allows us to characterize some comparative static effects in a straightforward manner. For example, in a portfolio model with one safe, and one risky asset and with a background risk (Ormiston and Schlee [11], Wagener [17]) the convexity of the indifference curve slope in σ, and thus properness, is sufficient for an agent to reduce his investment in the risky asset if the background risk increases or if the covariance of the risk sources increases.

2 Notation and preliminaries

Consider an individual whose preferences over lotteries are represented by a utility function $U : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, U = U(\sigma, \mu)$, where σ and μ are, respectively, the standard deviation and mean of a random variable. The linkages between mean-standard deviation utility function $U(\sigma, \mu)$ and mean-variance utility function $W(v, \mu)$ are specified by

$$U(\sigma, \mu) \equiv W(v, \mu) \tag{1}$$

with $v = \sigma^2$. We assume that the functions U and W are at least four times continuously differentiable with[2]

$$U_\mu(\sigma, \mu) = W_\mu(v, \mu) > 0, \tag{2a}$$

$$U_\sigma(\sigma, \mu) = 2\sigma \cdot W_v(v, \mu) < 0. \tag{2b}$$

Clearly, (2a) and (2b) reflect risk aversion and imply that indifference curves in (σ, μ)-space and in (v, μ)-space are upward sloping. Next we introduce

$$\alpha(\sigma, \mu) := -\frac{U_\sigma(\sigma, \mu)}{U_\mu(\sigma, \mu)} = -2\sigma \cdot \frac{W_v(v, \mu)}{W_\mu(v, \mu)} = 2\sigma \cdot \beta(v, \mu) > 0 \tag{2c}$$

where $\beta(v, \mu) := -W_v(v, \mu)/W_\mu(v, \mu)$. We denote by $\alpha(\sigma, \mu)$ $[\beta(v, \mu)]$ the marginal rate of substitution between σ $[v]$ and μ. Graphically, $\alpha(\sigma, \mu)$ and $\beta(v, \mu)$ measure the slope of (σ, μ)-indifference curves and (v, μ)-indifference curves, respectively. Ormiston and Schlee [11] identified $\alpha(\sigma, \mu)$ as the two-parameter analogue of the Arrow-Pratt concept of absolute risk aversion.

Following Tobin [15, p. 78] we impose that (σ, μ)-indifference curves hit the μ-axis with slope zero, i.e.

$$\alpha(0, \mu) = 0. \tag{2d}$$

[2] Subscripts denote partial derivatives.

Finally, indifference curve slopes are assumed to be decreasing in μ:

$$\alpha_\mu(\sigma,\mu) = 2\sigma \cdot \beta_\mu(v,\mu) < 0, \tag{3}$$

which diplays the concept of decreasing absolute risk aversion (DARA), see also Ormiston and Schlee [11].

3 Properness

Pratt and Zeckhauser [12] develop the concept of proper risk aversion or *properness*. This concept formalizes the idea that an undesirable lottery can never be made desirable by the presence of an independent undesirable lottery. Lajeri-Chaherli [8] transfers properness to the mean-variance approach and proves that[3] $W(v,\mu)$ *is proper if and only if it is quasi-concave and* $\beta(v,\mu)$ *is increasing in* μ (Lajeri-Chaherli [8, Proposition 3]).

To establish our result on properness and the indifference curve in the (σ,μ)-space, we need the definition of quasi-concavity. A necessary condition for W to be quasi-concave is that the bordered Hessian determinant[4]

$$B := \begin{vmatrix} 0 & W_\mu & W_v \\ W_\mu & W_{\mu\mu} & W_{\mu v} \\ W_v & W_{v\mu} & W_{vv} \end{vmatrix} = -W_\mu^2 \cdot W_{vv} + 2 \cdot W_\mu \cdot W_v \cdot W_{\mu v} - W_v^2 \cdot W_{\mu\mu} \tag{4}$$

is non-negative (Takayama [14, p. 127 Theorem 1.E.14]). Manipulation of (4) leads to

$$B = \frac{U_\mu^3}{4 \cdot \sigma^2} \cdot \left(\alpha_\sigma - \frac{\alpha}{\sigma} + \alpha \cdot \alpha_\mu\right). \tag{5}$$

Proof of (5). Differentiation of (1) yields

$$W_{\mu\mu} = U_{\mu\mu}, \qquad W_{\mu v} = \frac{U_{\mu\sigma}}{2 \cdot \sigma}, \qquad W_{vv} = \frac{1}{4 \cdot \sigma^2}\left(U_{\sigma\sigma} - \frac{U_\sigma}{\sigma}\right) \tag{6}$$

and differentiating (2c) we get

$$\alpha_\mu = -\frac{U_{\sigma\mu} + \alpha \cdot U_{\mu\mu}}{U_\mu}, \qquad \alpha_\sigma = -\frac{U_{\sigma\sigma} + \alpha \cdot U_{\mu\sigma}}{U_\mu}. \tag{7}$$

Next, we use (4) and (6) to obtain

$$B = \frac{U_\mu^2}{4 \cdot \sigma^2} \cdot \left[-\left(U_{\sigma\sigma} - \frac{U_\sigma}{\sigma}\right) + \frac{2 \cdot U_\sigma \cdot U_{\mu\sigma}}{U_\mu} - \frac{U_\sigma^2 \cdot U_{\mu\mu}}{U_\mu^2}\right] \tag{8}$$

[3] It should be noted that Lajeri-Chaherli uses the phrasing "$W(v,\mu)$ exhibits decreasing risk aversion" instead of "$\beta(v,\mu)$ is increasing in μ" . On page 50 she points out that decreasing risk aversion is equivalent to the indifference curve slope being decreasing in μ (or formally $\beta_\mu(v,\mu) < 0$) so that Lajeri-Chaherli's definition of decreasing risk aversion is equivalent to our definition of decreasing *absolute* risk aversion (see (3)).

[4] For notational convenience we suppress the arguments of the functions $U(\sigma,\mu)$, $W(v,\mu)$, $\alpha(\sigma,\mu)$ and $\beta(v,\mu)$.

which can be rearranged with the help of $\alpha = -U_\sigma/U_\mu$ to read

$$B = \frac{U_\mu^2}{4 \cdot \sigma^2} \cdot \left[-U_{\sigma\sigma} - \alpha \cdot U_{\sigma\mu} + \frac{\alpha \cdot U_\mu}{\sigma} - \alpha \cdot (U_{\sigma\mu} + \alpha \cdot U_{\mu\mu}) \right]. \tag{9}$$

Finally, we use (7) to obtain (5). □

In (5), $U_\mu > 0$ due to (2a) and for utility functions satisfying DARA we have $\alpha \cdot \alpha_\mu < 0$, as noted in (2c) and (3). Thus a necessary condition for $B \geq 0$ is $\alpha_\sigma - \alpha/\sigma > 0$. Since $\alpha(0, \mu) = 0$ for all μ, compare (2d), it is easy to verify[5]

$$\alpha_\sigma - \frac{\alpha}{\sigma} > 0 \quad \Longleftrightarrow \quad \alpha_{\sigma\sigma} > 0. \tag{10}$$

Now we are in the position to relate the properness-concept to the marginal rate of substitution between σ and μ.

Proposition 1. *The marginal rate of substitution $\alpha(\sigma, \mu)$ is increasing in μ and convex in σ if the mean-variance utility function $W(v, \mu)$ displays properness.*

Phrased geometrically, $\alpha_{\sigma\sigma} > 0$ says that the slope of the indifference curve is convex in σ.

4 Application: portfolio choice

In this section we illustrate the relevance of properness in comparative static analysis. For that purpose we consider the portfolio model with one safe and one risky asset which was initially analyzed by Fishburn and Porter [5] and recently discussed by Ormiston and Schlee [11]. In addition, we introduce a background risk as in Wagener [17]. To be more specific, final wealth y is given by

$$y(q) = w + q \cdot z + \epsilon \tag{11}$$

where w is the initial wealth (safe asset), z is the return of the risky asset, q is the amount invested in the risky asset and ϵ is the background risk. Then the agent's optimization problem can be written as

$$\begin{aligned} \max_q U(\sigma_y(q), \mu_y(q)) \quad \text{s.t.} \quad \sigma_y(q) &= \sqrt{q^2\sigma_z^2 + \sigma_\epsilon^2 + 2 \cdot q \cdot \mathrm{Cov}(z, \epsilon)}, \\ \mu_y(q) &= \mu_w + q \cdot \mu_z + \mu_\epsilon \end{aligned} \tag{12}$$

where the covariance of z and ϵ, written $\mathrm{Cov}(z, \epsilon)$, characterizes the dependency structure between the sources of randomness. The optimal investment amount $q^* > 0$ is determined by the first-order condition

$$C := \mu_z - \alpha(\mu_y(q^*), \sigma_y(q^*)) \cdot \frac{q^* \cdot \sigma_z^2 + \mathrm{Cov}(z, \epsilon)}{\sigma_y(q^*)} = 0. \tag{13}$$

[5] See Eichner and Wagener [3, Lemma 1] for a proof of a similar equivalence.

We assume that the second-order condition is satisfied everywhere, i.e. $C_q < 0$. To ensure the existence of an interior solution we assume

$$q \cdot \sigma_z^2 + \mathrm{Cov}(z, \epsilon) > 0. \tag{14}$$

Turning to comparative statics, we implicitly differentiate (13) with respect to σ_ϵ and $\mathrm{Cov}(z, \epsilon)$ to obtain

$$\frac{\partial q^*}{\partial \sigma_\epsilon} = -\frac{C_{\sigma_\epsilon}}{C_q} = \frac{1}{C_q} \cdot \left(\alpha_\sigma - \frac{\alpha}{\sigma_y(q)}\right) \cdot \frac{q \cdot \sigma_z^2 + \mathrm{Cov}}{\sigma_y(q)} \cdot \frac{\partial \sigma_y(q)}{\partial \sigma_\epsilon}, \tag{15a}$$

$$\frac{\partial q^*}{\partial \mathrm{Cov}} = -\frac{C_{\mathrm{Cov}}}{C_q} = \frac{1}{C_q} \cdot \left[\left(\alpha_\sigma - \frac{\alpha}{\sigma_y(q)}\right) \cdot \frac{q \cdot \sigma_z^2 + \mathrm{Cov}}{\sigma_y(q)} \cdot \frac{\partial \sigma_y(q)}{\partial \mathrm{Cov}} + \frac{\alpha}{\sigma_y(q)}\right] \tag{15b}$$

where

$$\frac{\partial \sigma_y}{\partial \sigma_\epsilon} = \frac{\sigma_\epsilon}{\sigma_y(q)} > 0, \qquad \frac{\partial \sigma_y}{\partial \mathrm{Cov}} = \frac{q}{\sigma_y(q)} > 0. \tag{16}$$

In light of (10) and Proposition 1, (14)-(16) give rise to[6]

Proposition 2.

(a) The optimal level q^* decreases upon an increase in the background risk σ_ϵ if and only if the marginal rate of substitution $\alpha(\sigma, \mu)$ is convex in σ.

(b) The optimal level q^* decreases upon an increase in the covariance $\mathrm{Cov}(z, \epsilon)$ if the marginal rate of substitution $\alpha(\sigma, \mu)$ is convex in σ.

Corollary 1.

(a) The optimal level q^* decreases upon an increase in the background risk σ_ϵ if the mean-variance utility function $W(v, \mu)$ displays properness.

(b) The optimal level q^* decreases upon an increase in the covariance $\mathrm{Cov}(z, \epsilon)$ if the mean-variance utility function $W(v, \mu)$ displays properness.

In the (μ, σ)-approach the convexity of $\alpha(\sigma, \mu)$ in σ is a simple sufficient condition for background risk and dependency structure to reduce the amount of investment. Moreover, due to Proposition 2(a) it is even necessary for $\partial q^*/\partial \sigma_\epsilon < 0$.[7] Proposition 1 allows an easy link to properness and thus Corollary 1 identifies properness as sufficient condition for both $\partial q^*/\partial \sigma_\epsilon < 0$ and $\partial q^*/\partial \mathrm{Cov} < 0$.

Presupposing that the mean-variance approach is in accordance with the expected utility approach, Wagener [17, Fact 6] shows that standardness (Kimball [7]), the combination of DARA and decreasing absolute prudence, is sufficient for $\partial q^*/\partial \sigma_\epsilon < 0$ if final wealth $y(q)$ is normally distributed. However, from Gollier and Pratt [6] we know that standardness is sufficient for properness such that Wagener's Fact 6 can be extended to elliptically symmetric distributions.

[6] For the special case of independent background risks Proposition 2(a) is established in Eichner and Wagener [4].

[7] It should be noted that Proposition 2(a) and Corollary 1(a) hold both for independent and dependent background risks. To get this comparative static effect we have to ensure that (14) is satisfied and that increasing σ_ϵ does not alter the dependency structure measured in terms of $\mathrm{Cov}(z, \epsilon)$.

Corollary 2. *Suppose that $y(q)$ is elliptically symmetric distributed, then*

(a) the optimal level q^* decreases upon an increase in the background risk σ_ϵ if the mean-variance utility function $W(v, \mu)$ displays standardness;

(b) the optimal level q^* decreases upon an increase in the covariance $\mathrm{Cov}(z, \epsilon)$ if the mean-variance utility function $W(v, \mu)$ displays standardness.

5 Concluding remarks

We identify the convexity of the indifference curve slope in σ, properness and standardness as sufficient conditions for some unambiguous comparative static effects in the portfolio choice model. We could have demonstrated analogous effects in Sandmo's [13] model of firm behavior under price uncertainty or in Ehrlich's and Becker's [2] model of insurance demand, since the setting of the aforementioned models is similar to (11).

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