Economic Theory 25, 703–710 (2005)
DOI: 10.1007/s00199-003-0454-4

Economic Theory

Imperfect recall and the relationships between solution concepts in extensive games

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Received: October 22, 2003; revised version: November 18, 2003

Summary. In a game of imperfect recall, a sequential equilibrium may not be a Nash equilibrium, and a perfect equilibrium may not be a sequential equilibrium. Sufficiency conditions weaker than perfect recall are given to ensure the standard relationships hold between perfect equilibrium, sequential equilibrium and Nash equilibrium.

Keywords and Phrases: Memory, Imperfect recall, Extensive games, Solutions.

JEL Classification Numbers: C72.

1 Introduction

Piccione and Rubinstein [7] showed a divergence between a time-consistent strategy and an optimal strategy in a one player game with imperfect recall. On the other hand, they showed that equivalence between these different optimality concepts is maintained if the player satisfies a memory condition known as occurrence memory (Okada [6]). All players with perfect recall have occurrence memory and some players without perfect recall have it. Kline [2] showed that equivalence is maintained to a weaker memory condition known as a-loss recall (Kaneko and Kline [1]).

In this note, I look at some implications of imperfect recall for the relationships between solution concepts in n-player games. In an n-player context, optimal strategies are replaced by the notion of a Nash equilibrium, and the notion of a time-consistent strategy combination is an appropriate generalization of a time-consistent strategy. Time consistent strategy combinations and sequential equilibria are closely related in much the same way that Nash equilibria and perfect equilibra are related. Every sequential equilibrium is a time consistent strategy combination, and every perfect equilibrium is a Nash equilibrium. The main difference between the concepts is that sequential equilibrium and perfect equilibrium restrict behavior at unreached information sets, while time-consistent strategy combinations and Nash equilibria do not. Because of these close relationships, and the divergence results for time-consistent and optimal strategies in one player games, one might wonder how far the standard relationships between perfect equilibrium, sequential equilibrium, and Nash equilibrium can be maintained for n-player games of imperfect recall.

We show that every perfect equilibrium is a sequential equilibrium and every sequential equilibrium is a Nash equilibrium provided every player has occurrence memory (Theorem 1(a)). While the standard relationship between sequential equilibrium and Nash equilibrium is maintained to players with a-loss recall (Theorem 1(b)), the standard relationship between perfect and sequential equilibrium is not (Fig. 1).

2 Extensive games and solution concepts

We follow Selten's [8] definition of a *finite extensive game* $\Gamma = ((K, P, U, C, p, h)$. The *chance player* (*nature*) is player 0, and $N = \{1, ..., n\}$ is the set of *personal players*. K is a *finite tree* partitioned into the set of *terminal nodes* denoted by Z and the set of *decision nodes* denoted by X. P is a *player partition*, and $U = \{U_0, U_1, ..., U_n\}$ is the *information pattern*. U_i is player *$i's$ information partition* and an element $u \in U_i$ is an *information set* of player i. U_0 is made up of singleton sets. C is a *choice partition* and C_u denotes the set of alternatives at an information set u.

For an information set u, a choice $c \in C_u$ and a node $x \in K$ we write $u \prec_c x$ iff $y \prec x$ for some $y \in u$ and c is the choice at y leading to x. For information sets u and v, we write $u \prec v$ iff $x \prec y$ for some $x \in u$ and some $y \in v$. Finally, p is a *probability assignment* to chance moves and h is a *payoff function* assigning a real vector $(h_1(z), ..., h_n(z))$ to each endnode $z \in Z$.

A *behavior strategy* of player i is a function b_i that assigns to each $u \in U_i$, a probability distribution b_{iu} over the set C_u of choices at u. We use B_i to denote the set of behavior strategies of player i, b_{iu} is a *local strategy,* and B_{iu} denotes the set of local strategies of player i at u. An n-tuple $b = (b_1, ..., b_n)$ of behavior strategies, one for each player, is called a *strategy combination*. We use (b'_i, b_{-i}) to denote the strategy combination obtained from b by replacing the behavior strategy b_i by b'_i. We also use (b'_{iu}, b_{-iu}) to denote the replacement of a local strategy in b.

For a strategy combination b, the *ex ante expected payoff* of player i is: $H_i(b) = \sum_{z \in Z} p(z, b) h_i(z)$, where $p(z, b)$ denotes the probability of reaching terminal node z when b is used. We say that b_i is *optimal* against b iff $H_i(b) \geq H_i(b'_i, b_{-i})$ for all $b'_i \in B_i$. A strategy combination $b = (b_1, ..., b_n)$ is a *Nash equilibrium* iff b_i is optimal against b for all $i \in N$.

A *system of beliefs* is a function μ on X satisfying: (a) $\mu(x) \in [0, 1]$ for all $x \in X$, and (b) $\sum_{x \in u} \mu(x) = 1$ for all $u \in \bigcup_{i=0}^{n} U_i$. An ordered pair (b, μ) where b is a strategy combination and μ is a system of beliefs is called an *assessment*. Given a strategy combination $b = (b_1, ..., b_n)$, a node $x \in X$, and a player $i \in N$, the *expected payoff of player i conditional on being at x* is: $H_{ix}(b) = \sum_{z \in Z_x} p(z \mid x, b) h_i(z)$ where $Z_x = \{z \in Z : x \prec z\}$ and $p(z \mid x, b)$ is the probability of reaching endnode z when currently at node x and b is used in the continuation of the game. For an assessment (b, μ) and an information set u belonging to player i, the *expected payoff of player i conditional on being at u* is denoted by $H_{iu}(b, \mu) = \sum_{x \in u} \mu(x) H_{ix}(b)$. An assessment (b, μ) is *sequentially rational at information set u* of personal player i iff $H_{iu}(b, \mu) \geq H_{iu}((b'_i, b_{-i}), \mu)$ for all $b'_i \in B_i$. An assessment (b, μ) is called *sequentially rational* iff (b, μ) is sequentially rational at each information set belonging to a personal player. An assessment (b, μ) is *consistent* iff there is a sequence of completely mixed strategy combinations $\{b^k\}_{k=1}^{\infty}$ satisfying both $\lim_{k \to \infty} b^k = b$, and for each u belonging to a personal player and each $x \in u$, $\mu(x) = \lim_{k \to \infty} \frac{p(x, b^k)}{\sum_{y \in u} p(y, b^k)}$.

An assessment (b, μ) is a *sequential equilibrium* iff (b, μ) is sequentially rational and consistent[3].

We give an n-person version of Piccione and Rubinstein's time-consistency in order to compare it to sequential equilibrium and Nash equilibrium. A strategy combination $b = (b_1, ..., b_n)$ is *time-consistent* iff for all $i \in N$, if $u \in U_i$ and $p(x, b) > 0$ for some $x \in u$, then $H_{iu}(b, \mu) \geq H_{iu}((b'_i, b_{-i}), \mu)$ for all $b'_i \in B_i$, where μ is any system of beliefs that satisfies $\mu(x) = \frac{p(x,b)}{\sum_{y \in u} p(y,b)}$ for all $x \in u$.

A *perturbed game* $\Gamma_\varepsilon = (\Gamma, \varepsilon)$ is a pair such that Γ is a finite extensive game and ε is a function assigning a minimum probability $\varepsilon_c > 0$ to each choice c at each personal information set u, and ε must satisfy the further restriction that for all personal information sets u, $\sum_{c \in C_u} \varepsilon_c < 1$. A behavior strategy $b_i \in B_i$ is *permissible* in the perturbed game Γ_ε iff at each $u \in U_i$, $b_{iu}(c) \geq \varepsilon_c$ for each $c \in C_u$. We let $B_{i\varepsilon}$ denote the set of permissible strategies of player i in the perturbed game Γ_ε. A strategy combination $b = (b_1, ..., b_n)$ is a *Nash equilibrium of the perturbed game* Γ_ε iff for all $i \in N$, $H_i(b) \geq H_i(b'_i, b_{-i})$ for all $b'_i \in B_{i\varepsilon}$.

A strategy combination b is a *perfect equilibrium* in Γ iff there is a sequence of perturbed games $\{\Gamma^k\}_{k=1}^{\infty}$ of Γ, and a sequence of strategy combinations $\{b^k\}_{k=1}^{\infty}$ such that (i) for each k, b^k is a Nash equilibrium of the perturbed game Γ^k, and (ii) $\Gamma^k \to \Gamma$ and $b^k \to b$ as $k \to \infty$.

3 Extending results

It is well known that for a game of perfect recall, every perfect equilibrium is a sequential equilibrium and every sequential equilibrium is a Nash equilibrium. We show in Theorem 1 that these standard relationships can be extended to some region of imperfect recall.

The regions of imperfect recall we will extend the results to are already known in the literature. The first region is due to Okada[6].

Occurrence Memory: The information partition U_i of a player i satisfies *occurrence memory* iff for all $u, v \in U_i$, and all $x, y \in u$, if $v \prec x$ then $v \prec y$.

This condition is interpreted as requiring a player to recall everything he observed, though he might forget what he did. His imperfect recall can only be about his own actions.

The second condition is due to Kaneko and Kline [1].

A-loss Recall: The information partition U_i satisfies *a-loss recall* iff for all $u, v \in U_i$, all $x, y \in u$, and all $c \in C_v$, if $v \prec_c x$, then either: (1) $v \prec_c y$ or (2) there exists $w \in U_i$ and distinct $d, e \in C_w$ satisfying $w \prec_d x$ and $w \prec_e y$.

This condition is a weakening of occurence memory since it allows a player to forget both things he observed in addition to forgetting his past actions. However, it is still restrictive in that any forgetfulness of what he learned must be accompanied by some forgetfulness of his past actions.

We will say that a game Γ has occurrence memory or a-loss recall when the conditions hold for each player $i \in N$.

Theorem 1.

(a) Let Γ be an extensive game with occurence memory. If b is a perfect equilibrium, then (b, μ) is a sequential equilibrium for some system of beliefs μ.
(b) Let Γ be an extensive game with a-loss recall. If (b, μ) is a sequential equilibrium, then b is a Nash equilibrium.

The standard results from the literature on extensive games of perfect recall are extended in this theorem to some games of imperfect recall. The converses of parts (a) and (b) do not hold in games of perfect recall, and do not hold in games of imperfect recall either.

Part (a) of this theorem states that the standard relationship between perfect equilibrium and sequential equilibrium holds for games of imperfect recall that satisfy occurence memory.

The one player game of Figure 1 shows that this result cannot be extended to games with A-loss recall. Here all moves are by player 1 with the information partition $U_1 = \{w, v, u\}$ and the game satisfies A-loss recall. The only perfect equilibrium is $b_{1w}(b) = b_{1v}(c) = b_{1u}(R) = 1$. This strategy combination is not part of a sequential equilibrium, however, since it is not sequentially rational at v for any beliefs.

Part (b) of this theorem states that the standard relationship between sequential equilibrium and Nash equilibrium can be extended to games of imperfect recall that satisfy A-loss recall.

When we move beyond A-loss recall, however, a sequential equilibrium may not be a Nash equilibrium. Consider an altered version of the game of Figure 1 where the root information set w belongs to a chance player who takes each action with probability 1/2, and the right-most payoff is reduced from 4 to 2. The only sequential equilibrium in this altered game is: $b_{1v}(c) = b_{1u}(R) = 0, \mu(x') = 1$, and $\mu(x) = 1/2$. This is not a Nash equilibrium, since it gives an expected payoff of $3/2$, while the alternative strategy $b'_{1v}(c) = 1$, $b'_{1u}(R) = 1$ yields an expected payoff

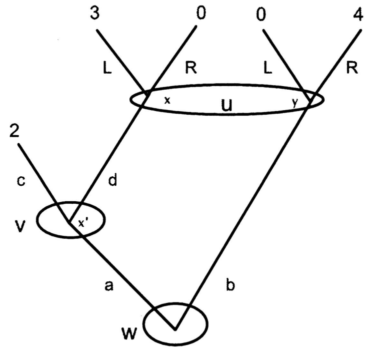

Figure 1

of 2. This example points out, or reminds us, that while a sequential equilibrium requires a player to maximize for the remainder of the game, there is no requirement that he maximize for the whole game.

Because of this finding, one might wonder if the relationship that every perfect equilibrium is a Nash equilibrium ever fails. Fortunately, Selten's proof ([8], first Lemma 3) that every perfect equilibrium is a Nash equilibrium did not use perfect recall, and thus every perfect equilibrium is a Nash equilibrium with or without perfect recall.

The one player game of Figure 1 is actually an example where a sequential equilibrium does not exist in a one-player game. Are there also one player games where a perfect equilibrium or a Nash equilibrium does not exist? The answer is no. Selten's [8] proof of the existence of a perfect equilibrium only used perfect recall to show existence of a Nash equilibrium in every perturbed game. Since a Nash equilibrium exists in every perturbed one player game, a perfect equilibrium exists in every one player game. By the above remark that every perfect equilbrium is a Nash equilibrium, we also have the existence of a Nash equilibrium in every one-player game.

We end this section with the proof of Theorem 1. Part (b) of the theorem is proved by the combination of the following two lemmas.

Lemma 1. *If* (b, μ) *is a sequential equilibrium, then* b *is time-consistent.*

Proof. Suppose that (b, μ) is a sequential equilibrium. Let u be an arbitrary information set of a personal player i. Suppose that $p(x, b) > 0$ for some $x \in u$. Consistency of (b, μ) implies that $\mu(x) = \frac{p(x,b)}{\sum\limits_{y \in u} p(y,b)}$. Sequential rationality of (b, μ) implies that at u, $H_{iu}((b), \mu) \geq H_{iu}((b'_i, b_{-i}), \mu)$ for all $b'_i \in B_i$. Since u was chosen arbitrarily, b is time-consistent. □

Lemma 2. *Let* Γ *be a game with a-loss recall. If* b *is time-consistent, then* b *is a Nash equilibrium.*

Lemma 2 is proved by simply applying one side of Kline's Theorem 1 [2] to each player.

Proof of Theorem 1. (a) Suppose that Γ satisfies occurence memory and b is a perfect equilibrium. There is a sequence of perturbed games $\{\Gamma^k\}$ of Γ, and a sequence of strategy combinations $\{b^k\}$ such that: (i) for each k, b^k is a Nash equilibrium of the perturbed game Γ^k, and (ii) $\Gamma^k \to \Gamma$ and $b^k \to b$ as $k \to \infty$.

For each $x \in X$, the sequence $\left\{ \frac{p(x,b^k)}{\sum_{y \in u} p(y,b^k)} \right\}$, where u is the information set contatining x, is bounded since each element of the sequence lies between zero and one. Thus, we can find a subsequence $\{b^{k_n}\}$ of $\{b^k\}$ such that for each $x \in X$, the belief $\mu(x) = \lim_{k_n \to \infty} \frac{p(x,b^{k_n})}{\sum_{y \in u} p(y,b^{k_n})}$ exists. We focus on this subsequence in the remainder of the proof. By the definition of μ, the pair (b, μ) is consistent. It suffices now to show that (b, μ) is sequentially rational.

Let u be an arbitrary information set of personal player i. For any perturbed game Γ^{k_n}, since b^{k_n} is a Nash equilibrium in Γ^{k_n}, it follows that:

$$H_i(b^{k_n}) \geq H_i(b'_i, b^{k_n}_{-i}) \text{ for all } b'_i \in B^{k_n}_i. \tag{3.1}$$

Here, $B^{k_n}_i$ denotes the set of permissible strategies of player i in Γ^{k_n}. If we let $Z_u = \{z \in Z : u \prec z\}$ and $Z_{-u} = Z - Z_u$, then we can rewrite (3.1) as follows:

$$\sum_{z \in Z_{-u}} p(z, b^{k_n}) h_i(z) + \sum_{x \in u} p(x, b^{k_n}) H_{ix}(b^{k_n}) \geq$$

$$\sum_{z \in Z_{-u}} p(z, (b'_i, b^{k_n}_{-i})) h_i(z) + \sum_{x \in u} p(x, (b'_i, b^{k_n}_{-i})) H_{ix}(b'_i, b^{k_n}_{-i}),$$

$$\text{for all } b'_i \in B^{k_n}_i. \tag{3.2}$$

Consider any strategy $b'_i \in B^{k_n}_i$ that coincides with $b^{k_n}_i$ everywhere except possibly at u and on $S(u) = \{v \in U_i : u \prec v\}$. Let $B^{k_n}_i(u, S(u))$ denote the set of all such strategies. By occurrence memory, it follows that for any $b'_i \in B^{k_n}_i(u, S(u))$, we have $\sum_{z \in Z_{-u}} p(z, (b'_i, b^{k_n}_{-i})) h_i(z) = \sum_{z \in Z_{-u}} p(z, b^{k_n}) h_i(z)$, and $p(x, (b'_i, b^{k_n}_{-i})) = p(x, b^{k_n})$ for all $x \in u$. By this we obtain from (3.2) that:

$$\sum_{x \in u} p(x, b^{k_n}) H_{ix}(b^{k_n}) \geq \sum_{x \in u} p(x, b^{k_n}) H_{ix}(b'_i, b^{k_n}_{-i}) \text{ for all } b'_i \in B^{k_n}_i(u, S(u)). \tag{3.3}$$

Now if we define $\mu(x, b^{k_n}) = \frac{p(x,b^{k_n})}{\sum_{y \in u} p(y,b^{k_n})}$ at each $x \in u$, and use the fact that $\sum_{y \in u} p(y, b^{k_n}) > 0$ since this is a perturbed game, then we find that (3.3) is equivalent to:

$$\sum_{x \in u} \mu(x, b^{k_n}) H_{ix}(b^{k_n}) \geq \sum_{x \in u} \mu(x, b^{k_n}) H_{ix}(b'_i, b^{k_n}_{-i}) \text{ for all } b'_i \in B^{k_n}_i(u, S(u)). \tag{3.4}$$

Now take the limit as $k_n \to \infty$ to obtain from (3.4) by continuity of the payoff function $H_{ix}(\cdot)$, and the definition of $\mu(x)$ that:

$$\sum_{x\in u}\mu(x)H_{ix}(b) \geq \sum_{x\in u}\mu(x)H_{ix}(b'_i, b_{-i}) \text{ for all } b'_i \in B_i(u, S(u)). \tag{3.5}$$

Observe that for any node $x \in u$, $H_{ix}(b'_i, b_{-i}) = H_{ix}(b''_i, b_{-i})$ for any two strategies b'_i and b''_i that agree on u and $S(u)$. Hence, from (3.5) we obtain:

$$\sum_{x\in u}\mu(x)H_{ix}(b) \geq \sum_{x\in u}\mu(x)H_{ix}(b'_i, b_{-i}) \text{ for all } b'_i \in B_i. \tag{3.6}$$

Since u was chosen arbitrarily, (b, μ) is sequentially rational.

(b) Suppose that Γ satisfies A-loss recall and (b, μ) is a sequential equilibrium. By Lemma 1, b is time-consistent. Since the game satisfies A-loss recall, b is also a Nash equilibrium by Lemma 2. □

4 Conclusion

We looked at the implications of imperfect recall for the standard relationships between solution concepts in an n-player extensive game. The motivation was the divergence results of Piccione and Rubinstein[7] for time-consistent and optimal strategies in one-player games of imperfect recall. On the positive side, we found in Theorem 1 that for games of imperfect recall with occurence memory, all the standard relationships between perfect equilibrium, sequential equilibrium, and Nash equilibrium hold. The relationship between sequential equilibrium and Nash equilibrium was extended even further to all games that satisfy A-loss recall. On the negative side, however, if the forgetfulness of a player does not satisfy A-loss recall, then a perfect equilibrium might not be a sequential equlibrium, and a sequential equilibrium might not be a Nash equilbrium. The main source of the divergence between these solution concepts seems to be that sequential equilibrium is based on maximizing for the remainder of the game, while perfect equilibrium and Nash equilibrium are based on maximizing for the entire game.

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