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# A costly state verification model with diversity of opinions

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**Summary.** In simple models of borrowing and lending with *ex-post* asymmetric information, Gale and Hellwig (1985) and Williamson (1986) have shown that optimal debt contracts are simple debt contracts where borrowers repay a fixed interest rate whenever possible and lenders seize all the profit when borrowers default. In this note, we depart from their works by assuming that borrowers and lenders have heterogeneous beliefs, and show that simple debt contracts do not necessarily survive as optimal contracts.

**Keywords and Phrases:** Costly state verification, Heterogeneity of beliefs.

**JEL Classification Numbers:** G3, D8.

## 1 Introduction

Consider a contracting problem between a borrower and a lender subject to a *costly state verification* (henceforth, CSV (see Townsend(1979)) problem, that is to say, the project return is costlessly observed by the borrower while the lender has to pay a monitoring cost in order to observe that return. Gale and Hellwig (1985) and Williamson (1986) have shown that optimal debt contracts among the class of incentive compatible and individually rational debt contracts are *simple debt contracts* where the borrower repays a fixed interest rate whenever possible and the lender seizes all the profit realized when the borrower defaults. Thus the CSV approach provides a rationale for debt-like financial contracts with costly bankruptcy such as debentures or corporate bonds.

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A common feature of the CSV approach is to maintain the common prior assumption. Borrowers and lenders share the same beliefs about the project profitability. The purpose of this note is to relax the common prior assumption in an otherwise standard CSV problem and to explore the optimality of simple debt contracts. We show that there may be no optimal contract in the class of simple debt contracts. We give a general first-order argument and some counter-examples.

The paper is organized as follows. In Section 2, we present a simple costly state verification model. In Section 3, we first, recall the result of Gale and Hellwig (1985) and Williamson (1986), and, second, state our main result, with some counter-examples. Finally, Section 4 concludes.

## 2 A simple costly state verification model

We consider a static two-period economy with a unique borrower and a unique lender. The lender is endowed with a single unit of a good that might be used for both investment and consumption while the borrower has no initial wealth. Moreover, they are assumed to be risk-neutral and to care only about second period consumption.

The borrower has access to an investment project requiring exactly one unit of the investment good to be undertaken. The project returns $\omega$ in the second period where $\omega$ is a realization of the random variable $\widetilde{\omega}$ with full support on $[0, \overline{\omega}]$. Furthermore, the project return $\omega$ is costlessly observable only to the entrepreneur while the lender has to pay an utility cost of $\gamma$ to perfectly monitor the return. The opportunity cost is set to $r$.

Finally, the borrower and the lender might differ in their beliefs about the project return. The borrower believes that the random variable $\widetilde{\omega}$ admits the probability density function $\mu$ while the lender believes in $\nu$. Density functions are common knowledge and continuous. In addition, we assume that the borrower believes the project profitable, that is $\int_0^{\overline{\omega}} \omega\mu(\omega)d\omega > r$.

Since the borrower has access to a high return project but does not have the wealth to undertake the project, he can propose a debt contract to the lender.

**Definition 1.** *A debt contract is a pair* $\left(\widehat{R}, M\right)$ *where*

- $\widehat{R} : [0, \overline{\omega}] \to \mathbb{R}_+,\ \omega \mapsto \widehat{R}(\omega)$ *is the repayment in state* $\omega$,
- $\widehat{R}(\omega) \leq \omega,\ \forall \omega \in [0, \overline{\omega}]$, *the limited liability constraint,*
- $M \subset [0, \overline{\omega}]$, *a subset of* $[0, \overline{\omega}]$ *where monitoring takes place.*

**Definition 2.** *A debt contract* $\left(\widehat{R}, M\right)$ *is truthtelling if and only if* $\forall \omega \in [0, \overline{\omega}]$, $\forall \omega' \notin M$,

$$\widehat{R}(\omega) \leq \widehat{R}(\omega').$$

A debt contract is thus said to be truthtelling when the borrower has no incentive to misreport the project return. Without loss of generality, we restrict ourselves to the class of debt contracts which satisfy the direct revelation mechanism (see Townsend

(1988)) and are truthtelling. Note that if $(\widehat{R}, M)$ is truthtelling then $\widehat{R}$ has to be constant on $[0, \overline{\omega}] \setminus M$, say with value $\overline{R}$, and obviously:

$$\left\{\widehat{R}(\cdot) < \overline{R}\right\} \subseteq M \subset \left\{\widehat{R}(\cdot) \leq \overline{R}\right\}.$$

Changing $(\widehat{R}, M)$ into the new truthtelling contract $(\widehat{R}, \{\widehat{R}(\cdot) < \overline{R}\})$ is therefore weakly Pareto improving since this transformation does not increase the expected monitoring cost. In other words, a truthtelling contract is simply determined by a pair $(R(\cdot), \overline{R})$ where the threshold $\overline{R}$ determines monitoring states $M = \{R(\cdot) < \overline{R}\}$ and $0 \leq R(\omega) \leq \omega$ for all $\omega \in [0, \overline{\omega}]$.

## 3 Simple debt contracts are not optimal in general

### *3.1 The borrower's program*

Given a contract $(R(\cdot), \overline{R})$, the borrower's expected payoff is

$$\int_{\{R(\cdot)<\overline{R}\}} (\omega - R(\omega))\mu(\omega)d\omega + \int_{\{R(\cdot)\geq\overline{R}\}} \left(\omega - \overline{R}\right) \mu(\omega)d\omega = \int_0^{\overline{\omega}} \omega\mu(\omega)d\omega - \int_0^{\overline{\omega}} \min\left(R(\omega), \overline{R}\right) \mu(\omega)d\omega,$$

while the lender's expected payoff is

$$\int_{\{R(\cdot)<\overline{R}\}} (R(\omega) - \gamma)\nu(\omega)d\omega + \overline{R} \int_{\{R(\cdot)\geq\overline{R}\}} \nu(\omega)d\omega = \int_0^{\overline{\omega}} \min\left(R(\omega), \overline{R}\right) \nu(\omega)d\omega - \gamma \int_{\{R(\cdot)<\overline{R}\}} \nu(\omega)d\omega.$$

Since the lender's outside option is $r$, a contract $\left(R(\cdot), \overline{R}\right)$ is accepted if the following participation constraint holds:

$$\int_0^{\overline{\omega}} \min\left(R(\omega), \overline{R}\right) \nu(\omega)d\omega - \gamma \int_{\{R(\cdot)<\overline{R}\}} \nu \geq r. \tag{1}$$

It follows that the borrower's program is given by

$$\inf_{\left(R(\cdot),\overline{R}\right)} F\left(R(\cdot), \overline{R}\right) := \int_0^{\overline{\omega}} \min\left(R(\omega), \overline{R}\right) \mu(\omega)d\omega \text{ subject to :} \\ 0 \leq R(\omega) \leq \omega, \text{ for all } \omega \in [0, \overline{\omega}], \left(R(\cdot), \overline{R}\right) \text{ satisfies (1)} \tag{2}$$

First, let us note that if $\left(R(\cdot), \overline{R}\right)$ solves (2) then the constraint (1) holds in equality. Indeed, if $\left(R(\cdot), \overline{R}\right)$ satisfies (1) with a strict inequality then the contract $(\left(\max(R(\cdot) - \varepsilon, 0), \overline{R} - \varepsilon\right)$ still satisfies (1) for small $\varepsilon > 0$ and strictly reduces the value of $F(\cdot, \cdot)$.

### *3.2 The common prior case*

The so-called *simple debt contracts* play a special role in both theory and practice. By definition, a simple debt contract is a truthtelling contract of the form $\left(\mathrm{id}, \overline{R}\right)$ with id the identity operator. Hence a simple debt contract specifies that the borrower repays a fixed "interest rate" $\overline{R}$ whenever possible and the lender seizes all the return $\omega$ when the borrower defaults (i.e., for $\omega \in M = [\,0, \overline{R}\,))$. It resembles corporate bonds or debentures. Assuming a *common prior* i.e., $\mu = \nu$, Gale and Hellwig (1985) and Williamson (1986) have shown that optimal contracts are necessarily simple debt contracts. To see this, start with a contract $\left(R\left(\cdot\right), \overline{R}\right)$ such that the set $\left\{\omega : R(\omega) < \omega\, , R(\omega) < \overline{R}\right\}$ has positive measure for $\mu = \nu$. Then there exists a simple debt contract $(\mathrm{id}, \overline{R}')$ with $\overline{R}' < \overline{R}$, and

$$\int_0^{\overline{\omega}} \min\left(\omega, \overline{R}'\right) \mu(\omega) d\omega = \int_0^{\overline{\omega}} \min\left(R(\omega), \overline{R}\right) \mu(\omega) d\omega,$$

so that $\left(\mathrm{id}, \overline{R}'\right)$ and $\left(R(\cdot), \overline{R}\right)$ yield the same utility to the borrower, but $\left(\mathrm{id}, \overline{R}\right)$ is strictly preferred by the lender since it implies smaller monitoring costs ($\mu\left(\left[0, \overline{R}'\right]\right) < \mu\left(\left[0, \overline{R}\right]\right) \leq \mu\left(\left\{R(\cdot) < \overline{R}\right\}\right)$). This proves that $\left(R(\cdot), \overline{R}\right)$ is not optimal. In the common prior case, optimal contracts have to minimize the expected monitoring cost, and thus to be simple debt contracts. If $\mu \neq \nu$ this particular property does not hold anymore.

### *3.3 Assumptions*

For the sake of tractability, we assume from now on that the belief of the lender is uniformly distributed on $[0, \overline{\omega}]$, that is

$$\nu(\omega) = \frac{1}{\overline{\omega}} \text{ for all } \omega \in [0, \overline{\omega}], \tag{3}$$

and that the expected return (under the lender's belief) of the risky project net of the monitoring cost exceeds the return of the outside option

$$\frac{1}{2}\overline{\omega} - \gamma > r. \tag{4}$$

This last assumption avoids the existence of corner solutions. With these assumptions, it follows that if a simple contract $(\mathrm{id}, \overline{R})$ solves (2), we have

$$r\overline{\omega} = -\frac{1}{2}\overline{R}^2 + (\overline{\omega} - \gamma)\overline{R}.$$

from (1) in equality. Since (4) holds, this quadratic equation has a unique root $R^* \in [0, \overline{\omega}]$ which is given by:

$$R^* = \overline{\omega} - \gamma - \sqrt{(\overline{\omega} - \gamma)^2 - 2r\overline{\omega}}. \tag{5}$$

We will see in the next paragraphs that $(\mathrm{id}, R^*)$ (the unique simple debt contract that satisfies (1) in equality, hence the only simple debt contract candidate to be optimal) does not solve (2) in general, that is for a general form of the borrower's belief $\mu$.

### 3.4 A necessary condition

The following Proposition gives a necessary condition for the simple debt contract $(\mathrm{id}, R^*)$ to be optimal.

**Proposition 1.** *If the simple contract* $(\mathrm{id}, R^*)$ *is optimal (i.e. solves (2)) then:*

$$\int_{R^*}^{\overline{\omega}} \mu(\omega)d\omega \geq \frac{C}{2}\mu(R^*), \tag{6}$$

*where* $R^*$ *is given by (5) and* $C$ *by*

$$C = 2\sqrt{(\overline{\omega}-\gamma)^2 - 2r\overline{\omega}} = 2(\overline{\omega}-\gamma-R^*). \tag{7}$$

*Proof.* For small $\varepsilon > 0$ let us construct a contract $(R_\varepsilon(.), \overline{R}_\varepsilon)$ as follows:

$$R_\varepsilon(\omega) = \begin{cases} \omega & \text{if } \omega \in [0, R^*-\varepsilon) \\ R^*-\varepsilon & \text{if } \omega \in [R^*-\varepsilon, R^*+\delta_\varepsilon) \\ R^*+\delta_\varepsilon & \text{if } \omega \in [R^*+\delta_\varepsilon, \overline{\omega}] \end{cases}$$

and $\overline{R}_\varepsilon := R^* + \delta_\varepsilon$ where $\delta_\varepsilon \in (0, \overline{\omega} - R^*)$ is chosen such that the participation constraint (1) is binding for the contract $\left(R_\varepsilon(.), \overline{R}_\varepsilon\right)$. From (5) and (7), this requirement is equivalent to

$$\delta_\varepsilon^2 - \delta_\varepsilon\left(\overline{\omega}-\gamma-R^*-\varepsilon\right) + \frac{1}{2}\varepsilon^2 = \delta_\varepsilon^2 - \delta_\varepsilon\left(\frac{C}{2}-\varepsilon\right) + \frac{1}{2}\varepsilon^2 = 0. \tag{8}$$

The smallest root of this quadratic equation is

$$\delta_\varepsilon = \frac{1}{2}\left(\left(\frac{C}{2}-\varepsilon\right) - \sqrt{\left(\frac{C}{2}-\varepsilon\right)^2 - 2\varepsilon^2}\right), \tag{9}$$

and in view of (7), $\delta_\varepsilon \in (0, \overline{\omega} - R^*)$, thus proving that the contract $\left(R_\varepsilon(.), \overline{R}_\varepsilon\right)$ satisfies the constraints of (2).

If $(\mathrm{id}, R^*)$ solves (2), this implies in particular that

$$F\left(R_\varepsilon(.), \overline{R}_\varepsilon\right) \geq F(\mathrm{id}, R^*), \text{ for } \varepsilon > 0 \text{ small enough .} \tag{10}$$

Since

$$F(\mathrm{id}, R^*) = \int_0^{R^*} \omega\mu(\omega)d\omega + R^* \int_{R^*}^{\overline{\omega}} \mu,$$

and

$$\begin{aligned} &F\left(R_\varepsilon(.), \overline{R}_\varepsilon\right) \\ &= \int_0^{R^*-\varepsilon} \omega\mu(\omega)d\omega + (R^*-\varepsilon)\int_{R^*-\varepsilon}^{R^*+\delta_\varepsilon} \mu + (R^*+\delta_\varepsilon)\int_{R^*+\delta_\varepsilon}^{\overline{\omega}} \mu, \end{aligned}$$

the inequality (10) reads as:

$$\int_{R^*-\varepsilon}^{R^*} (R^* - \omega)\mu(\omega)d\omega - \varepsilon \int_{R^*-\varepsilon}^{R^*+\delta_\varepsilon} \mu + \delta_\varepsilon \int_{R^*+\delta_\varepsilon}^{\overline{\omega}} \mu \geq 0. \tag{11}$$

From (9), we have $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, and we easily get

$$\delta_\varepsilon = \frac{\varepsilon^2}{C} + o(\varepsilon^2). \tag{12}$$

It follows that

$$\varepsilon \int_{R^*-\varepsilon}^{R^*+\delta_\varepsilon} \mu = \mu(R^*)\varepsilon^2 + o(\varepsilon^2), \tag{13}$$

and

$$\delta_\varepsilon \int_{R^*+\delta_\varepsilon}^{\overline{\omega}} \mu = \frac{\varepsilon^2}{C} \int_{R^*}^{\overline{\omega}} \mu + o(\varepsilon^2). \tag{14}$$

By the continuity of $\mu$ at $R^*$, we also get

$$\begin{aligned} \int_{R^*-\varepsilon}^{R^*} (R^* - \omega)\mu(\omega)d\omega &= \mu(R^*) \int_{R^*-\varepsilon}^{R^*} (R^* - \omega)d\omega + o(\varepsilon^2) \\ &= \frac{1}{2}\mu(R^*)\varepsilon^2 + o(\varepsilon^2). \end{aligned} \tag{15}$$

Using (13), (14) and (15) in (11) we get for small $\varepsilon > 0$

$$\varepsilon^2 \left[ \frac{1}{C} \int_{R^*}^{\overline{\omega}} \mu - \frac{1}{2}\mu(R^*) + o(1) \right] \geq 0.$$

Dividing the previous inequality by $\varepsilon^2$ and then letting $\varepsilon$ tend to $0$, we get exactly the desired inequality (6). □

### *3.5 Counter-examples*

In view of Proposition 1, it is easy to give special forms of $\mu$ for which (6) is violated, hence to give examples in which there exists no optimal simple debt contract. For instance, suppose that the density function $\mu$ is a member of the triangular density family parametrized by $\theta \in (0, \overline{\omega})$ with

$$\mu(\omega) = \begin{cases} \dfrac{2\omega}{\theta\overline{\omega}} & \text{for } \omega \in [0, \theta) \\ \dfrac{2(\overline{\omega} - \omega)}{(\overline{\omega} - \theta)\overline{\omega}} & \text{for } \omega \in [\theta, \overline{\omega}] \end{cases}$$

Observe that the mean of $\omega$ is $(\overline{\omega}+\theta)/3$, and we impose $\theta > 3r - \overline{\omega}$ for the project to be profitable (for the borrower beliefs). For $R^* > \theta$, we have

$$\int_{R^*}^{\overline{\omega}} \mu(\omega)d\omega = \frac{(\overline{\omega}-R^*)^2}{(\overline{\omega}-\theta)\overline{\omega}}, \tag{16}$$

and

$$\frac{C}{2}\mu(R^*) = 2\frac{(\overline{\omega}-R^*)(\overline{\omega}-R^*-\gamma)}{(\overline{\omega}-\theta)\overline{\omega}}. \tag{17}$$

From (16) and (17), it is easy to see that (6) is equivalent to

$$\gamma \geq \sqrt{(\overline{\omega}-\gamma)^2 - 2r\overline{\omega}}. \tag{18}$$

A simple inspection of this last condition shows that it is violated for small monitoring cost. [1] As a concrete example, take $\gamma = 0.1$, $r = 1$, $\overline{\omega} = 3$ and $\theta = 1$, then the expected project return is 3/2 for the lender beliefs and 4/3 for the borrower beliefs. The borrower is more optimistic than the lender. Moreover, we have $R^* = 1.35 > 1 = \theta$ as required, and (18) is violated (the left-hand side of (18) being 1.55). Thus the simple debt contract is not optimal.

As a second counter-example, consider the family of truncated exponentials on $[0, \overline{\omega}]$, that is

$$\mu(\omega) = \frac{\exp(-\omega/\theta)}{\theta(1-\exp(-\overline{\omega}/\theta))} \text{ for all } \omega \in [0, \overline{\omega}].$$

Then we have

$$\int_{R^*}^{\overline{\omega}} \mu(\omega)d\omega = \frac{\exp(-R^*/\theta) - \exp(-\overline{\omega}/\theta)}{1-\exp(-\overline{\omega}/\theta)}, \tag{19}$$

and

$$\frac{C}{2}\mu(R^*) = (\overline{\omega}-\gamma-R^*)\,\frac{\exp(-R^*/\theta)}{\theta(1-\exp(-\overline{\omega}/\theta))}. \tag{20}$$

After manipulating (19) and (20), a sufficient condition for (6) to be violated is

$$1 < \frac{1}{\theta}\sqrt{(\overline{\omega}-\gamma)^2 - 2r\overline{\omega}}. \tag{21}$$

As a numerical illustration, let $\gamma = 0.1$, $r = 1$, $\overline{\omega} = 3$ and $\theta = 1.5$. The expected project return is once again 3/2 for the lender but now 1.18 for the borrower. Thus the borrower is now more pessimistic than the lender. Moreover, the right-hand side of (21) is 1.03, and thus the simple debt contract is once again not optimal.

[1] Estimations of the monitoring cost suggest that they are relatively small. For instance, based on a sample of firm failures in New York State, White (1983) estimated the monitoring cost to be about 3% of assets for firms that liquidated.

## 4 Concluding remarks

In this note, we have shown that simple debt contracts do not necessarily survive as optimal contracts when the borrower and the lender have different beliefs. Thus heterogeneity of beliefs does matter in costly state verification models.

Our negative partial result opens many interesting questions such as the existence of a solution to (2), its uniqueness or the continuity and monotonicity of the optimal repayment. All these questions are left for future research.

Finally, let us point out that similar considerations should apply to insurance contracts with costly auditing.

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