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Economic Theory

Stationary measures for some Markov chain models in ecology and economics★

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Summary. Let $F \equiv \{f : f : [0,\infty) \to [0,\infty), f(0) = 0, f$ continuous, $\lim\limits_{x\downarrow 0} \frac{f(x)}{x} = C$ exists in $(0,\infty), 0 < g(x) \equiv \frac{f(x)}{Cx} < 1$ for x in $(0,\infty)\}$. Let $\{f_j\}_{j\geq 1}$ be an i.i.d. sequence from F and X_0 be a nonnegative random variable independent of $\{f_j\}_{j\geq 1}$. Let $\{X_n\}_{n\geq 0}$ be the Markov chain generated by the iteration of random maps $\{f_j\}_{j\geq 1}$ by $X_{n+1} = f_{n+1}(X_n),\ \ n \geq 0$. Such Markov chains arise in population ecology and growth models in economics. This paper studies the existence of nondegenerate stationary measures for $\{X_n\}$. A set of necessary conditions and two sets of sufficient conditions are provided. There are some convergence results also. The present paper is a generalization of the work on random logistics maps by Athreya and Dai (2000).

Keywords and Phrases: Population models, Random maps, Markov chains, Stationary measures.

JEL Classification Numbers: C22, D9.

1 Introduction

Many models of time series arising in population studies in ecology and growth models in economics are of the form

$$X_{t+1} = f_{t+1}(X_t), \quad t = 0, 1, 2, \cdots \tag{1}$$

Here X_t, the state of the system at time t, represents the population size or density in ecology and the total output in a one sector economy in economics. The function

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$f_{t+1}(\cdot)$ depends on the underlying dynamics in the period $[t, t+1]$. The functions $f_{t+1}(\cdot)$ are deterministic or stochastic depending on the underlying dynamics. In the deterministic case if f_t's are the same for all t one has a *discrete dynamical system*

$$X_{t+1} = f(X_t), \quad t = 0, 1, 2, \cdots \tag{1'}$$

In this case the initial value $X_0 = x$ gives rise to an orbit $\{x, f(x), f^{(2)}(x), ..., f^{(n)}(x), ...\}$ where for $n \geq 0$, $f^{(n+1)}(x) = f(f^{(n)}(x)$, $f^{(0)}(x) \equiv x$. The subject of discrete dynamical systems is concerned with the behavior of the orbits such as the existence of fixed points, periodic orbits, nonperiodic or chaotic behavior, existence of an equilibrium or stationary distribution π such that if X_0 is chosen to have distribution π then X_n will also have distribution π.

In the stochastic case the f_t's are random reflecting certain stochastic forces in the underlying evolutionary dynamics. In ecology these could be due to random patterns in climate, food web, predator-prey interactions, environmental changes etc. In economics these could represent stochastic shocks and or speculative behavior of the agents of the economy.

The stochastic analog of the discrete dynamical system (1') is the model (1) where $\{f_t\}$ are random but are i.i.d. or more generally a strictly stationary sequence.

When the $\{f_t\}_{t\geq 1}$ are i.i.d. and X_0 is chosen independently of $\{f_t\}_{t\geq 1}$ the sequence $\{X_t\}_{t\geq 0}$ defined by (1) becomes a *Markov chain* with *stationary transition probabilities* . The objects of interest are steady state distributions or stationary measures, convergence to them, laws of large numbers regarding the behavior of certain empirical averages etc.

In the present paper we focus on the case when the state space, i.e. the set of values of X_t is $R^+ \equiv [0, \infty)$ and the sequence $\{f_t\}_{t\geq 1}$ is a random sequence from a family F of maps from $R^+ \to R^+$ that possess two important features: (1) for small values of $x, f(x)$ is approximately linear in x reflecting the fact that ecological populations and fledgling economies grow exponentially when small and (2) for large values of $x, f(x)$ is sublinear reflecting the effect of density dependence or competition as the population grows or diminishing returns in an economy. Examples of such families include:

(i) the *logistic maps* (Athreya and Dai, 2000)

$$f_c(x) \equiv cx(1-x), \quad 0 \leq x \leq 1, \quad 0 \leq c \leq 4 \tag{2a}$$

(ii) the *Ricker maps* (Ricker, 1954))

$$f_{c,d}(x) = cxe^{-dx}, \quad 0 \leq c, \ d < \infty, \quad 0 \leq x < \infty \tag{2b}$$

(iii) the *Hassel maps* (Hassel, 1974)

$$(x) = cx(1+x)^{-d}, \quad 0 \leq c, \ d < \infty, \quad 0 \leq x < \infty \tag{2c}$$

(iv) the *Vellekoop-Hognas maps* (Vellekoop-Hognas[15])

$$f(x) = rx(h(x))^{-b}, \quad 0 < r, \ b < \infty$$
$$h(x) \geq 1 \text{ for } \quad x \geq 0, \quad h(0) = 1,$$

h is continuously differentiable and

$$\tilde{h}(x) \equiv \frac{xh'(x)}{h(x)} \text{ is strictly increasing} \tag{2d}$$

The main thrust of this paper is to investigate the existence of nontrivial stationary measures (ie, other than the delta measure at 0) for the case when the $\{f_t\}_{t\geq 1}$ sequence is i.i.d. with values in the set F. Our results here are generalizations of those of Athreya and Dai (2000) for the case of random logistic maps.

In the next section, as a preparation for Section 3, there is a brief review of Feller Markov chains and occupation measures. The main part of the paper is Section 3 where a set of necessary and two sets of sufficient conditions are provided for the existence of nontrivial stationary measures. Some open problems are mentioned in the last section.

2 Feller (Markov) chains, occupation and stationary measures

Let $\{X_n\}_{n\geq o}$ be a Markov chain with a metric state space (S, d) and a transition function $P(\cdot, \cdot)$.

Definition 1. $\{X_n\}_{n\geq 0}$ is called a Feller (Markov) chain (or P is called a Feller transition function) if $x_n \to x$ implies $P(x_n, \cdot) \to P(x, \cdot)$ in distribution or equivalently

$$E(k(X_1)|X_0 = x) \equiv \int_S k(y)P(x, dy) \equiv (Pk)(x) \tag{3}$$

is continuous in x for all functions $k : S \to R$ that are bounded and continuous.

If $\{X_n\}_{n\geq 0}$ is generated by an iteration scheme as in (1) with $\{f_t\}_{t\geq 1}$ i.i.d. with $f_1(\cdot)$ being *continuous* with probability one (w.p. 1) then it is Feller. Indeed, since $(Pk)(x) = Ek(f_1(x))$ and $f_1(\cdot)$ is continuous and k is bounded and continuous the assertion follows by the bounded convergence theorem. Note that all the four families listed in (2) consist of continuous functions.

Definition 2. Let $\{X_n\}_{n\geq 0}$ be a Markov chain with transition function P. Let for all $A \in \mathcal{S}$,

$$L_n(A) \equiv \frac{1}{n}\sum_0^{n-1} I_A(X_j) \qquad \text{and} \tag{4}$$

$$\mu_{n,x}(A) \equiv E(L_n(A)|X_0 = x) = \frac{1}{n}\sum_0^{n-1} P(X_j \in A|X_0 = x) \tag{5}$$

Then $L_n(\cdot)$ is called the *empirical measure* and $\mu_{n,x}(\cdot)$ the *occupation measure* for the chain $\{X_n\}$.

Definition 3. A sequence $\{\mu_n\}$ of probability measures on (S,d) is said to converge *weakly* or *in distribution* to a probability distribution μ if

$$\int k(x)\mu_n(dx) \to \int k(x)\mu(dx) \tag{6}$$

for all $k : S \to R$, bounded and continuous.

Definition 4. A sequence $\{\nu_n\}$ of subprobability measures on $(S,d)(i.e.\nu_n(S) \leq 1)$ is said to converge *vaguely* to a subprobability distribution ν if (6) holds for all $k : S \to R$, bounded, continuous and vanishing outside a compact set. See Chung (1974) for discussion on Definitions 3 and 4.

Definition 5. A measure μ on (S,d) is *stationary* for the transition function P if

$$\mu(A) = (\mu P)(A) \equiv \int P(x,A)\mu(dx) \text{ for all } A \in \mathcal{S} \tag{7}$$

One way of finding stationary measures for P is to consider all weak or vague limits of the occupation measures $\{\mu_{n,x}(\cdot)\}$. The following is well known but the next one is perhaps not so well known.

Proposition 1. Let $\{X_n\}$ be a Feller Markov chain with transition function P. Suppose for some intial distribution of X_0, there is a subsequence $\{n_k\}$ such that $\mu_{n_k,X_0}(\cdot)$ converges weakly, ie, in distribution to a probability measure μ. Then μ is stationary for P.

For a proof see Meyn and Tweedie (1993).

Proposition 2. Under the set up of Proposition 1 suppose that there is a subsequence $\{n_k\}$ such that $\mu_{n_k,X_0}(\cdot)$ converges *vaguely* to a subprobability measure μ (ie $\mu(S) \leq 1$) and that there exists an "*approximate identity*", ie, a sequence $\{g_r\}$ of continuous functions such that for each r, $g_r(x) \in [0,1] \forall\ x$ in S, $g_r(\cdot)$ has compact support and for each x in S, $g_r(x)$ increases to one as $r \to \infty$.

Then $\mu = \mu P$, ie, (7) holds.

Proof. By definition 4, for each $g : S \to R^+$ continuous and with compact support

$$\int g(y)\mu_{n_k,X_0}(dy) \to \int g(y)\mu(dy) \tag{8}$$

It is easy to check that if M is an upperbound for g on S then

$$\begin{aligned} |\textstyle\int g(y)\mu_{n_k,X_0}(dy) &- \textstyle\int g(y)\mu_{n_k+1,X_0}(dy)| \leq \tfrac{2M}{n_k} \to 0, \\ \text{and } |\textstyle\int g(y)\mu_{nx+1,X_0}(dy) &- \textstyle\int (Pg)(y)\mu_{n_k,X_0}(dy)| \leq \tfrac{2M}{n_k} \to 0. \end{aligned}$$

Also since $0 \leq g_r(\cdot) \leq 1$, and $g(\cdot) \geq 0$

$$\int (Pg)(y)\mu_{n_k,X_0}(dy) \geq \int (Pg)(y)g_r(y)\mu_{n_k,X_0}(dy).$$

Now $(Pg)(\cdot)$ is continuous since P is Feller.

Also $(Pg(y))g_r(y)$ is continuous with compact support. So

$$\int (Pg)(y)g_r(y)\mu_{n_k,x_0}(dy) \to \int (Pg)(y)g_r(y)\mu(dy). \tag{9}$$

Thus from (8) and (9) we get

$$\int g(y)\mu(dy) \geq \int (Pg)(y)g_r(y)\mu(dy)$$

Since $0 \leq g_r(\cdot) \uparrow 1$, by the monotone convergence theorem

$$\int (Pg)(y)g_r(y)\mu(dy) \uparrow \int (Pg)(y)\mu(dy).$$

Thus, for all $g : S \to R^+$ and continuous with compact support

$$\begin{aligned} \textstyle\int g(y)\mu(dy) &\geq \textstyle\int (Pg)(y)\mu(dy) \\ &= \textstyle\int g(y)(\mu P)(dy). \end{aligned}$$

This implies $\mu(A) \geq (\mu P)(A)$ for all $A \in \mathcal{S}$.

$$\text{But } \mu P(S) = \int_S P(x,S)\mu(dx) = \mu(S).$$
$$\text{Thus } \mu = \mu P.$$

□

Remark 1. If S is an interval in R then it has an approximate identity. For example, if $S = (0,1)$ then the sequence $\{g_r\}_{r\geq 2}$ by

$$g_r(x) = \begin{cases} 1 & \frac{1}{r} \leq x \leq 1 - \frac{1}{r} \\ 0 & 0 < x \leq \frac{1}{r+1} \text{ or } 1 > x \geq 1 - \frac{1}{r+1} \\ \text{linear in} & \left[\frac{1}{r+1}, \frac{1}{r}\right] \cup \left[1 - \frac{1}{r}, 1 - \frac{1}{r+1}\right] \end{cases}$$

is an approximate identity.

Similar construction works for any interval.

A natural problem is to find a sufficient condition for $\{\mu_{n,X_0}\}$ to have at least one vague limit point μ that is not the trivial measure 0. This is provided by the so called Foster-Lyaponov condition. See Meyn and Tweedie (1993).

Proposition 3. Suppose there exist a function $V : S \to R^+$, a set $K \subset S$ and constants $0 < \alpha, M < \infty$
such that

$$\left.\begin{aligned} &i)\ \forall\ x \notin K,\ E(V(X_1)|X_0 = x) - V(x) \leq -\alpha \\ &ii)\ \forall\ x \in S,\ E(V(X_1)|X_0 = x) - V(x)\ \leq M \end{aligned}\right\} \tag{10}$$

$$\text{Then, } \liminf \mu_{n,X_0}(K) \geq \frac{\alpha}{\alpha + M} \tag{11}$$

Proof. Let E_x stand for expectation when $X_0 = x$.

For $j \geq 1$,

$$\begin{aligned} E_x V(X_j) - E_x V(X_{j-1}) &= E_x(PV)(X_{j-1}) - V(X_{j-1}) \\ &\leq -\alpha P_x(X_{j-1} \notin K) + M P_x(X_{j-1} \in K) \\ &= -\alpha + (\alpha + M) P_x(X_{j-1} \in K). \end{aligned}$$

Adding over $j = 1, 2, ..., n$ and dividing by n yields

$$\frac{1}{n}(E_x V(X_n) - V(x)) \leq -\alpha + (\alpha + M)\mu_{n,x}(K).$$

Since $V(\cdot) \geq 0$, letting $n \to \infty$ yields (11). □

Remark 2. In many applications K would be a compact subset of S. From (11) it follows that for any vague limit point μ of μ_{n,X_0}, $\mu(K) > 0$ ensuring its nontriviality. Thus, Propositions 2 and 3 show that to establish the existence of a nontrivial stationary distribution it is not necessary to demand the *tightness* of $\{\mu_n\}$.

3 Stationary measures

Let the collection F of functions $f : [0, L) \to [0, L), L \leq \infty$
be such that

i) f is continuous
ii) $f(0) = 0$
iii) $\lim\limits_{x \downarrow 0} \frac{f(x)}{x} \equiv f'_+(0)$ exists and is positive and finite
iv) $g(x) \equiv \frac{1}{f'_+(0)} \frac{f(x)}{x}$ satisfies $0 < g(x) < 1$ for $0 < x < L$.

Let $(\Omega, \mathcal{B}, P)$ be a probability space.

Let $\{f_j(\omega, x\}_{j \geq 1}$ be a collection of random maps from $\Omega \times [0, \infty) \to [0, \infty)$ that are jointly measurable, ie, that are $(\mathcal{B} \times \mathcal{B}[0, \infty), \mathcal{B}[0, \infty))$ measurable and for each j, $f_j(\omega, \cdot) \in F$ with probability one. Consider the random dynamical system generated by the iteration scheme:

$$\begin{aligned} X_{t+1}(\omega, x) &\equiv f_{t+1}(\omega, X_t(\omega, x)), t \geq 0 \\ X_0(\omega, x) &\equiv x. \end{aligned} \tag{12}$$

Since $f_j(\omega, \cdot) \in F$ w.p. 1 the model (12) reflects the two features mentioned in the introduction, ie, for small values of X_t, X_{t+1} is proportional to X_t with proportionality constant $f'_{t+1}(0) \equiv C_{t+1}$, say, and for large values of X_t, this is reduced by the factor $g(X_t)$.

The class F includes the logistic, Ricker, Hassel, Vellekoop-Hognas families mentioned in (2a) – (2d).

For the logistic family $f_c(x) = cx(1-x), L = 1, f'_+(0) = c$, and $g(x) = 1-x$ for $0 \leq x \leq 1$.

For the Ricker family, $L = \infty, f_{c,d}(x) = cxe^{-dx},\ f'_+(0) = c, g(x) = e^{-dx}, 0 \leq x < \infty$.

For the Hassel family, $L = \infty, f_{c,d}(x) = cx(1+x)^{-d}, f'_+(0) = c$ and $g(x) = (1+x)^{-d}$.

For the Vellekoop-Hognas family, $L = \infty,\ f(x) = rx(h(x))^{-b}, f'_+(0) = r, g(x) = (h(x))^{-b}$.

Our first result gives a *necessary condition* for the existence of a nondegenerate stationary distribution π (ie, $\pi(0,L) > 0$) for the Markov chain $\{X_t\}$ in (12) generated by the case when $\{f_j\}_{j\geq 1}$ are i.i.d.

Theorem 1. *Let $\{f_j\}_{j\geq 1}$ be i.i.d. Let*

$$C_j(\omega) \equiv \lim_{x\downarrow 0} \frac{f_j(\omega,x)}{x} \in (0,\infty) \tag{13}$$

$$g_j(\omega,x) = \begin{cases} \frac{f_j(\omega,x)}{C_j(\omega)x} & x > 0 \\ \\ 1 & x = 0 \end{cases} \tag{14}$$

$$\textit{Assume} \quad E(lnC_1)^+ < \infty \tag{15}$$

Suppose there exists a stationary probability measure π for the Markov chain $\{X_t\}$ defined by (12) such that $\pi(0,\infty) > 0$.

Then

$$i) \quad E(lnC_1)^- < \infty, \int E|ln\ g_1(\omega,x)|\ \pi(dx) < \infty \tag{16}$$

$$\textit{and} \quad ii) \quad E(lnC_1) = -\int (E\, ln\ g_1(\omega,x))\ \pi(dx) \tag{17}$$

and hence is strictly positive.

Proof. Let X_0 have distribution π. Then, since π is a stationary measure for $\{X_n\}, X_1 = f_1(\omega, X_0)$ also has distribution π.

Since $X_1 = f_1(\omega, X_0)$ can be written as

$$X_1 = C_1(\omega) X_0 g_1(\omega, X_0) \tag{18}$$

taking logarithms yields (suppressing ω)

$$lnX_1 = lnC_1 + lnX_0 + lng_1(X_0). \tag{19}$$

Let

$$Z \equiv (lnC_1)^- + (-lng_1(X_0)). \tag{20}$$

Since $0 \leq g_1(\cdot) \leq 1, Z$ is a nonnegative random variable. From (19)

$$lnX_0 - lnX_1 + (lnC_1)^+ = Z. \tag{21}$$

If it was known that $E|lnX_0| < \infty$, then taking expectations in (21) and using (15) one could conclude that (16) and (17) hold. Since it is not known that $E|lnX_0| <$

∞ an alternate approach is required. A truncation argument works. Let, for $k = 1, 2......$,

$$\phi_k(x) = \begin{cases} x & \text{if} \quad |x| \leq \quad k \\ k & \text{if} \quad \;\; x > \quad k \\ -k & \text{if} \quad \;\; x < -k \end{cases}$$

It is clear that each $\phi_k(\cdot)$ is bounded and $|\phi_k(x) - \phi_k(y)| \leq |x - y|$ for all k, x, y.

It is easy to verify that if $\eta \geq 0$ and $x - y + \eta \geq 0$ then $\phi_k(x) - \phi_k(y) + \eta \geq 0$ (just by considering the nine possibilities arising out of x and y each being $< -k$, in $[-k, k]$ or $> k$).

Let

$$Z_k = \phi_k(lnX_0) - \phi_k(lnX_1) + (lnC_1)^+ \tag{22}$$

Since Z is ≥ 0 and $(lnC_1)^+ \geq 0$ it follows that $Z_k \geq 0$.

Also $Z_k \to Z$ w.p. 1 as $k \to \infty$. By stationarity of π and boundedness of ϕ_k and the hypothesis $E(lnC_1)^+ < \infty$ we get $EZ_k = E(lnC_1)^+$. Letting $k \to \infty$ and using Fatou's lemma yields

$$EZ \leq \underline{\lim} EZ_k = E(lnC_1)^+ < \infty. \tag{23}$$

Since $Z = (lnC_1)^- + (-lng_1(X_0)$ and both terms are nonnegative, (23) yields $E(lnC_1)^- < \infty$ and $E(-lng_1(X_0)) < \infty$. Thus (16) is established. Since $EZ < \infty$ and by hypothesis $E(lnC_1)^+ < \infty$ we get from (21) that

$$E|lnX_0 - lnX_1| < \infty. \tag{24}$$

Also $|\phi_k(lnX_0) - \phi_k(lnX_1)| \leq |lnX_0 - lnX_1|$ and $0 \leq Z_k \leq |lnX_0 - lnX_1| + (lnC_1)^+ \equiv \tilde{Z}$, say.

From (24), $E\tilde{Z} < \infty$ and so by the dominated convergence theorem we get

$$EZ_k \to EZ$$

$$\text{ie} \quad E(lnC_1)^+ = E(lnC_1)^- + E(-lng_1(X_0)).$$

All the terms involved being finite, this yields

$$\begin{aligned} E(lnC_1) &= -E(lng_1(X_0)) \\ &= -\textstyle\int Elng_1(x)\pi(dx) \end{aligned}$$

establishing (17). Since $\pi(0, L) > 0$ and w.p. 1 (15) holds, it follows that $ElnC_1 > 0$. $\square$

Corollary 1. *In the set up of Theorem 1 if $ElnC_1 \leq 0$ then*

i) the only stationary probability measure on $[0, L]$ is the delta measure at 0.

ii) For any initial distribution X_0, the occupation measure $\mu_{n,X_0}(A) \equiv \frac{1}{n}\sum\limits_{0}^{n-1} P(X_j \in A)$ converges to zero for all A such that its closure is $\subset (0, \infty)$ and hence for such A the empirical measure $L_n(A) = \frac{1}{n}\sum\limits_{0}^{n-1} I(X_j \in A) \to 0$ in probability.

Proof.

i) Suppose there is a stationary measure μ with $\mu(0,\infty) > 0$.
Let $\tilde{\mu}(A) \equiv \frac{\mu(A\cap(0,\infty))}{\mu(0,\infty)}$ for $A \in \mathcal{B}(0,\infty)$. Then $\tilde{\mu}$ is a probability measure on $(0,\infty)$. Also $\mu = \mu\{0\}\delta_0 + (1-\mu\{0\})\tilde{\mu}$. Since δ_0 and μ are both stationary for P so is $\tilde{\mu}$. By Theorem 6 this implies $ElnC_1 > 0$.

ii) Since $f_j(\omega,\cdot)$ are continuous w.p. 1 the Markov chain is Feller. Also $S = [0,\infty)$ admits an approximate identify in the sense of Theorem 4. So, if μ is any vague limit point of the occupation measures $\{\mu_{n,X_0}(\cdot)\}$ then μ is stationary for P. By (i), $\mu(0,\infty)$ must be zero.
Finally, since $EL_n(A) = \mu_{n,X_0}(A)$, and $\mu_{n,X_0}(A) \to 0$ for all $A \subset \overline{A} \subset (0,\infty)$, $L_n(A) \to 0$ in probability. □

Next we present two sets of sufficient conditions for the existence of a stationary measure π with $\pi(0,\infty) > 0$ for the Markov chain $\{X_t\}$ in (12).

Theorem 2. *Let $\{f_j\}\{C_j\}, \{g_j\}$ be as in Theorem 1. Let $D_j(\omega) \equiv \sup_{x\geq 0} f_j(\omega,x)$.*
Assume

i) $k(x) = -Elng_1(x) < \infty$ *for all* $0 < x < L$ *and bounded on every* $(a,b) \subset (0,L)$.
ii) $\lim_{x\downarrow 0} k(x) = 0$
iii) $k(\cdot)$ *be nondecreasing in* (T,L) *for some* $T < L$.
iv) $E|lnC_1| < \infty, ElnC_1 > 0$
v) $E(lnD_1)^+ < \infty$
vi) $E|k(D_1)| < \infty$

Then, there exists a stationary distribution π for the Markov chain $\{X_t\}$ defined by (12) such that $\pi(0,L) = 1$.

Proof. Suppressing ω, (12) becomes

$$X_{j+1} = C_{j+1}X_j g_{j+1}(X_j) \tag{25}$$

and so $\quad lnX_{j+1} - lnX_j = lnC_{j+1} + lng_{j+1}(X_j)$
Adding this over $j = 0, 1, ..., n-1$

$$lnX_n - lnX_0 = \sum_1^n lnC_j + \sum_1^n lng_j(X_{j-1}) \tag{26}$$

Since $X_j = f_j(X_{j-1}) \leq D_j$,

$$lnX_n \leq lnD_n \ .$$

Also $E|lng_j(X_{j-1})| = -Elng_j(X_{j-1}) = Ek(X_{j-1})$
(by independence of g_j and X_{j-1}).

For $\quad j \geq 1,$ by (iii) and (vi)

$$\begin{aligned} Ek(X_j) &= E(k(X_j) : X_j < T) + E(k(X_j) : X_j \geq T) \\ &\leq K_{0,T} + Ek(D_j) \quad < \infty \end{aligned}$$

where $K_{a,b} = \sup\{k(x) : a < x < b\}$.

Also $E|lnC_1| < \infty$ by (iv).

So the rightside of (26) has a finite expectation. Now choose X_0 such that $E|lnX_0| < \infty$, for eg, deterministic $X_0 \neq 0$.

Dividing (26) by n and taking expectations yields

$$\frac{1}{n}ElnX_n - \frac{1}{n}ElnX_0 = ElnC_1 - \frac{1}{n}\sum_1^n Ek(X_{j-1}) \; . \tag{27}$$

But $\frac{1}{n}ElnX_n \leq \frac{1}{n}E(lnD_n)^+ \to 0$ by (v)

and $\frac{1}{n}ElnX_0 \to 0$.

By hypothesis (iv) $ElnC_1 > 0$.

Let (H) be the condition that $\{\mu_{n,X_0}(\cdot)\}$ has no vague limit point μ with $\mu(0, L) > 0$. We shall show that if (H) holds then

$$\frac{1}{n}\sum_1^n Ek(X_{j-1}) \to 0 \; . \tag{28}$$

Thus in (27) the leftside is bounded above by a sequence that goes to zero but the rightside goes to a positive quantity. This contradiction shows that there is a vague limit point μ of $\{\mu_{n,X_0}(\cdot)\}$ with $\mu(0,\infty) > 0$. Here we use the fact that given any sequence of subprobability measures and therefore for any sequence of probability measures there is always a subsequence that converges vaguely to a subprobability measure (see Chung, 1974, p. 83). Then $\tilde{\mu}(A) \equiv (\mu(0,\infty))^{-1}\mu(A \cap (0,\infty))$ will be a stationary probability measure for P with $\tilde{\mu}(0,\infty) = 1$. It remains to establish (28).

Now fix $\epsilon > 0, \eta > 0$. Then

$$\begin{aligned} Ek(X_j) \leq\; & E(k(X_j) : X_j \leq \epsilon) + E(k(X_j) : \epsilon < X_j < T) \\ & + E(k(D_j) : X_j \geq T, |k(D_j)| \leq M_\eta) \\ & + E(k(D_j) : |k(D_j)| > M_\eta) \end{aligned}$$

where M_η is chosen so that

$$E(|k(D_j)| : |k(D_j)| > M_\eta) < \eta$$

(using hypothesis (vi)).

Thus,

$$\frac{1}{n}\sum_1^n Ek(X_{j-1}) \leq \sup_{x \leq \epsilon} k(x) + (M_\eta + K_{(\epsilon,L)})\mu_{n,X_0}(\epsilon, L) + \eta$$

implying that if (H) holds then

$$\overline{\lim_n}\; \frac{1}{n}\sum_1^n Ek(X_{j-1})$$

$$\leq \sup_{x \leq \in} k(x) + \eta$$

since, if (H) holds, $\overline{\lim_{n}}\ \mu_{n,X_0}(\epsilon, L) = 0\ \forall\ \ \in > 0$. By (ii) $\sup_{x \leq \in} k(x) \to 0$ as $epsilon \to 0$. Also $\eta > 0$ is arbitrary. Thus (28) is established and hence the theorem is proved. □

Remark 3. (Special cases).

1. If $f(\omega, x) = C(\omega)x(1-x), 0 \leq x \leq 1$ is a logistic map then $D(\omega) \equiv \sup_{x} f(\omega, x) = \frac{C(\omega)}{4}$ and $g(\omega, x) = (1-x)I_{[0,1]}(x)$. Thus, $k(D) = -ln(1-D) = -ln(1-\frac{C}{4})$. So if f_j is logistic w.p. 1 then the hypothesis i) - vi) of Theorem 2 reduce to $E - lnC_1 > 0$ and $-Eln(1-\frac{C}{4}) < \infty$ (see Athreya and Dai, 2000).

2. If $f(\omega, x) = C(\omega)xe^{-d(\omega)x}, 0 \leq x < \infty$ is a Ricker map then $D(\omega) = C(\omega)/d(\omega),\ g(\omega, x) = e^{-d(\omega)x}, k(x) = E(d(\omega))x$. So if f_j is Ricker w.p. 1 the hypothesis i) - vi) of Theorem 2 reduce to

$$Ed(\omega) < \infty, \qquad E\frac{C(\omega)}{d(\omega)} < \infty\ ,$$
$$E|lnd(\omega)|E|lnC(\omega)| < \infty,\ ElnC(\omega) > 0.$$

Similar reductions can be made in the other two cases, ie Hassel maps and Vellekoop-Hognas maps.

Now we give a second set of sufficient conditions.

Theorem 3. *Let $\{f_j\}, \{C_j\}, \{g_j\}$ be as in Theorem 1. Suppose*

i) $\lim_{x \to 0} ElnC_1 g_1(x) \equiv \beta_1$ *exists and is* > 0
ii) $\lim_{x \to 0} E(lnC_1 x g_1(x))^+ = 0$
iii) $\lim_{x \to L} ElnC_1 g_1(x) \equiv \beta_2$ *exists and is* < 0
iv) $\lim_{x \to L} E(lnC_1 x g_1(x))^- = 0$
v) $\tilde{k}(x) \equiv E|lnC_1 g_1(x)|$ *is bounded on* $[a, b]$ *for all* $0 < a < b < L$. *Then there exists a stationary measure* π *for* P *satisfying* $\pi(0, L) = 1$.

Proof. Since P is Feller we can apply Propositions 2 and 3. Let $V(x) \equiv |lnx|$.

We shall now show that there exists $\alpha, M, a, b \in (0, \infty)$ such that

$$E(V(X_1)|X_0 = x) - V(x) \leq -\alpha \text{ for all } x \notin [a, b] \tag{29}$$
$$\leq M \text{ for all } x \in [0, L] \tag{30}$$

Again suppressing ω and noting that

$$X_1 = C_1 X_0 g(X_0)$$

we see that

a) for $x < 1$
$E_x|lnX_1| - |lnx|$
$= -ElnC_1 g_1(x) + 2E(lnC_1 x g_1(x))^+$

and for $x > 1$
$E_x|lnX_1| - lnx|$
$= ElnC_1g_1(x) + 2E(lnC_1xg_1(x))^-$

By hypothesis (i) - (iv)

$$\lim_{x\to 0} E_x|lnX_1| - |lnx| = -\beta_1 < 0,$$
$$\lim_{x\to L} E_x|lnX_1| - |lnx| = \quad \beta_2 < 0$$

Choose $0 < a < b < \infty$ such that

$$\text{for } x \leq a,\ E_x|lnX_1| - |lnx| \leq \tfrac{-\beta_1}{2},$$
$$\text{for } x \geq b,\ E_x|lnX_1| - |lnx| \leq \tfrac{\beta_2}{2}.$$

Next for $a < x < b$

$$|E_x|lnX_1| - |lnx|| \leq E_x|lnX_1 - lnx|$$
$$= E_x|lnC_1g_1(x)| \quad = \tilde{k}(x)$$

which is bounded in $[a, b]$ by hypothesis (v).

Thus (29) and (30) are verified and so by Propositions 2 and 3 there exists a stationary measure $\tilde{\pi}$ for P such that $\tilde{\pi}(0, L) > 0$. Normalizing $\tilde{\pi}$ by $\tilde{\pi}(0, L)$ yields the desired measure π. □

Remark 4. In all the four special cases (logistic, Ricker, etc) the function

$$g_j(x) \equiv \frac{f_j(x)}{C_jx} \to 0 \text{ as } x \to \infty$$

This says that for large x the growth is sublinear. But in some ecological context such as arising in resource management procedures it is more realistic to keep $g_j(x)$ bounded away from zero as $x \to L$. Similarly in some growth models in economics the possibility of $f_j(x) \to \infty$ as $x \to L$ is not unrealistic.

The next corollary is easy to verify.

Corollary 2. *In the set up of Theorem 3 assume:*

i) $E|lnC_1| < \infty, ElnC_1 > 0.$
ii) With probability one $\lim_{x\downarrow 0} g_1(x) = 1, \lim_{x\uparrow L} g_1(x) = \eta > 0$ *and there exists* $0 < a$ *such that and* $a \leq \inf_x g_1(x) \leq \sup_x g_1(x) \leq 1$
iii) $ElnC_1 + Eln\eta < 0$

Then there exists a stationary π *for* P *satisfying* $\pi(0, L) = 1$.

Remark 5. We now comment briefly on a comparison of the two sets of sufficient conditions in Theorem 2 and Theorem 3. Two key hypothesis in Theorem 2 are that $D_1 \equiv \sup_{x\geq 0} f_1(\omega, x)$ is not only a finite random variable but satisfies the moment conditions v) and vi) and f_1 has to have a finite positive derivative at 0. This rules out growth functions f, such as $f(x) = x^\alpha, \alpha > 0$. On the otherhand, Theorem 3, allows for unbounded functions but with bounded linear growth rate such as when $\lim_{x\to L} g(x)$ exists and is positive. It still needs f_1 to have a finite positive derivative at 0. As mentioned in Remark 4 in some resource management problems arising in ecology the harvest policy may dictate that $g_1(x)$ not go to zero as $x \uparrow L$ but reach a factor η satisfying conditions (ii) and (iii) of Corollary 2. A rough guideline is that if $\overline{\lim}_{x\to L} f_1(\omega, x) < \infty$ then use Theorem 2 while if $\lim_{n\to L} f_1(\omega, x) = \infty$ but $\lim_{n\to L} \frac{f_1(\omega,x)}{x} < \infty$ then use Theorem 3. Both require f_1 to be in F and in particular f_1 to have a finite positive derivative at 0.

Remark 6. Growth models in economics where $f(x)$ is not approximately linear near zero do not belong to the class F. An example is $f(x) = Cx^{\alpha},\ 0 < \alpha < 1$. For such cases both Theorems 2 and 3 are not useful. But in many cases other methods are available. Here is one such example of some importance in economics. Let $\{X_n\}$ be defined by the iteration scheme

$$X_{n+1} = C_{n+1} X_n^{\alpha_{n+1}} \tag{31}$$

where $\{(C_n, \alpha_n)\}$ are i.i.d. rv in $(0, \infty)$ and $X_n \in (0, \infty)$ with X_0 independent of $\{(C_n, \alpha_n)\}$. This can be put in the framework of (12) if $P(\alpha_{n+1} \geq 1) = 1$. But in this case both Theorem 2 and 3 are not applicable because the function is unbounded violating (v) and (vi) of Theorem 2 and (iii) of Theorem 3. The special case of (31) with α_n deterministic equal to an $\alpha \in (0, 1)$ is treated in Bhattacharya and Majumdar (1980). See also Majumdar and Mitra (1982).

Taking logarithms in (31) leads to

$$lnX_{n+1} = lnC_{n+1} + \alpha_{n+1} lnX_n$$

$$\Longrightarrow \quad \frac{Y_{n+1}}{P_{n+1}} = \frac{d_{n+1}}{P_{n+1}} + \frac{Y_n}{P_n}$$

$$\text{where} \quad Y_n = lnX_n, \quad d_n = lnC_n, \quad P_n = \prod_1^n \alpha_j$$

$$\Longrightarrow \quad \frac{Y_n}{P_n} = \sum_{j=1}^{n} \frac{d_j}{P_j} + lnX_0 \tag{32}$$

Case 1. Assume $Eln\alpha_1 < 0$, and $E(ln|lnC_1|)^+ < \infty$. Then, (32)

$$\Longrightarrow \quad Y_n = \sum_{j=1}^{n} (d_j) \left(\prod_{j+1}^{n} \alpha_i \right) + (lnX_0) P_n \tag{33}$$

By the strong law, $P_n \to 0$ w.p. 1.

Since $\{(C_i, \alpha_i)\}^n$, are i.i.d. $Y_n \equiv \sum_{j=1}^{n} d_j \left(\prod_{j+1}^{n} \alpha_i \right)$ has the same distribution as $\tilde{Y}_n \equiv \sum_{j=1}^{n} d_j \prod_{i=1}^{(j-1)} \alpha_i$.

By the strong law, since $Eln\alpha_1 < 0, P_n = \prod_{i=1}^{n} \alpha_i = 0(\lambda^n)$ w.p. 1 for any $e^{Eln\alpha_1} < \lambda < 1$.

Next, $E(ln|lnC_1|)^+ = E(ln|d_1|)^+ < \infty \Longrightarrow \sum_n P(lnd_n > \gamma n) < \infty$ for any $0 < \gamma < \infty$ and hence by Borel Cantelli, w.p. 1.

$d_j \leq e^{\gamma j}$ for all large j. Thus, choosing $\gamma > 0$ such that $\lambda e^{\gamma} < 1$, we see that $\tilde{Y}_n \equiv \sum_{j=1}^{n} d_j \prod_{i=1}^{j=1} \alpha_i$ converges w.p. 1 to $\tilde{Y}$, say.

Thus, $Y_n \xrightarrow{d} \tilde{Y}$ for any initial $x_0 > 0$.

Case 2. $Eln\alpha_1 > 0$ and $E(ln|C_1|)^+ < \infty$. In this case, $\frac{1}{P_n} \to 0$ w.p. 1 at a geometric rate and arguing as in Case 1, one can show that

$$\sum_{1}^{\infty} \left| \frac{d_j}{P_j} \right| < \infty, \text{ w.p. } 1$$

and so $\frac{Y_n}{P_n} \to \sum_{1}^{\infty} \frac{d_j}{P_j}$ w.p. 1. and in particular, $|Y_n| \to \infty$ w.p. 1.

Case 3. $Eln\alpha_1 = 0$ is open.

Remark 7. The main thrust of this paper and this section has been to seek conditions for the existence of a stationary probability measure π for the Markov chain $\{X_n\}$ defined by (12) satisfying $\pi(0, L) = 1$. The questions of convergence of the distribution of X_n to this π, uniqueness or smoothness of π etc have not been addressed here. We now indicate briefly an approach to these questions.

Definition 6. A Markov chain $\{X_n\}$ with state space $(S, \mathcal{S})$ is said to be Harris irreducible with reference measure ϕ if $A \in \mathcal{S}, \phi(A) > 0 \Longrightarrow P(X_n \in A$ for some $n \geq 1 | X_0 = x) > 0$ for all $x \in S$.

The following is known (Meyn and Tweedie, 1993).

Proposition 4. Let $\{X_n\}$ be Harris irreducible. Suppose π is a stationary probability measure for $\{X_n\}$. Then:

a) π is unique,
b) for all x in S, the occupation measure sequence,

$$\mu_{n,x}(\cdot) \equiv \frac{1}{n} \sum_{0}^{n-1} P(X_j \in \cdot | X_0 = x)$$

converges to π in total variation for all x in S

c) for all A in $\mathcal{S}$, $L_n(A) \equiv \frac{1}{n} \sum_{0}^{n-1} I_A(X_j) \to \pi(A)$ w.p. 1.

Applying this to our set up we get the following.

Theorem 4. *Let $\{X_n\}$ be as in (12). Suppose:*

i) it is Harris irreducible in $S = (0, L)$.

ii) it admits a stationary probability measure $\pi(\cdot)$ *such that* $\pi(0, L) = 1$.
iii) the distribution of C_1 *has a positive absolutely continuous component in* $(0, K)$
iv) the random function $g_1(\cdot)$ *is such that if* X *is an absolutely continuous random variable with values in* $(0, L)$ *then* $Xg_1(X)$ *is also absolutely continuous.*
v) C_1 *and* $g_1(\cdot)$ *are independent.*

Then, in addition to a), b) and c) of Proposition 4, the following also holds.

d) π *is absolutely continuous.*

Proof. That (i) and (ii) imply (a) (b) and (c) follows from Proposition 4. It remains to prove (d).

Let X_0 have distribution π. Then X_1 also has distribution π. By (v), C_1 and $X_0 g_1(X_0)$ are independent. If r and s denote the weight of the absolutely continuous component of π and the distribution of C_1 (see definition below) then (iii) and (iv) and the relation $X_1 = C_1 X_0 g_1(X_0)$ imply that

$$(1 - r) = (1 - s)(1 - r).$$

Since $s > 0$, it follows that $r = 1$. □

By the Lebesgue decomposition theorem (Chung, 1974) every probability measure π on R can be written as $\pi = \alpha\pi_a + (1-\alpha)\pi_s$ where π_a is absolutely continuous and π_s is singular w.r.t. Lebesgue measure and $0 \leq \alpha \leq 1$.

Definition 7. The weight of the absolutely continuous component of π is α.

For some special cases of (12), in Theorem 4 the condition (iii) with some mild additional conditions imply (i), ie Harris irreducibility (see Athreya, 2002, for details).

5 Some open problems

The case of iteration of random logistic maps has been well studied by a number of authors (see Athreya and Bhattacharya, 2000, for a review). Many of those results have been extended to the general class F of Section 3 but many more remain. A few of them are outlined below.

i) *Harris irreducibility*: Find appropriate conditions on the distribution of $f_1(\omega, \cdot)$ and in particular $(C_1(\omega), g_1(\omega, \cdot))$ to ensure that $\{X_n\}$ is Harris irreducible (for some recent results see Athreya, 2002).
ii) *Nonuniqueness*: There are examples (see Athreya and Dai, 2002) in the random logistic case when C_1 takes only two values there are two nondegenerate stationary measures. It should be possible to extend that construction to the present more general setting.
iii) *Statistical inference*: Suppose the sequence $\{X_n\}$ has been observed for $0 \leq n \leq N$. Using this data one should be able to do statistical inference on the distribution of $(C_1, g_1(\cdot))$.
Also if it is known that it is supercritical and admits a unique stationary measure π then estimating π from the data $\{X_n = 0 \leq n \leq N\}$ would be very useful (see Athreya and Majumdar, 2001).

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