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# The Borda rule, Condorcet consistency and Condorcet stability

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**Summary.** The Borda rule is known to be the least vulnerable scoring rule to Condorcet inconsistency, Saari (2000). Such inconsistency occurs when the Condorcet winner (the alternative which is preferred to any other alternative by a simple majority) is not selected by the Borda rule. This note exposes the relationship between the Borda rule and the Condorcet $q$-majority principle as well as the Condorcet $q$-majority voting rule. The main result establishes that the Borda rule is Condorcet $q$-majority consistent when $q \geq (k-1)/k$ where $k$ is the number of alternatives. The second result establishes that $(k-1)/k$ is the minimal degree of majority decisiveness corresponding to the Borda rule under sincere voting. The same majority is required to ensure decisiveness under the Borda rule and to ensure that a $q$-rule (the generalized $q$-majority Condorcet rule) is a voting rule.

**Keywords and Phrases:** The Borda rule, $q$-rules, Condorcet consistency, Condorcet stability, Majority decisiveness.

**JEL Classification Numbers:** D71, D72.

## 1 Introduction

In this note we focus on the widely used special majority $q$-rules and on Borda's (1781) method of counts. We do not impose any structure on the finite set of alternatives the group faces and assume that the group contains a finite number of individuals who sincerely report their linear preferences. In the context of $q$-rules the choice function is inspired by Condorcet (1785) approach to social choice that requires the existence of a $q$-rule winner in all possible preference profiles. Such a winner is not 'beaten' by any other alternative by a $q$-majority. In this context it is

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known that, for simple majority, $q = \frac{1}{2}$, although the Borda rule is not Condorcet consistent, it is the least vulnerable scoring rule to Condorcet inconsistency, Saari (2000). Such inconsistency means that there exist preference profiles where the Condorcet winner (the alternative which is preferred to any other alternative by a simple majority) is not selected by the Borda rule. Our first objective is to prove that, when $q \geq (k-1)/k$, the Borda rule is Condorcet consistent.

The problem with $q$-rules is that they need not be voting rules, that is, they do not ensure the existence of a Condorcet $q$-rule winner under any preference profile. Our second objective is to prove that the minimal majority that ensures decisiveness under the Borda rule is precisely the majority required to ensure the decisiveness and, more importantly, the stability of a $q$-rule, namely, that the $q$-rule is a voting rule.

## 2 The setting

Let $N = \{1, \ldots, n\}, n \geq 3$, denote a finite set of voters and $A$ a finite set consisting of $k$ distinct alternatives, $k \geq 3$. Individual preference relations are defined over $A$ and are assumed to be strict (indifference is not allowed). Suppose that the preference relation $L_i$ of individual $i$, $i \in N$, is a strict linear order (complete, transitive and asymmetric relation) over $A$. The set of these orders is denoted $\mathcal{L}$. A preference profile is an $n$-tuple $L = (L_1, L_2, ..., L_n)$ of such linear orders. The set of preference profiles is denoted $\mathcal{L}^n$. A social choice rule $V$ is a mapping from $\mathcal{L}^n$ to the set of non-empty subsets of $A$. This rule specifies the collective choice for any preference profile, $V : \mathcal{L}^n \rightarrow A$. In this study we focus on the voting rules that are usually referred to as special majority rules or $q$-rules and on Borda's strict scoring rule.

Let $\{S_1, S_2, ..., S_k\}$ be a strictly monotone sequence of real numbers, $S_1 < S_2 < ... < S_k$. Each of the $n$ voters ranks the alternatives assigning $S_1$ points to the one ranked last, $S_2$ points to the one ranked next to the last, and so on. Under a strict **scoring rule** a candidate with a maximal total score is elected. The **Borda rule** is a strict scoring rule where $\{S_1, S_2, ..., S_{k-1}, S_k\} = \{0, 1, ..., k-2, k-1\}$.

Let $\frac{1}{2} \leq q \leq 1$. Under a special **q-majority rule**, a selected alternative is not defeated by a $q$-majority by any other alternative in $A$.[1]

## 3 Condorcet consistency

A selected alternative under a $q$-rule given some preference profile $L$ is called a Condorcet $q$-majority winner for $L$. A $q$-majority is a voting rule if every profile $L$ in $\mathcal{L}^n$ has a Condorcet $q$-majority winner. Even if a $q$-rule is not a voting rule, under many preference profiles it may possess a Condorcet $q$-majority winner. When $q = \frac{1}{2}$, no scoring rule is Condorcet consistent, including the Borda rule. However, as shown by Saari (2000), this latter rule is the most immune to Condorcet inconsistency among all scoring rules. The following result establishes that when $q = (k-1)/k$, the Borda rule is totally immune to Condorcet inconsistency.

[1] Such rules are sometimes referred to as supra-majority or qualified-majority rules.

**Theorem 1.** *If $q \geq (k-1)/k$, a Condorcet $q$-majority winner under a preference profile $L$ is selected by the Borda rule.*

*Proof.* Suppose that, under the preference profile $L$, alternative $x$ is a Condorcet $(k-1)/k$ winner. This means that $x$ is preferred by $(k-1)/k$ voters to any other altenative $y$. Under the Borda rule, the minimal difference between the total score assigned by the $(k-1)/k$-majority to alternatives $x$ and $y$ is $S^{maj}(x) - S^{maj}(y) = \frac{(k-1)}{k}n$. The maximal difference between the total score assigned by the $\frac{1}{k}$-minority to alternatives $y$ and $x$ is $S^{\min}(y) - S^{\min}(x) = \frac{1}{k}n(k-1)$. This maximal difference is obtained when $y$ is the most preferred alternative of the minority group members and $x$ is their least preferred alternative. Hence, $(S^{maj}(x) - S^{maj}(y)) - (S^{\min}(y) - S^{\min}(x)) \geq 0$, which implies that alternative $x$ is selected under the Borda rule. The same conclusion is obtained when $x$ is a Concorcet $q$-winner, where $q > (k-1)/k$. □

Note that an alternative selected by the Borda rule under a preference profile $L$ need not be unique or a Condorcet $q$-majority winner, $q \geq (k-1)/k$.

## 4 Majority decisiveness

Resorting to sincere voting[2], a majority coalition is called decisive if it can always impose its will (its most desired alternative), that is, ensure the selection of its consensus under any preference profile. That is, given a voting rule $V$, a coalition of voters $T$ is **decisive** if at every preference profile $L^{a(T)}$, where alternative $a$ is the $T$-majority consensus, $V(L^{a(T)}) = a$. Note that a decisive coalition $T$ can impose its will without resorting to voting manipulations and without having any knowledge on the preferences or the votes of the voters outside $T$. An $\alpha$-**majority decisiveness** means that there exists a coalition $T$, $|T| = \alpha n$, $\frac{1}{2} \leq \alpha < 1$, such that $T$ is decisive. The minimal size of $\alpha$ that depends on the voting rule $V$ applied by the voters can be referred to as the minimal degree of decisiveness.

The following result establishes that the minimal degree of decisiveness corresponding to the Borda rule is equal to $(k-1)/k$.[3]

**Theorem 2.** *Under the Borda rule any special $\alpha$-majority rule is decisive if $\alpha \geq \frac{k-1}{k}$.*

*Proof.* Consider a preference profile where all $\alpha n$ members of an $\alpha$-majority coalition[4], $\frac{1}{2} \leq \alpha < 1$, share the same preference regarding the best (most preferred) alternative $a$ and regarding the second best alternative $b$. Also suppose that at this profile the minority voters' best alternative is $b$ and their bottom (least preferred

[2] In the literature on the stability of $q$-rules, Austen-Smith and Banks (1999), sincere voting is the common assumption. This is the reason why we use it as a benchmark.

[3] This result is implied by Theorem 1 in Baharad and Nitzan (2002). For the sake of completeness we present its proof. When coordinated strategic voting is allowed, the minimal degree of decisiveness corresponding to the Borda rule is equal to $(2k-2)/(3k-2)$, as implied by Theorem 2 in Baharad and Nitzan (2002).

[4] To simplify the proof and with no loss of generality, we select $\alpha$ such that $\alpha n$ is an integer.

alternative) is $a$. Notice that under the Borda rule, if the majority consensus alternative $a$ is selected under this profile, namely, under a profile where the majority consensus $a$ gets minimal support from the minority and the challenger $b$ receives maximal support from the members of both the majority and minority coalitions, then it is selected under any other profile. By definition, an $\alpha$-majority decisiveness exists, if under the assumed profile the total score of alternative $a$ is equal to or larger than the total score of $b$. That is, the condition ensuring the decisiveness of an $\alpha$- majority under the Borda rule is:

$$\alpha n S_k + (1-\alpha)\, n S_1 \geq \alpha n S_{k-1} + (1-\alpha)\, n S_k$$

or, since $S_k = (k-1), S_{k-1} = (k-2)$ and $S_1 = 0$,

$$\alpha \geq \frac{k-1}{k}$$

which completes the proof. □

By definition, a special $\alpha$-majority rule is vulnerable to an $\alpha$-majority tyranny if the $\alpha$-majority rule is a voting rule. The necessary and sufficient condition for a $q$-majority to be a voting rule[5] is that $q \geq \frac{k-1}{k}$ (Greenberg, 1979). In fact, this condition ensures the non-existence of cycles in pair-wise voting as established by Usiskin (1964).[6] Hence, the minimal degree of decisiveness corresponding to the Borda rule is equal to the minimal majority $q$ required to ensure the decisiveness and, more importantly, the non-existence of voting cycles or the stability of the $q$-rule, namely, that it is a voting rule.

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[5] The requirement that a $q$-majority rule is a voting rule is sometimes referred to as the requirement for the existence of a $q$-rule choice function, the existence of a voting equilibrium under the special $q$-majority rule or the existence of a non-empty core in the $q$-majority voting game.

[6] We are indebted to an anonymous referee for turning our attention to the existence of this earlier important result.