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Non-cooperative games with a continuum of players whose payoffs depend on summary statistics*

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Summary. We consider static non-cooperative games with a continuum of small players whose payoffs depend on their own actions and finitely many summary statistics of the aggregate strategy profile. We prove the existence of an equilibrium in pure strategies without any convexity restrictions on payoffs or the common action space. We show that this result applies to a broad class of monopolistic competition models.

Keywords and Phrases: Large games, Summary statistics, Monopolistic competition.

JEL Classification Numbers: C72, D43.

1 Introduction

In many applied game-theoretic models, there is a continuum of small players whose payoffs depend on summary statistics of the aggregate strategy profile. For example, a general class of monopolistic competition models features a continuum of small firms with downward-sloping demand curves of the form

$$q(j) = D_j(p(j), \tilde{p}) \tag{1}$$

where $q(j)$ is firm j's quantity demanded, $p(j)$ is the price charged by firm j, and $\tilde{p}$ is a vector of summary statistics for the distribution of prices. The structure of the demand functions in (1) and the precise nature of the summary statistics $\tilde{p}$

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depends on the source of firms' market power. Vives (1999, 167-176) discusses several different versions of the canonical monopolistic competition model with differentiated products where such demand functions arise from a representative consumer's taste for variety. He also notes that

"In fact, in a monopolistically competitive industry, only aggregate statistics of firms' actions should be payoff relevant when consumers use search strategies based only on a few moments of the distribution of actions of firms" (p. 168). This class of monopolistic competition models, where firms' market power derives from search frictions, is analyzed in Rauh (1997).

In this paper we consider abstract static non-cooperative games with a continuum of small players whose payoffs depend on their own actions and finitely many summary statistics of the aggregate strategy profile. Players are small in the sense that the space of players is an atomless measure space so individual players have measure zero. We prove the existence of an equilibrium *in pure strategies* without any convexity restrictions on payoffs or the common action space. A more limited version of this theorem has already been invoked to prove the main result in Rauh (1997).

We now discuss some related work in the literature on large games. Schmeidler (1973, Theorem 2) proves the existence of an equilibrium in pure strategies when the space of players is the closed unit interval with Lebesgue measure, the common action space is finite, and payoffs depend on the mean of the strategy profile. Rath (1992) restricts the analysis to pure strategies (as we will do in this paper) which allows for a much simpler proof and extension to the case where the space of actions is a compact subset of n-dimensional Euclidean space. The counterexample in Khan, Rath, and Sun (1997, Section 4) shows that these results do not extend to general infinite-dimensional action spaces for Lebesgue measure spaces of players. Khan and Sun (1999, Theorem 2) show that such an extension can be achieved when the space of players is an atomless hyperfinite Loeb measure space. Their (nonstandard) framework also furnishes approximate results for the case where the number of players is large but finite.

All of these results involve the mean only and hence do not apply to monopolistic competition models with summary statistics different from the mean or several summary statistics. In this paper we show that the required extension can be achieved via relatively simple modifications to the model and proof in Rath (1992, Section 3). In so doing, we do not introduce any new techniques or proof strategies; instead, the contribution of the paper is to provide a ready-made, off-the-shelf existence result (for pure strategies) which applies to a much broader class of economic models.

2 The model and result

Let J be the set of players, $\mathcal{J}$ be a σ-algebra of subsets of J, and λ a probability measure on $\mathcal{J}$. We assume that $(J, \mathcal{J}, \lambda)$ is an atomless probability space so individual players are measure-theoretically negligible. For example, J could be the closed unit interval with Lebesgue measure.

Let $\mathbf{R}^k$ denote k-dimensional Euclidean space with all of its usual structure. Each player $j \in J$ chooses a pure strategy from P which is a nonempty compact subset of $\mathbf{R}^m$. For example, P might be the set of feasible prices a firm can charge. Furthermore, firms can have more than one choice variable; e.g., advertising level, spatial location, etc. A *strategy profile* is a measurable function $f : J \to P$.

In different monopolistic competition models, different summary statistics enter into firms' profit functions. A natural case, considered in Rauh (1997), is where $P \subseteq \mathbf{R}$ is the set of feasible prices and the summary statistics are the first n non-central moments of the distribution of prices. To formalize this, recall that a (Lebesgue) integrable function $f : J \to \mathbf{R}^k$ is integrated component-wise as follows. For each $1 \le i \le k$ let $\text{proj}_i : \mathbf{R}^k \to \mathbf{R}$ denote the i'th projection map defined by

$$\text{proj}_i(x_1, x_2, \ldots, x_k) = x_i. \tag{2}$$

If $f_i : J \to \mathbf{R}$ is the i'th coordinate function of f defined by $f_i = \text{proj}_i \circ f$ then

$$\int_J f \, d\lambda \equiv \left(\int_J f_1 \, d\lambda, \int_J f_2 \, d\lambda, \ldots, \int_J f_k \, d\lambda \right). \tag{3}$$

If we define $s : \mathbf{R} \to \mathbf{R}^n$ by $s(x) = (x, x^2, \ldots, x^n)$ then the first n moments of the price profile $f : J \to P \subseteq \mathbf{R}$ are given by $\int_J (s \circ f) \, d\lambda$. In Spence (1976a, b) [see also the discussion in Vives (1999, 167-176)] the set of firms is $[0, N]$ with Lebesgue measure and the summary statistic is $\tilde{q} = \int_0^N s(q(j)) \, dj$ where $q(j)$ is firm j's output and $s : \mathbf{R} \to \mathbf{R}$ is a strictly increasing continuous function.

We now construct a set Σ which contains all possible vectors of summary statistics for a more general model where players consider n summary statistics for each of m choice variables. For each $1 \le r \le m$ let $s_r : \mathbf{R} \to \mathbf{R}^n$ be a continuous function, for each $1 \le q \le n$ let $s_{rq} : \mathbf{R} \to \mathbf{R}$ be defined by $s_{rq} = \text{proj}_q \circ s_r$, and let $s : \mathbf{R}^m \to \mathbf{R}^{mn}$ be defined by $s = (s_1 \circ \text{proj}_1, \ldots, \text{proj}_r, \ldots, s_m \circ \text{proj}_m)$. The function s_{rq} corresponds to the q'th summary statistic of the r'th choice variable. For each $1 \le r \le m$ let $P_r = \text{proj}_r(P)$ (the image of P under proj_r); i.e., the set of all feasible values for the r'th choice variable. Each P_r is nonempty and compact since projection maps are continuous on $\mathbf{R}^k$. Let $m_{rq} = \min_{p_r \in P_r} s_{rq}(p_r)$, $M_{rq} = \max_{p_r \in P_r} s_{rq}(p_r)$, and

$$\Sigma = \prod_{\substack{1 \le r \le m \\ 1 \le q \le n}} [m_{rq}, M_{rq}] \subseteq \mathbf{R}^{mn}. \tag{4}$$

Integration preserves order in the sense of Theorem 18.7(2) in Aliprantis and Burkinshaw (1990) so it follows that $\sigma = \int_J (s \circ f) \, d\lambda \in \Sigma$ for any strategy profile f where σ is the generalized vector of summary statistics.

Let $\mathcal{P}$ denote the space of continuous payoff functions $P \times \Sigma \to \mathbf{R}$ with the supremum metric.

A *game* is then defined by a measurable function $\Gamma : J \to \mathcal{P}$ which assigns to each player $j \in J$ a continuous payoff function $\Gamma(j)(p, \sigma)$ which depends on the player's own action $p \in P$ and the vector $\sigma \in \Sigma$ of summary statistics.

An *equilibrium* (in pure strategies) for such a game is a strategy profile $f : J \to P$ such that each player is playing a best response against the induced vector of summary statistics; i.e.,

$$\Gamma(j)(f(j), \sigma_f) \geq \Gamma(j)(p, \sigma_f) \tag{5}$$

for all $j \in J$ and $p \in P$ where $\sigma_f = \int_J (s \circ f)\, d\lambda$.

Our proof of the existence of equilibrium relies on the following assumption.

Assumption. For each r there is a q such that $s_{rq} : \mathbf{R} \to \mathbf{R}$ is strictly increasing. Without loss of generality, we take $q = 1$.

This assumption allows us to back out the equilibrium strategy profile from the equilibrium summary statistics profile; see the proof below. Note that this assumption is satisfied in the monopolistic competition models discussed above.

Theorem. *Let $(J, \mathcal{J}, \lambda)$ be an atomless probability space. Let $P \subseteq \mathbf{R}^m$ be nonempty and compact (but not necessarily convex). For each $1 \leq r \leq m$ let s_{r1} satisfy the above assumption and Σ be defined as in (4). Let $\mathcal{P}$ denote the space of continuous functions $P \times \Sigma \to \mathbf{R}$ with the supremum metric. Then every game $\Gamma : J \to \mathcal{P}$ has an equilibrium in pure strategies.*

See Definitions 1.4.1 and 8.1.1 in Aubin and Frankowska (1990, pp. 38, 307) for the definitions of measurable and upper semicontinuous correspondence.

Lemma. *The best-response correspondence $B : J \times \Sigma \to P$ defined by*

$$B(j, \sigma) = \mathit{argmax}_{p \in P} \Gamma(j)(p, \sigma) \tag{6}$$

is nonempty-valued, closed-valued, measurable on J for each $\sigma \in \Sigma$, and upper semicontinuous on Σ for each $j \in J$.

The proof of the lemma follows along standard lines [e.g., see Rath (1992, Theorem 2)] and is thus omitted.

We now prove the theorem using standard results on integration of correspondences; e.g., see Aumann (1965), Hildenbrand (1974), and Aubin and Frankowska (1990).

Proof of Theorem. Let $F : J \times \Sigma \to \Sigma$ be the correspondence defined by $F(j, \sigma) = s(B(j, \sigma))$. Let $\Phi : \Sigma \to \Sigma$ be defined by $\Phi(\sigma) = \int_J F(j, \sigma)\, d\lambda$. We now show that Φ is (a) nonempty-valued, (b) convex-valued, and (c) upper semicontinuous.

(a) Let $\sigma \in \Sigma$. By the above lemma and the measurable selection theorem 8.1.3 in Aubin and Frankowska (1990, p. 308) there exists a measurable function $f : J \to P$ such that $f(j) \in B(j, \sigma)$ for all $j \in J$. Then the measurable function $g : J \to \Sigma$ defined by $g = s \circ f$ satisfies $g(j) \in F(j, \sigma)$ for all $j \in J$ which proves (a).

(b) F is nonempty-valued and closed-valued since s is continuous. Since λ is atomless, Φ is convex-valued by Theorem 8.6.3 in Aubin and Frankowska (1990, p. 308).

(c) It is easy to show that F is upper semicontinuous on Σ for each $j \in J$. Since Σ is nonempty and compact, Φ is upper semicontinuous by the lemma in Aumann (1976).

By the Kakutani fixed-point theorem there exists a $\sigma^* \in \Phi(\sigma^*)$. In other words, there exists a measurable function $g : J \to \Sigma$ such that $\sigma^* = \int_J g \, d\lambda$ and $g(j) \in F(j, \sigma^*)$ for all $j \in J$. Let $g_{rq} = \text{proj}_{rq} g$ denote the component function of g which corresponds to the q'th summary statistic of the r'th choice variable. Then it is clear that the strategy profile $f : J \to P$ defined by

$$f = \left(s_{11}^{-1}(g_{11}), s_{21}^{-1}(g_{21}), \ldots, s_{r1}^{-1}(g_{r1}), \ldots, s_{m1}^{-1}(g_{m1})\right) \tag{7}$$

is an equilibrium in pure strategies. Note that the Assumption ensures that the inverses in (7) are well-defined. □

3 Comments

(a) The construction of Σ ensures that $\int_J (s \circ f) \, d\lambda \in \Sigma$ for all strategy profiles f but some $\sigma \in \Sigma$ make no sense. For example, suppose that $P = [0, 1]$ and that players consider the first two moments of the strategy profile so that $\Sigma = [0, 1] \times [0, 1]$. Then $(0, 1) \in \Sigma$ makes no sense because there is no strategy profile with first moment zero and second moment one. For simplicity, we have assumed that players' payoff functions are well-defined for such nonsensical points in Σ but these are never equilibrium points by construction.

(b) The nature of the above equilibrium concept differs across monopolistic competition models. In product differentiation models where the tastes of a representative consumer dictate the summary statistics that enter into firms' demand functions, the equilibrium concept is no different from the usual Nash concept. In Rauh (1997), where consumers and firms approximate the true distribution of prices with finitely many of its moments, equilibrium involves non-rational expectations. In that model each agent has an exogenous "observation" or "expectations" map from moments to beliefs so the relevant equilibrium concept is that of *temporary equilibrium*.

(c) When J is the closed unit interval with Lebesgue measure, $P \subseteq \mathbf{R}$ is nonempty and compact, and payoffs depend on the distribution of actions (either cumulative distribution function or probability measure) then an equilibrium may not exist as shown in Rath, Sun, and Yamashige (1995) and Khan, Rath, and Sun (1997). Recall that any distribution with support contained in P is uniquely determined by all of its moments so the above theorem with $P \subseteq \mathbf{R}$ cannot be extended to the case where payoffs depend on all (infinitely many) moments unless J is taken to be an atomless hyperfinite Loeb measure space as in Khan and Sun (1999).

References

Aliprantis, C.D., Burkinshaw, O.: Principles of real analysis. London: Academic Press 1990

Aubin, J-P., Frankowska, H.: Set-valued analysis. Berlin: Birkhäuser 1990

Aumann, R.J.: Integrals of set-valued functions. Journal Mathematics and Analytical Application **12**, 1–12 (1965)

Aumann, R.J.: An elementary proof that integration preserves upper semicontinuity. Journal of Mathematical Economics **3**, 15–18 (1976)

Hildenbrand, W.: Core and equilibria of a large economy. Princeton: Princeton University Press 1974

Khan, M.A., Sun, Y.N.: Non-cooperative games on hyperfinite Loeb spaces. Journal of Mathematical Economics **31**, 455–492 (1999)

Khan, M.A., Rath, K.P., Sun, Y.N.: On the existence of pure strategy equilibria in games with a continuum of players. Journal of Economic Theory **76**, 13–46 (1997)

Rath, K.P.: A direct proof of the existence of pure strategy equilibria in games with a continuum of players. Economic Theory **2**, 427–433 (1992)

Rath, K.P., Sun, Y.N., Yamashige, S.: The nonexistence of symmetric equilibria in anonymous games with compact action spaces. Journal of Mathematical Economics **24**, 331–346 (1995)

Rauh, M.T.: A model of temporary search market equilibrium. Journal of Economic Theory **77**, 128–153 (1997)

Schmeidler, D.: Equilibrium points of nonatomic games. Journal of Statistical Phys. **7**, 295–300 (1973)

Spence, M.: Product differentiation and welfare. Papers and Proc. American Economic Review **66**, 407–414 (1976)

Spence, M.: Product selection, fixed costs, and monopolistic competition. Review Economic Studies **43**, 217–235 (1976)

Vives, X.: Oligopoly pricing. Cambridge: MIT Press 1999