Economic Theory **20**, 809–829 (2002)

Economic Theory

A leximin characterization of strategy-proof and non-resolute social choice procedures*

Donald E. Campbell[1] **and Jerry S. Kelly**[2]

[1] Department of Economics and The Program in Public Policy, The College of William and Mary, Williamsburg, VA 23187-8795, USA (e-mail: decamp@wm.edu)
[2] Department of Economics, Syracuse University, Syracuse, NY 13245-1090, USA (email: jskelly@maxwell.syr.edu)

Received: November 8, 1999; revised version: September 18, 2001

Summary. We characterize strategy-proof social choice procedures when choice sets need not be singletons. Sets are compared by leximin. For a strategy-proof rule g, there is a positive integer k such that either (i) the choice sets $g(r)$ for all profiles r have the same cardinality k and there is an individual i such that $g(r)$ is the set of alternatives that are the k highest ranking in i's preference ordering, or (ii) all sets of cardinality 1 to k are chosen and there is a coalition L of cardinality k such that $g(r)$ is the union of the tops for the individuals in L. There do not exist any strategy-proof rules such that the choice sets are all of cardinality k^* to k where $1 < k^* < k$.

Keywords and Phrases: Leximin, Non-resolute, Strategy-proof.

JEL Classification Numbers: D70.

1 Introduction

For *resolute* social choice procedures, i.e., when the choice set always contains just a single alternative, Gibbard (1973) and Satterthwaite (1975) have shown (subject to a range condition) that strategy-proofness implies dictatorship. Pattanaik (1973, 1974), Gärdenfors (1976), Kelly (1977), and Barberà (1977) began the investigation of strategy-proofness of non-resolute social choice rules. Ching and Zhou (1997), Baigent (1998), Barberà, Dutta, and Sen (1999), Duggan and

* We thank The Rockefeller Foundation for supporting us at the Bellagio Study Center where this work was initiated. We are also indebted to Matt Jackson, John Duggan, and Salvador Barberà for conversations on an early version of this paper.

Correspondence to: J.S. Kelly

Schwartz (2000), and Campbell and Kelly (1998, 1999, 2000) have resumed the inquiry. Each paper employs a different assumption about the way that individual preference over sets is generated from the primitive individual preference over alternatives, and all of them assume an unrestricted domain of profiles of individual preferences over X. Even though the choice set may contain more than one alternative, Ching and Zhou (1997), Barberà, Dutta, and Sen (1999), and Duggan and Schwartz (2000) assume that a single alternative will eventually be selected from the choice set by some random process. Therefore, they ground their extension principles in expected utility. In this paper, we extend results in Campbell and Kelly (2000) to complete a characterization of non-resolute social choice procedures.

There are two quite different reasons for relaxing resoluteness. First, even when resoluteness is desirable, non-dictatorship is even more important; and the Gibbard-Satterthwaite theorem tells us that resoluteness implies dictatorship for strategy-proof procedures. We then ask a trade-off question: If we relax resoluteness slightly, allowing small non-singleton choice sets, is there then a way to construct strategy-proof social choice rules that are far from dictatorial? This is the rationale treated in the earlier paper.

But sometimes we actually want to chose more than one alternative. First, we might be interested in a social choice procedure that represents a preliminary stage in a process that ultimately chooses a singleton outcome, but it is actually desirable to have several alternatives selected at the preliminary stage. For early phases of competitions, like piano competitions[1], or ice skating and other athletic contests, selecting several contestants to move on to a later stage is highly desired. In another context, a rules committee, reporting a set of competing bills for legislative consideration, typically has to leave some power to the legislature and has to report more than one proposed bill.

In cases where a single alternative will ultimately be selected, it might seem that we should have the individuals take into account the tie-breaking procedure that will be imposed and again apply Gibbard-Satterthwaite (or Gibbard, 1977, 1978). For example, the tie-break might be a random drawing from the chosen subset. But individuals may have no way to analyze the tie-break like this. Consider the William Kapell Piano Competition for which there are two juries. The first narrows the choice down to six finalists. Then a second jury selects a winner from the group of six. We contend that it doesn't make sense for the first panel to try to build a deterministic or probabilistic model of the tie-breaking by the second panel. They don't know the preferences of the members of that second jury and certainly don't know what manipulation strategies might be adopted by members of the second jury.

Second, even when we are modeling single-stage procedures, we may want more than one alternative to be selected. We distinguish between two possible situations. In the first sort, there is a some fixed $k > 1$ and we want the choice set to contain exactly k alternatives:

[1] See Horowitz (1990), for expressions of concern about strategic behavior by judges in early rounds of piano competitions.

1. Exactly k organs are available for transplant, and a hospital committee has to decide which of the many waiting recipients will have transplant surgery.
2. A committee of exactly k individuals is to be chosen from an organization's membership.

In the second sort of situation, the number of alternatives chosen can be a function of the preferences of the voters.

3. The International Mathematical Society will select up to four Fields Medalists to be announced at their next Congress.
4. In the first phase of a piano or ice skating competition, the judges are to choose anywhere from k^* to k competitors (where $1 < k^* < k$) to advance to the second round.

We will show in this paper, that if exactly k alternatives must be selected, then the only strategy-proof social choice rules require a "dictator" in the sense that there must be an individual such that the rule will always pick that person's topmost k alternatives. Where we want strategy-proof rules such that the choice sets contain at least k^*, but not more than k, alternatives (and $1 < k^* < k$), we obtain an impossibility result: no such rules exist.

Because strategy-proofness implies dictatorship when exactly k alternatives must be selected, it might be expected that we could modify the Gibbard-Satterthwaite proof suitably. One approach might be to re-interpret "outcome" as a set of alternatives and then see a social choice rule as a mapping from profiles of orderings of sets to one chosen set. An outcome alluded to in the Gibbard-Satterthwaite theorem is a subset of the feasible set X of alternatives. But we cannot apply the Gibbard-Satterthwaite theorem because it has a full domain assumption that is not appropriate here. If the members of X include w, x, y, and z, no reasonable hypothesis about the way an individual ordering of small sets is related to an ordering of supersets would allow $\{y, z\} \succ \{w, x\}$ when $\{w\} \succ \{x\} \succ \{y\} \succ \{z\}$. This disqualifies many transitive relations on (even small) subsets of X from membership in the individual preference domain when the outcomes are viewed as subsets of X. (Of course, such restrictions on set orderings become more severe as the number of alternatives in X increases.)

While the proofs of "impossibility theorems" often only use a fraction of the full domain anyway, the fact is that there appears to be no ready conversion of an existing proof of Gibbard-Satterthwaite to our problem. Consider, for example, the Barberà (1983) proof. A crucial step establishes that if the orderings R and R' have the same top-ranked alternative, then person 1's options at R (the set of outcomes that 1 can precipitate, given that everyone else reports R') are identical to person 1's options at R'. To prove this Barberà calls on a new ordering R'' that is related to R and R' in a way that would leave R'' outside the set of admissible preferences. Similar difficulties are encountered in translating other steps in the Barberà proof and also in the original Gibbard and Satterthwaite proofs. We can't even use the domain condition from Aswal, Chatterji, and Sen (1999). That paper has a simple domain condition on which every resolute and strategy-proof rule is dictatorial. There are domains satisfying their condition that have only $4 \mid X \mid -6$

members. Even so, their domain condition is too strong to be applicable to the theorems of this paper.

Moreover, in some cases it is implausible that individuals even *have* complete orderings over subsets of X. Imagine physicians asked to take part in the process of determining the recipients of the four available kidneys. They *may* feel comfortable writing down a ranking of the say 25 patients on the transplant list, but feel baffled by the request to rank the 12,650 quadruples of candidates.

Finally, even where individuals possess (lengthy) rankings over all sets of k alternatives, this information might be too unwieldy to cheaply and accurately gather and process. Cost considerations will then require us to define the social choice rule g so that the inputs are individual rankings of the alternatives.

2 Foundations

We take as given a set X of alternatives with $|X| > 2$ and a set $N = \{1, \ldots, n\}$ of individuals, with $n > 1$. A *strong order* $\succ$ (on X) is transitive and complete; in this paper $x \succ y$ only if $x \neq y$, and when we say that $\succ$ is *complete* we mean that either $x \succ y$ or $y \succ x$ holds if $x \neq y$. (Note that we do *not* have $x \succ x$.) Where $r(i)$ is a strong order, we designate the set containing just its top-most element (if it has one) by $r(i)[1]$, the set containing just its second by $r(i)[2]$, the set containing the top two by $r(i)[1, 2]$, the set containing the top k by $r(i)[1 : k]$, etc. A *profile* $r = (r(1), r(2), \ldots, r(n))$ assigns a complete strong order $r(i)$ to each $i \in N$. $L(X)^n$ is the collection of all possible profiles.

A *social choice function* g maps $L(X)^n$ into the family of nonempty subsets of X. For $H \subseteq N$, we say that g is *independent of H* if for any two admissible profiles p and r

$$p(i) = r(i) \text{ for all } i \in N\backslash H \text{ implies } g(p) = g(r).$$

The *range* of g is

$$X_g = \{Y \subseteq X : Y = g(r) \text{ for some } r \in L(X)^n\}.$$

We let $k_{min}(g)$ be the cardinality of the smallest set in X_g and $k_{max}(g)$ is the cardinality of the largest set in X_g. A social choice function g is *regular* if <u>all</u> subsets of X of cardinality $k_{min}(g)$ are in X_g. Regularity has three interpretations: It is a “sufficiently large range” condition, analogous to the one for resolute rules that requires each element to be chosen at one profile at least.[2] Regularity can also be thought of as a weak non-imposition requirement. Full non-imposition would imply that every non-empty subset of X is chosen at some profile. Regularity imposes a minimal size constraint (e.g., “the committee must contain at least three members”), although larger subsets may or may not be in the range of g. Third, regularity is a very weak neutrality condition. Full neutrality would say

[2] The original Gibbard (1973) proof made this assumption, but Satterthwaite (1975) had the general version of the theorem.

that if $g(r) = S$, then if $|T| = |S|$ and T is a subset of X there is a profile p (many, in fact) such that $g(p) = T$. In fact, full neutrality would tell us how to find profiles like p: Let θ be any $1-1$ (permutation) function from S to T. Construct p from r by changing each $r(i)$ to $p(i)$ by replacing each s in S by $\theta(s)$. Regularity weakens this in two ways: (i) We only require p for the case where S is a minimal size set in the range of g; (ii) For $|T| = |S|$, we only require one p with $f(p) = T$ and there need be no connection between p and permutations of elements.

If $k_{max}(g) = 1$, then g is resolute and the Gibbard-Satterthwaite result tells us there is an individual i such that $g(r) = r(i)[1]$ for all r in $L(X)^n$. We turn our attention then to rules where $1 < k_{max}(g)$. There are three cases to be considered depending on the cardinality of $k_{min}(g)$ relative to 1 and $k_{max}(g)$:

$$1 = k_{min}(g) < k_{max}(g); \tag{A}$$

$$1 < k_{min}(g) = k_{max}(g). \tag{B}$$

$$1 < k_{min}(g) < k_{max}(g); \tag{C}$$

The first of these cases is addressed in Campbell and Kelly (2000) and the result will be reported in Section 3 where examples will appear. Cases (B) and (C) are taken up in Sections 4 and 5, respectively.

An individual's input to the social choice function is an ordering on X; but for purposes of defining manipulability, we need an individual to use rankings on X_g. We assume that these rankings are derived from orderings on X by means of "extension principles". An *extension principle* D associates with each $r(i)$ a partial *strong* ordering $D(r(i))$ on non-empty subsets of X; each $D(r(i))$ is irreflexive, antisymmetric, and transitive. (Not every extended preference is complete.) For any $S \subseteq X$, and any profile r we let $r(i) \mid S$ denote the restriction of $r(i)$ to S, while $r \mid S$ represents the n-tuple $(r(1) \mid S, r(2) \mid S, \ldots, r(n) \mid S)$. Then the *restriction of* D *to* S, $D|S$, associates with each $r(i) \mid S$ the restriction of $D(r(i))$ to the power set of S.

Extension principle D is **at least as strong as** extension principle E if and only if, for all $r(i)$, $AE(r(i))B$ implies $AD(r(i))B$. An n-tuple of extension principles $D = (D_1, D_2, \ldots, D_n)$ is called a *context*. The set of all contexts is denoted by $\mathscr{D}$. The restriction of $\mathscr{D}$ to S, $\mathscr{D}|$S, is the collection of n-tuples $D|S \equiv (D_1|S, D_2|S, \ldots, D_n|S)$.

Profile r^* is an *i-variant* of profile r if r^* differs from r only in its value for individual i. We say that for a given $D = (D_1, D_2, \ldots, D_n)$, g is **manipulable** by i at r if there exists an i-variant r^* of r such that $g(r^*)D_i(r(i))g(r)$. We say g is **strategy-proof for** D if for every i and r, g is not manipulable by i at r. The designer of a social choice function must not only make it flexible enough to deal with a large domain of profiles of preferences – because we can not anticipate what people's preferences will be; the function must also be strategy-proof under many extension principles – because we can not anticipate what people's extension principles will be. g is *strategy-proof with respect to* $\mathscr{D}$ if it is strategy-proof for each $D \in \mathscr{D}$.

The results of this paper lean heavily on a collection $\mathscr{D}$ of contexts defined in terms of the following leximin extension principle L, which does generate a complete order on all non-empty subsets of X. Leximin sets $BL(r(i))C$ if the least-preferred element of B is preferred to the least-preferred element of C. If the least-preferred alternatives are identical then ignore them and rank B and C by comparing the second worst alternative in each set. If they are also identical, proceed inductively. Formally:

Leximin extension[3]. Given $r(i)$ and a subset Y of X, define $l^1(Y)$ to be the lowest ranking member of Y. With $l^t(Y)$ defined, let $l^{t+1}(Y)$ be the empty set if $Y = \{l^1(Y), l^2(Y), \ldots, l^t(Y)\}$, and otherwise $l^{t+1}(Y) = l^1(Y \setminus \{l^1(Y), l^2(Y), \ldots, l^t(Y)\})$. Given distinct subsets Y and Z of X, let t be the smallest integer such that $l^t(Y) \neq l^t(Z)$.

$$YL(r(i))Z \text{ if } l^t(Y)r(i)l^t(Z) \text{ or } l^t(Z) = \emptyset.$$

We will say $\mathscr{D}$ *supports leximin* when for every individual i, every profile r, and every pair A, B, of non-empty subsets of X, if $AL(r(i))B$, then there exists a $(D_1, D_2, \ldots, D_n) \in \mathscr{D}$ with $AD_i(r(i))B$. If $\mathscr{D}$ supports leximin on X, then $\mathscr{D}|S$ supports leximin on S. As is shown in Campbell and Kelly (2000), if g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin, then it is strategy-proof under the single $(L, L, \ldots, L)$. Accordingly, throughout this paper, without further comments, whenever we want to prove a characterization result that assumes g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin, we will confine our attention to just the single context $(L, L, \ldots, L)$.

3 Further background

In Campbell and Kelly (2000) we have shown that if g is a regular social choice function that is strategy-proof with respect to a context $\mathscr{D}$ that supports leximin and if $1 = k_{min}(g) < k_{max}(g)$, then there is a coalition H (i.e., a subset of N) of cardinality $k_{max}(g)$ such that for every admissible profile r

$$g(r) = \cup_{i \in H} r(i)[1].$$

For such rules all non-empty sets of cardinality $k_{max}(g)$ and all subsets of intermediate cardinalities will be in the range. We say that g is oligarchical in this case, and H is the oligarchy.[4]

We next establish some notation. Given two profiles r and r^*, we will often need to refer to the sequence of profiles

[3] Leximin was first used extensively in the study of strategy proofness by Pattanaik. For a survey of this work, see Pattanaik (1978).

[4] Demange (1987) identifies a fairly large family of non-oligarchical social choice rules that can not be manipulated by any individual or coalition. She employs an extension principal (*optimism*) that is strictly coarser than leximax, in the sense that when optimism ranks Y above Z then so does leximax, but the converse is not true.

$$\begin{aligned} r &= r_0 = (r(1), r(2), r(3), \ldots, r(n)) \\ & \quad\; r_1 = (r^*(1), r(2), r(3), \ldots, r(n)) \\ & \quad\; r_2 = (r^*(1), r^*(2), r(3), \ldots, r(n)) \\ & \quad\; \succ \qquad\qquad\qquad\qquad\quad \succ \\ r^* &= r_n = (r^*(1), r^*(2), r^*(3), \ldots, r^*(n)). \end{aligned}$$

We refer to this as the *standard sequence* from r to r^*. Two successive entries in this sequence are

$$\begin{aligned} r_{j-1} &= (r^*(1), r^*(2), \ldots, r^*(j-1), r(j), \ldots, r(n)), \text{ and} \\ r_j &= (r^*(1), r^*(2), \ldots, r^*(j-1), r^*(j), \ldots, r(n)) \end{aligned}$$

in which r_j is created from r_{j-1} by replacing the *jth* component of r_{j-1} by $r^*(j)$ and leaving all other components of r_{j-1} unchanged. For appropriately chosen profiles r and r^* we will show that g is manipulable by j by showing that either

$$\begin{aligned} & g(r_j)L(r(j))g(r_{j-1}) \text{ at } r_{j-1}; \text{ or} \\ & g(r_{j-1})L(r^*(j))g(r_j) \text{ at } r_j. \end{aligned}$$

In the first preliminary result we connect g values at r with the g values at r^* where r^* differs from r by bringing all the elements of $g(r)$ (and possibly a few more) to the top of everyone's ordering. Given a profile r and a subset S of X, $r(S)$ denotes a new profile constructed in the following way: For each $r(i)$ in r, the alternatives in S are raised above all the elements in $X \backslash S$. Within S and within $X \backslash S$, alternatives are ordered as they were in $r(i)$; that is $r(S)|S = r|S$ and $r(S)|X \backslash S = r|X \backslash S$. When S is small, we will drop the set bracket notation, referring to $r(x)$ and $r(x, y)$ instead of $r(\{x\})$ and $r(\{x, y\})$.

Lemma 1 (The Completeness Lemma). *If g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin and $g(r) \subseteq S \subseteq X$, then $g(r(S)) = g(r)$.*

We refer to this as the Completeness Lemma because of the central role played by the completeness of leximin extension, although the members of $D_i(r(i))$ may be far from complete.

The next preliminary lemma tells us that if we bring the elements of a sufficiently large proper subset Y to the top of everyone's ordering at a profile, then the elements chosen at that profile by g will all be in Y.

Lemma 2. *Suppose g is a regular social choice function that is strategy-proof with respect to a $\mathscr{D}$ that supports leximin. Let Y be a non-empty subset of X that contains a set Z in the range of g and let r be a profile such that for all i, all $y \in Y$ and $x \in X \backslash Y$, $yr(i)x$. Then $g(r) \subseteq Y$.*

We are going to use these lemmas later on, but they are also key in proving the following result from Campbell and Kelly (2000):

Theorem 3. *Suppose that g is a regular social choice function with $1 = k_{min}(g) < k_{max}(g)$ and g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin. Then there exists a subset H of N such that for every admissible profile r*

$$g(r) = \cup_{i \in H} r(i)[1].$$

Remark 4. There is a definition of *leximax* that is analogous to that for leximin and the rule $g(r) = \cup r(i)[1]$ is strategy-proof with respect to a $\mathscr{D}$ that supports leximax. But strategy-proofness with respect to a $\mathscr{D}$ that supports leximax does not force $g(r)$ to be the union of the tops of the member of a (small) coalition. In fact, g can be regular and strategy-proof but not independent of anyone's preferences. To see this, let $X = \{x, y, z\}$. Set $g(r) = \{z\}$ if everyone has z at the top of her ordering. Otherwise $g(r) \subseteq \{x, y\}$. If fact, $g(r) = \{x, y\}$ unless either (i) everyone prefers x to y, in which case $g(r) = \{x\}$, or (ii) everyone prefers y to x, in which case $g(r) = \{y\}$.

It is straightforward to show that g is strategy-proof with respect to leximax. But if even one person, say individual 1, employs the leximin extension principle there will be an opportunity for manipulation. Suppose

<u>1</u>	<u>2</u>	<u>3</u>	...	<u>n</u>
x	z	z		z
z	y	y		y
y	x	x		x

Then $g(r) = \{x, y\}$. But if person 1 reports z as her top-ranked alternative then g will select $\{z\}$. Because leximin gives $\{z\}$ $L(r(1))$ $\{x, y\}$, g is manipulable by 1 at r. However, we believe that the assumption that there are only three alternatives in the feasible set is essential to the existence of a regular rule with $k_{min}(g) = 1 < k_{max}(g)$ that is strategy-proof with respect to leximax.

4 Range condition: $1 < k_{min}(g) = k_{max}(g)$

In this section, we will show that if g is a regular social choice function that is strategy-proof with respect to a $\mathscr{D}$ that supports leximin and if $1 < k_{min}(g) = k_{max}(g)$, then there is an individual i who dictates g in the sense of the following

Theorem 5. *Suppose g is a regular social choice function, g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin, and $1 < k_{min}(g) = k_{max}(g)$. Then there is an i such that $g(r) = r(i)[1 : k_{max}(g)]$ for all r.*

Proof. If $k_{max}(g) = m = |X|$, the result is obvious; $g(r) = X = r(i)[1 : m]$ for *all* i. The remainder of the proof takes seven steps. Since we will do induction on the number $m > k_{max}(g)$ of alternatives, the first step consists of an induction basis. From now on, denote $k_{max}(g) = k_{min}(g)$ by k and the collection of all subsets of cardinality k by $k^*(X)$, all of which are in X_g.

Step 1. **Basis.** $m = k + 1$. The k-element sets that g can choose each eliminate just a single alternative. If an individual i has ordering $r(i)$, then $r(i)[1 : k]$

is the k-element set that excludes the alternative lowest ranked in $r(i)$ and $r(i)[1,2,\ldots,k-1,k+1]$ is the k-element set that excludes the alternative ranked next-to-last in $r(i)$ and so on. Associate with ordering $r(i)$ the leximin ordering $L(r(i))$ that ranks one set over another if the alternative excluded in the first is ranked lower in $r(i)$ than the alternative excluded in the second:

$$\begin{array}{c}\underline{\mathrm{L(r(i))}:}\\[1ex] r(i)[1,2,\ldots,k]\\ r(i)[1,2,\ldots,k-1,k+1]\\ \succ \qquad \succ\\ r(i)[2,3,\ldots,k,k+1]\end{array}$$

and with profile $r = (r(1), r(2), r(3),\ldots,r(n))$ the profile $R = (L(r(1)), L(r(2)), L(r(3)), \ldots, L(r(n)))$. Because this gives a $1-1$ correspondence between the profiles on X and the profiles on $k^*(X)$, we can associate with g the function G on profiles on $k^*(X)$ which takes $R = (L(r(1)), L(r(2)), L(r(3)),\ldots,L(r(n)))$ to $g(r(1), r(2), r(3),\ldots,r(n))$, i.e., $G(R) = g(r)$. Since g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin, G is strategy-proof in the Gibbard-Satterthwaite sense on the set of alternatives from $k^*(X)$. By Gibbard-Satterthwaite, G is dictatorial with dictator i; i.e., $G(R) = L(r(i))[1]$. But as we noted already in this step, $L(r(i))[1]$ is $r(i)[1,2,\ldots,k] = r(i)[1:k]$. Combining we have,

$$g(r) = G(R) = L(r(i))[1] = r(i)[1:k].$$

Induction stage. Suppose now the conclusion is true for $m = M-1$; we wish to prove that it is true for M.

Step 2. Fix an alternative α and look at the subdomain R_α that consists of all profiles such that α is at the bottom of everyone's ordering. Given a profile r on $X\backslash\{\alpha\}$, construct r_α by adding α at the bottom of everyone's ordering. Define $g_\alpha(r) = g(r_\alpha)$ so that $g_\alpha(r) \subseteq X\backslash\{\alpha\}$ by Lemma 2. g_α is a regular social choice function on $X\backslash\{\alpha\}$ with $k_{max}(g) = k_{min}(g) = k$ and is strategy-proof with respect to $\mathscr{D}|(X\backslash\{\alpha\})$ since g is strategy-proof with respect to $\mathscr{D}$. By the induction assumption, there is an individual $i(\alpha)$ such that, at each r, g_α selects the k-highest alternatives in $r(i)$.

We next show that for all $\alpha, \beta, i(\alpha) = i(\beta)$. This will require several steps (3 through 5). Notice that since our Basis step was for $m = k+1$, we now have $m \geq k+2$.

Step 3. We first treat the case $m > k+2$. Without loss of generality, assume that $i(\alpha) = 1$. Choose distinct $\beta, t \in X\backslash\{\alpha\}$ set $q = m-3$ and then let $X\backslash\{t,\alpha,\beta\} = \{y_1, y_2, \ldots, y_q, y_{q+1}\}$. Let r be the profile

$$\begin{array}{lll} r: & 1: & y_1y_2\ldots y_k t y_{k+1}\ldots y_q\beta\alpha\\ & 2: & t y_1y_2\ldots y_k y_{k+1}\ldots y_q\beta\alpha\\ & \succ & \qquad \succ\\ & n: & t y_1y_2\ldots y_k y_{k+1}\ldots y_q\beta\alpha\end{array}$$

where $r(j) = r(j')$ for $j, j' \geq 2$. Profile r^* is constructed from r by interchanging α and β:

$$\begin{array}{lll} r^*: & 1: & y_1 y_2 \ldots y_k t y_{k+1} \ldots y_q \alpha\beta \\ & 2: & t y_1 y_2 \ldots y_k y_{k+1} \ldots y_q \alpha\beta \\ & \succ & \quad \succ \\ & n: & t y_1 y_2 \ldots y_k y_{k+1} \ldots y_q \alpha\beta \end{array}$$

$g(r) = \{y_1, y_2, \ldots, y_k\}$ because $i(\alpha) = 1$. Consider the standard sequence from r to r^*. Suppose $g(r_j) = g(r)$. We have $g(r_{j+1}) \subset \{t, y_1, y_2, \ldots, y_k\}$ by Lemma 2. Now, $g(r) = g(r_j)$ is the lowest ranking k-element subset of $\{t, y_1, y_2, \ldots, y_k\}$ for the leximin extension of $r_j(j+1)$. Therefore, $g(r_{j+1}) = g(r)$, or else $j+1$ would manipulate at r_j. Therefore, $g(r) = g(r_n) = g(r^*)$, by induction. Therefore, $i(\alpha) = i(\beta)$ for all α, β.

Now we turn to the remaining case: $m = k + 2$. We first show that it is sufficient to analyze the case of two individuals:

Step 4. We will show that there exists a regular social choice function that is strategy-proof with respect to a D that supports leximin for $n > 2$ with $i(\alpha) \neq i(\beta)$ if and only if there is such a rule for $n = 2$. First suppose there is a regular and strategy-proof g for $n = 2$ such that $i(\alpha) \neq i(\beta)$ for some α, β. Then define

$$h(r(1), r(2), \ldots, r(n)) = g(r(1), r(2)).$$

Since g is regular and strategy-proof, so also is h. Clearly for this h, $i(\alpha) \neq i(\beta)$.

Next suppose there is a regular and strategy-proof rule h for $n > 2$ such that $i(\alpha) \neq i(\beta)$ for some α, β. Without loss of generality, let $i(\alpha) = 1$ and $i(\beta) = 2$. Then define

$$g(r(1), r(2)) = h(r(1), r(2), r(2), \ldots r(2)).$$

For $g, i(\alpha) \neq i(\beta)$. We show that g is regular and strategy-proof. Regularity follows from Lemma 2. To see that g is strategy-proof, suppose instead that g is manipulable. But if g were manipulable by 1, we would have

$$\begin{array}{l} g(r^*(1), r(2)) L(r(1)) g(r(1), r(2)) \text{ or} \\ h(r^*(1), r(2), r(2), \ldots r(2)) L(r(1)) h(r(1), r(2), r(2), \ldots r(2)) \end{array}$$

which immediately contradicts the strategy-proofness of h. So if g is manipulable, it must be by 2, i.e., we would have

$$\begin{array}{l} g(r(1), r^*(2)) L(r(2)) g(r(1), r(2)) \text{ or} \\ h(r(1), r^*(2), r^*(2), \ldots r^*(2)) L(r(2)) h(r(1), r(2), r(2), \ldots r(2)). \end{array}$$

Consider the standard sequence from

$$(r(1), r(2), r(2), \ldots r(2)) \text{ to } (r(1), r^*(2), r^*(2), \ldots r^*(2)).$$

Suppose that there is no i such that $h(r_{i+1})L(r(2))h(r_i)$. Then $h(r_0)L(r(2))$ $h(r_1)L(r(2))$ $h(r_2)\ldots L(r(2))$ $h(r_n)$, and thus $h(r_0)L(r(2))$ $h(r_n)$, because $L(r(2))$ is complete and transitive. This contradicts the fact that $h(r_n)L(r(2))h(r_0)$. Therefore, there must be an i such that $h(r_{i+1})L(r(2))h(r_i)$. But since at r_i individual i's set ranking is the same as $L(r(2))$, h would be manipulable by i at r_i.

Step 5. Now that Step 4 allows us, with $m = k + 2$, to focus on the case of two individuals, we wish to show that there do not exist two alternatives, α and β, such that $i(\alpha) \neq i(\beta)$. We prove this by assuming there are such α and β and finding a contradiction. Without loss of generality, $i(x) = 1$ and $i(w) = 2$. Since $m = k + 2 \geq 4$, one of the two individuals is $i(\alpha)$ for at least two alternatives; we suppose $i(z) = 1$ and $X = (x, z, w, y_1, \ldots, y_{k-1})$.

Let r^1 be the profile

$$\begin{array}{lll} r^1: & 1: & xzy_1,\ldots,y_{k-1}w \\ & 2: & zy_1,\ldots,y_{k-1}xw \end{array}$$

Since $i(w) = 2, g(r^1) = \{z, y_1, \ldots, y_{k-1}\}$. Then at profile

$$\begin{array}{lll} r^2: & 1: & xzy_1,\ldots,y_{k-1}w \\ & 2: & zy_1,\ldots,y_{k-1}wx \end{array}$$

we must also have $g(r^2) = \{z, y_1, \ldots, y_{k-1}\}$ or 2 will manipulate at r^2.

Consider

$$\begin{array}{lll} r^3: & 1: & xy_1,\ldots,y_{k-1}wz \\ & 2: & zy_1,\ldots,y_{k-1}wx \end{array}$$

$g(r^3) \neq \{x, y_1, \ldots, y_{k-1}\}$, otherwise person 1 would manipulate at r^2, because individual 1 prefers $\{x, y_1, \ldots, y_{k-1}\}$ to $g(r^2) = \{z, y_1, \ldots, y_{k-1}\}$ at r^2. We will show that each of the other possibilities for $g(r^3)$ leads to a contradiction, so we will be forced to conclude that $i(\alpha) = i(\beta)$.

First notice that $z \notin g(r^3)$. For otherwise 1 could change to

$$\begin{array}{lll} r^4: & 1: & y_1,\ldots,y_{k-1}wzx \\ & 2: & zy_1,\ldots,y_{k-1}wx \end{array}$$

and get $g(r^4) = \{w, y_1, \ldots, y_{k-1}\}$ (since $i(x) = 1$) which, under leximin, 1 prefers at r^3 to any set containing z.

So $g(r^3)$ is obtained by deleting one member from $X\backslash\{z\}$. Also, $g(r^3)$ doesn't contain both x and w or 2 would change to

$$\begin{array}{lll} r^5: & 1: & xy_1,\ldots,y_{k-1}wz \\ & 2: & xy_1,\ldots,y_{k-1}wz \end{array}$$

and get $g(r^5) = \{x, y_1, \ldots, y_{k-1}\}$ (since $i(z) = 1$) which 2 prefers at r^3 to any subset of $X\backslash\{z\}$ that contains both x and w. We know that $g(r^3) \neq \{x, y_1, \ldots, y_{k-1}\}$, and thus $g(r^3) = \{w, y_1, \ldots, y_{k-1}\}$. Consider

$$\begin{array}{lll} r^6: & 1: & xy_1,\ldots,y_{k-1}wz \\ & 2: & y_1,\ldots,y_{k-1}wxz \end{array}$$

$g(r^6) = \{x, y_1, \ldots, y_{k-1}\}$ (since $i(z) = 1$). But 2 would manipulate at r^6 (to r^3). Therefore we can exclude the possibility that there exist α, β in X with $i(\alpha) \neq i(\beta)$.

Step 6. From Step 5 we know there is an individual i such that for any profile r such that everyone has the same bottom alternative, $g(r) = r(i)[1 : k]$. Without loss of generality, $i = 1$. Now suppose there is an r (obviously one without a common bottom alternative) such that $g(r) \neq r(1)[1 : k]$. Construct an r^* from r which differs from r only for individuals 2 to n : for them, set $r^*(j) = r(j)(S)$ for $S = g(r)$. That is, $g(r)$ is brought to the top without disturbing the rankings within either $g(r)$ or $X\backslash g(r)$. Then $g(r^*) = g(r)$. For if they were different, construct the standard sequence from r to r^* and let t be the smallest integer such that $g(r_t) \neq g(r)$. g must be manipulable by t at r_t. Hence we will assume in the remainder that $g(r)$ is at the top for individuals 2 to n.

Step 7. We treat two cases.

Case 1. Suppose there is an alternative t in X that is not in $r(1)[1 : k] \cup g(r)$. Construct r^* from r by lowering t to the bottom for individuals 2 to n. It is easy to see that $g(r^*) = g(r)$. For if $g(r^*) \neq g(r)$, construct the standard sequence from r to r^* and let j be the least integer such that $g(r_j) \neq g(r); j > 0$. If $t \in g(r_j)$, g is manipulable by j at r_j. If $t \notin g(r_j)$, then $L(r(j))$ and $L(r^*(j))$ order $g(r_{j-1})[= g(r)]$ and $g(r_j)$ the same. Hence g is manipulable by j at either r_{j-1} or r_j. But with $g(r^*) = g(r)$, we see that at r^*, individual 1 would manipulate by lowering t to the bottom getting $r(1)[1 : k]$ which individual 1 prefers to $g(r)$.

Case 2. $X = r(1)[1 : k] \cup g(r)$. Since $g(r)$ and $r(1)[1, \ldots, k]$ both contain k elements, the sets $r(1)[1 : k]\backslash g(r) = X\backslash g(r)$ and $g(r)\backslash r(1)[1 : k] = X\backslash r(1)[1 : k]$ contain the same number of elements. But since $m > k + 1$, this number must be at least 2. Let $y = r(1)[m - 1]$ and $z = r(1)[m]$. Then $y, z \in g(r)\backslash r(1)[1 : k]$. Choose any $x \in r(1)[1 : k]\backslash g(r)$. Construct r^* from r by lowering x to the bottom for individuals 2 to n. Then $g(r^*) = g(r)$ by an argument that by now is quite familiar. Now, from r^*, construct r^{**} by bringing x to the bottom for individual 1. The resulting choice set $g(r^{**})$, while it may contain y, will not contain z and hence will be preferred by individual 1 to r^*. Hence g is manipulable by individual 1 at r^*. □

Remark 6. To see the importance of the regularity condition, that all $k_{min}(g)$ element sets are chosen somewhere, consider the following example. There is an odd number of individuals, each with strong preferences over the four alternatives in $X = \{w, x, y, z\}$. $g(r)$ is the two-element set that consists of the simple majority winner between w and x together with the simple majority winner between y and z. This g *is* strategy-proof, and there is no individual with the kind of power described in our theorem. But $\{w, x\}$ is not in the range of g, even when everyone ranks both of those above both y and z.

Remark 7. A collection $\mathscr{D}$ of contexts having the property that for every individual i, every profile r, and every pair of sets A, B, if the leximin extension of $r(i)$ has $AL(r((i))B$, then there exists a $D = (D_1, D_2, \ldots, D_n) \in \mathscr{D}$ with $AD_i(r(i))B$ may not have *any* strategy-proof social choice functions. All we are showing is that if one exists, then it must have the form

$$g(r) = r(i)[1 : k].$$

But if such a rule is manipulable at some other context in $\mathscr{D}$, then there don't exist *any* rules strategy-proof with respect to $\mathscr{D}$.

Remark 8. The rule $g(r) = r(i)[1 : k]$ is also strategy-proof with respect to *leximax*. Remark 4 of Section 3 presents a rule that is strategy-proof with respect to leximax, and everyone's preferences are taken into account. For that rule $g(r)$ is either a singleton or a pair, so $k_{min}(g) = 1 = k_{max}(g) - 1$. If we have a regular rule for which $k_{min}(g) = k_{max}(g) = m - 1$, then the proof that we employed for this case works for leximax as well. In fact, leximax coincides with leximin in this case – as do other extension principles that are weaker than leximin in general, but coincide with leximin when $k_{min}(g) = k_{max}(g) = m - 1$. Hence an example of a regular social choice rule with $k_{min}(g) = 2$ that is strategy-proof with leximax and that might take into account the preferences of more than one individual would have to involve an X with more than three alternatives. We have neither an example like this nor a proof that none exists. It is worthy of note, however, that our earlier leximax example required three alternatives and that we have been unable to provide a similar example with $m > 3$.

5 Range condition: $1 < k_{min}(g) < k_{max}(g)$

In Section 3 we described a class of strategy-proof rules with choice sets that contain anywhere from one to three alternatives: the union of the tops for a coalition of three individuals. In Section 4, we found a class of strategy-proof rules with all choice sets of cardinality two: the top two alternatives for some "dictatorial" individual. If we want to construct a social choice rule that generates choice sets that contain just two *or* three alternatives, we might consider a hybrid of these two types of rules:

$$g(r) = r(1)[1, 2] \cup r(2)[1].$$

This rule is regular; its range consists of all sets of two or three alternatives. However, g is *not* strategy-proof with respect to a $\mathscr{D}$ that supports leximin. Consider profile r with

$$\begin{array}{ll} 1: & abc \ldots yz \\ 2: & zabc \ldots y \end{array}$$

Then $g(r) = \{a, b, z\}$. If individual 1 were to submit the same ordering as individual 2, $zabc \ldots y$, then the choice set would be $\{a, z\}$. For $r(1)$, the leximin extension ranks $\{a, z\}$ above $\{a, b, z\}$ and so g is manipulable by individual 1 at r.

Here is a second example of a social choice rule g with range consisting of all two element sets and X. There are three alternatives and an odd number of individuals. $g(r)$ is the union of all the top-most ranked alternatives, unless that set is a singleton, say α; in that case, $g(r)$ is the set consisting of α together with the simple majority winner between the other two alternatives. This rule is regular; its range consists of all sets of two or three alternatives. However, g is *not* strategy-proof with respect to a $\mathscr{D}$ that supports leximin. Consider profile r with

$$\begin{aligned} 1:&\quad abc \\ i > 1:&\quad acb \end{aligned}$$

Then $g(r) = \{a, c\}$. If individual 1 were to submit *bac*, then the choice set would be $\{a, b\}$. For $r(1)$, the leximin extension ranks $\{a, b\}$ above $\{a, c\}$ and so g is manipulable by individual 1 at r. We will show in this section that there does not exist any strategy-proof social choice rule g with range consisting of all two element sets and at least one three element set.

In fact we have a very general impossibility result:

Theorem 9. *There does not exist a regular social choice function g with $1 < k_{min}(g) < k_{max}(g)$ that is strategy-proof with respect to a $\mathscr{D}$ that supports leximin.*

Proof. We will assume there does exist a g satisfying all the requirements of the theorem and derive a contradiction. We first show that the existence of such a g implies the existence of a rule h that also has those properties but where h has a very simple range. From X_g, the range of g, select a set S of next-to-smallest cardinality; i.e., $|S| > k_{min}(g)$ and there is no set in X_g of cardinality intermediate between $k_{min}(g)$ and $|S|$. Let p be a fixed ordering on $X\backslash$S. We now construct a regular and strategy-proof rule h that contains sets of only two cardinalities. Given any profile u on S, construct profile u^* on X by the rules:

$$\begin{aligned} &\text{(i)} && u^* = u^*(S); \\ &\text{(ii)} && u^* \mid S = u \mid S; \\ &\text{(iii)} && u^* \mid (X\backslash S) = (p, p, \ldots, p). \end{aligned}$$

$g(u^*) \subseteq S$ by Lemma 2, so we may define a social choice function h on S by

$$h(u) = g(u^*).$$

Then h is strategy-proof with respect to $\mathscr{D}|S$ since g is strategy-proof with respect to $\mathscr{D}$. Let T be any subset of S of cardinality $k_{min}(g)$. By the regularity assumption, there exists a profile r such that $g(r) = T$. By Lemma 1, $g(r(S)) = T$. Let r^* be constructed so that

$$\begin{aligned} &\text{(i)} && r^* = r^*(S); \\ &\text{(ii)} && r^* \mid S = r \mid S; \\ &\text{(iii)} && r^* \mid (X\backslash S) = (p, p, \ldots, p). \end{aligned}$$

By a standard sequence argument, $g(r^*) = T$. But then if u is r^* restricted to $S, h(u) = T$. A similar argument shows S to be in the range of h. Thus for Theorem 9, it is sufficient to prove the following lemma.

Lemma 10. *There does not exist a regular social choice function g with $1 < k_{min}(g) < k_{max}(g)$ such that either $g(r) = X$ or $|g(r)| = k_{min}(g)$ where g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin.*

Proof.
Step 1. We are first going to establish that some individual has substantial power on a subdomain of g. For each $x \in X$, consider g_x, the restriction of g to those profiles in which everyone has x at their bottom. By Lemma 2, $g_x(r) \subseteq X \backslash \{x\}$ and so by our range assumption, $|g_x(r)| = k_{min}(g)$ for all r. g_x is regular because for every set $V \subseteq X \backslash \{x\}$ with $|V| = k_{min}(g)$, there is a profile r with x at everyone's bottom and $r(V) = r$. Then $g_x(r) = V$. $D|(X \backslash \{x\})$ supports leximin on $X \backslash \{x\}$ since $\mathscr{D}$ supports leximin on X. g_x is strategy-proof with respect to $\mathscr{D}|(X \backslash \{x\})$ since g is strategy-proof with respect to $\mathscr{D}$. By Theorem 5 of the previous section, g_x has a dictator, $i(x)$. Observe that if $k_{min}(g) = m - 1$, *everyone* is a dictator for g_x. Accordingly we will defer until Step 5 the case where $k_{min}(g) = m - 1$; through Step 4, $m > k_{min}(g) + 1$. In this case, the dictator for g_x is unique and is labeled $i(x)$.

Step 2. We now show that the same individual dictates each g_x; i.e., for all x, y in $X, i(x) = i(y)$. It is useful to break our analysis into two cases:

$$\begin{array}{ll} \text{(I)} & k_{max}(g) > k_{min}(g) + 2; \\ \text{(II)} & k_{max}(g) = k_{min}(g) + 2 \end{array}$$

We start with (I) first and treat (II) in Step 3. To show $i(x) = i(y)$, we suppose $i(x) \neq i(y)$ and seek a contradiction. Without loss of generality, $i(x) = 1$ and $i(y) = 2$. Since $m > k_{min}(g) + 2$, we can find a set S of cardinality $k_{min}(g) + 1$ in $X \backslash \{x, y\}$. Let p be a strong order on S, let q be a strong order on $X \backslash (\{x, y\} \cup S)$, and construct profiles

$$\begin{array}{rrlrrl} r: & 1: & pqxy & \quad r^*: & 1: & pqxy \\ & 2: & p^{-1}qyx & & 2: & p^{-1}qyx \\ & i > 2: & pqyx & & i > 2: & pqxy \end{array}$$

where p^{-1} is p reversed. Now $g(r) = p[1 : k_{min}(g)]$ since individual 1 could force that by switching x and y. Consider the standard sequence from r to r^*. We have $r = r_0 = r_1 = r_2$. Suppose $g(r_j) = g(r)$. Then $g(r_{j+1}) = g(r)$: If $j < 2$ then $r_{j+1} = r_j$. If $j \geq 2$ then $g(r_{j+1}) = g(r_j)$, or $j + 1$ would manipulate at r_{j+1}. Therefore, $g(r^*) = g(r_n) = g(r)$ by induction. But $i(y) = 2$, so person 2 can get $p^{-1}[1 : k_{min}(g)]$ at r^* just by modifying $r^*(2)$ by sliding y to the bottom of his reported preference ordering. Therefore, $i(x) = i(y)$ for all x, y.

Step 3. Now we turn to Case (II): $k_{max}(g) = k_{min}(g) + 2$. We first establish the result for $n = 2$ and then extend it to the general case. So for the moment, the collection of individuals is just $\{1, 2\}$. Suppose $i(x) \neq i(y)$ say $i(x) = 1$ and $i(y) = 2$; where $X = \{a, b, c, d, \ldots, j, k, x, y\}$. Without loss of generality, also $i(a) = 1$. Consider

$$r^1: \quad \begin{array}{l} 1: abc\ldots jkxy \\ 2: bc\ldots jkxay \end{array}$$

$g(r^1) = \{b, c, \ldots, j, k, x\}$ because $i(y) = 2$.

$$r^2: \quad \begin{array}{l} 1: bc\ldots jkayx \\ 2: bc\ldots jkxay \end{array}$$

$y \notin g(r^2)$ or individual 2 brings x to bottom and gets $\{a, b, c, \ldots, j, k\}$ because $i(x) = 1$, and $\{a, b, c, \ldots, j, k\}$ is better for individual 2 than anything containing y. At r^1, the leximin extension of individual 1's ordering ranks the set chosen there, $\{b, c, \ldots, j, k, x\}$, lowest among all the $k_{min}(g)$ element sets not containing y. Hence, if $g(r^2) \neq \{b, c, \ldots, j, k, x\}$, individual 1 would manipulate from r^1 to r^2. Therefore, $g(r^2) = \{b, c, \ldots, j, k, x\}$.

$$r^3: \quad \begin{array}{l} 1: bc\ldots jkayx \\ 2: bc\ldots jkxya \end{array}$$

At r^3, $\{b, c, \ldots, j, k, x\} = g(r^2)$ is individual 2's highest ranked set (under leximin) of cardinality $k_{min}(g)$; if that weren't the choice set at r^3, individual 2 would manipulate to r^2; so $g(r^3) = \{b, c, \ldots, j, k, x\}$.

$$r^4: \quad \begin{array}{l} 1: bc\ldots jkyxa \\ 2: bc\ldots jkxya \end{array}$$

$g(r^4) = \{b, c, \ldots, j, k, y\}$ by $i(a) = 1$. At r^3, individual 1 prefers $g(r^4)$ to $g(r^3) = \{b, c, \ldots, j, k, x\}$; hence g is manipulable by individual 1 at r^3. Hence $i(x) = i(y)$ for all x, y.

We now use this result for $n = 2$ to extend the conclusion to general n. So suppose that for n individuals there exists a regular strategy-proof g with $k_{max}(g) = k_{min}(g) + 2$ such that $i(x) \neq i(y)$; say $i(x) = 1$ and $i(y) = 2$. Define a rule h for two individuals based on g:

$$h(p, q) = g(p, q, q, \ldots, q).$$

Then clearly $k_{min}(h) = k_{min}(g)$ and h is regular (just put an arbitrary set of cardinality $k_{min}(g)$ at the top of both p and q). For h, $i(x) = 1$ and $i(y) = 2$, since that is true for g. Suppose h is manipulable. If it is manipulable by individual 1, g is immediately seen to be manipulable by individual 1. So we assume h is manipulable by individual 2; i.e., there are p, q, and q^* with individual 2 ranking

$$h(p, q^*) = g(p, q^*, q^*, \ldots, q^*).$$

above

$$h(p, q) = g(p, q, q, \ldots, q).$$

in the leximin extension of q. But then in the standard sequence from $r = (p, q, q, \ldots, q)$ to $r^* = (p, q^*, q^*, \ldots, q^*)$, there must be a j such that $g(r_{j+1})$ is ranked above $g(r_j)$ in the leximin extension of q which means g is manipulable by j at r_j. Hence strategy-proofness of g implies strategy-proofness of h which we have shown is impossible if $i(x) \neq i(y)$. Hence $i(x) = i(y)$ for all x, y; without loss of generality, we assume $i(x) = i(y) =$ individual 1.

Step 4. Now we show that individual 1's power extends beyond the profiles on which everyone has the same last element; individual 1 dictates at every profile: That is, we will show that $g(r) = r(1)[1 : k_{min}(g)]$ at arbitrary profile r.

Let $z = r(1)[m]$, individual 1's bottom alternative, and let $r^* = r(X \backslash \{z\})$, i.e., bring z to the bottom of everyone's ordering, where $g(r^*) = r^*(1)[1 : k_{min}(g)] = r(1)[1 : k_{min}(g)]$, and consider the standard sequence from r^* to r. $z \notin g(r_0) = g(r^*)$. If $z \notin g(r_i)$, then $z \notin g(r_{i+1})$ or g would be manipulable by $i+1$ at r_{i+1}. Since $r(i)|(X \backslash \{z\}) = r^*(i)|(X \backslash \{z\})$, the leximin extensions of $r(i)$ and $r^*(i)$ order $g(r_i)$ and $g(r_{i+1})$ the same way. So if $g(r_i) \neq g(r_{i+1})$, g is manipulable. Therefore $g(r_i) = g(r_{i+1})$ for all i. But then $g(r) = g(r_n) = g(r_0) = g(r^*) = r(1)[1 : k_{min}(g)]$.

Step 5. Having completed all three steps for the case where $m > k_{min}(g) + 1$, we must deal with the possibility that $m = k_{min}(g)+1$. So $k_{max}(g) = m$, $k_{min}(g) = m-1$, and the range of g consists of X and all subsets of X of cardinality $m-1$. We first establish that if g is strategy-proof with respect to a $\mathscr{D}$ that supports leximin, then $g(r) = X$ only if for every $x \in X$ there exists an i such that $r(i)[1] = \{x\}$.

Suppose $g(r) = X$, but that there is an alternative x such that $r(i)[1] \neq \{x\}$ for any individual i. Let r^* be the profile $r(X \backslash \{x\})$ that looks like r except that x is brought to the bottom of everyone's ordering. $g(r^*) \subseteq X \backslash \{x\}$ by Lemma 2 and so $g(r^*) = X \backslash \{x\}$ since $k_{min}(g) = m - 1$. Construct the standard sequence from r to r^* and let j be the largest integer such that $g(r_j) = X$. Almost every subset of X of cardinality $m-1$ is preferred at r by $j+1$ to X. If any such subset were $g(r_{j+1})$, then g would be manipulable by j+1 at r_j. The only exception, the only subset of cardinality $m - 1$ that is *not* preferred to X by $j + 1$ at r_j is $r(j + 1)[2 : m]$. Hence $g(r_{j+1}) = r(j + 1)[2 : m]$. But since $\{x\} \neq r(j + 1)[1]$, $r^*(j + 1)[2 : m] = r(j + 1)[2 : m]$. But with $g(r_{j+1}) = r^*(j + 1)[2 : m]$, g is manipulable at r_{j+1} by $j + 1$ who prefers $X = g(r_j)$ to $g(r_{j+1})$.

As an easy corollary to this step we have confirmed Lemma 10 for the case $n < m$:

If $n < m = k_{max}(g)$, there does not exist a regular social choice function g with $k_{min}(g) = m - 1$ that is strategy-proof with respect to a $\mathscr{D}$ that supports leximin.

Step 6. We now extend this non-existence claim to cases with $n \geq m$. Our proof will be by induction on n and in this step we carry out the basis argument for the induction:

With $m = n = k_{max}(g)$, there does not exist a regular social choice function g with $k_{min}(g) = m - 1$ that is strategy proof with respect to a $\mathscr{D}$ that supports leximin.

Let r be a profile with $g(r) = X = \{x_1, x_2, \ldots, x_m\}$. By Step 5, every alternative must be at someone's top. Without loss of generality (we can just re-label alternatives), we assume $\{x_i\} = r(i)[1]$.

Let r^* be the same as r except (possibly) that, for each individual, alternatives ranked second through last are ordered by subscript:

$$r^*(i) : x_i x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_{m-1} x_m .$$

If $g(r^*) \neq X$ then $|g(r^*)| = m - 1$. In that case, construct the standard sequence from r to r^*. Let j be the largest integer such that $g(r_j) = X$. Then $\mid g(r_{j+1}) \mid = m - 1$. But at r, individual $j+1$ prefers all but one of the $(m-1)$-element subsets to X. So $g(r_{j+1})$ must be $r(j+1)[2 : m]$. But $r^*(j+1)[2 : m] = r(j+1)[2 : m]$, so g must be manipulable by $j+1$ at r_{j+1}. So $g(r^*) = X$.

Now construct r' from r^* by lowering x_m in individual m's ordering to second place:

$$\begin{array}{lrl} r' : & 1: & x_1x_2x_3 \ldots x_{m-1}x_m \\ & 2: & x_2x_1x_3 \ldots x_{m-1}x_m \\ & \succ & \succ \\ & m-1: & x_{m-1}x_1x_2 \ldots x_{m-2}x_m \\ & m: & x_1x_mx_2 \ldots x_{m-2}x_{m-1} \end{array}$$

By Step 5, $\mid g(r') \mid = m - 1$. But the only k-element set individual m doesn't prefer to X at r^* to X is $r^*(m)[2 : m]$. Therefore $g(r') = r^*(m)[2 : m] = \{x_1, x_2, x_3, \ldots, x_{m-2}, x_{m-1}\}$.

Since at r', individual $m-1$ is getting his most preferred k-element set, any rearrangement within his top k alternatives must leave the choice unchanged. Hence also $g(r'') = \{x_1, x_2, x_3, \ldots, x_{m-2}, x_{m-1}\}$ at

$$\begin{array}{lrl} r'' : & 1: & x_1x_2x_3 \ldots x_{m-1}x_m \\ & 2: & x_2x_1x_3 \ldots x_{m-1}x_m \\ & \succ & \succ \\ & m-1: & x_1x_2 \ldots x_{m-2}x_{m-1}x_m \\ & m: & x_1x_mx_2x_3 \ldots x_{m-2}x_{m-1} \end{array}$$

Now go back to r^* and construct r^{**} by lowering x_{m-1} in individual $m-1$'s ordering to next-to-last place:

$$\begin{array}{lrl} r^{**} : & 1: & x_1x_2x_3 \ldots x_{m-1}x_m \\ & 2: & x_2x_1x_3 \ldots x_{m-1}x_m \\ & \succ & \succ \\ & m-1: & x_1x_2 \ldots x_{m-2}x_{m-1}x_m \\ & m: & x_mx_1x_2 \ldots x_{m-2}x_{m-1} \end{array}$$

By Step 5, $\mid g(r^{**}) \mid = m - 1$. But the only $(m-1)$-element set individual $m-1$ doesn't prefer at r^* to X is $r^*(m-1)[2 : m]$. Therefore $g(r^{**}) = \{x_1, x_2, x_3, \ldots, x_{m-2}, x_m\}$.

Now notice that r^{**} and r'' differ only in the ordering for individual m. At r'', individual m prefers $g(r^{**}) = \{x_1, x_2, x_3, \ldots, x_{m-2}, x_m\}$ to $g(r'') = \{x_1, x_2, x_3, \ldots, x_{m-2}, x_{m-1}\}$. Therefore, g is manipulable by m at r''.

Step 7. This is the induction stage of our argument. Suppose Lemma 10 is true for $n = n^*$ and we wish to prove it for n^*+1. Let r be a profile at which $g(r) = X$. By Step 5, every alternative must be at someone's top. Since $n^* + 1 > m$, some alternative must be at the top for two individuals. Without loss of generality, assume $r(n^*)[1] = r(n^* + 1)[1]$. Construct r^* as follows:

$$\begin{array}{ll}(1) & r^*(i) = r(i) \text{ for } i \leqslant n^*;\\ (2) & r^*(n^*+1) = r^*(n^*) = r(n^*).\end{array}$$

Then $g(r^*) = X$ or g will be manipulable at r^* if $g(r^*) = r(n^* + 1)[2 : m]$, and at r otherwise.

Consider the restriction g^* of g to profiles where the orderings for individuals n^* and $n^* + 1$ are identical. By the first paragraph, X is in the range of g^* and by Lemma 2, all sets of cardinality $m - 1 = k_{min}(g^*)$ are in the range of g^*. g^* is strategy-proof with respect to $\mathscr{D}$ if g is. Now define a social choice function for n^* individuals: at profile $u = (u(1), u(2), \ldots, u(n))$,

$$h(u) = g^*(u(1), u(2), \ldots, u(n^*), u(n^*)).$$

Then $k_{max}(h) = m$ and $k_{min}(h) = m - 1$ and h is regular. We show h is strategy-proof. For suppose h were manipulable by individual j' at r; if $j' < n^*$ there is an j'-variant $r^* = (r(1), \ldots, r(j' - 1), r(j'), r^*(j' + 1), \ldots, r(n^*))$ of r with

$$h(r(1), \ldots, r(j' - 1), r^*(j'), r(j' + 1), \ldots, r(n^*))L(r(j'))h(r(1), \ldots,$$
$$r(j' - 1), r(j' + 1), r(j'), \ldots, r(n^*)).$$

That is,

$$g^*(r(1), r(2), \ldots, r(j' - 1), r^*(j'), r(j' + 1), \ldots, r(n^*), r(n^*))L(r(j'))$$
$$g^*(r(1), r(2), \ldots, r(j' - 1), r(j'), r(j' + 1), \ldots, r(n^*), r(n^*))$$

but that means g^* is manipulable contrary to our assumption. So it would have to be $j' = n^*$ so

$$h(r(1), r(2), \ldots, r^*(n^*))L(r(n^*))h(r(1), r(2), \ldots, r(n^*)).$$

That is, $g(r(1), r(2), \ldots, r(j' - 1), r(j'), r(j' + 1), \ldots, r^*(n^*), r^*(n^*))$ is preferred under the leximin extension of $r(n^*)$ to $g(r(1), r(2), \ldots, r(j' - 1), r(j'), r(j' + 1), \ldots, r(n^*), r(n^*))$. For this to be true, either $g(r(1), r(2), \ldots, r(j' - 1), r(j'), r(j' + 1), \ldots, r^*(n^*), r^*(n^*))$ is preferred to $g(r(1), r(2), \ldots, r(j' - 1), r(j'), r(j' + 1), \ldots, r(n^*), r^*(n^*))$ or $g(r(1), r(2), \ldots, r(j' - 1), r(j'), r(j' + 1), \ldots, r(n^*), r^*(n^*))$ is preferred to $g(r(1), r(2), \ldots, r(j' - 1), r(j'), r(j' + 1), \ldots, r(n^*), r(n^*))$

In either case, g is manipulable; since that violates our strategy-proof assumption on g, h must not be manipulable. But this violates our induction hypothesis. □

Remark 11. As was true with Theorem 3, this result does not assume that the range of g contains any sets of cardinality k for $k_{min}(g) < k < k_{max}(g)$, or even more than one set of cardinality $k_{max}(g)$. Actually, we did not require full regularity. It will be sufficient that for some set S in the range of g, with $|S| > k_{min}(g)$, all the subsets of S of cardinality $k_{min}(g)$ are in the range of g.

But we do need some degree of regularity. Consider the rule

$$g(r) = \{a, b\} \cup r(1)[1].$$

The range of g consists of $\{a, b\}$ and all three element supersets of $\{a, b\}$. g is strategy-proof with respect to leximin. Of course, this rule still represents a poor trade-off since it is independent of the preferences of all but one individual. It is possible to construct non-regular rules that select just sets of cardinality two or three that are strategy-proof with respect to leximin but for which *everyone* has a say. As an example, suppose $g(r) = \{a, b\}$ unless at r everyone has c as their top-most alternative, then $g(r) = \{a, b, c\}$. Of course this rule has an extremely constrained set of outcomes.

Remark 12. Theorem 9 is much stronger in its conclusion than most Gibbard-Satterthwaite type results; there isn't even a "dictatorial" exception.

Remark 13. As with Theorem 3, Theorem 9 will not work if we modify it by replacing leximin by leximax. Consider the following rule g: If, at r, everyone has the same top two elements, then $g(r)$ is the set containing those two elements; otherwise $g(r) = X$. This rule is manipulable with respect to a $\mathscr{D}$ that supports leximin. But, if everyone uses leximax, this rule is strategyproof.

Remark 14. Although it has nothing to do with "social" choice, even with $n = 1$, there does not exist any regular social choice function g with $1 < k_{min}(g) < k_{max}(g)$ that is strategy-proof with respect to a $\mathscr{D}$ that supports leximin. For suppose r is a "profile", $r(1)$, at which $g(r)$ has cardinality $k_{max}(g)$. Let $r^*(1)$ be a "profile" at which $g(r^*) = r(1)[1 : k_{min}(g)]$. Then at r, individual 1 would manipulate by submitting $r^*(1)$.

6 Conclusion

We say that a rule g is *oligarchical* if there is a coalition $H \subseteq N$ such that for every profile r we have $g(r) = \cup_{i \in H} \{Y \in X_g$: no member of X_g ranks above Y in the leximin extension of $r(i)\}$. Theorems 3, 5 and 9 together establish that if g is regular and strategy-proof with respect to a $\mathscr{D}$ that supports leximin then g is oligarchical. The converse is not true, because strategy-proofness puts severe restrictions on the range of g. We know that if the range of g does not contain all singleton subsets of X, but it is strategy-proof and regular, then there is an individual i and a positive integer k such that $g(r) = r(i)[1 : k]$ at every profile r.

References

Aswal, N., Chatterji, S., Sen, A.: Dictatorial domains. Indian Statistical Institute Working Paper (1999)

Baigent, N.: Strategy-proofness of social choice correspondences with restricted domains. University of Graz (1998)

Barberà, S.: The manipulation of social choice mechanisms that do not leave too much to chance. Econometrica **45**, 1573–1588 (1977)

Barberà, S.: Strategy-proofness and pivotal voters: a direct proof of the Gibbard-Satterthwaite theorem. International Economic Review **24**, 413–418 (1983)

Barberà, S., Dutta, B., Sen, A.: Strategyproof social choice correspondences, Indian Statistical Institute (1999)

Campbell, D. E., Kelly, J. S.: The incompatibility of strategy-proofness and the Condorcet condition. Social Choice and Welfare **15**, 583–592 (1998)

Campbell, D. E., Kelly, J. S.: Trade-offs for democratic preference revelation. Journal of Public Economic Theory **1**, 465–473 (1999)

Campbell, D. E., Kelly, J. S.: A trade-off result for preference revelation. Journal of Mathematical Economics **34**, 129–142 (2000)

Ching, S., Zhou, L.: Multi-valued strategy-proof social choice rules. Duke University (1997)

Demange, G.: Nonmanipulable cores. Econometrica **55**, 1057–1074 (1987)

Duggan, J., Schwartz, T.: Strategic manipulation without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized. Wallis Institute of Political Economy, University of Rochester (1997)

Gärdenfors, P.: Manipulation of social choice functions. Journal of Economic Theory **13**, 217–228 (1976)

Gibbard, A.: Manipulation of voting schemes: a general result. Econometrica **41**, 587–601 (1973)

Gibbard, A.: Manipulation of schemes that mix voting with chance. Econometrica **4**, 665–681 (1977)

Gibbard, A.: Straightforwardness of game forms with lotteries as outcomes. Econometrica **46**, 595–614 (1978)

Horowitz, J.: The ivory trade. New York: Summit Books 1990

Kelly, J. S.: Strategy-proofness and social choice functions without single-valuedness. Econometrica **45**, 439–446 (1977)

Pattanaik, P. K.: On the stability of sincere voting situations. Journal of Economic Theory **6**, 558–574 (1973)

Pattanaik, P. K.: Stability of sincere voting under some classes of non-binary group decision procedures. Journal of Economic Theory **8**, 206–224 (1974)

Pattanaik, P. K.: Strategy and group choice. Amsterdam: North Holland 1978

Satterthwaite, M. A.: Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory **10**, 187–217 (1975)