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Stability and limit cycles in one-dimensional dynamic optimisations of competitive agents with a market externality

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Abstract. This paper considers low dimensional (more precisely, one state variable) dynamic optimisation problems of competitive agents. These individual decisions lead to a dynamic externality for the evolution of the system. However, the impact of an individual and competitive agent is negligible and thus each agent considers this evolution as exogenous data. This leads, assuming rational expectations (perfect foresight due to the deterministic set up), to motions in the three dimensional space of state, costate and externality. Considering the fact that such externalities are widespread, e.g., R&D in the literature on new growth theory, pollution in environmental economics, etc., the incorporation of such externalities due to competitive markets is important, yet this incorporation may alter the stability of the system. Indeed, complex policies such as stable limit cycles are sustainable in such a low-dimensional economy, even for a separable and strictly concave model.

Key words: Limit cycles – Intertemporal optimisation – Externality

JEL-classification: C61; C62; D62

1 Introduction

The purpose of this investigation is to obtain conditions for limit cycles in the space with the lowest possible dimension and to characterise the stability properties of the associated system. From Hartl (1987) we know that one-state variable, continuous-time dynamic optimisations allow only for a

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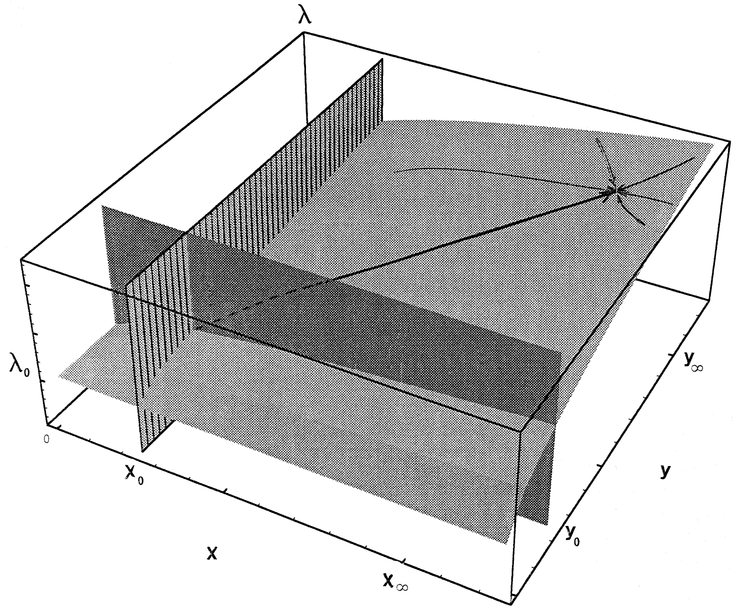

Fig. 1. Stable manifold for a one-dimensional control problem with a dynamic externality

monotonic solution. On the other hand, it is well known that higher-order dimensional optimisation problems allow for complex strategies. See for example the conjecture in Ryder-Heal (1973) about limit cycles in consumption, and Benhabib-Nishimura (1979) for limit cycles in a two sector growth model. Wirl (1992) gives four necessary conditions for a Hopf bifurcation (and thus for limit cycles) in two dimensional optimal control problems and establishes the existence of stable limit cycles for separable and thus potentially very simple models. Recently, Greiner-Hanusch (1994) and Greiner-Semmler (1996) presented one-dimensional optimisation problems with a market externality (learning by doing) and proved, using the Hopf bifurcation theorem, that stable limit cycles describe an intertemporal competitive, perfect foresight equilibrium.[1] The underlying geometry of the possibility of limit cycles is shown in Fig. 1. The optimal policy of each competitive firm is characterised by a two-dimensional manifold in the three dimensional Euclidean space of the state (x), the costate (λ) and the externality (y). Given initial conditions of the state, x_0, and the externality, y_0, one can determine an unique value of the costate, λ_0, that ensures convergence to the steady state $(x_\infty, y_\infty, \lambda_\infty)$; initial conditions outside this plane cannot converge to the equilibrium, and this conditional stability ensures the uniqueness of the optimal policy. Figure 1 constructs the op-

[1] For the sake of completeness, Rauscher (undated) showed in a paper on renewable resource harvesting that limit cycles may be optimal if aggregate market conditions are introduced. However, this example does not fit exactly the following framework; in particular, it is not amenable to the Hopf bifurcation theorem due to a singularity.

timal starting value of the costate λ_0 and the entire paths for the given initial conditions, and shows some examples of other flows.[2]

The task of this paper is to study one-dimensional dynamic optimisation problems with an externality and to investigate the stability of the associated competitive equilibrium, that is, to study the possible motions within this manifold, to establish conditions that allow for complex behavior, and, in particular, to derive conditions for limit cycles. This is the content of Section 3. Section 2 introduces the optimisation problem and the intertemporal competitive equilibrium. Section 4 presents a simple example that induces limit cycles as a competitive equilibrium. Concluding remarks complete this investigation.

2 The agents' intertemporal optimisation problem and the perfect foresight, competitive equilibrium

We consider a competitive economy where the representative agent solves the following dynamic optimisation problem:

$$\max_{\{u(t)\}} \int_0^\infty \exp(-rt)v(x(t), u(t), y(t))dt, \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t), y(t)), \quad x(0) = x_0. \tag{2}$$

That is, each agent chooses a trajectory $\{u(t), t \in [0, \infty)\}$ such that the present value aggregate of the individual profits is maximised. The agents consider the evolution of y(t) as exogenous data, because y is negligibly affected by the representative agent's actions due to the supposition of a competitive market.

We assume that the associated Hamiltonian,[3]

$$H = v(x, u, y) + \lambda f(x, u, y), \tag{3}$$

is strictly concave in the control, i.e., $H_{uu} < 0$, and jointly concave in control and state, i.e., $H_{xx} \leq 0$ and $(H_{uu}H_{xx} - Hux^2) \leq 0$. Therefore, the following conditions are sufficient for an optimal, interior (e.g., because control is not constrained) decision, denoted u^*:

$$H_u = 0 => u^* = U(x, \lambda; y), \quad U_x = -H_{ux}/H_{uu},$$
$$U_\lambda = -f_u/H_{uu}, \quad U_y = -H_{xy}/H_{uu}, \tag{4}$$

$$\dot{\lambda} = r\lambda - H_x, \tag{5}$$

$$\lim_{t\to\infty} \exp(-rt)x(t)\lambda(t) = 0. \tag{6}$$

Although H is concave with respect to state and control in order to ensure that the first order conditions are sufficient for optimality, it need not be jointly concave with respect to all three variables, u, x and y. In particular,

[2] It is well known that a plane allows for more complex flows than the stable node shown, such as damped, undamped, and stable limit cycles.

[3] From now on we drop the time subscript.

the inclusion of the externality y may give rise to increasing returns, thus to a nonconcavity in all three variables, which is a crucial characteristic of the intertemporal competitive equilibria that are studied in the new theory of economic growth (See e.g., Lucas (1988) and Romer (1990)).

The agents take y as exogenous data but neglect the impact of their actions on y due to the supposition of competition. The following analysis assumes rational expectations of agents, i.e., perfect foresight due to the deterministic framework: Each agent accounts for the entire evolution of this externality, $\{y(t), t \in [0, \infty)\}$, which is a solution of the following differential equation:[4]

$$\dot{y} = g(x, u, y), \quad y(0) = y_0. \tag{7}$$

3 Stability analysis of the competitive equilibrium

The optimality conditions of the representative agent's optimisation problem amended for the evolution of the externality result in three differential equations:

$$\begin{aligned} \dot{x} &= f(x, U(x, \lambda, y), y),\ x(0) = x_0, \\ \dot{\lambda} &= r\lambda - H_x(x, U(x, \lambda, y), y),\ \lim_{t\to\infty} \exp(-rt)x(t)\lambda(t) = 0, \\ \dot{y}\ &= g(x, U(x, \lambda, y), y), y(0) = y_0. \end{aligned} \tag{8}$$

Assuming the existence of a stationary solution of (8), the local stability of this dynamic system depends on the eigenvalues of the Jacobian (evaluated at the steady state)

$$J = \frac{1}{H_{uu}} \begin{bmatrix} f_x H_{uu} - f_u H_{ux} & -f_u^2 & f_y H_{uu} - f_u H_{uy} \\ H_{ax}^2 - H_{xx} H_{uu} & (r - f_x) H_{uu} + f_u H_{ux} & H_{ux} H_{uy} - H_{xy} \\ g_x H_{uu} - g_u H_{ux} & -f_u g_u & g_y H_{uu} - g_u H_{uy} \end{bmatrix}. \tag{9}$$

The eigenvalues, $e_i, i = 1, \ldots, 3$, are the roots of the following characteristic polynomial:

$$P(e) = e^3 - tr(J)e^2 + we - \|J\|, \tag{10}$$

with the coefficients:

$$tr(J) = r + g_y - g_u H_{uy}/H_{uu'} \tag{11}$$

[4] Examples fitting into this framework, (1)–(2) and (7), may be found in the literature on the new economic growth theory that considers a full range of externalities and spillovers, some of them dynamic such as learning by doing, see Greiner-Hanusch (1994) and Greiner-Semmler (1996), and the environment, see e.g., Marrewijk-Ploeg-Verbeek (1993) and Withagen (1995). Indeed, environmental economics and renewable resource extraction, where agents operating under laissez faire neglect externalities of their actions on the environmental commons, are other potential and topical areas of applications.

$$w := \left\| \begin{matrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial x} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{matrix} \right\| + \left\| \begin{matrix} \frac{\partial \dot{\lambda}}{\partial \lambda} & \frac{\partial \dot{\lambda}}{\partial y} \\ \frac{\partial \dot{y}}{\partial \lambda} & \frac{\partial \dot{y}}{\partial y} \end{matrix} \right\| + \left\| \begin{matrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{matrix} \right\| => $$

$$\begin{aligned} w = &- (f_x^2 H_{uu} - 2 f_u f_x H_{ux} + f_u^2 H_{xx})/H_{uu} \\ &+ r(f_x + g_y) \\ &- f_y g_x \\ &+ (H_{ux}/H_{uu})(f_y g_u - r f_u) + (H_{uy}/H_{uu})(f_u g_x - r g_u) \\ &- (H_{xy}/H_{uu}) f_u g_u, \end{aligned} \tag{12}$$

$$\begin{aligned} \|J\| = &(r - f_x)(f_x g_y - f_y g_x) \\ &-(H_{ux}/H_{uu})[(r - f_x)(f_u g_y - f_y g_u) + f_u(f_y g_x - f_x g_y)] \\ &-(H_{uy}/H_{uu})(r - f_x)(f_x g_u - f_u g_x) \\ &-(H_{xx}/H_{uu}) f_u (f_u g_y - f_y g_u) \\ &-(H_{xy}/H_{uu}) f_u (f_x g_u - f_u g_x). \end{aligned} \tag{13}$$

These coefficients of the characteristic polynomial, tr(J), w and $\|J\|$, determine the signs and the properties (real, complex or purely imaginary) of the eigenvalues $\{e_1, e_2, e_3\}$. On the other hand, assuming the eigenvalues, the coefficients of the characteristic polynomial can be expressed in terms of these eigenvalues. In other words, solving the following linear equation system (linear in terms of the coefficients of the polynomial p)

$$e_1^3 - tr(J)e_1^2 + we_1 - \|J\| = 0, \tag{14.1}$$

$$e_2^3 - tr(J)e_2^2 + we_2 - \|J\| = 0, \tag{14.2}$$

$$e_3^3 - tr(J)e_3^2 + we_3 - \|J\| = 0, \tag{14.3}$$

for the coefficients of (10), i.e., for tr(J), w and $\|J\|$, yields these coefficients as functions of the eigenvalues. (See Table 1 for real roots, and Table 2, for a pair of complex conjugate roots of (10).) This in turn establishes some helpful relations between the coefficients and the eigenvalues.

The roots of (10), i.e., the eigenvalues of J, determine the stability of the system. We exclude in the following global stability of (8) because of the associated indeterminacy of the optimal policy[5]. Given this restriction to unique optimal policies, the discussion can be limited to those cases where at least one eigenvalue is positive. Now if the other two eigenvalues are both negative and real, then this ensures not only saddlepoint stability but also (local) monotonicity. A pair of complex conjugate eigenvalues with negative real parts still ensures saddlepoint stability, but damped oscillations become optimal. A pair of complex conjugate eigenvalues with positive real parts destroys the asymptotic stability of the steady state and the motions become locally exploding spirals. These spirals may either diverge (the first kind of instability) or may converge to a limit cycle, which requires a nonlinear

[5] However, the issue of indeterminate optimal policies in particular in the context of (endogenous) growth models receives considerable attention in the recent literature. See e.g., Boldrin-Rustichini (1994), Benhabib-Perli (1994) and Greiner-Semmler (1996).

Table 1. Relation between the real eigenvalues e_i and the coefficients of the characteristic polynomial

		$e_1>0$ $e_2>0$ $e_3>0$	$e_1>0$ $e_2>0$ $e_3<0$	$e_1>0$ $e_2<0$ $e_3<0$	$e_1<0$ $e_2<0$ $e_3<0$
$tr(J)$	$=e_1+e_2+e_3$	+	?	?	–
w	$=e_1(e_2+e_3)+e_2e_3$	+	?	?	+
$\|J\|$	$=e_1e_2e_3$	+	–	+	–

Table 2. Implications of the existence of a pair of conjugate complex eigenvalues ${}_2e_3=\rho \pm i\omega$ on the coefficients of the characteristic polynomial

		$e_1>0$ $\rho>0$	$e_1<0$ $\rho>0$	$e_1>0$ $\rho<0$	$e_1<0$ $\rho<0$
$tr(J)$	$=e_1+2\rho$	+	?	?	–
w	$=2\rho e_1+(\rho^2+\omega^2)$	+	?	?	+
$\|J\|$	$=e_1(\rho^2+\omega^2)$	+	–	+	–

system (8). A second positive real eigenvalue reduces the stability domain from the plane of initial conditions for x and y to a one-dimensional manifold, and aside from this set with Lebesque measure zero no trajectory can reach the steady state (the second kind of an instability for (8)).

From Tables 1 and 2 and a few additional considerations (see the Appendix) follow some characterisations of the stability of (8) in terms of the coefficients of the characteristic polynomial (and thus implicitly in terms of the model parameters):

Proposition 1. *A positive determinant of the Jacobian J and a negative coefficient w, are sufficient for saddlepoint stability, i.e., a unique two-dimensional manifold determines the set of stable flows that converge to the steady state.*

Proposition 2. *A positive determinant and a negative trace of the Jacobian are sufficient for saddlepoint stability.*

Proposition 3. *A negative determinant of the Jacobian implies instability, i.e., except for a one dimensional manifold of initial conditions* (x_0, y_0), *it is impossible to reach the steady state.*

Proposition 4. *The existence of a pair of purely imaginary eigenvalues requires that the following conditions are simultaneously satisfied:*

$$(i)\ tr(J) > 0,\ \ (ii)\ \|J\| > 0,\quad (iii)\ w > 0,\quad (iv)\ \|J\| = tr(J)w.$$

Moreover, stable limit cycles as optimal long run strategies exist.

The proof of these propositions 1 to 4 is relegated to the Appendix. Propositions 1 and 2 give sufficient conditions for saddlepoint stability, but do not rule out the possibility of transient oscillations. Proposition 4 addresses the most crucial condition for a Hopf bifurcation (for details see Guckenheimer-Holmes (1983)), the existence of a pair of purely imaginary

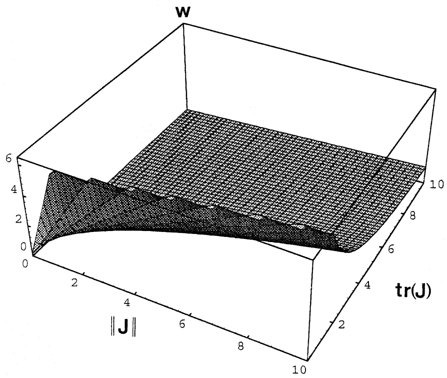

Fig. 2. Set of purely imaginary eigenvalues in the space $(\|J\|, tr(J), w)$

eigenvalues. Conditions i-iv are shown in Fig. 2 in the space of the coefficients of $(\|J\|, tr(J), w)$; the equality condition (iv), which represents a plane in this space, must be constrained to the positive orthant since all three terms must be simultaneously positive to facilitate purely imaginary eigenvalues.

From Proposition 4, it follows that w>0 is a crucial necessary condition for a Hopf bifurcation, similar to the coefficient "K" in two-dimensional control problems (see e.g. Dockner-Feichtinger (1991)). In fact, the formal definition of w and K is identical (both are the sums of the leading minors of dimension two), and moreover, the presentation of w in (12) is similar to this coefficient K in Wirl (1992).[6] The first term between the brackets in the first row of w calculated in (12) is a negative semi-definite quadratic form such that the first row is always negative. The second row in (12) is only positive when growth is present (at the steady state), either in x or y, and the growing factor is dominating. The third row is only positive for predator-prey type interactions, i.e., x is beneficial for the growth of y but y is harmful for the growth of x (or vice versa). The fourth row may become positive for interactions of the control with the state or the externality, and the last row accounts for the interactions between state and externality. From this characterisation follows immediately

Proposition 5. *The existence of a pair of purely imaginary eigenvalues (which is a necessary condition for a Hopf bifurcation) requires tr(J)>0, thus*

$$r > g_u H_{uy}/H_{uu} - g_y,$$

and w>0 such that at least one of the following four terms must be positive and must outweigh other possibly negative elements (including the quadratic form in (12)):

1. $r(f_x + g_y) > 0$, *i.e., growth.*
2. $-f_y g_x > 0$, *i.e., predator-prey interactions.*

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[6] Dockner (1985) gives the formula for the eigenvalues of a two dimensional optimal control problem in terms of r, K and the determinant of the Jacobian, see Appendix.

3. *$H_{ux}(rf_u - f_y g_u) + H_{uy}(rg_u - f_u g_x) > 0$, i.e., nonlinear interactions between the control and either the state or the externality (e.g., due to endogenous preferences).*
4. *$f_u g_u H_{xy} > 0$, i.e., nonlinear interactions between the state and the externality.*

The crucial coefficient w is defined as the sum of the leading minors of dimension two. These three determinants (according to the listing in (15)) measure the interactions between state and costate, costate and externality, and finally between the state and the externality, and w aggregates all these impacts. Now considering this definition and assuming a stable optimal solution for the competitive agent given a constant value of y, the first determinant in (12) must be negative:

$$\left\| \begin{matrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial x} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{matrix} \right\| = f_x(r - f_x) - (rf_u H_{ux} - 2f_u f_x H_{ux} + f_u^2 H_{xx})/H_{uu} < 0. \tag{15}$$

Note that the inequality (15), which is necessary for the stability of the optimal policy for a constant y, needs not hold for a competitive equilibrium with an externality. In short, *market interactions may help to stabilise otherwise unstable policies!*

4 A simple, separable example for a Hopf bifurcation

An example fitting to the above general framework and allowing for limit cycles is already given in Greiner and Hanusch (1994). It is not the purpose here to repeat this example, nor to look for another equally complicated example, which can be easily picked from the new growth theory. Instead, I look for a very simple example that is capable of sustaining limit cycles. The most obvious way simplifying the elements in the Jacobian (9) and thus the necessary calculations is to consider a separable model such that all mixed second order derivatives vanish. This example is in stark contrast to the mechanism yielding limit cycles in Greiner and Hanusch (1994) that applies a route along the mixed second order derivatives, 'adjacent complementarity', dating back to Ryder-Heal (1973). In addition to separability, I introduce nonlinearity only at the required degree and at the easiest point, i.e., at the level of the control, such that all second-order derivatives other than H_{uu} vanish. More precisely, I consider the following formal model:

$$\max_u \int_0^\infty \exp(-rt)[pu - (k^2/3)u^3 + x]dt, \tag{16.1}$$

$$\dot{x} = a + mx + ny - u, \quad x(0) = x_0, \tag{16.2}$$

$$\dot{y} = qx - \delta y, \quad y(0) = y_0. \tag{16.3}$$

Although the model (16) serves primarily a formal purpose, it is easy to give economic interpretations.[7] For example, the optimisation problem (16) describes the decision of a firm harvesting a renewable resource stock x. The harvest, which is sold at the market price p, incurs the costs $c(u) = (k^2/3)u^3$, and this cubic function is the only nonlinearity in (16).[8] The resource x grows at a natural rate $a + mx$, $m>0$, but this growth is affected by the externality, y. And vice versa, the private stocks x affect the growth of the externality y. This interdependence between x and y may be symbiotic – increasing the private stocks x increases the external stock y, which in turn raises x (algebraically $q>0$, $n>0$) – or a predator-prey relation – $qn<0$, for example, $n<0$ means that the firms harvest the prey – and y depreciates at the constant rate $\delta \geq 0$. The resource stock x provides directly private benefits, e.g., due to multiple use (forests are a familiar example because they provide for logging and recreation).

Proposition 6. *Sufficiently strong predator-prey interactions such that $-qn > \delta^2$ ensure a Hopf bifurcation[9] for positive growth of x, $r > m > 0$, at the critical value $m^{crit} = \delta$. Therefore, concave and separable models can sustain limit cycles as optimal strategies. Positive growth $m>0$ and $r>m$ violates the inequality (15), so that the firms' programs are unstable for any given and constant y, yet stable competitive equilibria, accounting for the evolution of y, exist.*

Proof. The optimisation problem (16.1) and (16.2), given the externality (16.3), leads to the following system of differential equations for a competitive, rational expectation equilibrium:

$$\dot{x} = a + mx + ny - (\sqrt{p - \lambda}/k), \tag{17.1}$$

$$\dot{\lambda} = (r - m)\lambda - 1, \tag{17.2}$$

$$\dot{y} = qx - \delta y. \tag{17.3}$$

The term in (17.1) in parentheses represents the optimal control u^*. It is straightforward to calculate the steady states[10]

[7] For example, adding a further impact of the harvest on the externality such that $\dot{y} = \alpha u + qx - \delta y$ has no effect on the stability analysis and thus, for reasons of simplicity, we restrict the analysis to $\alpha = 0$.

[8] Quadratic costs are insufficient because the associated optimality conditions lead to a system of linear differential equations, which cannot give rise to limit cycles but only to centers. However, any other power function for the costs will do the trick too.

[9] That is, a pair of purely imaginary eigenvalues exists and the imaginary axis is crossed at non-zero velocity. This guarantees the existence of a cycle, which may be attracting or repelling. The verification of the stability of the limit cycle requires tedious calculations to determine the sign of the coefficients of the so-called normal form; see Guckenheimer-Holmes (1983). Hence, the existence of a stable limit cycle will be proved numerically.

[10] The steady-state is well-defined whenever the root can be taken, i.e., for $p>1/(r-m)$, and nonnegative for careful chosen parameters; in particular, a must be positive for $(\delta m + qn) < 0$.

$$x_\infty = \frac{\delta[\sqrt{p + 1/(m-r)} - ak]}{k(\delta m + qn)},$$
$$y_\infty = \frac{q[\sqrt{p + 1/(m-r)} - ak]}{k(\delta m + qn)}, \quad \lambda_\infty = 1/(r-m), \tag{18}$$

the Jacobian,

$$J = \begin{bmatrix} m & [2k\sqrt{p - 1/(r-m)}]^{-1} & n \\ 0 & r-m & 0 \\ q & 0 & -\delta \end{bmatrix}, \tag{19}$$

and the coefficients of the characteristic polynomial:

$$tr(J) = r - \delta, \tag{20.1}$$
$$w = m(r-m) - qn - r\delta, \tag{20.2}$$
$$\|J\| = (m-r)(\delta m + qn). \tag{20.3}$$

The first and crucial condition for a Hopf bifurcation concerns the existence of a pair of purely imaginary eigenvalues. This in turn requires, according to condition (iv) of Proposition 4, a solution of the following equation:

$$\|J\| - tr(J)w = (\delta - m)[qn + \delta r + r(m-r)] = 0. \tag{21}$$

From this arrangement (21) of the bifurcation condition follows immediately a solution for $m = m^{crit} = \delta$. The second root of (21), $m = (r - \delta) - qn/r$, implies $\|J\| = (\delta - r)(qn + \delta r)^2$ and $w = -(qn + \delta r)^2/r^2$. This root violates the simultaneous positivity of $\|J\|$, tr(J) and w, in particular, $w<0$. As a consequence, this root cannot lead to purely imaginary eigenvalues. However, the first root of (21), $m^{crit} = \delta$, facilitates a pair of purely imaginary eigenvalues. The trace is positive for $r > \delta$. A positive coefficient w requires at $m = m^{crit}$ that $qn < -\delta^2$, thus $qn < 0$, signifying predator-prey interactions of a sufficient degree. This last condition ensures a positive determinant of the Jacobian for $r > m$ due to (20.3). Hence, we have proven that a pair of purely imaginary eigenvalues exists for $m = \delta$.

Now we prove the second condition of the Hopf bifurcation theorem, namely that the imaginary axis is crossed at non-zero velocity with respect to the bifurcation parameter m at $m = m^{crit} = \delta$. The first real eigenvalue of (19), $e_1 = (r - m)$, is positive for the economically sensible case $r > m$ (otherwise no finite steady state exists anyway due to (18) and thus no indeterminacy can arise). The complex eigenvalues of the Jacobian (19) can be explicitly calculated:

$$_2e_3 = 1/2[(m-\delta) \pm \sqrt{(m-\delta)^2 + 4qn}]. \tag{22}$$

Differentiating the real part of (22) with respect to m yields

$$\frac{dRe(_2e_3)}{dm} = 1/2 \neq 0 \text{ for all m and in particular for } m = m^{crit} = \delta. \tag{23}$$

This verifies the second condition of the Hopf bifurcation theorem and ensures the existence of a cycle.

The verification of the stability of limit cycles requires tedious calculations that are here suppressed, similar to the bulk of the economics literature on limit cycles; for exceptions see Feichtinger-Novak-Wirl (1994). However, the existence of stable limit cycles is assured for the differential equation system (8), which is proved numerically using the computer package LOCBIF (see Khibnik-Kuznetsov-Levitin-Nikolaev (1992)). More precisely, the parameters $r = 0.5$, $p = 5$, $k = 1$, $a = 2$, $n = -0.1$, $q = 1$, $\delta = 0.05$, lead to a Hopf bifurcation at $m^{crit} = \delta = .05$, with steady states $x_\infty = 0.17094$, $y_\infty = 3.4188$, $\lambda_\infty = 2.2$, and with a negative Lyapunov number such that the arising limit cycles are stable (i.e., a supercritical Hopf bifurcation). *In passing, it is worth mentioning that the controlled system (the system after a central planner internalises this externality y) yield a Hopf bifurcation at the same critical value of the parameter, $m^{crit} = \delta$, see Appendix 2. Hence, government intervention and a thus proper internalisation of the externality will not eliminate the limit cycle. Persistent oscillations can result for both the competitive equilibrium and the social optimum.*

Finally, it remains to prove the claim that a stable competitive equilibrium exists despite the fact that the solution of the optimisation problem (1) subject to (2) and a constant y is unstable. Consider the above example (16) with $r > m > 0$. The corresponding determinant (15) is positive, because all the mixed second order derivatives are zero so that (15) simplifies to $f_x(r - f_x) = m(r - m) > 0$. The optimal policy is unstable for any given constant value of y, yet the competitive equilibrium is stable for $0 < m < \delta$ since, according to (22), two eigenvalues with negative real parts exist. ■

This example (16) demonstrates that simple and separable economic models permit complex strategies, in particular, limit cycles. In contrast to Greiner-Hanusch (1994), this example exploits the conditions of positive growth (presumably present in Greiner-Hanusch (1994) due to the spillovers and aggregate increasing returns to scale) and predator-prey type interactions. In fact, both conditions must hold simultaneously since growth alone cannot support limit cycles. This is in contrast to the associated social optimum where the evolution of y may be directly controlled such that a two-dimensional optimisation problem results and where growth itself can generate cycles (see Wirl (1992)). Another implication of this study is that apparently different routes to limit cycles exist, even for such low dimensional problems. Moreover, from the above proof follows immediately the corollary that growth need not take place in the private stocks x but may characterise the evolution of the externality y.

Corollary. *Although the two conditions – growth and predator-prey interactions–are crucial for a Hopf bifurcation, it is possible to obtain purely imaginary eigenvalues for the competitive economy (16) for $m<0$ if the externality y exhibits growth with respect to y (instead of depreciation), i.e., for $\delta < 0$.*

5 Concluding remarks

This paper analysed the stability of general one-dimensional, concave dynamic optimisation problems of competitive agents where the agents' actions create spillovers or an externality which in turn affects the agents' profits. This externality, albeit important for the evolution, is not internalised in the economy, neither through government interventions nor private arrangements. In particular, it was shown that a competitive equilibrium may stabilise otherwise unstable policies and that the evolution may be complex - with damped, undamped and persistent oscillations, i.e., limit cycles - even for a one dimensional optimisation problem. The existence of limit cycles was already proven in Greiner-Hanusch (1994). In contrast to that example, this paper considered a general one-dimensional optimisation problem, investigated the stability, derived a set of necessary conditions for a Hopf bifurcation, and proved the existence for separable and thus potentially very simple models. The assumption of separability implies that growth and a predator-prey type interaction between the state and the externality are necessary for limit cycles. The social optimum achieved through government interventions (or a proper internalisation) need not eliminate the complexities of the competitive equilibrium, but may in fact enlarge the domain of complex strategies. Besides deriving and characterising the stability properties of such intertemporal competitive equilibria, these results are a warning to the recent literature on 'new growth theory' that emphasises balanced growth paths, although the actual evolution may be less smooth than implicitly assumed. Indeed, the recent literature seems to account for this. For example, Mulligan-Sala-i-Martin (1993) investigate the motions off the balanced growth paths, but restricted to saddlepoint paths, and Boldrin-Rustichini (1994), Benhabib-Perli (1994), and Greiner-Semmler (1996) focus on indeterminate solutions, which may also be possible within the presented framework; this question is left for future research. Other possible applications of the framework presented in this paper are to environmental economics, characterised by externalities with increasing recognition of dynamic externalities (such as global warming); see e.g., Wirl (1994) and Withagen (1995). Another and theoretical extension of this paper is to investigate which of the four conditions supports on its own limit cycles, similar to Wirl (1996) for two-dimensional optimal control problems.

Appendix
1. Calculation of the eigenvalues of (8)

Ruling out global stability, which is neither attainable nor desirable because of the implied indeterminacy of the optimal policy, one eigenvalue must be positive and real, and w.l.i.g. $e_1 > 0$. Hence, the characteristic polynomial P(e) can be written in the following way:

$$P(e) = (e - e_1)(e^2 + \alpha e + \beta), \tag{A1}$$

where the other two eigenvalues e_2 and e_3 are the roots of the quadratic polynomial in (A1) and these roots,

$${}_2e_3 = 1/2[-\alpha \pm \sqrt{\alpha^2 - 4\beta}], \tag{A2}$$

determine the local stability of the system (8). Comparing the coefficients of (A1) with (10) gives for the coefficients α and β in (A1):

$$\alpha = e_1 - tr(J), \tag{A3}$$

$$\beta = \|J\|/e_1 \text{ or } \beta = w + e_1\alpha = w + e_1^2 - e_1 tr(J). \tag{A4}$$

Now the following constellations are possible (for $e_1 > 0$):

1. The two eigenvalues ${}_2e_3$ are both real and negative <=> $\alpha > 0$, $(\alpha^2 - 4\beta) > 0$, $\beta \geq 0$ (thus $\|J\| \geq 0$).
2. The two eigenvalues ${}_2e_3$ are real, and one is positive and the other is negative <=> $\alpha < 0$, $(\alpha^2 - 4\beta) > 0$, $\beta < 0$ (thus $\|J\| < 0$). This case implies that the stability is restricted to a one dimensional manifold of initial conditions, while it is impossible to reach the steady state for all other initial conditions in the (x, y) plane. Hence, the generic property is **instability**.
3. The two eigenvalues ${}_2e_3$ are complex with negative real parts < = > $\alpha > 0$ and $(\alpha^2 - 4\beta) < 0$, thus $\beta > 0$ and $\|J\| > 0$.
4. The two eigenvalues ${}_2e_3$ are complex with positive real parts < = > $\alpha < 0$ and $(\alpha^2 - 4\beta) < 0$, which implies again $\beta > 0$ and $\|J\| > 0$.
5. The two eigenvalues ${}_2e_3$ are purely imaginary <=> $\alpha = 0$ and $\beta > 0$.

The following tables summarise the various criteria. Table A1 starts with the above addressed stability properties, their characterisation in terms of the coefficients α and β and their implications on the coefficients of the characteristic polynomial. Table A2 in contrast starts with these coefficients of the characteristic polynomial, derives the implications for α and β, and determines, if possible, the associated stability properties. Therefore, Table A2 provides a direct proof of the Propositions 1 and 2.

Table A1. Stability properties and implications for the coefficients

Assumptions (e_1>0) roots	**Implications** α	β	$(\alpha^2–4\beta)$	tr(J)	w	$\|J\|$
real, negative	+	+	+	?	?	+
complex, negative real parts	+	+	–	?	?	+
complex, negative real parts	–	+	–	+	+	+
real, one positive, one negative	–	–	+	+	?	–

Table A2. The Coefficients of the characteristic polynomial and the implications on stability

Assumptions tr(J)	w	(e_1>0) $\|J\|$	**Implications** α	β	$(\alpha^2–4\beta)$	eigenvalues and stability properties
+	+	+	?	+	?	saddlepoint or exploding spirals
–	+	+	+	+	?	negative reals or real parts, saddlepoint
+	–	+	+	+	?	negative reals or real parts, saddlepoint
–	–	+	+	+	?	negative reals or real parts, saddlepoint
+	+	–	–	–	+	one positive, one negative, unstable
+	–	–	?	–	+	one positive, one negative, unstable
–	+	–				impossible for e_1>0
–	–	–	+	–	+	one positive, one negative, unstable

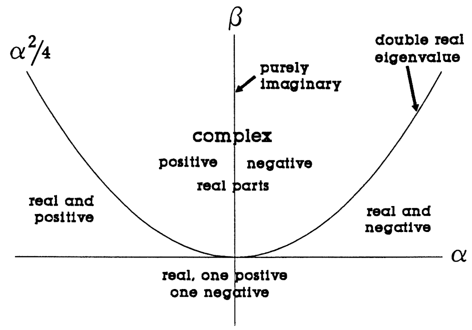

Fig. A1. Stability properties in terms of $\alpha := e_1 - tr(J)$ and $\beta := \|J\|/e_1$ for $e_1 > 0$

The condition for purely imaginary eigenvalues implies immediately the condition of Proposition 4, since $\alpha = 0$ and (A3) imply $e_1 = tr(J)$ such that, according to (A4), $\beta = \|J\|/tr(J) = w$. The geometry of the roots $_2e_3$ and the associated stability properties in terms of α and β are shown in Fig. A1.

2 Analysis of the controlled system (16)

The socially optimal control of the external evolution of y leads to the optimisation of (16.1) subject to (16.2) and (16.3) with the associated Hamiltonian H',

$$H' = pu - (k^2/3)u^3 + x + \lambda'(a + mx + ny - u) + \mu(qx - \delta y). \qquad (A5)$$

(Primes are used to differentiate between the controlled and uncontrolled system.) The costates evolutions are independent of the control:

$$\dot{\lambda}' = (r - m)\lambda' - 1 - \mu q, \qquad (A6.1)$$

$$\dot{\mu} = (r + \delta)\mu - \lambda n. \qquad (A6.2)$$

The Hamiltonian maximising condition remains unchanged. Hence, (17.1), (17.3) and (A6) describe the four-dimensional system that characterises the optimal evolution. The steady state can be again explicitly computed and has a formal structure similar to (18)

$$x'_\infty = \frac{\delta[\sqrt{p + 1/A} - ak]}{k(\delta m + qn)}, \quad y_\infty' = \frac{q[\sqrt{p + 1/A} - ak]}{k(\delta m + qn)},$$
$$\lambda'_\infty = -(r + \delta)/A, \ \mu_\infty = -n/A, \qquad (A8)$$

with the exception of the denominator A under the square root:

$$A := (\delta + r)(m - r) + qn, \qquad (A9)$$

instead of (m–r) in (18).

The eigenvalues ϵ_i, i=1 to 4, of the associated Jacobian

$$J' = \begin{bmatrix} m & n & [2k\sqrt{p-1}]^{-1} & 0 \\ q & -\delta & 0 & 0 \\ 0 & 0 & r-m & q \\ 0 & 0 & -n & \delta + r \end{bmatrix}, \tag{A10}$$

can be calculated with the following formula

$${}^{3}_{1}\epsilon^{4}_{2} = (r/2) \pm \sqrt{(r/2)^2 - (K/2) \pm 1/2\sqrt{K^2 - 4\|J\|}}, \tag{A11}$$

given in Dockner (1985); K is defined similarly to w as the sum of the leading minors of dimension 2 of the Jacobian J'.

A Hopf bifurcation results according to Dockner-Feichtinger (1991), if

$$\|J'\| - (K^2/4 + r^2K/2) = 0 \text{ and simultaneously } K > 0. \tag{A12}$$

It is straightforward to compute the crucial coefficients in the formula (A11):

$$K = m(r - m) - \delta(r + \delta) - 2qn, \tag{A13}$$

$$\|J'\| = (\delta m + qn)[(\delta + r)(m - r) + qn]. \tag{A14}$$

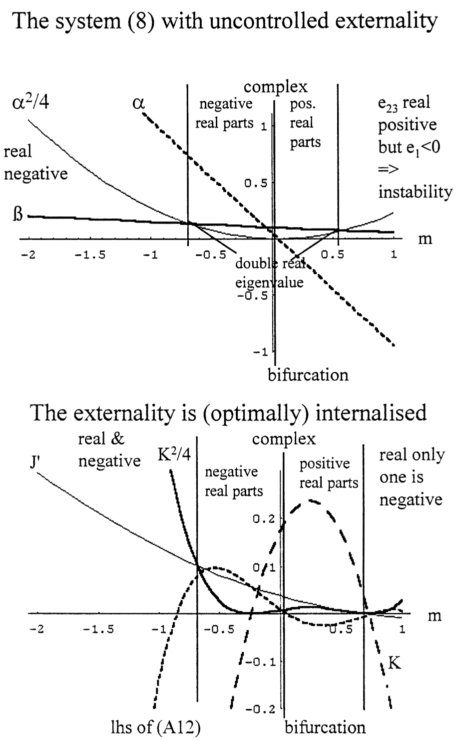

Fig. A2. Stability analysis for variations in m for the reference example, $r = 0.5$, $\delta = 0.05$, $p = 5, k = 1, a = 2, n = -0.1, q = 1$

The bifurcation condition (A12) can be arranged in manner amenable to explicit solutions:

$$\|J'\| - (K^2/4 + r^2K/2) = (m - \delta)(\delta + 2r - m)[(\delta + m)^2 + 4qn - r^2]. \qquad (A15)$$

Hence, according to (A15), (A12) has four roots, but only one root, $m = \delta$, allows as in Section 4 for a Hopf bifurcation (since the determinant and the coefficient K must be simultaneously positive). *As a consequence, both the controlled and the uncontrolled system lead to a Hopf bifurcation at the same critical parameter value* $m = m^{crit} = \delta$.

Figure A2 compares the stability of the uncontrolled system (differential equations (8)) with the system, where the externality is optimally internalised (the subject of this appendix) for variations in the parameter m and the reference example in the paper. More precisely, Fig. A2 determines (qualitatively) the eigenvalues of the stable manifold according to Fig. A1 and the criteria outlined in Dockner (1985). Observe that the eigenvalue $e_1 = r - m$ changes the sign above $m = r$ and becomes negative. This does not lead to indeterminacy because the other two eigenvalues are positive, but to a domain of instability (except for a one dimensional manifold associated with the negative eigenvalue e_1). This instability is not surprising if the resource grows faster than the agents discount. In fact, the system is already unstable for $m > r = 0.5$ due to (18).

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