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REGULAR ARTICLE

Law of the jungle: firm survival and price dynamics in evolutionary markets

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Abstract In this paper I develop a simple, and general model of supply and demand within which almost any theory of consumer and producer behaviour may be integrated by varying parameters. I then investigate the dynamics of this model and its implications for the theory of market evolution, and show that it unifies a number of insights from evolutionary economics. I extend upon these evolutionary theories and also characterise the distribution of prices across the market and investigate its evolution over time.

Keywords Differential survival and growth · Price theory · Firm strategy · Innovation · Competitive selection

JEL Classification D00 · D21 · D40 · D49 · C00

1 Markets as evolutionary systems

Who trusted God was love indeed
And love Creation's final law
Tho' Nature, red in tooth and claw
With ravine, shriek'd against his creed

Lord Tennyson's words contrast the story we tell children to make them sleep about a benevolent and benign "fair" world with the violent and brutal reality of natural

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selection. In the jungle or the savannah "God" doesn't care whether or not you're a lion with a particularly good work ethic, if that doesn't translate into an ability to kill for food you will die, and if your pheromones aren't attractive enough you'll die without scion.

"Survival of the fittest" has always seemed more than just an analogy for the market process and slogan of social Darwinism. It is very rare indeed for a firm to survive for more than fifty years, and it is well known that the vast majority of businesses fail within a year. The very words "market competition" conjure to the mind firms competing against each other for growth by "undercutting" their rivals' price, offering a better product than them, being more efficient in their production etc. The market system has been viewed as an evolutionary system of selection through differential growth amongst individuals and organisations using different rules for economic behaviour since Thorstein Veblen (Veblen 1898) and Alfred Marshall (Foster 1993; Raffaelli 2003; Metcalfe 2007). Unfortunately, it is well known that if an economic system is populated by individuals using a vastly varied set of rules and routines in their economic decision-making this presents problems for analysis using the conventional tools of equilibrium comparative-static analysis using "rational" agents (Alchian 1950; Nelson and Winter 1982; Hayek 1989).

The theory of the market as an evolving system took a quantum leap in formalism and rigour with Nelson and Winter's *Evolutionary Theory of Economic Change* (1982).[1] Within the "evolutionary economics" literature which grew up around this book there are a number of formal models which provide fascinating insights into the market process. However, given the variety of firm strategies and consumer behavioural rules there has been a tendency within this literature toward agent-based simulation simulations of non-analytically solvable theories (in the style of Cyert and March (1963) and their successors) for particular economic phenomena or data sets within the market.

There are exceptions to this tendency to turn away from analytical theories of evolutionary dynamics. For instance, Downie (1958), Metcalfe (1998), Metcalfe et al. (2006), and Andersen and Holm (2013) provide analytical theories of the evolution of the market system with tantalising potential by - to different extents - using formalism borrowed from evolutionary biology. Yet there remains no provision of a common mathematical grammar and logic for analysing the evolution of the economic system with a level of generality comparable to, say Debreu (1959). An outstanding question within evolutionary economics then is: amongst all the complexity of the economic system generated by the massive variety of behavioural rules and routine, can we

[1]The unawareness of this book within economics and the lack of a Nobel Prize for Richard Nelson and Sidney Winter is surprising given that a cursory internet search reveals that as of August 2014 the book has received well over 25,000 citations, and is often taken as the foundational document for an entire research program in economics (evolutionary economics) which itself is regularly used as a theoretical basis for business and management research. A similarly cursory search - by way of comparison - reveals as of August 2014 approximately a little under 20,000 citations for George Akerlof's Nobel Prize winning article which introduced the notion of adverse selection and contributed to the development of the economics of imperfect information.

find a common, and analytical theory of evolutionary dynamics? Put differently, do we have to sacrifice analytical rigour and rely on agent-based simulation to say anything about an evolutionary system populated by heterogeneous agents? Must we confine ourselves to studying models of equilibria between effectively homogeneous "agents"[2] which are fundamentally at odds with the fact that in the real economy we have a population of heterogeneous firms using different decision rules to try and be "selected" by consumers?

A similarly troubling lacuna in this research is the lack of a theory of prices, which was identified within a paper presented by Stan Metcalfe and Harry Bloch to the 2014 International Schumpeterian Society Conference in Jena, Germany. Where partial and general equilibrium economics provides an explanation for little else but price determination, evolutionary economics provides genuine explanations of economic dynamics and the interaction of heterogeneous agents, but hasn't much to say at the present stage about prices. The current paper aims to move some way toward addressing the first issue, and to provide a starting point for addressing the second.

In this paper I develop a simple but general model of market dynamics which allows for the analysis of almost any economic scenario by varying three behavioural parameters - the elasticity of demand, and two functions for firm behaviour with respect to prices and product attributes. This model allows for just about any theory of economic behaviour to be nested within it, from the hyper-intelligent demon of game theory to the habitual human being of institutional economics. On the demand side of the market we may include individuals maximising a complete and transitive preference ordering, individuals simply choosing alternatives at random, or any mix in between. Similarly on the supply side we can incorporate any mix of decision rules across the population from perfectly rational profit maximising firms to firms using simple mark-up rules. While infinitely many distributions of behavioural rules across the population may be input into the model, the model imposes an order on the interaction of these behavioural rules through the selective processes of market competition.

I then analyse the implications of the model for the theory of firm growth and survival (the process by which selection occurs) and demonstrate that this model nests some major formal theories in evolutionary economics developed over the past thirty years, - conceptually if not in exact functional form, - and in fact in that somewhat generalises them. With this I proceed to suggest a solution to the problem of what we mean by "theory of prices" in evolutionary economics, characterise the distribution of prices across the market and provide some preliminary theoretical results. Before concluding, I consider the implications of the model dynamics of supply and demand for some market aggregates using the Price equation. Not only can we find a fairly acceptable general model for evolutionary dynamics across the market which can handle a heterogeneous population of firms using different decision rules, we can say interesting things about firm size and price dynamics without recourse to agent-based simulation.

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[2]Insofar as all agents in a standard Walrasian model use exactly the same constrained optimisation decision rule and differ only in their endowments and the preference pre-ordering of their alternatives space.

Before proceeding further, I should acknowledge that there is debate in evolutionary economics around whether evolution is an analogy for economic processes or an inherently economic process (Witt 2008), though this is now largely settled in favour of the latter (Vromen 2012). Much like Joseph Schumpeter (Hodgson 1997) I hold to the view that the economy is and "is like" the economy first and foremost, any analogy is incidental, one of those iso-morphic laws which complex systems tend to share (von Bertalanffy 1950). In this paper I work from economic principles and intuitions only.[3] An economy is a system in which consumer demand functions to select successful entities from a population of firms experimenting with different rules for pricing and product attributes.

Also, while I would argue that the model presented here has some ability to generalise, unify and extend evolutionary economic theory, this model is only as useful as it is when applied to the analysis of real situations in history. I am in agreement with Nelson and Winter (1982), Malerba et al. (1999) and Foster (2011) that the proper place of formal general theories such as the one presented here is to provide a common grammar and logic out of which research into real economic phenomena can be conducted. It may be interesting - as I do here - to tinker with the assumptions of the model and arrive deductively at *a priori* theories of what governs firm growth and survival, and what happens to the distribution of prices over time. But the proper goal of science is to understand real phenomena in their historical setting (in the social sciences this last is extremely important), not exclusively to tease out the implications of a logical system; this is what distinguishes science from mathematics. To that end I hope that this paper will be of some use for economists delving into the annals of history, collecting data and trying to appreciate it by using the logical system presented below.

2 Dynamics of an evolutionary market

Suppose we have a market consisting of N consumers and F firms. Let $\left\{q_i^c\right\}_{i=1}^F$ be the schedule of consumer c's demand for firm i's output, such that $\left\{\left\{q_i^c\right\}_{i=1}^F\right\}_{c=1}^N$ is the overall demand schedule for the market. The demand for firm i's output, q_i^d, is then the summation of these consumer demands

$$q_i^d = \sum_{c=1}^{N} q_i^c \tag{2.1}$$

It seems fairly uncontroversial to assume that demand is a function of prices per unit charged for firm output $\{p_i\}_{i=1}^F$ and also of the number of consumers (given that q_i^d is a summation of demand across them). Following Lancaster (1966) and Ironmonger (1972) however, we could also argue that consumer demand is also some

[3] Those who have sympathies otherwise are directed to the excellent papers of Knudsen (2002; 2004) and Hodgson and Knudsen (2004).

function of product attributes. This is something broadly recognised in choice modelling (McFadden 2013) if not adequately addressed on a theoretical level through random utility or Dixit and Stiglitz (1977) models. We can describe the objective attributes (as opposed to a consumer's perception thereof - an important distinction) of firm i's output as a vector $\alpha_i = \left\{\alpha_i^k\right\}_{k=1}^{N_A}$ in a characteristic space $A = \Pi_{k=1}^{N_A} A_k \subset \mathbb{R}_+^{N_A}$.

As a function of price, attributes and population, consumer demand could then be approximated as

$$q_i^d = \sum_{c=1}^{N} q_i^c \approx q_i^d \left(N \quad \left\{ p_j \quad \left\{\alpha_j^k\right\}_{k=1}^{N_A} \right\}_{j=1}^{F} \right) \tag{2.2}$$

This specification contains a rather strong assumption that consumer tastes (preferences) vis-a-vis attributes or their perception of those attributes do not develop through time (perhaps due to the influence of marketing) sufficiently to change the function form or function parameters of demand with respect to price, attributes or market size. This can be relaxed at a later stage, but it remains a significant behavioural assumption for this model, though one which it may be better at this preliminary stage to keep given the empirical and theoretical troubles of interpreting $\alpha_i = \left\{\alpha_i^k\right\}_{k=1}^{N_A}$ as reported or percieved attributes.

On the production side of the market we have a schedule $\{q_i\}_{i=1}^{F}$ of firm supply, assuming that any one firm produces only one output in this particular market. To be more precise, q_i is output *sold*, which is *not* necessarily the same as output produced. We can think of output produced either to-order (i.e. produced after a sale has been made), or as a capacity limit on firm supply $\bar{q}_i$ similarly to Downie (1958) and Metcalfe (1998). This distinction between output supplied to consumers and output actually produced is intellectually neater and closer reality than conflating the two, and allows for the existence of inventories.[4] For simplicity the production process generating this output is suppressed for now and it is assumed that $\bar{q}_i \geq q_i$ for all intents and purposes in what follows, though it is not difficult to imagine nesting such a process within the model. Firms can rightly be said to exist or not depending on whether the material inputs for such a production process are available to the organisation, such that mergers would consist of the transfer of the material inputs of the parties to the merger from one organisation into another, though here the entry/exit of the firm from the market shall be taken to be reflected in the output sold by the firm. Focusing on *supply* rather than output does not lose us a great deal of generality if we assume firms produce to order or always have spare capacity, and an extension of the model below to explicitly incorporate the production process which determines

[4]Note that this also opens up a link between an evolutionary model of the market and the notion of unplanned inventories due to effective demand shortfalls which lies at the heart of the macroeconomic analysis of Keynes (1936).

output produced such that capacity constraints are allowed to bind and to evolve over time should prove interesting.[5]

Firms have two decisions to make. Firstly, they must decide upon the price of their output, a decision which can be represented by a pricing rule $p_i = f_i(x_i)$ which transforms a vector of firm-specific decision factors x_i via a firm specific rule $f_i(\cdot)$ into a set price for output. The vector of decision factors may include, for instance, firm growth rates relative to aspirations, costs of production, profits relative to desired profits etc. It may seem rather *ad hoc* (an extremely easy accusation to make), but this specification is both intuitively appealing and empirically supported by studies of pricing by firms[6] and allows us to model a population of heterogeneous firms without getting mired in the peculiar forms of the rules.

These pricing rules may take a variety of different forms, in fact there are infinitely many combinations of decision factors x_i and pricing rules $f_i(\cdot)$ which may reflect firm behaviour. For instance, prices may be a simple mark-up over some function $h[c_i(\bar{q}_i)]$ of costs[7] such that $p_i = \lambda h[c_i(\bar{q}_i)]$ where λ is a scalar and $c_i(\cdot)$ the cost of producing output q_i. Setting $\lambda = 1$ and $h = \frac{\partial \cdot}{\partial \bar{q}_i}$ gives us the standard marginal cost-pricing rule of firms in a neoclassical perfect competition equilibrium while $\lambda \geq 1$ gives us the marginal cost mark-up pricing rule most commonly used in Post-Keynesian economics (Lavoie 1996). We could even go further and specify (in the spirit of Nelson and Winter (1982) and Downie (1958))[8] that costs themselves are some function of past profits, reflecting the ability of firms to invest in lowering costs, so $p_i^t = \lambda h\left[c_i\left(\bar{q}_i^t \quad \pi_{t-1}\right)\right]$, where $\pi_{t-1} = p_i^{t-1} q_i^{t-1} - c_i^{t-1}\left(\bar{q}_i^{t-1} \quad \pi_{t-2}\right)$. Alternatively (or additionally, it is perfectly possible for x_i to be quite a large vector though this may make $f_i(x_i)$ rather involved, possibly even algorithmic rather than purely functional) in the spirit of Cyert and March (1963), Simon (1955a), Selten (1998) and Kahneman and Tversky (1979) the decision factors x_i might include firm growth rates relative to some aspiration level so that prices are set as a reaction to differences of outcomes to aspirations. For instance, firms may target a certain growth rate $[\partial q_i/\partial t]^*$ and set prices to achieve that growth rate or better so that

$$\frac{\partial f_i(x_i)}{\partial\left\{\left[\frac{\partial q_i}{\partial t}\right]^* - \left[\frac{\partial q_i}{\partial t}\right]\right\}} < 0 \,\forall \left[\frac{\partial q_i}{\partial t}\right]^* > \left[\frac{\partial q_i}{\partial t}\right] \tag{2.3}$$

[5]This extension of the model would allow, amongst other things, for the analysis of the role within the market of organisational slack and inventory management, and the definition of firm existence by the factors of production rather than the size of the firm in the market.

[6]See the Oxford price studies around Hall and Hitch (1939); Andrews (1949, 1950, 1964) surveyed in Earl (1995, Ch.8 and 9) and the later studies of Blinder et al. (1998) in which it is axiomatic that firms set prices, and that they set prices according to a variety of different rules.

[7]For instance, if $h(\cdot) = \frac{\cdot}{\bar{q}}$ we are talking about an averaging of costs, if $h(\cdot) = \cdot$ then we are talking about total costs, and if $h(\cdot) = \frac{\partial \cdot}{\partial \bar{q}_i}$ we are talking about marginal costs.

[8]Downie rarely uses the word "evolution" in *The Competitive Process* (1958), much less "Darwinism", and his focus is more on the production process which generates the capacity for sales of firms, though as Nightingale (1997) has also noted, he does note the importance of past profits to expanding capacity for sales and the ability to lower costs of production.

reflecting a belief that firm growth is decreasing in prices. There is absolutely no reason either why strategic concerns cannot enter into the pricing rule, for saying that firms are being strategic with respect to rivals' prices (*a la* game theoretical Bertrand competition) merely means that $p_{j\neq i} \in x_i$, and, in a fairly simple case where the strategy is to undercut one's rivals, $p_i = f\left(\{p_j\}_{j=1}^{F}\right) = \min_{j\neq i}\{p_j\}_{j=1}^{F} - \epsilon$, or in a more general specification to shift prices in line with others so that $\partial f_i(x_i)/\partial p_{j\neq i} > 0$. Alternatively, firms may be strategic with respect to rivals' growth rates (a twist on game theoretical Cournot competition), which can be represented by substituting other firms' growth rates for the "aspiration level" in Eq. 2.3, if firms are of the belief that they will increase firm growth by decreasing prices. The phenomenon of "stock take sales" might be incorporated if we were to include a roll-over of unused capacity from previous time periods within the pricing decision rule, i.e. $\left(\bar{q}_i^{t-i} - q_i^{t-k}\right) \in x_i = \arg f_i(x_i)$. This approach does not even necessarily exclude analysis of auction markets, which can be thought of as a special case of pricing where consumer demand is discrete (that is $q_i^c = \{a_c \quad 0\}$ about some cut-off price) and firms implement either an English (ascending bid) or Dutch (descending bid) auction in the setting of their prices, reflected in the pricing rule $f_i(\cdot)$ and, for example, the decision factor $x_i = \{sale \quad no\,sale\}$.[9]

This being said, the question of which types of rules are actually used by which firm is ultimately an empirical one. Most interesting analysis of evolutionary market systems will of course lie in understanding the exact make-up of x_i and $f_i(\cdot)$, but our first challenge in developing an analytical theory of the evolution of the market system is to place pricing rules in the context of market dynamics. Setting aside the complexity and variety of firm pricing strategies for the moment in favour of finding the role of prices in the market will allow us in future to investigate how specific distributions of rules play out in firm growth rates.

In addition to deciding which price to set, firms must also decide on the attributes α_i of their product, a decision which may similarly be represented by a rule $\alpha_i = \left\{g_i^k(x_i)\right\}_{k=1}^{N_A}$ which transforms the firms' decision factors into the attributes of its output. This rule can be thought of as representing a process of investment in developing certain product attributes, a process which is determined by the firms' decision factors x_i - in other words, a rule for R&D. For instance, we might expand $\left\{g_i^k(\cdot)\right\}_{k=1}^{N_A}$ so that an investment process in developing attribute k is explicitly included, so that $\alpha_i^k = g_i^k(x_i) = g_i^k(I_k(x_i))$, where $I_k(\cdot)$ transforms firm decision factors into investment in attribute k. Given that attributes characterise the product being produced, we can see this rule $g_i(x_i)$ as being the firms' rule for product innovation, the "Schumpeter Mark I" kind of innovation, the more radical innovation compared with "Schumpeter Mark II" innovation which here can be understood to be incorporated in the pricing rule.

[9] This view would actually constitute a generalisation of auction models in neoclassical microeconomic theory such as the seminal Myerson (1981) and Myerson and Satterthwaite (1983) mechanism design problems which solve for optimal pricing schemes by assuming the seller knows the distribution of types. The present model requires no such assumption for the analysis of market outcomes.

The behavioural assumptions of the model therefore are contained in the shape of demand and the firms' decision rules vis-a-vis pricing and attributes. These specifications of behaviour are quite general and will allow the investigation of a variety of different behaviours at a market level simply by nesting, for instance, a model of rational optimising agents or boundedly rational satisficers within q_i^d, and profit maximising or simple mark-up-pricers within $f_i(\cdot)$. It is important to note that the specification of such behavioural rules allows necessarily for the possibility of purposive adaptation of firms to market conditions. Indeed the cases where subsets of the market conditions, $\{q_i\}_{i=1}^F, \{p_i\}_{i=1}^F, \left\{\{\alpha_k\}_{k=1}^{N_A}\right\}_{i=1}^F$ and their time derivatives enter $\{x_i\}_{i=1}^F$ are the only really interesting rules. If sales, profits, rivals' sales or any other variable determined by the market enters into the firms' decision factors, their decision rules for prices and product attributes will typically dictate that they adapt to market conditions as these variables change. Under conditions where decision factors and rules are limited by imperfect information and bounded cognitive ability, adaptability of supply to market conditions becomes crucial to firm survival and growth (Witt 1986).

The relation of supply and demand by which selection is brought about by market dynamics is the act of exchange. The act of exchange in a market context simply consists of an object being transferred from a firm to a consumer, so every unit of output sold must logically be purchased and vice versa. Hence all exchanges within the market between a particular firm and consumers can be summarised by a simple and familiar equation

$$q_i = q_i^d \tag{2.4}$$

This is *not* an "equilibrium" condition in any sense[10] but merely a description of the two sides to an exchange relationship expressed formally. It may be the case that for any given firm, the amount it produces (its capacity limit) is *strictly* greater than the amount it sells so that $\bar{q}_i > q_i = q_i^d$ and the firm adds to inventories, but we will assume only that $\bar{q}_i \geq q_i^d$. Now, expanding q_i^d by equating (2.2) and (2.4) we can say that

$$q_i = q_i^d \left(N \quad \left\{ p_j \quad \left\{ \alpha_j^k \right\}_{k=1}^{N_A} \right\}_{j=1}^{F} \right) \tag{2.5}$$

which expresses the volume of exchange as a function of firm behaviour. Assuming for simplicity, that this function is differentiable we can identify the dynamics of the market by finding the differential of firm demand in time across all firms

$$\frac{\partial q_i}{\partial t} = \frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + \sum_{j=1}^{F}\left[\frac{\partial q_i^d}{\partial p_j}\frac{\partial p_j}{\partial t} + \nabla_{\alpha_j} q_i^d \left(\nabla_t \alpha_j\right)'\right] \tag{2.6}$$

[10] A recent paper, Richter and Rubenstein (2015) going back to the basics of equilibrium serves to show that equilibrium in psychological sciences must involve much greater assumptions upon the psychology (in economics, the preferences) underlying exchange.

where

$$\nabla_{\alpha_j} q_i^d = \left\{ \frac{\partial q_i^d}{\partial \alpha_j^k} \right\}_{k=1}^{N_A} \qquad \nabla_t \alpha_j = \left\{ \frac{\partial \alpha_j^k}{\partial t} \right\}_{k=1}^{N_A} \tag{2.7}$$

Multiplying Eq. 2.6 by dt would give us the familiar total differential. Now, expanding Eq. 2.6 using these Jacobeans and using the definition of demand elasticity,[11] we get a surprisingly elegant equation for the output dynamics of firm i

$$\frac{\partial q_i}{\partial t} = \frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + \sum_{j=1}^{F} \frac{q_i^d}{p_j} \varepsilon_{d_i}^{p_j} \frac{\partial p_j}{\partial t} + \sum_{j=1}^{F}\sum_{k=1}^{N_A} \frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t} \tag{2.8}$$

We do not enforce a strict substitutability between price and attributes in Eq. 2.6 for it is (but for one step) a total differential. If we dig into the cross-derivatives of, for example, price elasticity $\left\{ \frac{\partial \varepsilon_{d_i}^{p_j}}{\partial \alpha_j^k} \right\}_{k=1}^{N_A}$ we might find that demand is more sensitive to price changes in certain regions of attribute space than others. For instance, consumers would be more sensitive to price changes when the products they are considering are of very high or very low quality. Alternatively, we might upon examining the cross-derivative of the response of demand to attributes $\left\{ \frac{\partial^2 q_i^d}{\partial \alpha_j^k \partial p_j} \right\}_{k=1}^{N_A}$ find that these are positive, indicating sensitivity to attributes increases as prices increase, so that high quality goods are sought for when prices are high.

Taken together, the differential equations $\left\{ \frac{\partial q_i}{\partial t} \right\}_{i=1}^{F}$ characterise the evolution of firm output as determined by the interaction of supply and demand across the market. In combination with the behavioural assumptions of the model these equations allow us to analyse the evolution of the firm size and price distributions across the market and thus the selection through differential "survival" (i.e. output growth) of certain firms and the rules they use. Ultimately the vector of prices $\{p_i\}_{i=1}^{F}$ is the outcome of firm decision making and so subject to the specification of x_i and $f_i(\cdot)$. However the "incidence" of particular prices, the number of transactions occurring at that price, can evolve through time. This idea provides the basis for the evolutionary theory of prices: myriad different rules for setting prices exist, but market selection through differential survival on the basis of prices means that certain price rules will have a greater incidence and therefore prevalence in the market than others.

For the purpose of analysing the dynamics of the price distribution across the market over time it will be beneficial to define its first and second moments. The market

[11] Assuming differentiability, (though this can be relaxed) the elasticity of demand for firm i with respect to the price set by firm j is $\varepsilon_{d_i}^{p_j} = \frac{p_j}{q_i^d}\frac{\partial q_i^d}{\partial p_j}$.

situation at any given point in time can be characterised by the tuple $\{p_i \quad q_i\}_{i=1}^{F}$, so it would make sense to define the distribution of prices according to the number of transactions in the market at which that price prevails. In that case, the average price (weighted according to the prices "incidence") across the market can be defined as

$$E_q(p) = \frac{\sum_{i=1}^{F} q_i p_i}{\sum_{j=1}^{F} q_j} \tag{2.9}$$

Also interesting will be the variance of the price distribution with respect to output

$$\mathrm{Var}_q(p) = \frac{\sum_{i=1}^{F} q_i \left[p_i - E_q(p)\right]^2}{\sum_{j=1}^{F} q_j} \tag{2.10}$$

The mathematical objects (2.8), (2.9), (2.10) together with the behavioural assumptions with respect to $\left\{\left\{\varepsilon_d^{p_j}\right\}_{j=1}^{F}\right\}_{i=1}^{F}$, $f(\cdot)$ and $g(\cdot)$ are sufficient to analyse the evolution of the firm size and price distribution in an evolutionary market. That said, they have what might be called an "anything goes" quality about them. It should be fairly obvious that simply by varying the behavioural assumptions of the model we can generate any evolutionary path of prices and firm size distribution we choose. However, it is important to realise that this would only be possible by varying the behavioural assumptions, and so these equations serve the purpose of guiding our thinking about what kind of tendencies on the part of firms with respect to pricing and differentiation strategies would give rise to what dynamics in prices and their distribution over time.

3 Theory of firm survival and flourishing

It might not be a particularly new exercise, but it is interesting to note immediately some properties of the evolutionary equation which give a theoretical insight into the dynamics of firm size. Using Eq. 2.8 we can find express the difference in what might be called "differential survival rates" for firms i and n, expressed as the difference between their growth rates

$$\begin{aligned}\frac{\partial q_i}{\partial t} - \frac{\partial q_n}{\partial t} = \frac{\partial N}{\partial t}\left(\frac{\partial q_i^d}{\partial N} - \frac{\partial q_i^n}{\partial N}\right) + \sum_{j=1}^{F} \frac{1}{p_j}\frac{\partial p_j}{\partial t}\left(q_i^d \varepsilon_{d_i}^{p_j} - q_n^d \varepsilon_{d_n}^{p_j}\right) \\ + \sum_{j=1}^{F}\sum_{k=1}^{N_A} \frac{\partial \alpha_j^k}{\partial t}\left(\frac{\partial q_i^d}{\partial \alpha_j^k} - \frac{\partial q_n^d}{\partial \alpha_j^k}\right)\end{aligned} \tag{3.1}$$

This confirms a fairly intuitive point that differential survival between firms is not dependent on factors common to both firms, but their differences. The factors common to any two firms serve merely to amplify their differences. To be more specific, a closer examination of Eq. 3.1 tells us that what matters most is how much better than others firms are at translating growth factors common to the both of them into demand for their own product. For instance, the growth rate in the population merely serves to amplify the affect on differential survival of how good one firm is at transforming that population growth into growth in demand compared with another. Similarly, what matters is not so much price or attribute dynamics across the market, but how well one firm can transform those dynamics into growth in demand for its product compared with another (reflected in the elasticity of demand to price and response of demand to attributes respectively). The percentage change in prices $\frac{1}{p_j}\frac{\partial p_j}{\partial t}$ serves only to amplify the effect of the difference in cross price elasticities of i and n's demand weighted for their customer bases. Setting aside constraints on the capacity of sales imposed by the production process, demand therefore plays a selective role in the market, determining which firms amongst a heterogeneous population will survive and grow relative to others and which will not. As Jack Downie put it:

> "If we are to regard the desire to grow as common to all firms in an industry, the explanation of different rates of growth must lie in differences in command over the means to growth. The means to growth are customers and capacity."
>
> (Downie 1958, p.64)

Returning to the basic equation for the dynamics of firm size over time we can manipulate it to reveal an interesting relationship between firm growth and the difference in prices, and the percentage change in prices. If we take Eq. 2.8 and apply some tautologies we obtain

$$\frac{\partial q_i}{\partial t} = \frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + \sum_{j\neq i}\frac{1}{F-1}\frac{q_i^d}{p_i}\varepsilon_{d_i}^{p_i}\frac{\partial p_i}{\partial t} + \sum_{j\neq 1}\frac{q_i^d}{p_j}\varepsilon_{d_i}^{p_j}\frac{\partial p_j}{\partial t} + \sum_{j=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t} \tag{3.2}$$

which, subtracting from each term in the first sum each corresponding term in the second sum and simplifying gives us

$$\frac{\partial q_i}{\partial t} = \frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + q_i^d\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] + \sum_{j=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t} \tag{3.3}$$

This equation relates the growth of the firm within the market to the adjusted difference (using own-price and cross-price elasticity of demand) between the *rates* of change of prices (i.e. the percentage growth over time) of any two firms across the market. It contains a number of interesting theoretical insights vis-a-vis the growth of the firm. They are admittedly not new, but these insights are captured here within a single equation.

First and foremost, suppose that we have in a market of normal goods which are substitutes for each other (which covers most markets for a particular good), so that $\varepsilon_{d_i}^{p_i} < 0$ and $\varepsilon_{d_i}^{p_j} > 0\,\forall\, j \neq i$. In this case we can see that, if prices are falling, firm growth depends positively on what might be called the "adjusted" differential of inverse prices, or the terms

$$\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\frac{\partial p_i}{\partial t}\right)\frac{1}{p_i}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\frac{\partial p_j}{\partial t}\right)\frac{1}{p_j} \tag{3.4}$$

contained within the second sum in Eq. 3.3. That is, *ceteris paribus,* a firm will grow faster if it lowers its price relative to its rivals, which we can see by the fact that in the expression (3.4) for "adjusted" differential of prices, a decrease of p_i under the assumptions above will increase the overall term which enters monotonically increasingly into the firm survival equation.

Changing the focus to price *dynamics* rather than statics, we can see that in a market with normal goods which are substitutes for each other, the firm growth rate is increasing in the "adjusted" (by the elasticities of demand) difference in percentage change in prices

$$\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t} \tag{3.5}$$

If firm i's price is falling faster *ceteris paribus* than firm j's, then the first term in the above expression will increase relative to the second and since it enters monotonically increasingly into firm size growth, so too will the growth of the firm increase.

This of course is not new, but what it shows is that in Eq. 3.3 we have something of a generalisation of the equations for firm survival and growth developed by Metcalfe (1998) and Metcalfe et al. (2006). In these equations the argument of the firm growth equation is typically the simple difference of prices $p_i - p_j$. While keeping the intuition of these models of evolutionary markets, Eq. 3.3 above also demonstrates the importance of the cross-price elasticities of demand for the affect of these price differentials. The affects of relative prices described above are amplified by elasticities of demand, so that if own-price elasticity is greater in magnitude then decreases in price have a more positive affect than they would otherwise, while if cross-price elasticity is lower in magnitude, then the effect of competition is dampened. It is small wonder then that rent-seeking and anti-competitive practices are so common throughout the economy given that such predictions arise even from this fairly innocuous model (with respect to the strength of its assumptions).

Another interesting implication of this model for the evolution of the market is that it generates a prediction on the firm size distribution which seems well aligned with the empirical literature on the shape of this distribution. It is well known (and has been well known since (Simon 1955b)) that this distribution is highly skewed to the right, indicating the existence of a high density of small firms and a low density of large ones. It is easily verified by looking at Eq. 3.3 that firm growth is increasing in the size of the firm, provided that the sum total of price affects on firm growth is positive - that is

$$\frac{\partial^2 q_i}{\partial t \partial q_i} > 0 \iff \sum_{j \neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] > 0 \tag{3.6}$$

assuming that $\frac{\partial}{\partial q_i}\left[\frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t}\right] = 0$ and $\frac{\partial}{\partial q_i}\left[\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t}\right] = 0\,\forall k \in \{1, \ldots, N_A\}$, which are admittedly not weak assumptions (though the latter is much stronger than the first, implying that the effects of attribute changes on demand are independent of overall supply). They suffice however to show that mathematically, this equation is more or less identical to the "fitness-adjusted" preferential attachment rule common in complex systems science introduced by Simon (1955b) and popularised by Barabasi and Albert (1999) as an explanation for skewed distributions. Bianconi and Barabasi (2001) demonstrated[12] that with such a rule for the growth of firm size as we have here, even a normal distribution for the affects of adjusted price differentials across firms generates a distribution in firm size which exhibits a high degree of skew. It may be argued that this model is not aligned with the empirical literature because it violates Gibrat's "law" that growth rates of firms are independent of their size (see Gabaix (2009)). Setting aside that it is theoretically highly doubtful that firm growth rates are entirely independent of size, this would indeed be true, if the empirical literature on Gibrat's law were in fact settled in favour of its validity. What literature there is on this "law" is certainly far from agreement that problems with data collection and representativeness have even been addressed to a sufficient degree to even begin to settle the question in the affirmative or negative (Mata 2008). It would seem then this model of evolutionary markets nonetheless will often generate a firm size distribution consonant with that observed empirically. In fact, it gives this result with very weak assumptions on consumer and firm behaviour - weaker certainly than the assumptions of rationality in the complicated Luttmer (2007) model of productivity growth through Brownian motion which can be taken to constitute the neoclassical microeconomic explanation of the firm size distribution.

Continuing on this "Matthew affect" theme[13] another nice property of Eq. 3.3 is that a little manipulation provides us with a micro-founded replicator dynamic for economic phenomena. Replicator dynamics constitute a formalisation of the three step Darwinian process of evolution whereby variety is generated, that variety is resolved through selection of the "fittest" species, and then that species retained by natural selection (Witt 2008; Hodgson 2004; Dawkins 1976). Again by applying a manipulative tautology to Eq. 3.3 we see that

$$\frac{\partial q_i}{\partial t} = \frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + \left(q_i^d + \bar{q} - \bar{q}\right)\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] + \sum_{j=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t} \tag{3.7}$$

[12]Though with physical interpretations of the variables rather than economic.

[13]"For unto every one that hath shall be given, and he shall have abundance: but from him that hath not shall be taken even that which he hath" - Matthew 25:29, King James Version.

and, collecting some uninteresting terms for this problem and rearranging a little we arrive at what can be seen as a generalisation upon a replicator dynamic equation

$$\frac{\partial q_i}{\partial t} = \kappa + \left(q_i^d - \bar{q}\right) \sum_{j \neq i} \left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1} \right) \frac{1}{p_i} \frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)} \right) \frac{1}{p_j} \frac{\partial p_j}{\partial t} \right] + \bar{q} \sum_{j \neq i} \left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1} \right) \frac{1}{p_i} \frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)} \right) \frac{1}{p_j} \frac{\partial p_j}{\partial t} \right] \tag{3.8}$$

where $\kappa = \frac{\partial q_i^d}{\partial N} \frac{\partial N}{\partial t} + \sum_{j=1}^{F} \sum_{k=1}^{N_A} \frac{\partial q_i^d}{\partial \alpha_j^k} \frac{\partial \alpha_j^k}{\partial t}$ and $\bar{q}$ is an average of firm sizes.[14] In typical replicator dynamic equations, the "fitness" (typically measured in a gene's/firm's share of the population/market) of any particular unit of selection grows at a rate which depends positively on the fitness of that unit of selection relative to the average fitness of the population. In economics, this typically translates into a firms growth rate being dependent on its size relative to the market average, which is exactly what Eq. 3.8 expresses in its second term - again provided that the effect of adjusted price dynamic differentials across the market is positive. Formally, this is an "augmented" replicator dynamic because amongst other things we have

$$\frac{\partial^2 q_i}{\partial t \partial \left(q_i^d - \bar{q}\right)} > 0 \iff \sum_{j \neq i} \left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1} \right) \frac{1}{p_i} \frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)} \right) \frac{1}{p_j} \frac{\partial p_j}{\partial t} \right] > 0 \tag{3.9}$$

The "augmentation" of the replicator dynamic is twofold, in that in addition to the replicator dynamic term we are also including the affects of attributes and the growth in the population of consumers, but due to the manipulation we have another augmentation which has an interesting interpretation. The final term of Eq. 3.8 tells us that the average size of firms in the market amplifies the affect of price differentials on firm growth. Roughly speaking, this term which augments the replicator dynamics of the model tells us that the greater the size of the market (more strictly the average size of the firms within) the greater the affect of the firms price dynamics relative to all other firms in the market. Hence not only does the size of the firm relative to the average directly affect its growth, but also the average size of firms across the market *directly* impacts upon the growth rate of firms. In this sense this model is a generalisation upon the replicator dynamics presented in the evolutionary economics literature, prominently by Metcalfe (1998), Metcalfe et al. (2006) and Silverberg and Verspagen (2005), which take a more simple and digestible form like $\partial q_i / \partial t = \delta \left(q_i - \bar{q}\right)$. The current model specifies a form for the δ term and illustrates the dependence of the growth of the firm on differential prices even within a replica-

[14] This expression is in fact robust to the exact type of average used, for instance, it does not matter whether $\bar{q}$ is an arithmetic or geometric mean, or a weighted arithmetic or a weighted geometric mean.

tor dynamic process, and also the importance of the (roughly speaking again) overall size of the market for firm growth.

It is well known (Metcalfe 1998; Joosten 2006) that systems described by replicator dynamics are essentially convergence processes and tend toward a stable "equilibrium" of sorts of "saturation" of the market by the fittest firms (I will return to this property below).[15] Indeed, in the simpler forms of replicator dynamics, where parameters other than the relative fitness such as $\{p_i\}_{i=1}^{F}$ and $\left\{\{\alpha_i^k\}_{k=1}^{N_A}\right\}_{i=1}^{F}$ are constant over time, Joosten (2006) has demonstrated that the conditions for equilibrium and stability thereof are in fact mathematically analogous to the conditions for stability in Walrasian general equilibrium economies, and it is well known (Metcalfe 1998; Nelson and Winter 1982) such systems tend to an equilibrium where the "fittest" firm totally dominates. It is important however to note the distinction of the present model from replicator dynamics in that the aforementioned parameters, or their dynamics, only appear exogenous and constant because their emergence from firm decisions has been suppressed for the purposes of placing pricing and innovation strategies within a market context.

The conditions which generate convergence to a saturation of the market and stability of that point are unlikely to hold in general, especially when the pricing and attribute strategies of firms exhibit an ability to adapt to and learn from changing market conditions reflected by the presence within x_i (hence $f_i(x_i)$ and $\{g_i^k(x_i)\}_{k=1}^{N_A}$) of market conditions observed in the present and the past.[16] Under such conditions, the competitive process will be one of feedback loops from firm strategies into market dynamics, and market dynamics into firm strategies. The model represented by Eq. 3.8 and the firm pricing and attribute strategies nested within it should allow us not only to identify under what conditions there will be convergence to a point-equilibrium or a limit cycle of firm dynamics, but also under what conditions the market will be in a state of constant chaotic evolution. In fact, it should not only allow us to identify such conditions, it should allow us to characterise chaotic evolutionary processes, if conditions are such that they shall prevail. The model therefore extends from simple replicator dynamics to a model of economic self-organisation where the active behaviour of firms interacts with the selective properties of consumer demand to generate evolutionary dynamics (Foster 1997; Witt 1999).

An intriguing implication of the model represented by Eq. 3.3 concerns the affect of increases in the number of competitors in the market. If we take the limiting case of a large number of firms in the market we find that

$$\lim_{F\to\infty}\frac{\partial q_i}{\partial t}=\frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t}-q_i^d\sum_{j\neq i}\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}+\sum_{j=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial\alpha_j^k}\frac{\partial\alpha_j^k}{\partial t} \tag{3.10}$$

[15] I thank Ulrich Witt for reminding me of this property of replicator dynamic models.

[16] We could incorporate any number on assumptions on the firms' knowledge and behaviour here from the firm knowing the exact functional form of demand as in neoclassical economics, to the bare minimum that the firm can only observe the growth in its own size.

Now provided that $\sum_{j \neq i} \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)} \right) \frac{1}{p_j} \frac{\partial p_j}{\partial t}$ converges[17] so that the growth rate of firms is in fact defined, the model therefore implies that as the number of firms in the market becomes large, the degree to which a firms *own* price affects its *own* growth rate approaches zero. That is, as the number of firms in the market becomes large, only competitor's price dynamics have an effect on the firm's growth rate. One would also expect that the requirements for $\sum_{j \neq i} \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)} \right) \frac{1}{p_j} \frac{\partial p_j}{\partial t}$ to converge as $F \to \infty$ would imply that only a finite number of terms in the expression are permitted to be non-infinitesimal. Hence either only a select few competitor's prices will in fact be able to affect the firms' growth or that each and every competitor's price can only have an infinitesimal effect on firm growth. In fact, this same argument also applies to the sum $\sum_{j=1}^{F} \sum_{k=1}^{N_A} \frac{\partial q_i^d}{\partial \alpha_j^k} \frac{\partial \alpha_j^k}{\partial t}$, so that as the number of firms in the market becomes large, firm growth is only mathematically well-defined if either competition becomes "localised" or the only variable to have a non-infinitesimal affect on growth is population growth.

These are all interesting enough accounting exercises and high theories of the evolutionary tendencies of the market system, but much economic - and business for that matter - research consists of asking the question; what conditions will allow a firm to achieve growth? Up until now I have left prices alone and taken them effectively as given, as with attributes, in order to place firm decisions within the context of market dynamics. The beauty of Eq. 3.3 is that it allows us to input any combination of an infinite space of possible pricing and attribute decision rules, as well as an infinite space of possible decision factors. Recognising that $p_i = f_i(x_i)$ and $\alpha_j = \left\{ g_i^k(x_i) \right\}$ are firm decisions and inputting this into Eq. 3.3, again for the sake of argument assuming additive separability, gives us

$$
\begin{aligned}
\frac{\partial q_i}{\partial t} &= \frac{\partial q_i^d}{\partial N} \frac{\partial N}{\partial t} \\
&+ q_i^d \sum_{j \neq i} \left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1} \right) \frac{1}{f_i(x_i)} \nabla_{x_i} f_i(x_i) (\nabla_t x_i)' - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)} \right) \frac{1}{f_j(x_j)} \nabla_{x_j} f_j(x_j) (\nabla_t x_j)' \right] \\
&+ \sum_{j=1}^{F} \sum_{k=1}^{N_A} \frac{\partial q_i^d}{\partial \alpha_j^k} \nabla_{x_i} g_j^k(x_j) (\nabla_t x_j)' \qquad (3.11)
\end{aligned}
$$

The apparent complexity of this equation vanishes when we place in context what it tells us, which is that in order to understand the impact of certain behaviours on the survival and flourishing of a particular firm and more broadly an evolutionary market system we need to work out a select few aspects of consumer and firm decision making processes. To specify the consumer side all we need ask ourselves is the

[17] That is, $\lim_{F \to \infty} \sum_{j \neq i} \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)} \right) \frac{1}{p_j} \frac{\partial p_j}{\partial t} < \infty$.

response of demand for the firm's output to increases in population ($\frac{\partial q_i^d}{\partial N}$), and the sensitivity of demand to changes in price and attributes ($\varepsilon_{d_i}^{p_i}$ and $\varepsilon_{d_i}^{p_j}$ respectively). On the firm side, all we really need to analyse the dynamics of firm size and hence market evolution is ask what is the response of firms' pricing to changes in decision factors ($\frac{\partial f_i(x_i)}{\partial x_i}$), and what are the changes in those factors over time ($\frac{\partial x_i}{\partial t}$). We can make the specific assumptions as granular and intricate or wide ranging and simple as we wish. Clearly the dynamics of the system of firms and consumers under different behavioural assumptions on consumers and firms represented by Eq. 3.11 are in general not analytically solvable (being an N-body problem (Qiu-Dong 1991)) and are therefore candidates for simulation studies in the spirit of Nelson and Winter (1982). However, equations do not need to be analytically solvable in order to be analysed.[18] After arriving at a particular mix of assumptions about firm behaviour and consumer behaviour we need only feed these into the equation above (Eq. 3.11) in order to analytically derive theoretical predictions about the growth rate of any particular firm.

3.1 Some theorems on firm growth strategy

Stepping away again from delving into the exact details of any particular firms' pricing and attribute decisions, we can derive a few informative theorems (albeit perhaps trivial) on what *any* pricing/attribute decision making strategy must satisfy in order to maintain firm growth. For the proofs of all theorems that follow see Appendix B.

Theorem 1 *Suppose that* $p_j = 0, \forall j \in \{1, \ldots, F\}$, *or* $\partial p_j / \partial t = 0, \forall j \in \{1, \ldots, F\}$. *Then the growth of any arbitrarily selected firm i is*

$$\frac{\partial q_i}{\partial t} = \frac{\partial q_i^d}{\partial N} \frac{\partial N}{\partial t} + \sum_{j=1}^{F} \sum_{k=1}^{N_A} \frac{\partial q_i^d}{\partial \alpha_j^k} \frac{\partial \alpha_j^k}{\partial t} \tag{3.12}$$

This seemingly trivial theorem is actually quite interesting, for it illustrates the fundamental constraint placed upon the growth rate of firms by demand. If we assume that we have a market for a normal good (so that $\frac{\partial q_i}{\partial p_i} \leq 0$, $\frac{\partial q_i}{\partial p_{j \neq i}} \geq 0 \forall i, j \in \{1, \ldots, F\}$) and can meaningfully sum across these goods, demand when prices across the market are zero $K = \sum_{i=1}^{F} q_i^d \left(N \quad \{0\}_{j=1}^{F} \quad \left\{ \left\{ \alpha_j^k \right\}_{k=1}^{N_A} \right\}_{j=1}^{F} \right)$ can be taken to represent the "capacity" of demand within the system for a given consumer population and attribute spread beyond which price can have no further affect on demand if we are to forbid negative prices. Theorem 1 tells us what occurs theoretically to

[18]For instance, Einstein's equations of general relativity, generate interesting insights even without applying the particular set of assumptions (such as those made by Oppenheimer to demonstrate the existence of black holes) which yield an analytical solution.

firm growth rates as we tend toward a zero price scenario, or perhaps more intuitively, a scenario in which firms no longer compete on the basis of price, perhaps because they cannot do so. This might be the case, for instance, when firms across the market have through "process innovation" (*a la* Nelson and Winter (1982) and Jack Downie (1958) (Nightingale 1997)) cut prices as low as possible in a market already without incurring losses and cease to be able to compete on the basis of price without doing so. If this happens, firms will only achieve growth through the growth in the population of consumers, and/or through a change in their attributes, in short, through exogenous factors and also factors over which they have control to increase the carrying capacity of the market, the K-limit.

This sounds very similar to the idea behind a "diffusion curve" common in studies of evolutionary markets that the "carrying capacity", or K-limit of the market acts as an attractor toward which the market tends as it "matures" through price competition facilitated by cost-cutting (Dopfer et al. 2004; Foster 2005; Dopfer and Potts 2007). Technically speaking of course, a diffusion curve (in our grammar of the economy), a staple of models of self-organisational systems, should be of the functional form $\partial q_i/\partial t = g\,(t)\,F\,(q_i/K)\,q_i$, which typically manifests in a logistic "s-shape" type growth path "levelling out" as the K-limit is approached. It may or may not be possible to manipulate Eq. 3.3 into such a form, but the idea still stands that in the evolutionary market above there is a constraint placed upon growth by the capacity of demand, and as that capacity is reached by supply, or we come close enough to it that firms can no longer compete on the basis of price, growth rates converge to the growth rate of the demand capacity of the market.[19] What makes this model slightly different also is that this K-limit can be influenced by firms through their choice of attributes for their products, and is not necessarily entirely exogenous to the model.

It is interesting in particular to note the theory suggests that if we have arrived at a state where across the market price competition is no longer possible, the only way to maintain growth through factors upon which the firm can exert some influence (assuming firms cannot seek out new markets for their product) is through product innovation. Should, for instance, $\partial q_i/\partial t \in x_i$ and $\exists\,\{\eta\} \in \{1, \ldots, N_A\}$: $\partial g_i^k\,(x_i)\,/\partial x_i < 0\,\forall\,k \in \{\eta\}$, the *ceteris paribus* slowdown in growth as price competition ceases to have an affect will trigger what might be thought of as investment in "improving" the attributes of the products. This reflects in the mathematics of the evolutionary market represented by Eq. 3.3 the argument of Dopfer et al. (2004) that innovation becomes most intense as the market becomes concentrated and saturated (with respect to demand).

We will return shortly to competition on the basis of product innovation, but let us now consider the process of competition on the basis of price. If the firm is trying to compete on the basis of its price in a normal good market it is fairly straightforward to identify the conditions required for price competition to contribute to its growth.

[19]Strictly speaking of course, an individual growth rate $\frac{\partial q_i}{\partial t}$ would converge to the growth rate of $q_i^d\left(N \quad \{0\}_{j=1}^F \quad \left\{\left\{\alpha_j^k\right\}_{k=1}^{N_A}\right\}_{j=1}^F\right)$.

Theorem 2 *Suppose that we are in a market of normal goods which are substitutes for one another and also that population growth is zero and product attributes are unchanging. If this is the case then, provided that* $q_i \neq 0$

$$\frac{\partial q_i}{\partial t} > 0 \iff \exists B \subset \{1, \ldots, F\}$$

where B *satisfies*

$$\begin{cases} \left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t} > 0 \, \forall j \in B & (1) \\ \sum_{j \in B}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] > \sum_{j \notin B}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] & (2) \end{cases}$$

This theorem is a little more subtle than may first appear. What it tells us is that a firm must set prices such that its adjusted price dynamics relative to the adjusted price dynamics of its rivals must be favourable for a sufficient number of its rivals. However, at the same time, the firm must have a "foothold" in the market - a base level of demand/output - in order for these price differentials to have any affect whatsoever on its growth rate. That is, a non-zero supply is a necessary, but not sufficient condition for a positive growth rate, but once this necessary condition is satisfied, it is necessary and sufficient for the firm's price to lead to sufficient number of favourable adjusted price differentials. It suggests therefore that a good strategy for a firm with limited information about its rivals prices and about the sensitivity of demand to its own and to other's prices would be simply to compile an ordered list of its competitor's prices, have a guess at how far "down" that list it must go until it is competitive on the basis of price, and then adjust its prices accordingly, provided it has a "foothold" in the market in the form of a non-zero demand for its output in the first instance.

What is also interesting about the conditions of Theorem 2 is that if price dynamics across the market are quite small $\frac{\partial p_i}{\partial t} \to 0$, then price differentials (p_i compared with p_j) will only have an affect provided that demand elasticities are quite high. This might at first appear a little problematic given many evolutionary economic models such as those of Metcalfe (1998) relate absolute price differentials to growth, not dynamics. However, on a second look the difference is instructive, for what Theorem 2 is telling us here is that if price dynamics are small, then price differentials will only have an affect on survival and growth if own and cross-price elasticities - the willingness of consumers to take their custom elsewhere - are quite high. This merely serves to elaborate the coefficient of price differentials in prior evolutionary models and refine it to be a coefficient for inverse price differentials.[20] It also makes sense on an intuitive level, if preferences have not changed, and prices have not changed and there was prior awareness of lower prices,[21] what would make consumers change

[20] Specifically, this coefficient would be equal to $\varepsilon\left(\frac{F}{1-F}\right)\frac{\partial p}{\partial t}$, and we would have to assume that elasticities for each firms' demand and price dynamics are identical across the market so that growth equations can be written in the form $\delta\left(\frac{1}{p_i} - \frac{1}{p_j}\right)$.

[21] Note that the inclusion of q_i as an amplifying effect for price differentials could be taken as a proxy for the "availability" of information about a particular firm for consumers.

their behaviour? In effect, for firms to be able to compete and grow on the basis of price differentials in absolute terms and not their dynamics, consumers need to be fairly non-habitual in their decision making. This is not out of line with the theory of habitual decision making, which suggests individuals will only change their habits once they have sufficient evidence that their current habits of decision making are not generating an acceptable outcome (Hodgson 2004; 2010). Consumers coming into the market, or more strictly speaking, entering into the factoring of firm i's demand curve would rightly enter into the dynamics through its being a function of the population (that is, it would be reflected in $\frac{\partial q_i}{\partial N}$), and were this to be on the basis of price differentials, we would find this to depend on price differentials.

If we dig a little deeper still into the conditions of Theorem 2 we can see that achieving growth through price competition becomes more difficult *ceteris paribus* the more firms there are in the market. This is because the adjustment made to the inverse price differential in the equation for firm growth includes the number of firms in the market, and if we return to the condition which determines whether firms are included in the set B for which adjusted price differentials are favourable for the firm

$$\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}>0$$

we can see that, provided we are in a market for a normal substitutable goods where prices are trending down over time this condition becomes more and more difficult to satisfy as F grows. That is, for competitors to be included in the set B as the market size (in terms of number of firms) grows requires *ceteris paribus* lower and lower prices, and/or faster and faster rates of price reduction for firm i. Hence it becomes more and more difficult for firms to compete on the basis of price as the number of firms in the market increases.

Firms can and do compete on the basis of price, but as Schumpeter (1911) so vividly demonstrated, this is not the only basis upon which competition operates. The model above reflects Joseph Schumpeter's insight that firms can also compete through product innovations - new "combinations" (here) of attributes - as well as "process" innovations which lower costs. They can compete on the basis of their product attributes as much as they can on price. It is therefore interesting to see what is required for firms to be able to maintain growth through the dynamics of their product attributes alone.

Theorem 3 *Suppose that we have a market for normal, substitutable goods. It is the case that*

$$\frac{\partial q_i}{\partial t}>0 \iff \sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_i^k}\frac{\partial \alpha_i^k}{\partial t}>q_i^d\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}\right]$$
$$+\sum_{j\neq i}^{F}\sum_{k=1}^{N_A}\left[(-1)\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t}\right]-\frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} \tag{3.13}$$

Again, this theorem may seem trivial, but it actually contains a some interesting insights into the theory of firm survival and competition on the basis of attributes. As might be expected, it tells us that the growth in demand from attributes must be sufficient to outweigh the affects of price differentials, the affect on demand of competitors' product attributes and the growth rate of the population. It also tells us that growth is easier to maintain through changing attributes - less innovation is required to ensure growth - the greater is the population growth rate (provided that this increases demand), and the more favourable the adjusted price differentials for the growth of the firm. But closer inspection of condition (3.13) reveals some more subtle requirements for the firm to compete on the basis of attributes to the extent that it grows. Theorem 3 emphasises that for the firm to maintain growth it not only requires a sufficiently high degree of innovativeness (manifest in a high $\frac{\partial \alpha_i^k}{\partial t} = \frac{\partial g_i^k(x_i)}{\partial x_i}\frac{\partial x_i}{\partial t}$) but also a sufficiently high degree of responsiveness of demand to that innovation (that is, a high $\frac{\partial q_i^d}{\partial \alpha_i^k}$). So not only must the firm be highly innovative to guarantee growth on the basis of product innovation, but also consumers must be sufficiently receptive to that innovation. Put another way, a firm must be competing on the basis of its strengths, where demand is most responsive to its innovations.

Conversely, theorem 3 serves to demonstrate that if price differentials - adjusted for price elasticity of demand - are sufficiently unfavourable (i.e. the relative prices and sensitivity of demand to them are too high) it will be very difficult indeed for the firm to maintain growth through product innovation. As we can see by the fact that in such a situation the right hand side of condition (3.13) grows as firms become less competitive with respect to price, there is a limit to how much incremental product innovation (i.e. smaller rather than substantial or radical) can guarantee growth of the firm, without even taking into account that other firms are also likely to be innovating and making it difficult to compete on this basis. The situation only becomes worse as the growth rate of the population approaches its lower limit of zero. The intensity and multidimensionality of the struggle to survive in a market system - "capitalism red in tooth and claw" - comes very much alive in these equations. Even an Apple led by the prophet Jobs would not be able maintain growth if the population of computer users were not growing and its prices (adjusted for consumer demand factors) were too extravagant - relative to Microsoft.

An intriguing implication of Theorems 1 and 3 therefore arises if we suppose that products are characterised by a *region* of the characteristic space of attributes. The first tells us that as we approach the zero lower bound on prices, or a nonzero lower bound, growth can only be maintained by product innovation and the growth of the consumer population - i.e. increasing the capacity of the market. The second tells us that the more unfavourable the price differentials in the market, and the slower the growth of the population, the greater product innovation is required to maintain growth.

It stands to reason that within a particular product dimension a certain object only can be said to exist within a certain region. For instance, within the dimension of "number of wheels" a car is something which has more than two wheels, but once it has more than, say six, it becomes more sensible to speak of a "truck". Similarly, on the dimension "quality of photography" (measured, say in pixels of photographs taken), a land-line phone typically will have an attribute of zero, an early camera phone will have some non-zero quality while an iPhone or Samsung Galaxy will be "in a different realm entirely" within this dimension. Now supposing then that we can identify a region[22] $A_q \subset A$ of the characteristic space which defines what is (any vector $\alpha_i \in A_q$) and what is not (any vector $\alpha_i \notin A_q$) product q, we can see that if innovation even in one attribute dimension $\frac{\partial \alpha_i^k}{\partial t} = \frac{\partial g_i^k(x_i)}{\partial x_i} \frac{\partial x_i}{\partial t}$ is radical enough , we might end up with a scenario for firm i in which $\exists k \in \{1, \ldots, N_A\} : \alpha_i^k \notin A_{q_i}$. Such a firm will have introduced a new product by investing in developing the attributes of its product, for instance upgrading from a mobile phone to a "smart" mobile phone or developing a luxury car around the engine of a fighter plane (as was the case with twice with BMW post WWI and WWII).

Again therefore we can see that the equations for firm survival in an evolutionary market reflect the intuition of Dopfer et al. (2004) and Foster (2005) that pressure toward radical innovations builds toward the end of the diffusion process - the "saturation" point or K-limit - where price competition across the market becomes infeasible. But we also see that radical innovations may be a strategy pursued by a growth-seeking firm which has become uncompetitive with respect to price, or which exists in an intensely competitive market where favourable price differentials are hard to maintain, and other firms are innovative. True to the tale of Schumpeter (1911), competition is not a static, "circular flow in equilibrium" situation, competition is a process of trying new combinations of attributes, trying new production processes to lower prices, trying to break with the past in order to get ahead in the future, and the process whereby economic development is brought about.

3.2 Firm strategy and market share

The above results pertain to the growth of the firm as an absolute variable, but we may also be interested in the dynamics of the "structure" of the market as characterised by market share. One final theorem on the theory of firm survival which will be quite useful in analysing the possible theoretical dynamics of the price distribution concerns the requirements for the growth of market share.

[22] A bounded, typically connected and convex subspace - that is, a multidimensional interval of A.

Theorem 4 *A firms' market share grows if and only if*

$$\frac{\partial N}{\partial t}\left[\frac{\partial q_i^d}{\partial N}-\frac{s_i}{1-s_i}\sum_{j\neq i}\frac{\partial q_j^d}{\partial N}\right]+q_i^d\sum_{j\neq i}\left\{w_i\frac{1}{p_i}\frac{\partial p_i}{\partial t}-w_j\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right\}$$
$$+\sum_{k=1}^{N_A}\frac{\partial \alpha_i^k}{\partial t}\left[\frac{\partial q_i^d}{\partial \alpha_i^k}-\frac{s_i}{1-s_i}\sum_{j\neq i}\left(\sum_{k=1}^{N_A}\frac{\partial q_j^d}{\partial \alpha_i^k}-\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial \alpha_i^k}\right)\right]$$
$$>\frac{s_i}{1-s_i}\sum_{j\neq i}\left\{q_j^d\sum_{n\neq j,i}\left[\left(\frac{\varepsilon_{d_j}^{p_j}}{F-1}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}-\left(\frac{\varepsilon_{d_j}^{p_n}}{(-1)}\right)\frac{1}{p_n}\frac{\partial p_n}{\partial t}\right]+\sum_{n\neq i}\sum_{k=1}^{N_A}\frac{\partial q_j^d}{\partial \alpha_n^k}\frac{\partial \alpha_n^k}{\partial t}\right\} \tag{3.14}$$

where $w_i=\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)+\frac{s_j}{1-s_i}\frac{q_j^d}{q_i^d}\left(\frac{\varepsilon_{d_j}^{p_i}}{(-1)}\right)\right]$ *and* $w_j=\left[\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)-\frac{s_j}{1-s_i}\frac{q_j^d}{q_i^d}\left(\frac{\varepsilon_{d_j}^{p_j}}{F-1}\right)\right]$ *act to adjust price dynamic differentials for behavioural factors, market share and market size by number of firms.*

This theorem on market share growth is more informative than the *prima facie* complexity of Eq. 3.14 would obscure. Firm i can only directly influence (i.e. through its pricing and attribute decisions rather than through rent-seeking) the terms on the left hand side of the inequation. On the right hand side of the equation we have grouped together, and weighted for firm i's market share the growth of all other firms due directly to their price differentials with respect to each other, and the growth of these firms due to growth in their attributes. In a sense, the right hand side of condition (3.14) keeps track of what is going on vis-a-vis the interplay of the "rest" of the market with respect to itself, while the left hand side accounts for the competitive dynamics with respect to population, prices and attributes over which the firm i has direct influence through its choices.

It seems rather obvious then to say that the growth factors over which firm i has direct control through its pricing and attribute decisions, taken together, must therefore outweigh the dynamics of the rest of the market. However, what Eq. 3.14 implies is sufficient for this to be the case is rather interesting and intricate. Notice first that the term $\frac{s_i}{1-s_i}$ enters as a coefficient on a number of terms, which indicates that the market share of firm i, *as a fraction of the other firms' share of the market* will amplify certain market factors on growth of firm i's market share. If market share of firm i grows therefore, this effect is in fact greater than it would be if firm i's market share alone entered the inequation (since $1-s_i \leq 1\,\forall\, s_i \in [0, 1]$). And in particular, this term enters on terms which make it more difficult for the theorem to be satisfied the greater they are.

The first term of the inequality condition in Theorem (3.14) tells us that for the firm's market share to grow, it must be relatively "better" than *all* its rivals taken together at transforming population growth into demand for its products, when those rivals ability to transform population growth into demand is weighted for firm i's market share as a fraction of the market share of all other firms. Interestingly, this means that the greater the firms' market share, the more the firm needs to be able to convert new customers into new custom. I will for now supress consideration of this interesting property but I will return to it shortly below.

The second term on the left hand side of condition (3.14) confirms that the general effect of replicator dynamics and preferential attachment (a "Matthew effect") applies to the dynamics of market share as well. *Ceteris paribus* the greater the size of the firm, the easier it is for it to satisfy Theorem 4 and the greater is the growth rate of its market share - provided that the firm is sufficiently good at outperforming its rivals on the basis of price. It also indicates that the implications of theorem 2 with respect to the affects of price differentials on firm growth more or less apply directly also to the growth of market share - insofar as that, in a market for normal goods, provided that the firm's prices are lower and falling with respect to a sufficient number of its competitors it will grow in terms of market share.

This reflects in the mathematics a fascinating irony at the heart of modern evolutionary economics since Nelson and Winter (1982) and in direct contradiction with neoclassical economics - competition destroys itself. The essence of the process of competition is that tends, in the absence of any great change, to one firm, or a subset of firms, coming to dominate the market over time through the growth of their market share. As again Joseph Schumpeter said (in as many words), nobody gets into a market to make "normal" (which to a classical or neoclassical economist oddly means "zero") economic profits, and rarely would we see this in reality.[23] In seeking profits through competition on the basis of price, the differential survival therefore of more "competitive" (high quality, low cost, preferably both) firms will lead to their dominance. In competition are the seeds for monopoly.

However, the existence of attribute and population effects will confound this growth from price. It is by no means obvious that the "most" competitive firm with respect to price, or for that matter attributes will come to dominate the market. Even ignoring one of these two growth factors at a time, competition is on the basis of what I have called "adjusted" price differentials - not price differentials alone. With respect to prices, even ignoring attributes and population growth confounding factors with respect to the survival of the lowest price firm or firms are the own-price and cross-price elasticities of demand, market shares and firm sizes. For instance, if cross-price elasticities are quite small then it is perfectly feasible for a high relative price firm to survive and even grow in share over time.

[23]Though ultimately, whether normal profits prevail in reality is an assumption, since "economic" profits are supposed to include "opportunity" costs of courses of action not taken, and these are vague enough conceptually to be quite malleable in estimating economic profit.

To elucidate some properties of the third term of condition (3.14) term I will restate it for convenience and rearrange it a little

$$\sum_{k=1}^{N_A} \frac{\partial \alpha_i^k}{\partial t} \left[\frac{\partial q_i^d}{\partial \alpha_i^k} - \frac{s_i}{1-s_i} \sum_{j \neq i} \left(\sum_{k=1}^{N_A} \frac{\partial q_j^d}{\partial \alpha_i^k} - \frac{\partial q_i^d}{\partial \alpha_j^k} \frac{\partial \alpha_j^k}{\partial \alpha_i^k} \right) \right]$$
$$= \sum_{k=1}^{N_A} \frac{\partial \alpha_i^k}{\partial t} \left[\frac{\partial q_i^d}{\partial \alpha_i^k} - \sum_{j \neq i} \sum_{k=1}^{N_A} \frac{\partial q_j^d}{\partial \alpha_i^k} + \frac{s_i}{1-s_i} \sum_{j \neq i} \sum_{k=1}^{N_A} \frac{\partial q_i^d}{\partial \alpha_j^k} \frac{\partial \alpha_j^k}{\partial \alpha_i^k} \right] \tag{3.15}$$

This shows us that not only must firms have a sufficiently high rate of improvement in its product attributes through product innovation, it must also be able to transform this innovation into custom. This is reflected by the fact that this term is only positive - provided improvement of firm i's attributes deprives firm j of demand at the margin (so that $\frac{\partial q_j^d}{\partial \alpha_i^k} > 0$) - if the growth in its demand and its ability to deprive other firms of demand through product innovation outweighs (after some adjustments) other firms' ability to deprive *it* of demand through innovation, $\frac{\partial q_i^d}{\partial \alpha_j^k}$. The inclusion of what might be called a "reaction function" or more simply a relative rate of innovation $\frac{\partial \alpha_j^k}{\partial \alpha_i^k}$ as an adjustment to this term is instructive, because it demonstrates that the firm must be innovating sufficiently *given* that other firms are innovating also. Again and perhaps surprisingly, as the firm grows in share, it becomes difficult for the firm to maintain further growth in market share provided that other firms' innovations *ceteris paribus* deprive it of demand ($\frac{\partial q_i^d}{\partial \alpha_j^k} < 0$) and the general trend of the market is to innovate ($\frac{\partial \alpha_j^k}{\partial \alpha_i^k} > 0$). The inclusion of these behavioural and market share factors such that market share growth does not depend only on relative attribute strengths means that it is by no means obvious that the "best" product will become dominant in the market, even without the confounding influence of prices and population growth.

Notice that within the discussion above, with some caveats it is the case that the greater the firms' market share, the greater the difficulty of maintaining growth of demand. A rather fascinating property of in Eq. 3.14 is that as $s_i \to 1$, provided we have a market for normal substitutable goods and sufficiently many price dynamics in the same direction across the market, two positive infinities develop in w_i and w_j which cancel out, so that prices cease to affect the growth of market share. Then the existence of what might be called the "market share weight" under most conditions therefore makes the left hand side tend to either a negative infinity or zero and the right hand side tend to a positive infinity, provided the other firms in the market are tending to grow also. This may seem alarming until we recognise that it merely means that as a firm comes to *completely* dominate the market it becomes impossible for it to continue to increase its market share - which merely states a mathematical fact. The exact properties of the model as $s_i \to 1$ I will leave for future research, but the model therefore can be said to confirm the intuitive implications of theorem 1 again that in many markets as one firm tends toward saturating the market, it becomes increasingly difficult for that firm to maintain growth through factors under its control. It is not much of a stretch to imagine that if firms care about growth of market *share* and not just growth overall that this could help us to understand yet another reason why political rent-seeking is so pervasive throughout the economy, and also provide an explanation for why even seemingly totally dominant firms engage in it. Indeed, it seems rather eerily to confirm on a theoretical level and from a different basis Marx's idea that when capitalists run out of customers they will seek new ones, to the extent that like the East India Company, they may start to found empires.

The eeriness of these theorems in their conceptual similarity to the struggle for survival amongst the species in evolutionary biology is rather interesting, and perhaps even disturbing. Conjecture has it that George R. Price, one of the pioneers of evolutionary mathematics whose theorems I will make use of below, found the violence of the concepts represented by equations such as these so disturbing that they eventually contributed to his suicide. Certainly, what comes through very clearly in the proliferation of "relative" terms is that in the market as in the jungle, even the institutional structure of modern civilisation is not enough to prevent the existence of a "red in tooth and claw" environment. Firm survival is a matter of constantly needing to beat one's rivals and "get ahead", and if one cannot keep "winning" in certain areas then one needs to switch to a different theatre to overwhelm any rivals who may appear "fitter". In this race, furthermore

> "I returned, and saw under the sun, that the race is not to the swift, nor the battle to the strong, neither yet bread to the wise, nor yet riches to men of understanding, nor yet favour to men of skill; but time and chance happeneth to them all."
>
> Ecclesiastes 9:11 (King James Version)

The theorems do not give survival courtesy of selection in the market to the "fittest" overall. The environment the firm happens to find itself in is as much a determinant of its survival as its own performance on price and product. A firm that may appear to be brutally fit in one environment may be completely unfit in another. Growth is driven almost always by *relative* fitness, and if it is not driven by relative fitness, when a firm's own price/attributes have little effect on its own demand then growth is anyway taken out of the direct hands of the firm. The firm must be fit in the context of its competitors. But it must also be fit in the particular market context outside its competitors. It must face favourable demand curves, or be fit relative to these demand curves, able to provide what the particular customers it faces desire, and do this better than any other firm in the market.

If one firm can do that consistently, it will drive all others eventually into complete submission even while like a dying star struggling increasingly to maintain its own growth. Depending on one's politics this struggle for survival is either the glory, horror or some mix therein of violent capitalistic evolution.

4 Dynamics of the distribution of prices

There is no "clear-cut" theory of prices in the model above, in the sense that there is no equation which prices are the solution to and which we can analyse using comparative statics. Indeed, if we acknowledge the Oxford price studies Hall and Hitch (1939), Andrews (1949), Andrews (1950), and Andrews (1964) and all studies which stem from them Blinder et al. (1998) and Lavoie (1996), there can really be no "one" theory of prices of a fashion such as the theory of general equilibrium (Mas-Collel et al. 1995) which approximates reality to an acceptable level given the vast variety of pricing rules of firms and market interactions between firms and customers. This has been well known at least since Janos Kornai (1971) pointed out that there is no one method by which prices are determined. What we *can* have however, with the model set out above, is a theory of how the *distribution* of prices evolves over time. We can put some order on the population of rules for prices even without knowing the exact distribution of these rules, because we know how the outputs of those rules - prices - interact in the evolutionary market.

In the model above, firms ultimately set their own prices in a firm-specific manner and with respect to firm-specific decision factors. The positive theory of price determination is on this level a question of the theory of the firm, and one which must be based on an empirical analysis of firm price-setting. However, the demand for firms' output evolves over time in a manner dependent upon their prices, and so the market determines the incidence (how often that price is paid for output) in the market. The theory of prices in evolutionary markets then consists of the analysis of how firm pricing rules $\{p_i = f_i(x_i)\}_{i=1}^F$, firm sizes and their growth over time, $\left\{q_i \quad \frac{\partial q_i}{\partial t}\right\}_{i=1}^F$ interact to generate the dynamics of the distribution of prices across the market. We can analyse this distribution using its moments, which for the sake of convenience, are the average price and variance in prices weighted by firm size, or the "incidence" of the price in the market

$$E_q(p) = \frac{\sum_{i=1}^F q_i p_i}{\sum_{j=1}^F q_j} \tag{4.1}$$

$$\mathrm{Var}_q(p) = \frac{\sum_{i=1}^F q_i \left[p_i - E_q(p)\right]^2}{\sum_{j=1}^F q_j} \tag{4.2}$$

An interesting question well worth answering here is whether or not this distribution converges to a particular shape under certain conditions over time. It is for this purpose that Theorem 4 is quite useful, as it allows us to characterise the dynamics for the *share* of the market transactions at which a particular price prevails. One particularly important type of price distribution is the degenerate price distribution, for if the price distribution is degenerate, it is very easy to see this implies a uniform price is charged across the market.

Definition 1 The price distribution is degenerate if and only if for any set of firms $\Xi(p) = \{i \in \{1, \ldots, F\} : p = f_i(x_i)\} \subset \{1, \ldots, F\}$

$$\begin{cases} q_i \geq 0 \, \forall \, i \in \Xi(p) & (1) \\ q_i = 0 \, \forall \, i \notin \Xi(p) & (2) \end{cases}$$

Immediately following this definition is the implication that a degenerate price distribution represents a market scenario in which a uniform price prevails for any transaction in the market.

Lemma 1 *A uniform price prevails such that* $p_i = E_q(p) \, \forall \, i : q_i > 0$*, and* $Var_q(p) = 0$ *if and only if the price distribution is degenerate.*

This lemma is mathematically quite trivial, but economically it is quite important, because the definition of the price distribution serves to specify the necessary and sufficient conditions that the dynamics of firm survival must satisfy for a market to evolve to such a state that (incurred) price differentials are eliminated. If we can identify conditions using the theorems on firm size and growth which will imply convergence to the distributional conditions of Lemma 1, we then will have completely characterised the theoretical sufficient conditions for a uniform price market.

Theorem 5 *A uniform price* $p_i = E_q(p) \, \forall \, i : q_i > 0$ *will prevail in the market if and only if for some set of firms with identical prices* $\Xi\left(E_q(p)\right) \subset \{1, \ldots, F\}$ *there is a set of time periods*

$$T_{i \notin \Xi(E_q(p))} = \left\{ t \in \mathbb{R}_+ : \frac{\partial s_i}{\partial t} < 0 \right\} \subset [1 \quad T] \subset \mathbb{R}_+$$

in which Theorem (3.14) is violated for each and every firm $i \notin \Xi\left(E_q(p)\right)$ *such that for an initial market share* s_i*,* $T_{i \notin \Xi(E_q(p))}$ *is sufficiently large for each and every firm* $i \notin \Xi\left(E_q(p)\right)$ *that*

$$-lim_{T \to \infty} \int_{t \in T_{i \notin \Xi(E_q(p))}} \frac{\partial s_i}{\partial t} \partial t = s_i + lim_{T \to \infty} \int_{t \notin T_{i \notin \Xi(E_q(p))}} \frac{\partial s_i}{\partial t} \partial t$$

That is, for each and every firm given the market conditions into the future there are sufficiently many periods for which Theorem (3.14) is violated not only to reduce market size to zero but also to overwhelm any periods for which Theorem (3.14) holds.

It should be fairly obvious that these conditions are quite difficult to satisfy. They require that *any* firm with a price above, *or below* the uniform price to which the market will converge will suffer a price/attribute mix which under the market demand conditions will allow for sufficiently many periods of time in which they lose market share that not only their current market share is wiped out, but also any share gain that they may make in the future. Such a scenario is unlikely to hold unless there is some factor which enforces adherence to a uniform price across the market in the pricing rules $\{p_i = f_i(x_i)\}_{i=1}^F$ - be that regulation, strategic concerns within a cartel, implicit collusion or “keeping in line with the market” (Hall and Hitch (1939) and Andrews (1949) and Andrews (1950) noted such concerns were quite common empirically) or a situation in which firms have competed to the lowest price that they can.

This last is quite similar to the idea of perfect competition in the long run if $\Xi\left(E_q(p)\right)$ is quite large, given one would in all likelihood need fairly homogeneous firms with respect to attributes given theorem 3 that non-price competitive but innovative firms can grow. This theorem however would point out the limitations of a model based on the idea of perfect competition, given the strength of the conditions implied by Theorem 5 and the need for a lack of innovativeness (even without additional assumptions with respect to market power) which confound the selection of a uniform price. In any case, almost any circumstances where such a uniform price distribution would be begin to emerge would be highly unstable given the results of theorem 2 that firms *ceteris paribus* will increase their growth by lowering prices (perhaps through “process” innovation to lower costs), theorem 1 that even in such situations where firms cannot lower prices growth can be maintained and theorem 3 that “product” innovation can hypothetically maintain growth for even highly price uncompetitive firms.

We can conclude that outside particular populations of rules for prices $\{p_i = f_i(x_i)\}_{i=1}^F$, we are unlikely to see a uniform price emerge across the market through selection and differential survival as perfect competition would demand in standard neoclassical economics. Perhaps ironically, it is much easier to imagine that a uniform price distribution would prevail naturally as a result of the emergence of a completely dominant, hyper-competitive, firm than through the competition of many smaller non-dominant firms. The conditions for this are simply a special case of theorem 5.

Corollary 1 *A uniform price* $p_i = E_q(p)\ \forall i : q_i > 0$ *will prevail in the market if there is a firm* $j = \Xi(p)$ *such that if*

$$T_{i\notin\Xi(E_q(p))} = \left\{t \in \mathbb{R}_+ : \frac{\partial s_i}{\partial t} < 0\right\} \subset [1 \quad T] \subset \mathbb{R}_+$$

in which Theorem (3.14) is violated for each and every firm $i \notin \Xi\left(E_q(p)\right)$ *such that for an initial market share* s_i, $T_{i\notin\Xi(E_q(p))}$ *is sufficiently large for each and every firm* $i \notin \Xi\left(E_q(p)\right)$ *that*

$$lim_{T\to\infty} \int_{t\notin T_{i\notin\Xi(E_q(p))}} \frac{\partial s_i}{\partial t}\partial t = 1 - s_i + lim_{T\to\infty} \int_{t\in T_{i\notin\Xi(E_q(p))}} \frac{\partial s_i}{\partial t}\partial t$$

With reference to theorem (3.14) it should be fairly clear that these sufficient conditions for a uniform price distribution are much easier to satisfy than the conditions of theorem 5. What is required for a uniform price to emerge through the emergence of a monopoly is merely that the firm be either extremely price competitive, extremely innovative, or a combination of both - think Walmart, or IBM at its zenith - such that theorem 5 holds for enough time periods that the firm's market share will grow sufficiently large to ensure the non-survival of all competitors. The moment we wish for the same uniform price to hold with more than one firm however, we need the stronger conditions of 5 to hold, and if we imagine (as is not hard to do) that heterogeneity of product, and decision rules (which likely rely partly on production processes) grows with the number of firms in the market, it will become more and more difficult for the conditions of that theorem to hold. This would seem to confirm the idea again that in competition are the seeds for its destruction. Amongst a collection of firms in a market, without additional assumptions on the population of pricing rules, it is much easier to imagine that a single price will emerge because one hyper-competitive firm comes to dominate all others than because a collection of different and almost certainly heterogeneous firms come to dominate all others.

Looking at the dynamics of the price distribution in general however, we can say that "anything goes" with respect to the distribution of prices. Even in a market for normal substitutable goods it is possible to conceive of situations in which the expected price *increases* over time. For example, the average market price may will increase if there is a firm with a higher price than all others but one which is sufficiently innovative to maintain growth in market share and which has a sufficiently low cross-price elasticity.

In the other diretion, we can imagine that if sufficiently many firms seek growth ($\frac{\partial q_i}{\partial t} \in x_i$), and sufficiently many believe that decreasing prices (perhaps permitted only by investment out of profits in lower costs) will lead to growth, they will *ceteris paribus* seek to lower their prices if growth decreases so that $\frac{\partial f_i(x_i)}{\partial \frac{\partial q_i}{\partial t}} \leq 0$. In the absence of other confounding factors (innovation, population growth) the requirements for market share growth collapse to

$$q_i^d \sum_{j \neq i} \left\{ w_i \frac{1}{p_i} \frac{\partial p_i}{\partial t} - w_j \frac{1}{p_j} \frac{\partial p_j}{\partial t} \right\} > \frac{s_i}{1 - s_i} \sum_{j \neq i} \left\{ q_j^d \sum_{n \neq j,i} \left[\left(\frac{\varepsilon_{d_j}^{p_j}}{F - 1} \right) \frac{1}{p_j} \frac{\partial p_j}{\partial t} - \left(\frac{\varepsilon_{d_j}^{p_n}}{(-1)} \right) \frac{1}{p_n} \frac{\partial p_n}{\partial t} \right] \right\}$$

which, it is not hard to imagine is not particularly difficult to satisfy with respect to what is required of consumer demand for this to be the case. If sufficiently many growth-seeking firms are satisfying this condition, then the general dynamics of prices across the market will be downward over time even if there is not a uniform price across the market.

Even from this rudimentary analysis we can see that "everything goes", but with the caveat that "everything" can only be achieved by a certain mixes of assumptions - some which will be far stronger in the light of historical data than others. Hence the mathematics of the evolutionary market here can capture all the chaos of the world, but also place some order on what generates that chaos. Obviously, the dynamics of the distribution are complex, and very much a candidate for agent based simulations, though there is probably much that can be said analytically without recourse to such "black box" and less transparent methods. There is such a universe of possible dynamics for the distribution of prices that even beginning to investigate them justifies a great deal of further research.

5 The growth of the market - some macro concerns

Before concluding, I will point out some accounting which allows us to aggregate the "within-market" dynamics laid out above into growth at the market level. We already have a means in the price distribution for characterising prices and their dynamics across the market, but using some general evolutionary mathematics we can also characterise in a manner of speaking the growth in output across the whole market system. This general evolutionary mathematics (one of the many "iso-morphic laws" (von Bertalanffy 1950) occurring across science) takes the form of the famed Price equation, named for its creator whom I have already mentioned, which when applied to economic phenomena will give us a decomposition in the growth of average firm size across the market, and in some sense the growth rate of the market as a whole.

$$\frac{\partial E_q(q)}{\partial t} = \frac{1}{E_q(q)}\mathrm{Cov}_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t} \quad q\right) + E_q\left(\frac{\partial q}{\partial t}\right)$$

A derivation of this equation using purely economic intuition can be found in Appendix A. What it says is that across any given period of time ∂t, the growth in average firm size across the market is directly proportional to the covariance of firm growth rates over that time with their sizes plus the average growth rate of firms when weighted for firm size. Immediately, it demonstrates that if there is only one completely dominant firm in the market so that $\exists i : s_i = 1$, we have that $E_q(q) = q_i$, $E_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t}\right) = \frac{\partial t}{q_i}\frac{\partial q_i}{\partial t}$ and $q_j = 0 \,\forall\, j \neq i$ so that $\mathrm{Cov}_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t} \quad q\right) = 0$ and

$$\frac{\partial E_q(q)}{\partial t} = E_q\left(\frac{\partial q}{\partial t}\right) = \frac{\partial q_i}{\partial t}$$

This is fairly trivial. Of course if there is one firm in the market any growth of average market size will be equivalent to this firms' growth. In fact, the same result obtains if all firms are completely homogeneous, i merely becomes redefined as the representative firm amongst them. What is less trivial is what occurs compared to this basic scenario once other firms enter the market (that is $\exists j \neq i : q_j > 0$), for if this is the case, and these firms are not identical, then $\mathrm{Cov}_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t} \quad q\right) \neq 0$. Now, if firms are seeking growth in the market, and trying to compete on the basis of prices, we know from Eq. 3.3 for firm growth and the cross-partial derivative (3.6) that the correlation (covariance) between firm growth and firm sizes is in all likelihood positive. This is very interesting for

$$\mathrm{Cov}_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t} \quad q\right) > 0 \implies \frac{\partial E_q(q)}{\partial t} > E_q\left(\frac{\partial q}{\partial t}\right)$$

That is, firms seeking to compete on prices, leading to the covariance between their growth and their size being positive, means that growth in average size across the market is actually greater than average growth rates, and so easily in many scenarios greater than if we had a totally homogeneous or monopolistic market. In any case, growth in the market (which could be said to be reflected in the growth of average firm size) will be directly correlated to the covariance of firm sizes and firm growth rates

$$\frac{\partial E_q(q)}{\partial t} \propto \mathrm{Cov}_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t} \quad q\right)$$

That this growth in the average output across the market depends on heterogeneous firms being sufficiently competitive with respect to price mathematically demonstrates a key insight of economics since the times of Adam Smith in addition to the general evolutionary principle that growth is correlated with variety.[24] It demonstrates that competition between firms, the struggle to survive and grow through price and innovation strategies, because it serves to bring a positive correlation between firm size and firm growth, is a driver of economic growth. It is true that if a monopoly grows, the markets size will also grow, but when firms are competing on the basis of price, this growth will be less than if the market were more competitive. The insight of the Price equation could be effectively summarised by the following statement: *growth in output depends positively on the covariance between firm growth rates and firm size, and this in turn depends positively upon the competitive efforts of firms reflected in their price and innovation strategies*. Or even simpler, the greater the efforts exerted in the struggle to out-compete rivals, the greater the overall growth of the market. Prices do more than just alert consumers and firms (in theory) to the value of the objects they are purchasing and thereby co-ordinate their actions (the role emphasised by Hayek (1945) and the famous *I Pencil* example of Read (1958)). By being one of the core platforms on which firms compete, they co-ordinate firms' efforts to outdo each other into economic growth.

6 What is the point of all this?

Let me restate the four major equations arrived at in this paper by way of summary. First we have equations for each firm i in the market which governs its growth rate, and which together characterise the dynamics of the market

$$\frac{\partial q_i}{\partial t} = \frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + q_i^d \sum_{j \neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] + \sum_{j=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t}$$

[24] Fisher's notoriously opaque "fundamental theorem" of natural selection - another core principle within evolutionary mathematics - says that in the absence of other confounding factors growth in average fitness is equal to the variance of fitness (Price 1972b).

second, we have two summary statistics for the distribution of prices across the market with respect to their incidence

$$E_q(p) = \frac{\sum_{i=1}^{F} q_i p_i}{\sum_{j=1}^{F} q_j}$$

$$\mathrm{Var}_q(p) = \frac{\sum_{i=1}^{F} q_i \left[p_i - E_q(p)\right]^2}{\sum_{j=1}^{F} q_j}$$

and finally, the Price equation representing the growth of the average size in the market, and thus in a sense the general growth of the market

$$\frac{\partial E_q(q)}{\partial t} = \frac{1}{E_q(q)} \mathrm{Cov}_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t} \quad q\right) + E_q\left(\frac{\partial q}{\partial t}\right)$$

The point of these equations is to relate three behaviours together; the behaviour of consumers (reflected in the demand function derivatives), and the pricing and innovation behaviour of firms. I have provided some results on the market dynamics implied by this model at a highly general and abstract level. This may be interesting *a priori*, and there are many *a priori* theoretical results to be derived with respect to this model which I have not delved into here or even foreseen. The model itself can and should at some point be extended to incorporate change in consumer tastes. But the real test of the usefulness of this model is whether or not it helps to give us a common grammar for talking about economic evolution and a logic through which to run our intuitions about the reasons why particular economic phenomena have occurred at particular points in history. It is only as useful then as it is when used for what Nelson and Winter (1982) called "appreciative theory", using formal, abstract, general theory to help fully appreciate and therefore understand the economic events of a particular interval in history.

I believe that this model goes some way toward providing the common grammar and logic for this type of theory. I have showed that it nests, and in fact somewhat generalises insofar as it does not coincide exactly in functional form with, many theoretical ideas within evolutionary economics such as the role of replicator dynamics in market processes, diffusion and the associated notion of pressure to innovate especially as the market tends toward capacity. It confirms theoretically and reflects mathematically a number of interesting insights economics and business studies has to offer with respect to the theory of firm survival and growth in the market while also placing these within a single model. But also, it provides us with a means of placing firm pricing strategies (and the production processes which they represent) and innovation strategies (and the investment processes they represent) within an evolutionary market context to provide a means by which we can understand the evolution of prices. I foresee that many interesting theoretical studies of particular pricing and innovation strategies can be arrived at by running the mathematical formulation of these strategies through the model above.

Finally, the model also confirms two extremely important insights into the nature of the process of evolution in the economy. The economy as an evolutionary system is one where competition - the "law of the jungle" - reigns supreme. It is not enough for a firm to have a reasonable price for a fairly good quality output. Under almost any market demand conditions, this will not save a firm from non-survival. The economy is a system where survival and growth depend on being better than rivals, and given that they will be also trying to outperform their rivals, survival, let alone growth can easily become a continuous and rather brutal struggle. But, and this is the second important insight, this process is not entirely zero-sum, for competition (under certain conditions) is what drives greater growth of the market overall. Without variety and competition amongst the various different firms which make up that variety, growth can occur, but it will be very likely to be the weaker for the absence of the struggle to survive and thrive.

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Appendix A: An intuitive derivation of the Price equation for economics

One problem with applications of the Price equation to evolutionary economics is that the economic intuition behind it can become difficult to follow. Here I provide a re-derivation (based upon Price (1972a)) of the Price equation using only the terminology of economics. The Price equation serves to decompose the growth rate in average fitness across a population, here this means a decomposition of the growth across the population of the average size of firms. We will begin with the change over one "time period"

$$E^{t+1}_{q(t+1)}(q) - E^{t}_{q(t)}(q) = \frac{\sum_{i=1}^{F} q_i(t+1)\, q_i(t+1)}{\sum_{j=1}^{F} q_j(t+1)} - \frac{\sum_{i=1}^{F} q_i(t)\, q_i(t)}{\sum_{j=1}^{F} q_j(t)} \quad (6.1)$$

it is important to note that the average here is weighted with respect to the $\{q_i(t)\}_{i=1}$, much the same as the distribution of prices, which may seem unusual (weighting firm sizes with firm sizes), though it is less unusual once one grasps that the "second", or "weighting" firm size included in each term of the above expression is being normalised *relative* to the size of the market - in a sense, but absolutely not exactly being "cancelled out". Now, we can define what is often called a "selection coefficient" z_i for each individual firm within the market Price (1972a, p.486) which reflects the growth rate of the firms within the market[25]

$$z_i(t) = \frac{q_i(t+1)}{q_i(t)} = 1 + \frac{\partial q_i(t)}{q_i(t)} \tag{6.2}$$

Now, using this selection coefficient to expand the average size of firms within the market at time $t+1$, and using the fact that $q_i(t+1) = q_i(t) + \Delta q_i(t)$ we obtain

$$\begin{aligned} E^{t+1}_{q(t+1)}(q) &= \frac{1}{\sum_{j=1}^{F} q_i(t) z_i(t)} \sum_{i=1}^{F} q_i(t) z_i(t) q_i(t+1) \\ &= \frac{1}{\sum_{j=1}^{F} q_i(t) z_i(t)} \sum_{i=1}^{F} q_i(t) z_i(t) \left(q_i(t) + \Delta q_i(t)\right) \end{aligned} \tag{6.3}$$

Expanding this last equality and re-substituting for $q(t+1)$ we get

$$E^{t+1}_{q(t+1)}(q) = \frac{1}{\sum\limits_{j=1}^{F} q_i(t) z_i(t)} \sum_{i=1}^{F} q_i(t) z_i(t) q_i(t) + \frac{1}{\sum_{j=1}^{F} q_i(t+1)} \sum_{i=1}^{F} q_i(t+1) \Delta q_i(t) \tag{6.4}$$

Now employing some further mathematical tautologies to manipulate the first term of this expression gives us the following

$$\begin{aligned} E^{t+1}_{q(t+1)}(q) = {} & \frac{1}{\sum_{j=1}^{F} q_i(t) z_i(t)} \frac{\sum_{j=1}^{F} q_i(t)}{\sum_{j=1}^{F} q_i(t)} \sum_{i=1}^{F} q_i(t) \left[z_i(t) - E^t_{q(t)}(z)\right] \left[q_i(t) - E^t_{q(t)}(q)\right] \\ & + \frac{1}{\sum_{j=1}^{F} q_i(t+1)} \sum_{i=1}^{F} q_i(t+1) \Delta q_i(t) \\ & + \frac{1}{\sum_{j=1}^{F} q_i(t) z_i(t)} \frac{\sum_{j=1}^{F} q_i(t)}{\sum_{j=1}^{F} q_i(t)} \left[\sum_{i=1}^{F} q_i(t) z_i(t) E^t_{q(t)}(q) \right. \\ & \left. + \sum_{i=1}^{F} q_i(t) \left[q_i(t) - E^t_{q(t)}(q)\right] E^t_{q(t)}(z) \right] \end{aligned} \tag{6.5}$$

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[25] In the original evolutionary biology interpretation, these coefficients reflect the ability of organisms with a certain trait to reproduce (Price 1970).

Notice that when we expand this last term out carefully it collapses to $E^{t}_{q(t)}(q)$. We can now recognise that this equation is reducible using the quotidian statistical concepts of covariance and expectation

$$E^{t+1}_{q(t)}(q) = \frac{1}{E^{t}_{q(t)}(z)}\mathrm{Cov}^{t}_{q(t)}(z \quad q) + E^{t}_{q(t+1)}(\Delta q(t)) + E^{t}_{q(t)}(q) \tag{6.6}$$

Now subtracting $E^{t}_{q(t)}(q)$ from both sides gives us

$$E^{t+1}_{q(t)}(q) - E^{t}_{q(t)}(q) = \frac{1}{E^{t}_{q(t)}(z)}\mathrm{Cov}^{t}_{q(t)}(z \quad q) + E^{t}_{q(t+1)}(\Delta q(t)) \tag{6.7}$$

Which can be further reduced to the elegant Price equation by generalising to an arbitrary time period and (for the sake of it) assuming differentiability in time

$$\frac{\partial E_q(q)}{\partial t} = \frac{1}{E_q(q)}\mathrm{Cov}_q(z \quad q) + E_q\left(\frac{\partial q}{\partial t}\right) \tag{6.8}$$

Now, inputting $z_i = 1 + \frac{1}{q}\frac{\partial q}{\partial t}\partial t$ into the definition of covariance gives us

$$\frac{\partial E_q(q)}{\partial t} = \frac{1}{E_q(q)}\mathrm{Cov}_q\left(\frac{\partial t}{q}\frac{\partial q}{\partial t} \quad q\right) + E_q\left(\frac{\partial q}{\partial t}\right)$$

Appendix B: Proofs of theorems

Proof of Theorem 1

Proof If it is the case that $p_j = 0\,\forall\, j$ then by the definition of demand elasticity $\varepsilon^{p_j}_{d_i} = \frac{p_j}{q^d_i}\frac{\partial q^d_i}{\partial p_j} = 0\,\forall\, j$. This implies that $\left(\frac{\varepsilon^{p_i}_{d_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} = 0$ and $\left(\frac{\varepsilon^{p_j}_{d_i}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t} = 0\,\forall\, j \neq i$, since by induction $0 \times \lim_{x\to\infty} x = 0$. Inserting this into Eq. 3.3 yields the result. Similarly, we need only input directly the assumption $\partial p_j/\partial t = 0$, $\forall\, j \in \{1,\ldots,F\}$ into Eq. 3.3 to obtain the result. □

Proof of Theorem 2

Proof First take Eq. 3.3 and input the assumptions of normal and substitutable goods, a zero population growth rate and unchanging product attributes. Then the growth rate of the firm can be expressed as

$$\frac{\partial q_i}{\partial t} = q^d_i \sum_{j\neq i}\left[\left(\frac{\varepsilon^{p_i}_{d_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon^{p_j}_{d_i}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] \tag{6.9}$$

Now, it is a mere fact of logic that any output bought must have been sold, so $q^d_i = q_i$. But note that $q_i = 0 \implies \frac{\partial q_i}{\partial t} \neq 0 \Leftrightarrow$

$\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right]\neq 0$. Since the laws of physics decree that $q_i\geq 0$ therefore, it is the case that

$$\frac{\partial q_i}{\partial t}>0\Leftrightarrow\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right]>0 \tag{6.10}$$

Now suppose that we can partition the set of firms $\{1,\ldots,F\}$ into two sets B and $\neg B$ such that $\forall\, j\in B$ we have $\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right]>0$. Then condition (6.10) holds if and only if

$$\sum_{j\in B}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right]>\sum_{j\notin B}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] \tag{6.11}$$

□

Proof of Theorem 3

Proof This condition consists of a simple rearrangement of Eq. 3.3 when restricted to be greater than zero, so

$$\frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t}+q_i^d\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right]+\sum_{j=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial\alpha_j^k}\frac{\partial\alpha_j^k}{\partial t}>0 \tag{6.12}$$

when rearranged gives us

$$\begin{aligned}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial\alpha_i^k}\frac{\partial\alpha_i^k}{\partial t}&>q_i^d\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}-\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}\right]\\&-\sum_{j\neq i}^{F}\sum_{k=1}^{N_A}\left[\frac{\partial q_i^d}{\partial\alpha_j^k}\frac{\partial\alpha_j^k}{\partial t}\right]-\frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t}\end{aligned} \tag{6.13}$$

□

Proof of Theorem 4

Proof The market share of firm i with respect to output is given by $s_i=\frac{q_i}{\sum_{j=1}^{F}q_j}$, and its dynamics can be readily obtained by simple application of the quotient rule

$$\frac{\partial s_i}{\partial t}=\frac{\frac{\partial q_i}{\partial t}\sum_{j=1}^{F}q_j-q_i\frac{\partial\sum_{j=1}^{F}q_j}{\partial t}}{\left(\sum_{j=1}^{F}q_j\right)^2} \tag{6.14}$$

It is fairly straightforward then to confirm that market share only grows if the firm grows at a faster rate than output across the market

$$\frac{\partial s_i}{\partial t} > 0 \Leftrightarrow \frac{1}{q_i}\frac{\partial q_i}{\partial t} > \frac{1}{\sum_{j=1}^{F} q_j}\frac{\partial \sum_{j=1}^{F} q_j}{\partial t} \tag{6.15}$$

using the additive law of differential calculus we can expand this into a slightly more tractable (in view of the model above) expression

$$\frac{\partial q_i}{\partial t} > \frac{s_i}{1-s_i}\sum_{j\neq i}\frac{\partial q_j}{\partial t} \tag{6.16}$$

Now substituting in Eq. 3.3 for the growth rate of firm size we obtain

$$\begin{aligned}
&\frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + q_i^d \sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right] + \sum_{j=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t}\\
&\quad > \frac{s_i}{1-s_i}\sum_{j\neq i}\left\{\frac{\partial q_j^d}{\partial N}\frac{\partial N}{\partial t} + q_j^d\sum_{n\neq j}\left[\left(\frac{\varepsilon_{d_j}^{p_j}}{F-1}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t} - \left(\frac{\varepsilon_{d_j}^{p_n}}{(-1)}\right)\frac{1}{p_n}\frac{\partial p_n}{\partial t}\right]\right.\\
&\quad\left. + \sum_{n=1}^{F}\sum_{k=1}^{N_A}\frac{\partial q_j^d}{\partial \alpha_n^k}\frac{\partial \alpha_n^k}{\partial t}\right\}
\end{aligned} \tag{6.17}$$

We can reduce this inequation by isolating behavioural and firm strategy terms which refer to firm i as follows

$$\begin{aligned}
&\frac{\partial q_i^d}{\partial N}\frac{\partial N}{\partial t} + \frac{s_i}{1-s_i}\sum_{j\neq i}\frac{\partial q_j^d}{\partial N}\frac{\partial N}{\partial t}\\
&\quad + q_i^d\sum_{j\neq i}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t} - \left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}\right]\\
&\quad - \sum_{j\neq i}\frac{s_i}{1-s_i}q_j^d\left[\left(\frac{\varepsilon_{d_j}^{p_j}}{F-1}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t} - \left(\frac{\varepsilon_{d_j}^{p_i}}{(-1)}\right)\frac{1}{p_i}\frac{\partial p_i}{\partial t}\right]\\
&\quad - \sum_{j\neq i}\frac{s_i}{1-s_i}\left\{q_j^d\sum_{n\neq j,i}\left[\left(\frac{\varepsilon_{d_j}^{p_j}}{F-1}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t} - \left(\frac{\varepsilon_{d_j}^{p_n}}{(-1)}\right)\frac{1}{p_n}\frac{\partial p_n}{\partial t}\right]\right\}\\
&\quad + \sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_i^k}\frac{\partial \alpha_i^k}{\partial t} + \sum_{j\neq i}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial t} - \frac{s_i}{1-s_i}\sum_{j\neq i}\sum_{k=1}^{N_A}\frac{\partial q_j^d}{\partial \alpha_i^k}\frac{\partial \alpha_i^k}{\partial t}\\
&\quad - \frac{s_i}{1-s_i}\sum_{j\neq i}\sum_{n\neq i}\sum_{k=1}^{N_A}\frac{\partial q_j^d}{\partial \alpha_n^k}\frac{\partial \alpha_n^k}{\partial t} > 0
\end{aligned} \tag{6.18}$$

if we multiply and divide every term in $\sum_{j\neq i}^{F}\sum_{k=1}^{N_A}\frac{\partial q_i^d}{\partial \alpha_i^k}\frac{\partial \alpha_i^k}{\partial t}$ by $\frac{\partial \alpha_i^k}{\partial \alpha_i^k}$ we can by grouping like terms obtain

$$\begin{aligned}
&\frac{\partial N}{\partial t}\left[\frac{\partial q_i^d}{\partial N}-\frac{s_i}{1-s_i}\sum_{j\neq i}\frac{\partial q_j^d}{\partial N}\right]\\
&+q_i^d\sum_{j\neq i}\left\{\frac{1}{p_i}\frac{\partial p_i}{\partial t}\left[\left(\frac{\varepsilon_{d_i}^{p_i}}{F-1}\right)+\frac{s_i}{1-s_i}\frac{q_j^d}{q_i^d}\left(\frac{\varepsilon_{d_j}^{p_i}}{(-1)}\right)\right]-\frac{1}{p_j}\frac{\partial p_j}{\partial t}\left[\left(\frac{\varepsilon_{d_i}^{p_j}}{(-1)}\right)-\frac{s_i}{1-s_i}\frac{q_j^d}{q_i^d}\left(\frac{\varepsilon_{d_j}^{p_j}}{F-1}\right)\right]\right\}\\
&-\frac{s_i}{1-s_i}\sum_{j\neq i}\left\{q_j^d\sum_{n\neq j,i}\left[\left(\frac{\varepsilon_{d_j}^{p_j}}{F-1}\right)\frac{1}{p_j}\frac{\partial p_j}{\partial t}-\left(\frac{\varepsilon_{d_j}^{p_n}}{(-1)}\right)\frac{1}{p_n}\frac{\partial p_n}{\partial t}\right]\right\}\\
&+\sum_{k=1}^{N_A}\frac{\partial \alpha_i^k}{\partial t}\left[\frac{\partial q_i^d}{\partial \alpha_i^k}-\frac{s_i}{1-s_i}\sum_{j\neq i}\left(\sum_{k=1}^{N_A}\frac{\partial q_j^d}{\partial \alpha_i^k}-\frac{\partial q_i^d}{\partial \alpha_j^k}\frac{\partial \alpha_j^k}{\partial \alpha_i^k}\right)\right]-\frac{s_i}{1-s_i}\sum_{j\neq i}\left\{\sum_{n\neq i}\sum_{k=1}^{N_A}\frac{\partial q_j^d}{\partial \alpha_n^k}\frac{\partial \alpha_n^k}{\partial t}\right\}>0
\end{aligned}\tag{6.19}$$

Which when we rearrange to isolate on the left hand side any market forces over which firm i has influence gives us our result. □

Proof of Lemma 1

Proof This proof is rather trivial and well known to mathematicians and statisticians, but will be included for the sake of rigour. Firstly, if the price distribution is degenerate then

$$E_q\left(p\right)=\frac{\sum_{i=1}^{F}q_i p_i}{\sum_{j=1}^{F}q_j}=\frac{\sum_{i\in\Xi(p)}q_i p_i}{\sum_{i\in\Xi(p)}q_j}\tag{6.20}$$

and by the definition of $\Xi\left(p\right)$, $p_i=p\,\forall\, i\in\Xi\left(p\right)$ and so $\forall\, i\in\Xi\left(p\right),\ E_q\left(p\right)=p_i$, while by the definition of degeneracy this must be true for any $i\,:\,q_i>0$. The zero variance of the distribution follows immediately by inputting $p_i=E_q\left(p\right)\,\forall\, i\,:\,q_i>0$ into the definition of $\mathrm{Var}_q\left(p\right)$. Conversely if it is to be the case that $p_i=E_q\left(p\right)\,\forall\, i\,:\,q_i>0$, and $\mathrm{Var}_q\left(p\right)=0$ then by the definition of the first and second moments it must be the case that $\exists\,\{i\}\subset\{1,\ldots,F\}\,:\,p_i=E_q\left(p\right)\,\forall\, i\in\{i\}$ and for any i not in this set $q_i=0$, which is the definition of degeneracy. □

Proof of Theorem 5

Proof In order for a uniform price to prevail across the market the price distribution must be degenerate by Lemma 1. But for the price distribution to be degenerate it can seen that we will need the dynamics of the market to lead to convergence to a situation in which $\sum_{i \in \Xi(p)} s_i = 1$, since

$$\sum_{i \in \Xi(E_q(p))} s_i = 1 \iff \sum_{i \in \Xi(E_q(p))} \frac{q_i}{\sum_{j=1}^{F} q_j} = \frac{1}{\sum_{j=1}^{F} q_j} \sum_{i \in \Xi(E_q(p))} q_i = 1 \iff \sum_{i \in \Xi(E_q(p))} q_i = \sum_{j=1}^{F} q_j$$

But given that $\Xi\left(E_q(p)\right) \subset \{1, \ldots, F\}$, for each q_i on the left hand side there is a corresponding q_i on the right hand side, and so for this to be the case requires that $q_i = 0 \, \forall \, i \notin \Xi\left(E_q(p)\right)$, which is the key to the distribution being degenerate by the definition referred to in Lemma 1. So saying that $\sum_{i \in \Xi(p)} s_i = 1$ is identical to saying that the price distribution is degenerate. Hence if it is to be the case that in the market there is convergence to a situation in which $\sum_{i \in \Xi(E_q(p))} s_i = 1$, it must also be the case by definition of market share that there is a convergence to a situation in which $s_i = 0 \, \forall \, i \notin \Xi\left(E_q(p)\right)$. But then for it to be the case that $\lim_{t \to \infty} s_i = 0 \, \forall \, i \notin \Xi\left(E_q(p)\right)$, for each and every $i \notin \Xi\left(E_q(p)\right)$ from an initial market share of s_i we must have

$$s_i + \lim_{T \to \infty} \int_t^T \frac{\partial s_i}{\partial t} \partial t = 0$$

$$\implies s_i = -\lim_{T \to \infty} \int_t^T \frac{\partial s_i}{\partial t} \partial t$$

It should be fairly clear that this will hold if and only if there are a sufficient number of time periods for which $\frac{\partial s_i}{\partial t} < 0$ so that $\lim_{T \to \infty} \int_t^T \frac{\partial s_i}{\partial t} \partial t < 0$. That is, there must exist a set of time periods $T_{i \notin \Xi(p)} \subset \mathbb{R}_+$ for each and every $i \notin \Xi\left(E_q(p)\right)$ such that

$$\frac{\partial s_i}{\partial t} < 0 \, \forall \, t \in T_{i \notin \Xi(E_q(p))}$$

that is, for these periods Theorem 4 are violated for each firm i, and

$$s_i = -\lim_{T \to \infty} \left\{ \int_{t \in T_{i \notin \Xi(E_q(p))}} \frac{\partial s_i}{\partial t} \partial t - \int_{t \notin T_{i \notin \Xi(E_q(p))}} \frac{\partial s_i}{\partial t} \partial t \right\}$$

□

Proof of Corollary 1

Proof This is simply the reverse argument of Theorem 5 for the emergence of a dominant firm rather than the non-survival of those which do not conform the uniform price. □

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